

# Solutions to Bain and Engelhardt's Introduction to Probability and Mathematical Statistics

## 06.01

**Given: the pdf of x**  $f_x(x) = \begin{cases} 4x^3 & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases}$

**Find: PDF of a)  $Y = X^4$**

**Setup:** Use the CDF technique to get the CDF of Y in terms of a CDF of X  
 $F_Y(y) = P[Y \leq y] = P[X^4 \leq Y] = P[-y^{\frac{1}{4}} \leq X \leq y^{\frac{1}{4}}] = F_X(y^{\frac{1}{4}}) - F_X(-y^{\frac{1}{4}})$

**Steps: i)** Differentiate with respect to y to find an equation given in terms of the pdf of x:  
 $f_y(y) = \frac{d}{dy}F_X(y^{\frac{1}{4}}) - \frac{d}{dy}F_X(-y^{\frac{1}{4}}) = f_x(y^{\frac{1}{4}})\frac{d}{dy}y^{\frac{1}{4}} - f_x(-y^{\frac{1}{4}})\frac{d}{dy}(-y^{\frac{1}{4}}) = f_x(y^{\frac{1}{4}})\frac{y^{-\frac{3}{4}}}{4} - f_x(-y^{\frac{1}{4}})\frac{-y^{-\frac{3}{4}}}{4}$

**ii)** Plug in the original limits and function for the pdf of x, and compute the cdf for y

**Result:**  $f_y(y) = \begin{cases} 4y^{\frac{3}{4}}\frac{1}{4y^{\frac{3}{4}}} & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases} = \begin{cases} 1 & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases}$

**Find: PDF of b)  $W = e^X$**

**Setup:** Use the CDF technique to get the CDF of W in terms of a CDF of X  
 $F_W(w) = P[W \leq w] = P[e^X \leq W] = P[X \leq \ln W] = F_X(\ln W)$

**Steps: i)** Differentiate with respect to w to find an equation given in terms of the pdf of x:  
 $f_w(w) = \frac{d}{dw}F_X(\ln W)\frac{d}{dw}(\ln w) = f_x(\ln w)\frac{1}{w}$

**ii)** Plug in the original limits and function for the pdf of x, and compute the cdf for y

**Result:**  $f_w(w) = \begin{cases} \frac{4(\ln w)^3}{w} & , \quad 1 < w < e \\ 0 & , \quad o/w \end{cases}$

**Find: PDF of c)  $Z = \ln x$**

**Setup:** Use the CDF technique to get the CDF of Z in terms of a CDF of X  
 $F_Z(z) = P[Z \leq z] = P[\ln x \leq z] = P[X \leq e^z] = F_X(e^z)$

**Steps: i)** Differentiate with respect to z to find an equation given in terms of the pdf of x:  
 $f_z(z) = \frac{d}{dz}F_X(e^z) = f_x(e^z)\frac{de^z}{dz}$

ii) Plug in the original limits and function for the pdf of x, and compute the cdf for y

**Result:**  $f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \leq z < 0 \\ 0 & , \quad o/w \end{cases}$

**Find: PDF of d)  $U = (X - 0.5)^2$**

**Setup:** Use the CDF technique to get the CDF of U in terms of a CDF of X  
 $F_U(u) = P[U \leq u] = P[(X - 0.5)^2 \leq u] = P[|X - 0.5| \leq u^{0.5}] = F_X(u^{1/2} + 1/2) - F_X(-u^{1/2} + 1/2)$

**Steps: i)** Differentiate with respect to u to find an equation given in terms of the pdf of x:  
 $f_U(u) = \frac{d}{du} F_X(u^{1/2} + 1/2) = f_x(u^{1/2} + 1/2) \frac{d}{du} (u^{1/2} + 1/2) - f_x(-u^{1/2} + 1/2) \frac{d}{du} (-u^{1/2} + 1/2)$   
 $f_x(u^{1/2} + 1/2) 1/2 u^{-1/2} - f_x(-u^{1/2} + 1/2) 1/2 u^{-1/2}$

ii) INCOMPLETE

**Result:**  $f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \leq z < 0 \\ 0 & , \quad o/w \end{cases}$

06.02

**Given:**  $X \sim Unif(0, 1)$

**Find: a) PDF of  $Y = X^{1/4}$**

**Setup:**  $F_Y(y) = P[Y \leq y] = P[X^{1/4} \leq y] = P[X \leq y^4] = F_X(y^4)$

**Steps: i)** find the pdf of x. Because X is a Uniform distribution with parameters 1 and 0, the pdf, which for Unif(a,b) is  $1/(b-a)$  where  $a < x < b$ . Here, Unif(0,1) gives  $1/1-0 = 1$

ii) Differentiate with respect to y to find an equation given in terms of the pdf of x:  
 $f_Y(y) = \frac{d}{dy} F_X(y^4) = 4y^3$

**Result:**  $f_Y(y) = \begin{cases} 4y^3 & , \quad 0 < y < 1 \\ 0 & , \quad o/w \end{cases}$

**Find: b) PDF of  $W = e^{-X}$**

**Setup:**  $F_W(z) = P[W \leq w] = P[e^{-X} \leq w] = P[-X \leq \ln w] = P[X \geq -\ln w] = 1 - F_x(-\ln w)$

**Steps: i)** find the pdf of x. See part a) for an explanation of why it is 1 when  $a < x < b$

ii) Differentiate with respect to  $w$  to find an equation given in terms of the pdf of  $x$ :  
 $f_W(w) = -\frac{d}{dw}F_X\frac{d}{dw}(-\ln w) = -f_X(-\ln w)\frac{-1}{w}$  for  $e^{-1} < w < 1 = -\frac{1}{w}$

**Result:**  $f_W(w) = \begin{cases} \frac{1}{w} & , \quad e^{-1} < w < 1 \\ 0 & , \quad o/w \end{cases}$

**Find: c) PDF of  $Z = 1 - e^{-X}$**

**Setup:**  $F_Z(z) = P[Z \leq z] = P[1 - e^{-X} \leq z] = P[-e^{-X} \leq z - 1] = P[e^{-X} \geq 1 - z] = P[-X \geq \ln(1 - z)] = P[X \leq -\ln(1 - z)] = F_x(-\ln(1 - z))$

**Steps: i)** find the pdf of  $x$ . See part a) for an explanation of why it is 1 when  $a < x < b$

ii) Differentiate with respect to  $w$  to find an equation given in terms of the pdf of  $x$ :  
 $f_Z(z) = -\ln(1 - z) = -\frac{-1}{1-z} = \frac{1}{1-z}$  for  $0 < z < e^{-1}$

**Result:**  $f_W(w) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < e^{-1} \\ 0 & , \quad o/w \end{cases}$

**Find: d) PDF of  $U = X(1 - X)$**

**Setup:**  $F_U(u) = P[U \leq u] = P[X(1 - x) \leq u] = P[-X^2 + X \leq u] = P[-(X - 1/2)^2 \leq u - 1/4] = P[(X - 1/2)^2 \geq 1/4 - u] = P[|(X - 1/2)| \geq (1/4 - u)^{1/2}] =$

**Steps: i)** find the pdf of  $x$ . See part a) for an explanation of why it is 1 when  $a < x < b$

ii) INCOMPLETE:

$f_Z(z) = -\ln(1 - z) = -\frac{-1}{1-z} = \frac{1}{1-z}$  for  $0 < z < e^{-1}$

**Result:**  $f_W(w) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < e^{-1} \\ 0 & , \quad o/w \end{cases}$

06.03

**Given: PDF**  $f_R(r) = \begin{cases} 6r(1 - r) & , \quad 0 < r < 1 \\ 0 & , \quad o/w \end{cases}$

**Find: Distribution of the circumference**

**Setup:** The circumference is  $c = 2\pi r$  - we have the pdf in terms of  $x$ , so this is the transformation

$F_C(c) = P[C \leq c] = P[2\pi r \leq c] = P[r \leq c/2\pi] = F_x(c/2\pi)$

**Steps: i)** Differentiate with respect to  $c$  to find an equation given in terms of the pdf of  $x$ .  
 $f_C(c) = \frac{d}{dc} F_R(c/2\pi) = f_R(c/2\pi) \frac{d}{dc}(c/2\pi) = f_R(c/2\pi)(1/2\pi)$

**ii)** Plug the original pdf back into this new form:

$$f_C(c) = \frac{6c}{2\pi}(1 - (c/2\pi))(1/2\pi) = \frac{6c(2\pi-c)}{2\pi^3} \quad \text{if } 0 < c < 2\pi$$

**Result:** 
$$f_C(c) = \begin{cases} \frac{6c(2\pi-c)}{2\pi^3} & , \quad 0 < c < 2\pi \\ 0 & , \quad o/w \end{cases}$$

## Find: Distribution of the area

**Setup:** The area is  $a = \pi r^2$  so the cdf  $F_A(a) = P[A \leq a] = P[\pi r^2 \leq a] = P[r^2 \leq a/\pi] = P[|r| \leq (a/\pi)^{1/2}] = P[-(a/\pi)^{1/2} \leq r \leq (a/\pi)^{1/2}] = F_r(a/\pi)^{1/2} - F_r(-(a/\pi)^{1/2})$

**Steps: i)** Differentiate with respect to  $a$  to find an equation in terms of the pdf of  $x$ .

$$f_A(a) = \frac{d}{da} F_R(a/\pi)^{1/2} - \frac{d}{da} F_R(-(a/\pi)^{1/2}) = f_R[(a/\pi)^{1/2}] \frac{d}{da}(a/\pi)^{1/2} - f_R[-(a/\pi)^{1/2}] \frac{d}{da}(-(a/\pi)^{1/2})$$

**Result:** 
$$f_A(a) = \begin{cases} \frac{3(\sqrt{\pi}-\sqrt{a})}{\pi^{3/2}} & , \quad 0 < a < \pi \\ 0 & , \quad o/w \end{cases}$$

06.14

**Given: Joint PDF** 
$$f(x, y) = \begin{cases} 4e^{-2(x+y)} & , \quad 0 < x < \infty, 0 < y < \infty \\ 0 & , \quad o/w \end{cases}$$

**Find: a) CDF of  $W=X+Y$**

**Setup:**  $F_w(w) = P[W \leq w] = P[X + Y \leq w]$

**Steps:**

**i)** Express as a sum of probabilities, replace probabilities with binomials

**ii)** Simplify and Use Combinatorial Identity

**Result:**  $\binom{n+m}{k}$  **06.15** This is a simplified version of example 6.4.5.

$X_1, X_2 \sim POI(\lambda)$  so the MGF of both is  $e^{\lambda(e^t-1)}$ . Thus by theorem 6.4.4

$$M_Y(t) = e^{\lambda(e^t-1)} e^{\lambda(e^t-1)} = e^{2\lambda(e^t-1)} \sim POI(2\lambda)$$

The pdf then of Y is

$$f_Y(y) = \begin{cases} \frac{e^{-2\lambda}(2\lambda)^y}{y!} & y = 0, 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

**06.16** Note: the pdf of  $f_{x_1, x_2} = \frac{1}{x_1^2} \frac{1}{x_2^2}$

a) We need to find  $f_{u,v} = f_{x_1, x_2}(x_1(u, v), x_2(u, v))|J|$  where J is our jacobian. First we let  $u = x_1 x_2$  and  $v = x_1$  thus  $x_1 = v$  and  $x_2 = \frac{u}{v}$ , now we can find J.

$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & 0 \end{vmatrix} = \frac{1}{v}$$

Finally, our pdf is:

$$\begin{aligned} f_{U,V}(u, v) &= f_{x_1, x_2}\left(v, \frac{u}{v}\right) \left| \frac{1}{v} \right| \\ &= \frac{1}{v^2} \frac{1}{\left(\frac{u}{v}\right)^2} \left| \frac{1}{v} \right| \\ &= \frac{1}{u^2 v}, 1 < v < u < \infty \end{aligned}$$

**06.23**

**Given:**  $X_1, X_2, \dots, X_k$  are independent RVs, let  $Y = X_1 + X_2 + \dots + X_k$ , and  $X_i \sim \text{Geo}(p)$

**Find:** The MGF and distribution of Y

**Setup:**  $F_w(w) = P[W \leq w] = P[X + Y \leq w]$

**Steps:**

- i) Express as a sum of probabilities, replace probabilities with binomials
- ii) Simplify and Use Combinatorial Identity

**Result:**  $\binom{n+m}{k}$

**06.29**

**Given: PDF**  $f(x) = \begin{cases} \frac{1}{x^2} & , \quad 1 \leq x < \infty, 0 < y < \infty \\ 0 & , \quad o/w \end{cases}$

**Find: a) Joint PDF of the order statistics**

**Setup:**  $F_w(w) = P[W \leq w] = P[X + Y \leq w]$

**Steps: i)** Differentiate with respect to  $a$  to find an equation in terms of the pdf of  $x$ .  
 $f_A(a) = \frac{d}{da} F_R(a/\pi)^{1/2} - \frac{d}{da} F_R - (a/\pi)^{1/2} = f_R[(a/\pi)^{1/2}] \frac{d}{da} (a/\pi)^{1/2} f_R[-(a/\pi)^{1/2}] \frac{d}{da} - (a/\pi)^{1/2}$

**ii)** Simplify and Use Combinatorial Identity

**Result:**  $\binom{n+m}{k}$

**Find: b) PDF of the smallest order statistic  $Y_1$**

**Setup:**

**Steps: i)**

**Result:**

**Find: c) PDF of the largest order statistic  $Y_n$**

**Setup:**

**Steps: i)**

**Result:**

**Find: d) PDF of the sample range  $R = Y_n - Y_1$ , for  $n = 2$**

**Setup:** The area is  $a = \pi r^2$  so the cdf  $F_A(a) = P[A \leq a] = P[\pi r^2 \leq a] = P[r^2 \leq a/\pi]$   
 $= P[|r| \leq (a/\pi)^{1/2}] = P[-(a/\pi)^{1/2} \leq c \leq (a/\pi)^{1/2}] = F_r(a/\pi)^{1/2} - F_r-(a/\pi)^{1/2}$

**Steps: i)**

**Result:**

**Find: e) PDF of the sample median  $R = Y_r - Y_1$ , for  $n$  odd so that  $r = (n+1)/2$**

**Setup:**

**Steps: i)**

**Result:**