Solutions to Bain and Engelhardt's Introduction to Probability and Mathematical Statistics

06.01 This is where the first solution will go.
06.02 This is where the second solution will go.
06.15 This is a simplified version of example 6.4.5.

$X_1, X_2 \sim POI(\lambda)$ so the MGF of both is $e^{\lambda(e^t-1)}$. Thus by theorem 6.4.4

$$M_Y(t) = e^{\lambda(e^t-1)}e^{\lambda(e^t-1)} = e^{2\lambda(e^t-1)} \sim POI(2\lambda)$$

The pdf then of Y is

$$f_Y(y) = \begin{cases} \frac{e^{-2\lambda}(2\lambda)^y}{y!} & y = 0, 1, 2, ... \\ 0 & otherwise. \end{cases}$$

06.16 Note: the pdf of $f_{x_1,x_2} = \frac{1}{x_1^2}\frac{1}{x_2^2}$
a) We need to find $f_{u,v} = f_{x_1,x_2}(x_1(u,v), x_2(u,v))|J|$ where J is our jacobian. First we let $u = x_1x_2$ and $v = x_1$ thus $x_1 = v$ and $x_2 = \frac{u}{v}$, now we can find J.

$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & 0 \end{vmatrix} = \frac{1}{v}$$

Finally, our pdf is:

$$\begin{aligned} f_{U,V}(u,v) &= f_{x_1,x_2}(v, \frac{u}{v}) \left|\frac{1}{v}\right| \\ &= \frac{1}{v^2}\frac{1}{(\frac{u}{v})^2}\left|\frac{1}{v}\right| \\ &= \frac{1}{u^2v}, 1 < v < u < \infty \end{aligned}$$

06.23 We will use the property that independant identically distributed random variables has the form of 6.4.4, $M_Y(t) = [M_X(t)]^n$ where $Y = X_1+X_2+...+X_n$. then since $X_i \sim GEO(p)$

$$\begin{aligned} Mgf(Y) &= M_{X_1}(t)M_{X_2}(t)...M_{X_k}(t) \\ &= (M_X(t))^k \\ &= (\frac{pe^t}{1-qe^t})^k \sim NegativeBinomial(k,p) \end{aligned}$$

06.25 First note, X_1, X_2, X_3, X_4 are all independant, but they are not IID as only $X_2, X_3, X_4 \sim POI(5)$ with X_1 not being listed. So formula 6.4.5 does not hold. 6.4.4 does though.

A)

$$\begin{aligned}
Mgf(Y) &= M_{X_1}(t)M_{X_2+X_3+X_4}(t) \\
&= M_{X_1}(t)(M_{X_i}(t))^3
\end{aligned}$$

Since X_2, X_3, X_4 are iid 6.4.5 holds for moving to this mgf

$$\begin{aligned}
&= M_{X_1}(t)(e^{\mu(e^t-1)})^3 \\
&= M_{X_1}(t)e^{3\mu(e^t-1)} \\
&= M_{X_1}(t)e^{15(e^t-1)} \\
e^{25(e^t-1)} &= M_{X_1}(t)e^{15(e^t-1)} \\
\frac{e^{25(e^t-1)}}{e^{15(e^t-1)}} &= M_{X_1}(t) \\
e^{10(e^t-1)} &= M_{X_1}(t) \sim POI(10)
\end{aligned}$$

B) For $W = X_1 + X_2$ we have $X_1 \sim POI(10)$ and $X_2 \sim POI(5)$. So $POI(10+5) = POI(15)$