

Solutions to Bain and Engelhardt's Introduction to Probability and Mathematical Statistics

06.01

Given: the pdf of x $f_x(x) = \begin{cases} 4x^3 & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases}$

Find: PDF of a) $Y = X^4$

Setup: Use the CDF technique to get the CDF of Y in terms of a CDF of X
 $F_Y(y) = P[Y \leq y] = P[X^4 \leq Y] = P[-y^{\frac{1}{4}} \leq X \leq y^{\frac{1}{4}}] = F_X(y^{\frac{1}{4}}) - F_X(-y^{\frac{1}{4}})$

Steps: i) Differentiate with respect to y to find an equation given in terms of the pdf of x:
 $f_y(y) = \frac{d}{dy}F_X(y^{\frac{1}{4}}) - \frac{d}{dy}F_X(-y^{\frac{1}{4}}) = f_x(y^{\frac{1}{4}})\frac{d}{dy}y^{\frac{1}{4}} - f_x(-y^{\frac{1}{4}})\frac{d}{dy}(-y^{\frac{1}{4}}) = f_x(y^{\frac{1}{4}})\frac{y^{-\frac{3}{4}}}{4} - f_x(-y^{\frac{1}{4}})\frac{-y^{-\frac{3}{4}}}{4}$

ii) Plug in the original limits and function for the pdf of x, and compute the cdf for y

Result: $f_y(y) = \begin{cases} 4y^{\frac{3}{4}}\frac{1}{4y^{\frac{3}{4}}} & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases} = \begin{cases} 1 & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases}$

Find: PDF of b) $W = e^X$

Setup: Use the CDF technique to get the CDF of W in terms of a CDF of X
 $F_W(w) = P[W \leq w] = P[e^X \leq W] = P[X \leq \ln W] = F_X(\ln W)$

Steps: i) Differentiate with respect to w to find an equation given in terms of the pdf of x:
 $f_w(w) = \frac{d}{dw}F_X(\ln W)\frac{d}{dw}(\ln w) = f_x(\ln w)\frac{1}{w}$

ii) Plug in the original limits and function for the pdf of x, and compute the cdf for y

Result: $f_w(w) = \begin{cases} \frac{4(\ln w)^3}{w} & , \quad 1 < w < e \\ 0 & , \quad o/w \end{cases}$

Find: PDF of c) $Z = \ln x$

Setup: Use the CDF technique to get the CDF of Z in terms of a CDF of X
 $F_Z(z) = P[Z \leq z] = P[\ln x \leq z] = P[X \leq e^z] = F_X(e^z)$

Steps: i) Differentiate with respect to z to find an equation given in terms of the pdf of x:
 $f_z(z) = \frac{d}{dz}F_X(e^z) = f_x(e^z)\frac{de^z}{dz}$

ii) Plug in the original limits and function for the pdf of x, and compute the cdf for y

Result: $f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \leq z < 0 \\ 0 & , \quad o/w \end{cases}$

Find: PDF of d) $U = (X - 0.5)^2$

Setup: Use the CDF technique to get the CDF of U in terms of a CDF of X
 $F_U(u) = P[U \leq u] = P[(X - 0.5)^2 \leq u] = P[|X - 0.5| \leq u^{0.5}] = F_X(u^{1/2} + 1/2) - F_X(-u^{1/2} + 1/2)$

Steps: i) Differentiate with respect to u to find an equation given in terms of the pdf of x:
 $f_U(u) = \frac{d}{du} F_X(u^{1/2} + 1/2) = f_x(u^{1/2} + 1/2) \frac{d}{du} (u^{1/2} + 1/2) - f_x(-u^{1/2} + 1/2) \frac{d}{du} (-u^{1/2} + 1/2)$
 $f_x(u^{1/2} + 1/2) 1/2 u^{-1/2} - f_x(-u^{1/2} + 1/2) 1/2 u^{-1/2}$

ii) INCOMPLETE

Result: $f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \leq z < 0 \\ 0 & , \quad o/w \end{cases}$

06.02

Given: $X \sim Unif(0, 1)$

Find: a) PDF of $Y = X^{1/4}$

Setup: $F_Y(y) = P[Y \leq y] = P[X^{1/4} \leq y] = P[X \leq y^4] = F_X(y^4)$

Steps: i) find the pdf of x. Because X is a Uniform distribution with parameters 1 and 0, the pdf, which for Unif(a,b) is $1/(b-a)$ where $a < x < b$. Here, Unif(0,1) gives $1/1-0 = 1$

ii) Differentiate with respect to y to find an equation given in terms of the pdf of x:
 $f_Y(y) = \frac{d}{dy} F_X(y^4) = 4y^3$

Result: $f_Y(y) = \begin{cases} 4y^3 & , \quad 0 < y < 1 \\ 0 & , \quad o/w \end{cases}$

Find: b) PDF of $W = e^{-X}$

Setup: $F_W(z) = P[W \leq w] = P[e^{-X} \leq w] = P[-X \leq \ln w] = P[X \geq -\ln w] = 1 - F_x(-\ln w)$

Steps: i) find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

ii) Differentiate with respect to w to find an equation given in terms of the pdf of x :
 $f_W(w) = -\frac{d}{dw}F_X\frac{d}{dw}(-\ln w) = -f_X(-\ln w)\frac{-1}{w}$ for $e^{-1} < w < 1 = -\frac{1}{w}$

Result: $f_W(w) = \begin{cases} \frac{1}{w} & , \quad e^{-1} < w < 1 \\ 0 & , \quad o/w \end{cases}$

Find: c) PDF of $Z = 1 - e^{-X}$

Setup: $F_Z(z) = P[Z \leq z] = P[1 - e^{-X} \leq z] = P[-e^{-X} \leq z - 1] = P[e^{-X} \geq 1 - z] = P[-X \geq \ln(1 - z)] = P[X \leq -\ln(1 - z)] = F_x(-\ln(1 - z))$

Steps: i) find the pdf of x . See part a) for an explanation of why it is 1 when $a < x < b$

ii) Differentiate with respect to w to find an equation given in terms of the pdf of x :
 $f_Z(z) = -\ln(1 - z) = -\frac{-1}{1-z} = \frac{1}{1-z}$ for $0 < z < e^{-1}$

Result: $f_W(w) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < e^{-1} \\ 0 & , \quad o/w \end{cases}$

Find: d) PDF of $U = X(1 - X)$

Setup: $F_U(u) = P[U \leq u] = P[X(1 - x) \leq u] = P[-X^2 + X \leq u] = P[-(X - 1/2)^2 \leq u - 1/4] = P[(X - 1/2)^2 \geq 1/4 - u] = P[|(X - 1/2)| \geq (1/4 - u)^{1/2}] =$

Steps: i) find the pdf of x . See part a) for an explanation of why it is 1 when $a < x < b$

ii) INCOMPLETE:

$f_Z(z) = -\ln(1 - z) = -\frac{-1}{1-z} = \frac{1}{1-z}$ for $0 < z < e^{-1}$

Result: $f_W(w) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < e^{-1} \\ 0 & , \quad o/w \end{cases}$

06.03

Given: PDF $f_R(r) = \begin{cases} 6r(1 - r) & , \quad 0 < r < 1 \\ 0 & , \quad o/w \end{cases}$

Find: Distribution of the circumference

Setup: The circumference is $c = 2\pi r$ - we have the pdf in terms of x , so this is the transformation

$F_C(c) = P[C \leq c] = P[2\pi r \leq c] = P[r \leq c/2\pi] = F_x(c/2\pi)$

Steps: i) Differentiate with respect to c to find an equation given in terms of the pdf of x .
 $f_C(c) = \frac{d}{dc} F_R(c/2\pi) = f_R(c/2\pi) \frac{d}{dc}(c/2\pi) = f_R(c/2\pi)(1/2\pi)$

ii) Plug the original pdf back into this new form:

$$f_C(c) = \frac{6c}{2\pi} (1 - (c/2\pi))(1/2\pi) = \frac{6c(2\pi-c)}{2\pi^3} \quad \text{if } 0 < c < 2\pi$$

Result:
$$f_C(c) = \begin{cases} \frac{6c(2\pi-c)}{2\pi^3} & , \quad 0 < c < 2\pi \\ 0 & , \quad o/w \end{cases}$$

Find: Distribution of the area

Setup: The area is $a = \pi r^2$ so the cdf $F_A(a) = P[A \leq a] = P[\pi r^2 \leq a] = P[r^2 \leq a/\pi] = P[|r| \leq (a/\pi)^{1/2}] = P[-(a/\pi)^{1/2} \leq r \leq (a/\pi)^{1/2}] = F_r(a/\pi)^{1/2} - F_r(-(a/\pi)^{1/2})$

Steps: i) Differentiate with respect to a to find an equation in terms of the pdf of x .

$$f_A(a) = \frac{d}{da} F_R(a/\pi)^{1/2} - \frac{d}{da} F_R(-(a/\pi)^{1/2}) = f_R[(a/\pi)^{1/2}] \frac{d}{da}(a/\pi)^{1/2} - f_R[-(a/\pi)^{1/2}] \frac{d}{da}(-(a/\pi)^{1/2})$$

Result:
$$f_A(a) = \begin{cases} \frac{3(\sqrt{\pi}-\sqrt{a})}{\pi^{3/2}} & , \quad 0 < a < \pi \\ 0 & , \quad o/w \end{cases}$$

06.10 Suppose X has pdf $f_X(x) = \frac{1}{2}e^{-|x|}$ for all real x .

(a) Find the pdf of $Y = |X|$.

CDF Method

$$F_Y(y) = P[Y \leq y] = P[|x| \leq y] = P[-y \leq X \leq y] = F_X(y) - F_X(-y)$$

$$f_Y(y) = \frac{dF_X(y)}{dy} - \frac{dF_X(-y)}{dy}$$

$$f_Y(y) = f_X(y) \frac{dy}{dx} - f_X(-y) \left(\frac{-dy}{dy} \right)$$

$$f_y = \frac{1}{2}e^{-y} + \frac{1}{2}e^{-y} = e^{-y} \quad y > 0$$

(b) Let $W = 0$ if $X \leq 0$ and $W = 1$ if $X > 0$. Find the CDF of W

$$F_W(w) = P[W = 0] = \frac{1}{2}$$

$$F_W(w) = P[W = 1] = \frac{1}{2}$$

$$F_W(w) =$$

$$\begin{cases} 0 & w \leq 0 \\ \frac{1}{2} & 0 < w \leq 1 \\ 1 & w > 1 \end{cases}$$

06.14

Given: Joint PDF
$$f(x, y) = \begin{cases} 4e^{-2(x+y)} & , \quad 0 < x < \infty, 0 < y < \infty \\ 0 & , \quad o/w \end{cases}$$

Find: a) CDF of $W=X+Y$

Setup: $F_w(w) = P[W \leq w] = P[X + Y \leq w]$

Steps:

- i) Express as a sum of probabilities, replace probabilities with binomials
- ii) Simplify and Use Combinatorial Identity

Result: $\binom{n+m}{k}$ **06.15** This is a simplified version of example 6.4.5.

$X_1, X_2 \sim POI(\lambda)$ so the MGF of both is $e^{\lambda(e^t-1)}$. Thus by theorem 6.4.4

$$M_Y(t) = e^{\lambda(e^t-1)} e^{\lambda(e^t-1)} = e^{2\lambda(e^t-1)} \sim POI(2\lambda)$$

The pdf then of Y is

$$f_Y(y) = \begin{cases} \frac{e^{-2\lambda}(2\lambda)^y}{y!} & y = 0, 1, 2, \dots \\ 0 & otherwise. \end{cases}$$

06.16 Note: the pdf of $f_{x_1, x_2} = \frac{1}{x_1^2} \frac{1}{x_2^2}$

a) We need to find $f_{u,v} = f_{x_1, x_2}(x_1(u, v), x_2(u, v))|J|$ where J is our jacobian. First we let $u = x_1 x_2$ and $v = x_1$ thus $x_1 = v$ and $x_2 = \frac{u}{v}$, now we can find J.

$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & 0 \end{vmatrix} = \frac{1}{v}$$

Finally, our pdf is:

$$\begin{aligned} f_{U,V}(u, v) &= f_{x_1, x_2}\left(v, \frac{u}{v}\right) \left| \frac{1}{v} \right| \\ &= \frac{1}{v^2} \frac{1}{\left(\frac{u}{v}\right)^2} \left| \frac{1}{v} \right| \\ &= \frac{1}{u^2 v}, 1 < v < u < \infty \end{aligned}$$

6.18 It is given that X and Y have a joint pdf given by

$$f(x, y) = e^{-y} \quad \text{if } 0 < x < y < \infty. \quad (1)$$

(a): Find the joint pdf of $S = X + Y$ and $T = X$.

This can be done using the joint transformation method. By rearranging the above formulas we get $X = T$ and $Y = S - T$. Then it is easy to get the jacobian

$$J = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad (2)$$

whose determinant is clearly one. Note that the order in which you take partial derivatives is unimportant provided you are consistent - you will get the same determinant either way. Then

we substitute in $X = T$ and $Y = S - T$ into the pdf and multiply by the determinant of the jacobian:

$$f_{S,T}(s,t) = f_{X,Y}(x(s,t),y(s,t)) \times 1 = \begin{cases} e^{t-s} & \text{if } 0 < t < s/2 \\ 0 & \text{otherwise} \end{cases}. \quad (3)$$

The bounds of the function can be found in a few different ways. One way is to consider the bounds of the original function, $0 < x < y < \infty$. We can substitute in the new formulas for X and Y to get

$$0 < t < s - t < \infty. \quad (4)$$

Then it is apparent that

$$0 < 2t < s < \infty, \quad (5)$$

which then yields

$$0 < t < s/2, \quad (6)$$

the bounds of our new function.

(b): Find the marginal pdf of T.

The easiest way to do this is to "integrate out" S from the joint pdf we derived:

$$\begin{aligned} f_T(t) &= \int_{-\infty}^{\infty} f_{S,T}(s,t) ds = \int_{2t}^{\infty} e^{t-s} ds \\ &= e^t \int_{2t}^{\infty} e^{-s} ds = e^t (-e^{-s}|_{2t}^{\infty}) \\ &= e^{-t} \quad \text{if } t > 0. \end{aligned} \quad (7)$$

(c): Find the marginal pdf of S.

This is just like part (b), except this time "integrate out" T :

$$\begin{aligned} f_S(s) &= \int_{-\infty}^{\infty} f_{S,T}(s,t) dt = \int_0^{s/2} e^{t-s} ds \\ &= e^{-s} \int_0^{s/2} e^t dt = e^{-s} (e^t|_0^{s/2}) \\ &= e^{-s} (e^{s/2} - 1) \quad \text{if } s > 0. \end{aligned} \quad (8)$$