Solutions to Bain and Engelhardt's Introduction to Probability and Mathematical Statistics

06.01

Given: the pdf of x $f_x(x) = \begin{cases} 4x^3 & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases}$

Find: PDF of a) $Y = X^4$

Setup: Use the CDF technique to get the CDF of Y in terms of a CDF of X
$F_Y(y) = P[Y \leq y] = P[X^4 \leq Y] = P[-y^{\frac{1}{4}} \leq X \leq y^{\frac{1}{4}}] = F_X(y^{\frac{1}{4}}) - F_X(-y^{\frac{1}{4}})$

Steps: i) Differentiate with respect to y to find an equation given in terms of the pdf of x:
$f_y(y) = \frac{d}{dy}F_X(y^{\frac{1}{4}}) - \frac{d}{dy}F_X(-y^{\frac{1}{4}}) = f_x(y^{\frac{1}{4}})\frac{d}{dy}y^{\frac{1}{4}} - f_x(-y^{\frac{1}{4}})\frac{d}{dy}-y^{\frac{1}{4}} = f_x(y^{\frac{1}{4}})\frac{y^{-\frac{3}{4}}}{4} - f_x(-y^{\frac{1}{4}})\frac{-y^{-\frac{3}{4}}}{4}$

ii) Plug in the original limits and function for the pdf of x, and compute the cdf for y

Result: $f_y(y) = \begin{cases} 4y^{\frac{3}{4}}\frac{1}{4y^{\frac{3}{4}}} & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases} = \begin{cases} 1 & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases}$

Find: PDF of b) $W = e^X$

Setup: Use the CDF technique to get the CDF of W in terms of a CDF of X
$F_W(w) = P[W \leq w] = P[e^X \leq W] = P[X \leq lnW] = F_X(lnW)$

Steps: i) Differentiate with respect to w to find an equation given in terms of the pdf of x:
$f_w(w) = \frac{d}{dw}F_X(lnW)\frac{d}{dw}(lnw) = f_x(lnw)\frac{1}{w}$

ii) Plug in the original limits and function for the pdf of x, and compute the cdf for y

Result: $f_W(w) = \begin{cases} \frac{4(lnw)^3}{w} & , \quad 1 < w < e \\ 0 & , \quad o/w \end{cases}$

Find: PDF of c) $Z = \ln x$

Setup: Use the CDF technique to get the CDF of Z in terms of a CDF of X
$F_Z(z) = P[Z \leq z] = P[lnx \leq z] = P[X \leq e^z] = F_X(e^z)$

Steps: i) Differentiate with respect to z to find an equation given in terms of the pdf of x:
$f_z(z) = \frac{d}{dz}F_X(e^z) = f_x(e^z)\frac{de^z}{dz}$

ii) Plug in the original limits and function for the pdf of x, and compute the cdf for y

Result: $f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \leq z < 0 \\ 0 & , \quad o/w \end{cases}$

Find: PDF of d) $U = (X - 0.5)^2$

Setup: Use the CDF technique to get the CDF of U in terms of a CDF of X
$F_U(u) = P[U \leq u] = P[(X-0.5)^2 \leq u] = P[|X - 0.5| \leq u^{0.5}] = F_X(u^{1/2} + 1/2) - F_X(-u^{1/2} + 1/2)$

Steps: i) Differentiate with respect to u to find an equation given in terms of the pdf of x:
$f_U(u) = \frac{d}{du} F_X(u^{1/2} + 1/2) = f_x(u^{1/2} + 1/2)\frac{d}{du}(u^{1/2} + 1/2) - f_x(-u^{1/2} + 1/2)\frac{d}{du}(-u^{1/2} + 1/2)$
$f_x(u^{1/2} + 1/2)1/2u^{-1/2}) - f_x(-u^{1/2} + 1/2)1/2u^{-1/2}$

ii) INCOMPLETE

Result: $f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \leq z < 0 \\ 0 & , \quad o/w \end{cases}$

06.02

Given: $X \sim Unif(0,1)$
Find: a) PDF of $Y = X^{1/4}$

Setup: $F_Y(y) = P[Y \leq y] = P[X^{1/4} \leq y] = P[X \leq y^4] = F_X(y^4)$

Steps: i) find the pdf of x. Because X is a Uniform distribution with parameters 1 and 0, the pdf, which for Unif(a,b) is 1/b-a where $a < x < b$. Here, Unif(0,1) gives 1/1-0 =1

ii) Differentiate with respect to y to find an equation given in terms of the pdf of x:
$f_Y(y) = \frac{d}{dy} F_X(y^4) = 4y^3$

Result: $f_Y(y) = \begin{cases} 4y^3 & , \quad 0 < y < 1 \\ 0 & , \quad o/w \end{cases}$

Find: b) PDF of $W = e^{-X}$

Setup: $F_W(z) = P[W \leq w] = P[e^{-X} \leq W] = P[-X \leq lnw] = P[X \geq -lnw] = 1 - F_x(-lnw)$

Steps: i) find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

ii) Differentiate with respect to w to find an equation given in terms of the pdf of x:
$f_W(w) = -\frac{d}{dw}F_X\frac{d}{dw}(-lnw) = -f_X(-lnw)\frac{-1}{w} \quad for \quad e^{-1} < w < 1 = -\frac{1}{w}$

Result: $f_W(w) = \begin{cases} \frac{1}{w} & , \quad e^{-1} < w < 1 \\ 0 & , \quad o/w \end{cases}$

Find: c) PDF of $Z = 1 - e^{-X}$

Setup: $F_Z(z) = P[Z \le z] = P[1 - e^{-X} \le z] = P[-e^{-X} \le z - 1] = P[e^{-X} \ge 1 - z] = P[-X \ge ln(1-z)] = P[X \le -ln(1-z)] = F_x(-ln(1-z))$

Steps: i) find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

ii) Differentiate with respect to w to find an equation given in terms of the pdf of x:
$f_Z(z) = -ln(1-z) = -\frac{-1}{1-z} = \frac{1}{1-z} \quad for \quad 0 < z < e^{-1}$

Result: $f_W(w) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < e^{-1} \\ 0 & , \quad o/w \end{cases}$

Find: d) PDF of $U = X(1 - X)$

Setup: $F_U(u) = P[U \le u] = P[X(1-x) \le u] = P[-X^2 + X \le u] = P[-(X-1/2)^2 \le u - 1/4] = P[(X-1/2)^2 \ge 1/4 - u] == P[|(X-1/2)| \ge (1/4 - u)^{1/2}] =$

Steps: i) find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

ii) INCOMPLETE:
$f_Z(z) = -ln(1-z) = -\frac{-1}{1-z} = \frac{1}{1-z} \quad for \quad 0 < z < e^{-1}$

Result: $f_W(w) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < e^{-1} \\ 0 & , \quad o/w \end{cases}$

06.03

Given: PDF $f_R(r) = \begin{cases} 6r(1-r) & , \quad 0 < r < 1 \\ 0 & , \quad o/w \end{cases}$

Find: Distribution of the circumference

Setup: The circumference is $c = 2\pi r$ - we have the pdf in terms of x, so this is the transformation
$F_C(c) = P[C \le c] = P[2\pi r \le c] = P[r \le c/2\pi] = F_x(c/2\pi)$

Steps: i) Differentiate with respect to c to find an equation given in terms of the pdf of x.
$f_C(c) = \frac{d}{dc}F_R(c/2\pi) = f_R(c/2\pi)\frac{d}{dc}(c/2\pi) = f_R(c/2\pi)(1/2\pi)$

ii) Plug the original pdf back into this new form:
$f_C(c) = \frac{6c}{2\pi}(1-(c/2\pi))(1/2\pi) = \frac{6c(2\pi-c}{2\pi^3} \quad if \quad 0 < c < 2\pi$

Result: $f_C(c) = \begin{cases} \frac{6c(2\pi-c}{2\pi^3} & , \quad 0 < c < 2\pi \\ 0 & , \quad o/w \end{cases}$

Find: Distribution of the area

Setup: The area is $a = \pi r^2$ so the cdf $F_A(a) = P[A \leq a] = P[\pi r^2 \leq a] = P[r^2 \leq a/\pi]$
$= P[|r| \leq (a/\pi)^{1/2}] = P[-(a/\pi)^{1/2} \leq c \leq (a/\pi)^{1/2}] = F_r(a/\pi)^{1/2} - F_r - (a/\pi)^{1/2}$

Steps: i) Differentiate with respect to a to find an equation in terms of the pdf of x.
$f_A(a) = \frac{d}{da}F_R(a/\pi)^{1/2} - \frac{d}{da}F_R - (a/\pi)^{1/2} = f_R[(a/\pi)^{1/2}]\frac{d}{da}(a/\pi)^{1/2} f_R[-(a/\pi)^{1/2}]\frac{d}{da} - (a/\pi)^{1/2}$

Result: $f_A(a) = \begin{cases} \frac{3(\sqrt{\pi}-\sqrt{a}}{\pi^{3/2}} & , \quad 0 < a < \pi \\ 0 & , \quad o/w \end{cases}$

06.04 *Please double check the results of this solution*
For $X \sim WEI(\theta,\beta)$ we have the CDF as $F_X = 1 - e^{\frac{x}{\theta}^\beta}$ and the pdf is $f(x) = \frac{\beta}{\theta^\beta}x^{\beta-1}e^{-\frac{x}{\theta}^\beta}$
a) We make the transformation by the CDF method:

$$\begin{aligned} \Pr(Y \leq y) &= \Pr\left(\frac{X}{\theta}^\beta \leq y\right) \\ &= \Pr\left(X \leq \theta y^{\frac{1}{\beta}}\right) \\ &= F_X\left(\theta y^{\frac{1}{\beta}}\right) \\ &= 1 - e^{-\frac{\theta y^{\frac{1}{\beta}}}{\theta}^\beta} \\ &= 1 - e^{-y}, \text{ where } 0 < y \end{aligned}$$

So we have our CDF. For the pdf we simply take the derivative of the above. So $pdf = e^{-y}$ where $0 < y$

b) $W = \ln X$. Again, the most simply method to get the CDF, and in turn the pdf is the CDF method.

$$\begin{aligned} \Pr(W \leq w) &= \Pr(\ln X \leq w) && (1) \\ &= \Pr(X \leq e^w) && (2) \\ &= F(e^w) && (3) \\ &= 1 - e^{-\frac{e^w}{\theta}^\beta} \text{ where } 0 < w && (4) \\ & && (5) \end{aligned}$$

Again we simply differentiate to get the pdf. which turns out to be $\beta e^{\beta w}\theta^{-\beta}e^{-\frac{e^{w\beta}}{\theta}}$, $0 < w$
c)
06.10 Suppose X has pdf $f_X(x) = \frac{1}{2}e^{-|x|}$ for all real x.

(a) Find the pdf of $Y = |X|$.
CDF Method
$F_Y(y) = P[Y \leq y] = P[|x| \leq y] = P[-y \leq X \leq y] = F_X(y) - F_X(-y)$
$f_Y(y) = \frac{dF_X(y)}{dy} - \frac{dF_X(-y)}{dy}$
$f_Y(y) = f_X(y)\frac{dy}{dx} - f_X(-y)(\frac{-dy}{dy})$
$f_y = \frac{1}{2}e^{-y} + \frac{1}{2}e^{-y} = e^{-y}\ y > 0$

(b) Let $W = 0$ if $X \leq 0$ and $W = 1$ if $X > 0$. Find the CDF of W
$F_W(w) = P[W = 0] = \frac{1}{2}$
$F_W(w) = P[W = 1] = \frac{1}{2}$
$F_W(w) =$

$$\begin{cases} 0 & w \leq 0 \\ \frac{1}{2} & 0 \leq w \leq 1 \\ 1 & w > 1 \end{cases}$$

06.13 X has pdf

$$f(x) = \begin{cases} \dfrac{x^2}{24} & -2 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

We want pdf of the CDF $Y = X^2$ with regions: $(-2, 0) \cup [0, 4)$

$$[F_x(\sqrt{y}) - F_x(-\sqrt{y})] = \left[f_x(\sqrt{y})(\frac{1}{2}\sqrt{y}) - f_x(-\sqrt{y})(-\frac{1}{2}\sqrt{y})\right]$$

$$f_y(y) = \begin{cases} \dfrac{y}{48\sqrt{y}} + \dfrac{y}{48\sqrt{y}} & 0 < y < 4 \\ \dfrac{y}{48\sqrt{y}} & 4 \leq y \leq 16 \\ 0 & \text{otherwise} \end{cases}$$

$$f_y(y) = \begin{cases} \dfrac{\sqrt{y}}{24} & 0 < y < 4 \\ \dfrac{\sqrt{y}}{48} & 4 \leq y \leq 16 \\ 0 & \text{otherwise} \end{cases}$$

06.14

Given: Joint PDF $f(x,y) = \begin{cases} 4e^{-2(x+y)} & , \quad 0<x<\infty, 0<y<\infty \\ 0 \quad , & o/w \end{cases}$

Find: a) CDF of W=X+Y

Setup: $F_w(w) = P[W \leq w] = P[X + Y \leq w]$

Steps:

i) Express as a sum of probabilities, replace probabilities with binomials

ii) Simplify and Use Combinatorial Identity

Result: $\binom{n+m}{k}$

06.15 This is a simplified version of example 6.4.5.

$X_1, X_2 \sim POI(\lambda)$ so the MGF of both is $e^{\lambda(e^t-1)}$. Thus by theorem 6.4.4

$$M_Y(t) = e^{\lambda(e^t-1)}e^{\lambda(e^t-1)} = e^{2\lambda(e^t-1)} \sim POI(2\lambda)$$

The pdf then of Y is

$$f_Y(y) = \begin{cases} \frac{e^{-2\lambda}(2\lambda)^y}{y!} & y = 0,1,2,... \\ 0 & otherwise. \end{cases}$$

06.16 Note: the pdf of $f_{x_1,x_2} = \frac{1}{x_1^2}\frac{1}{x_2^2}$

a) We need to find $f_{u,v} = f_{x_1,x_2}(x_1(u,v), x_2(u,v))|J|$ where J is our jacobian. First we let $u = x_1x_2$ and $v = x_1$ thus $x_1 = v$ and $x_2 = \frac{u}{v}$, now we can find J.

$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & 0 \end{vmatrix} = \frac{1}{v}$$

Finally, our pdf is:

$$\begin{aligned} f_{U,V}(u,v) &= f_{x_1,x_2}(v, \frac{u}{v})\left|\frac{1}{v}\right| \\ &= \frac{1}{v^2}\frac{1}{(\frac{u}{v})^2}\left|\frac{1}{v}\right| \\ &= \frac{1}{u^2v}, 1 < v < u < \infty \end{aligned}$$

06.17 **6.18** It is given that X and Y have a joint pdf given by

$$f(x,y) = e^{-y} \quad if \quad 0 < x < y < \infty. \tag{6}$$

(a): Find the joint pdf of $S = X + Y$ and $T = X$.

This can be done using the joint transformation method. By rearranging the above formulas we get $X = T$ and $Y = S - T$. Then it is easy to get the jacobian

$$J = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \tag{7}$$

whose determinant is clearly one. Note that the order in which you take partial derivatives is unimportant provided you are consistent - you will get the same determinant either way. Then we substitute in $X = T$ and $Y = S - T$ into the pdf and multiply by the determinant of the jacobian:

$$f_{S,T}(s,t) = f_{X,Y}(x(s,t), y(s,t)) \times 1 = \begin{cases} e^{t-s} & if \quad 0 < t < s/2 \\ 0 & otherwise \end{cases} . \tag{8}$$

The bounds of the function can be found in a few different ways. One way is to consider the bounds of the original function, $0 < x < y < \infty$. We can substitute in the new formulas for X and Y to get

$$0 < t < s - t < \infty. \tag{9}$$

Then it is apparent that

$$0 < 2t < s < \infty, \tag{10}$$

which then yields

$$0 < t < s/2, \tag{11}$$

the bounds of our new function.

(b): Find the marginal pdf of T.

The easiest way to do this is to "integrate out" S from the joint pdf we derived:

$$\begin{aligned} f_T(t) &= \int_{-\infty}^{\infty} f_{S,T}(s,t) ds = \int_{2t}^{\infty} e^{t-s} ds \\ &= e^t \int_{2t}^{\infty} e^{-s} ds = e^t (-e^{-s}|_{2t}^{\infty}) \\ &= e^{-t} \quad if \quad t > 0. \end{aligned} \tag{12}$$

(c): Find the marginal pdf of S.

This is just like part (b), except this time "integrate out" T:

$$\begin{aligned} f_S(s) &= \int_{-\infty}^{\infty} f_{S,T}(s,t) dt = \int_{0}^{s/2} e^{t-s} ds \\ &= e^{-s} \int_{0}^{s/2} e^t dt = e^{-s} (e^t|_0^{s/2}) \\ &= e^{-s}(e^{s/2} - 1) \quad if \quad s > 0. \end{aligned} \tag{13}$$

6.21 Let X and Y be continuous random variables with a joint density function given by

$$f_{X,Y}(x,y) = 2(x+y) \quad if \quad 0 < x < y < 1 \quad and \quad 0 \quad otherwise. \tag{14}$$

(a) Find the joint density function of $S = X$ and $T = XY$.

We can solve for X and Y in terms of the new variables, to get $X = S$ and $Y = T/S$. Then the jacobian is given by

$$J = \begin{pmatrix} 1 & 0 \\ -T/S^2 & 1/S \end{pmatrix}. \tag{15}$$

Then the new pdf is given by

$$f_{T,S}(s,t) = f_{X,Y}(x(s,t), y(s,t)) \times |1/s| = \begin{cases} 2(s + t/s)\,|1/s|\,, & 0 < s^2 < t < s < 1 \\ 0 & otherwise. \end{cases} \tag{16}$$

The bounds of this equation can be interpreted in the following way: the old triangular region in the xy plane got transformed to the region in the st plane between the lines $T = S$ and $T = S^2$.

(b) Find the marginal pdf of T.

To find the marginal of T, s needs to be "integrated out."

$$\begin{aligned} f_T(t) &= \int_{-\infty}^{\infty} f_{S,T}(s,t)ds \\ &= \int_t^{\sqrt{t}} 2(s + t/s)\,|1/s|\,ds = \int_t^{\sqrt{t}} 2(1 + t/s^2)ds \\ &= 2\int_t^{\sqrt{t}} ds + 2t\int_t^{\sqrt{t}} 1/s^2 ds = 2(\sqrt{t} - t) + 2t(-1/s|_t^{\sqrt{t}} \\ &= 2\sqrt{t} - 2t + 2 - 2\sqrt{t} = \begin{cases} 2 - 2t & t \in (0,1) \\ 0 & otherwise. \end{cases} \end{aligned} \tag{17}$$

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06.23 We will use the property that independant identically distributed random variables has the form of 6.4.4, $M_Y(t) = [M_X(t)]^n$ where $Y = X_1 + X_2 + ... + X_n$. then since $X_i \sim GEO(p)$

$$\begin{aligned} Mgf(Y) &= M_{X_1}(t)M_{X_2}(t)...M_{X_k}(t) \\ &= (M_X(t))^k \\ &= (\frac{pe^t}{1 - qe^t})^k \sim NegativeBinomial(k,p) \end{aligned}$$

06.25 First note, X_1, X_2, X_3, X_4 are all independant, but they are not IID as only $X_2, X_3, X_4 \sim POI(5)$ with X_1 not being listed. So formula 6.4.5 does not hold. 6.4.4 does though.

A)

$$\begin{aligned} Mgf(Y) &= M_{X_1}(t)M_{X_2+X_3+X_4}(t) \\ &= M_{X_1}(t)(M_{X_i}(t))^3 \end{aligned}$$

Since X_2, X_3, X_4 are iid 6.4.5 holds for moving to this mgf

$$
\begin{aligned}
&= M_{X_1}(t)(e^{\mu(e^t-1)})^3 \\
&= M_{X_1}(t)e^{3\mu(e^t-1)} \\
&= M_{X_1}(t)e^{15(e^t-1)} \\
e^{25(e^t-1)} &= M_{X_1}(t)e^{15(e^t-1)} \\
\frac{e^{25(e^t-1)}}{e^{15(e^t-1)}} &= M_{X_1}(t) \\
e^{10(e^t-1)} &= M_{X_1}(t) \sim POI(10)
\end{aligned}
$$

B) For $W = X_1 + X_2$ we have $X_1 \sim POI(10)$ and $X_2 \sim POI(5)$. So $POI(10+5) = POI(15)$
06.29

Given: PDF $f(x) = \begin{cases} \frac{1}{x^2} & , \quad 1 \leq x < \infty, 0 < y < \infty \\ 0 & , \quad o/w \end{cases}$

Find: a) Joint PDF of the order statistics

Setup: $F_w(w) = P[W \leq w] = P[X + Y \leq w]$

Steps: i) Differentiate with respect to a to find an equation in terms of the pdf of x.
$f_A(a) = \frac{d}{da}F_R(a/\pi)^{1/2} - \frac{d}{da}F_R - (a/\pi)^{1/2} = f_R[(a/\pi)^{1/2}]\frac{d}{da}(a/\pi)^{1/2} f_R[-(a/\pi)^{1/2}]\frac{d}{da} - (a/\pi)^{1/2}$

ii) Simplify and Use Combinatorial Identity

Result: $\binom{n+m}{k}$

Find: b) PDF of the smallest order statisticY_1

Setup:

Steps: i)

Result:

Find: c) PDF of the largest order statistic Y_n

Setup:

Steps: i)

Result:

Find: d) PDF of the sample range $R = Y_n - Y_1, \quad forn = 2$

Setup: The area is $a = \pi r^2$ so the cdf $F_A(a) = P[A \leq a] = P[\pi r^2 \leq a] = P[r^2 \leq a/\pi]$
$= P[|r| \leq (a/\pi)^{1/2}] = P[-(a/\pi)^{1/2} \leq c \leq (a/\pi)^{1/2}] = F_r(a/\pi)^{1/2} - F_r - (a/\pi)^{1/2}$

Steps: i)

Result:

Find: e) PDF of the sample median $R = Y_r - Y_1, \quad fornoddsothatr = (n+1)/2$

Setup:

Steps: i)

Result: **06.35** Suppose X_1, X_2 are independent exponentially distributed random variables $X_i \sim$EXP(θ), and let $Y = X_1 - X_2$.
(a) Find the MGF of Y.
We can think of $Y = X_1 - X_2$ as $Y = X_1 + (-1)X_2$. Then using Theorem 6.4.1,

$$\begin{aligned}
M_Y(t) &= (M_{X_1}(t))(M_{-X_2}(t)) \\
M_Y(t) &= (M_{X_1}(t))(M_{X_2}(-t)) \\
M_Y(t) &= \left(\frac{1}{1-\theta t}\right)\left(\frac{1}{1-\theta(-t)}\right) \\
M_Y(t) &= \left(\frac{1}{1-\theta t}\right)\left(\frac{1}{1+\theta t}\right) \\
M_Y(t) &= \frac{1}{1-\theta t+\theta t-\theta^2 t^2} \\
M_Y(t) &= \frac{1}{1-\theta^2 t^2}
\end{aligned}$$

(b) What is the distribution of Y?
Since $\frac{1}{1-\theta^2 t^2}$ is the MGF of a double exponential, $Y \sim$ DE(θ,0).

07.01 Consider a random sample of size n from a distribution with $CDFF(x) = 1 - \frac{1}{x} if 1 \leq x \leq \infty$

(a) Derive the CDF of the smallest order statistic, $X_{1:\,n}$

Solution: $G_1(y_1) = 1 - [1 - F_X(y_1)]^n = 1 - [1 - [1 - \frac{1}{y_1}]]^n = 1 - [\frac{1}{y_1}]^n$

$$G_1(y_1) = \begin{cases} 1 - \frac{1}{[y_1]^n} & \text{if } 1 \leq y_1 \\ 0 & \text{if } 0 > y_1. \end{cases}$$

(b) Find the limiting distribution of $X_{1:\,n}$ Solution:

$\lim_{n\to\infty} 1-\frac{1}{{y_1}^n} = \begin{cases} 1 & \text{if } y_1 > 1 \\ 0 & \text{if } y_1 \le 1 \end{cases}$

The limiting distribution of $X_{1:\,n}$ is degenerate at $y = 1$

(c) Find the limiting distribution of ${X^n}_{1:\,n}$

Solution:

$F_{{X^n}_{1:\,n}}(y) = P({X^n}_{1:\,n} \le y) = P(X_{1:\,n} \le y^{\frac{1}{n}}) = F_{X_{1:\,n}}(y^{\frac{1}{n}}) = 1 - \frac{1}{{y^{\frac{1}{n}}}^n} = 1 - \frac{1}{y_n}$

then, the limiting distribution of ${X^n}_{1:\,n} = \begin{cases} 1-\frac{1}{y_n} & \text{if } y > 1 \\ 0 & \text{if } otherwise \end{cases}$

07.02

$$F(x) = \begin{cases} -\dfrac{1}{x^2}, \text{all real x} \end{cases}$$

2a. $F_{X_{n:n}}(y) = (\frac{1}{1+e^{-y}})^n; \lim_{n\to\infty}(\frac{1}{1+e^{-y}})^n$ has no limiting distribution.

2b. $F_{X_{n:n}-\ln(n)}(y) = P[X_{n:n} - \ln(n) \le y] = P[X_{n:n} \le y + \ln(n)]$

$= F_{X_{n:n}}(y+\ln(n))^n = (\frac{1}{1+e^{-(y+\ln(n))}})^n = (\frac{1}{1+\frac{e^{-y}}{n}})^n;$

$\lim_{n\to\infty}(\frac{1}{1+\frac{e^{-y}}{n}})^n = e^{-e^{-y}}$

07.03 3a. $F(x) = \begin{cases} 1-\dfrac{1}{x^2}, x>1 \\ 0, x \le 0 \end{cases}$

$F_{X_{1:n}}(y) = P[X_{1:n} \le y] = 1 - P[X_{1:n} \ge y] = 1 - \frac{1}{y^{2n}}, y > 1$

$\lim_{n\to\infty}(1-\frac{1}{y^{2n}}) = 1 - 0 = \begin{cases} 1, y>1 \\ 0, y \le 0 \end{cases}$

3b.
$F_{X_{n:n}}(y) = P[X_{n:n} \le y] = 1 - P[X_{n:n} \ge y] = 1 - (1-\frac{1}{y^2})^n = \frac{1}{y^{2n}}; lim?_{n\to\infty}\frac{1}{y^{2n}} = 0,$
Therefore $F_{X_{n:n}}(y)$ has no limiting distribution.
3c.
$F_{n^{-\frac{1}{2}}X_{n:n}}(y) = P[\frac{1}{\sqrt{n}}X_{n:n} \le y] = P[X_{n:n} \le \sqrt{n}y] = F_{X_{n:n}}(\sqrt{n}y) = (1-(\sqrt{n}y)^{-2})^n, fory > \frac{1}{\sqrt{n}}$

$\lim_{n\to\infty}(1-(\sqrt{n}y)^{-2})^n = (1-\frac{1}{ny^2})^n = \begin{cases} e^{-y^{-2}}, y>0 \\ 0, y \le 0 \end{cases}$

07.07 The WEI $(1,2)$ distribution has pdf $f(x) = 2xe^{-x^2}$ for $x > 0$,mean $\mu = \Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$ and variance $\sigma^2 = \Gamma(2) - \Gamma(\frac{3}{2})^2 = 1 - \frac{\pi}{4}$
(a)According to the central limit theorem, this holds with $a = \mu - \frac{1.96\sigma}{\sqrt{n}}$ and $b = \mu + \frac{1.96\sigma}{\sqrt{n}}$, so if $n = 35$ we have $a = 0.7328$ and $b = 1.0397$

(b)For odd n, $X_{\frac{n+1}{2}:n}$ is approximately $N(x_{\frac{1}{2}}, \frac{c^2}{n})$, where $c^2{=}\frac{1}{4f(x_{\frac{1}{2}})^2}$. Now $F(x_{\frac{1}{2}}) = \frac{1}{2}$, because $F(x) = 1 - e^{-x^2}$, it implies that $x_{\frac{1}{2}} = \sqrt{ln2}$. Also because $c^2 = \frac{1}{4ln2}$, we have $a = x_{\frac{1}{2}} - \frac{1.96c}{\sqrt{n}}$ and $b = x_{\frac{1}{2}} + \frac{1.96c}{\sqrt{n}}$, so when $n = 35$ we have $a = 0.6336$ and $b = 1.0315$

07.11 a) First we need to know the μ and the σ. For a Uniform variable with $a = 0, b = 1$ we have $\mu = 1/2$ and $\sigma = 1/\sqrt{12}$ (Note: it is not σ^2). We also need to know that $n = 20$ from there we can use the CLT:

$$\begin{aligned}
\Pr\left(\sum^{20} X_i < 12\right) &= \Pr\left(\frac{\sum X_i - 10}{\sqrt{20}\frac{1}{\sqrt{12}}} < \frac{12-10}{\sqrt{20}\frac{1}{\sqrt{12}}}\right) \\
&= \Phi\left(\frac{12-10}{\sqrt{20}\frac{1}{\sqrt{12}}}\right) \\
&\approx .9394
\end{aligned}$$

b) We let $Y = \sum^{20} X_i$, let Y' be our 90th percentile that we want to find. So we setup our probability as $\Pr(Y \leq Y') = .9$, .9 as we are interested in the 90th percentile. Using μ, σ, and n from part (a) we solve with CLT:

$$\begin{aligned}
\Pr\left(Y \leq Y'\right) &= \Pr\left(\frac{Y - \mu n}{\sigma\sqrt{n}} \leq \frac{Y' - \mu n}{\sigma\sqrt{n}}\right) \\
&= \Pr\left(Z \leq \frac{Y' - 10}{\sqrt{20}\frac{1}{\sqrt{12}}}\right) \text{ Note: Z is standard normal due to CLT} \\
.9 &= \Phi\left(\frac{Y' - 10}{\sqrt{20}\frac{1}{\sqrt{12}}}\right)
\end{aligned}$$

We now solve for Y'. We know (from a chart or list) that .9 from Φ is $z \approx 1.285$. So we set our final equation for finding out Y' with that in mind.

$$\begin{aligned}
\frac{Y' - 10}{1.291} &= 1.285 \\
Y' &\approx 11.658
\end{aligned}$$

07.12 a) First, an understanding that the wording here implies that X is actually "failures" of weapons. So the given p would normally be q in other contexts. So using the binomial theorem we would have $p = .05$ and $q = .95$. Knowing that we can use the Binomial theorem easily:

$$\begin{aligned}
\Pr\left(X \geq 1\right) &= 1 - \Pr\left(X < 1\right) \\
&= 1 - \binom{n}{0}(.05)^0(.95)^n
\end{aligned}$$

We now solve for n from the above equation knowing that the desired probability is .99

$$\begin{aligned}
.99 &= 1 - (.95)^n \\
\ln .95^n &= \ln .01 \\
n &= \frac{\ln .01}{\ln .95}
\end{aligned}$$

So n, since it must be an integer, is rounded to 90.
b)
07.13 From the hint we know that $Y_n = \sum^n X_i$ where $X_i \sim Geo(p)$. So for $\sum^n X_i$ the $\mu = \frac{n}{p}$ and $\sigma^2 = \frac{nq}{p^2}$. Then by the CLT:

$$\begin{aligned}
\Pr\left(Y_n \leq y\right) &= \Pr\left(\sum^n X_i \leq y\right) \\
&= \Pr\left(\frac{\sum^n X_i - \frac{n}{p}n}{\sqrt{n}\sqrt{\frac{nq}{p^2}}} \leq \frac{y - \frac{n}{p}n}{\sqrt{n}\sqrt{\frac{nq}{p^2}}}\right) \\
&= \Phi\left(\frac{y - \frac{n}{p}n}{\sqrt{n}\sqrt{\frac{nq}{p^2}}}\right)
\end{aligned}$$

07.16 a) We need two things for this proof. First, we need to know μ and σ^2 of $\overline{X}$. We know this is $\mu = \mu$ and $\sigma^2 = \frac{\mu^2}{n}$ from facts of the sample mean distribution of $POI(\mu)$. Next the theorems from section 7.6, namely 7.6.2 and from 7.7, 7.7.2. These will let us prove the following:

$$\begin{aligned}
\Pr\left[|\overline{X_n} - \mu| < \epsilon\right] &\geq 1 - \frac{\mu^2}{\epsilon^2 n} \\
\lim_{n\to\infty} \Pr\left[|\overline{X_n} - \mu| < \epsilon\right] &= 1
\end{aligned}$$

From this we now know that $\overline{X} \xrightarrow{P} \mu$ from 7.6.3. For our goal, $e^{\overline{X_n}}$ we simply need to know 7.7.2. Since $\overline{X} \xrightarrow{P} \mu$ then $e^{\overline{X}} \xrightarrow{P} e^{\mu}$

b) It has been shown elsewhere in the text that any $\overline{X_n}$ will converge to $N(0,1)$ if standardized. The theorem we need to use then, is 7.7.6 which states that a function of an already convergent series also converges to an asymptotic normal distribution. (For an almost direct example see Example 7.7.3)

Our $g(y)$ here is $e^{-\overline{X_n}}$ where $g(y) = e^y$. So then $g'(y) = -e^{-y}$ and using 7.7.6 we can find our distribution if $\frac{d}{d\mu}e^{-\mu} = -e^{-\mu}$ then $N(e^{\mu}, \frac{-e^{-2\mu}\mu^2}{n})$

c) From parts (a) we know that $\overline{X_n} \xrightarrow{P} \mu$ and $e^{\overline{-X}} \xrightarrow{P} e^{-\mu}$. So we can use theorem 7.7.3 via section (2), which states that $X_nY_n \xrightarrow{P} cd$. In our case we have the prior two found distributions. So then by the theorem $\overline{X_n}e^{\overline{X_n}} \xrightarrow{P} \mu e^{-\mu}$

08.01 $X_i \sim N(101, 4)$ so then $\mu = 101$ and $\sigma^2 = 4$ or $\sigma = 2$. We just use the CLT to solve this:

$$\begin{aligned}
\Pr\left(20 \text{ bags will weigh at least 1 ton}\right) &= \Pr\left(\sum^{20} X_i \geq 2000\right) \\
&= \Pr\left(\frac{\sum^{20} X_i - 2020}{2\sqrt{20}} \geq \frac{2000-2020}{2\sqrt{20}}\right) \\
&= 1 - \Phi\left(-2.23\right) \\
&\approx 0.987
\end{aligned}$$

08.02 a) Since both S and B are normal variables we may transform them, via theorem 8.3.1 into a new normal variable. The values we need are μ and σ of both S and B. For S we have $\mu = 1$, $\sigma^2 = .0004$, for B $\mu = 1.01$ and $\sigma^2 = .0009$. For the question, we want the probability that $S > B$ so in other words $\Pr(S - B > 0)$. This means $S - B = Y$ is a new normal variable (by theorem), with values $\mu = -.01$ and $\sigma^2 = .0013$. We will use the CLT to solve the probability, so we need $\sigma = .036$. Using that we solve:

$$\begin{aligned}
\Pr\left(S - B > 0\right) &= \Pr\left(Y > 0\right) \\
&= \Pr\left(\frac{Y - (-.01)}{.036} > \frac{0 - (-.01)}{.036}\right) \\
&= \Phi\left(\frac{.01}{.036}\right) \\
&\approx 0.39
\end{aligned}$$

b) We now assume that for S and B that σ^2 are identical for each, but unknown. We do know our desired probability, .95 so we will solve for that instead. Very similar in approach to part (a), we just solve for σ now. Important fact is that the $N(-.01, \sigma^2 + \sigma^2) = N(-.01, 2\sigma^2)$ so $\sigma = \sigma\sqrt{2}$

$$\Phi\left(\frac{.01}{\sigma\sqrt{2}}\right) = .95$$

So we find the value in our table, 1.65 and solve for σ

$$\begin{aligned}
\frac{.01}{\sigma\sqrt{2}} &= 1.65 \\
&\approx .00428
\end{aligned}$$

Problem 8.3 Let $X_1, X_2, ..., X_n$ be a random sample of size n from a normal distribution, X_i is appoximately $N(\mu, \sigma^2)$, and define $U = \sum_{i=1}^{n} X_i$ and $W = \sum_{i=1}^{n} X_i^2$

a) Find a statistic that is a function of U and W and unbiased for the parameter $\theta = 2\mu - 5\sigma^2$:
First we find for μ

$$\begin{aligned}\mu &= E(\bar{x}) \\ &= E(\frac{\sum_{i=1}^{n} X_i}{n}) \\ &= \frac{1}{n}E(U)\end{aligned}$$

Then find for σ^2

$$\begin{aligned}\sigma^2 &= E(S^2) \\ &= E(\frac{\sum_{i=1}^{n} X_i^2 - n\bar{x}^2)}{n-1}) \\ &= \frac{1}{n-1}E(W - n(\frac{(\sum_{i=1}^{n} X_i)^2}{n^2}) \\ &= \frac{1}{n-1}E(W - \frac{U^2}{n}) \\ &= E(\frac{1}{n-1}(W - \frac{U^2}{n}))\end{aligned}$$

thus

$$\begin{aligned}\theta &= 2\mu - 5\sigma^2 \\ &= 2(\frac{1}{n}E(U)) - 5E(\frac{1}{n-1}(W - \frac{U^2}{n})) \\ &= E(\frac{2}{n}(U) - \frac{5}{n-1}(W - \frac{U^2}{n})) \\ &= \frac{2U}{n} - \frac{5}{n-1}(W - \frac{U^2}{n})\end{aligned}$$

which is an unbiased estimator for θ

b) Find a statistic that is unbiased for $\sigma^2 + \mu^2$

$$\mu = E(\frac{\sum_{i=1}^{n} X_i}{n})$$

$$\begin{aligned}
\mu^2 &= (E(\frac{\sum_{i=1}^{n} X_i}{n}))^2 \\
&= E((\frac{\sum_{i=1}^{n} X_i}{n})^2) - Var(\frac{\sum_{i=1}^{n} X_i}{n}) \\
&= E(\frac{1}{n^2}(\sum_{i=1}^{n} X_i)^2) - \frac{1}{n^2} Var(\sum_{i=1}^{n} X_i) \\
&= E(\frac{U^2}{n^2}) - \frac{1}{n^2} n\sigma^2 \\
&= E(\frac{U^2}{n^2}) - \frac{\sigma^2}{n}
\end{aligned}$$

From this then, $\sigma^2 = \frac{1}{n-1}E(W - \frac{U^2}{n})$. then finally the following:

$$\begin{aligned}
\sigma^2 + \mu^2 &= \sigma^2 + E(\frac{U^2}{n^2}) - \frac{\sigma^2}{n} \\
&= \frac{n-1}{n}\sigma^2 + E(\frac{U^2}{n^2}) \\
&= \frac{n-1}{n}\frac{1}{n-1}E(W - \frac{U^2}{n}) + E(\frac{U^2}{n^2}) \\
&= E(\frac{W}{n} - \frac{U^2}{n}) + E(\frac{U^2}{n^2}) \\
&= E(\frac{W}{n}) \\
&= \frac{W}{n}
\end{aligned}$$

which is an unbiased estimator for $\sigma^2 + \mu^2$

c) Let c be a constant, and define $Y_i = 1$ if $X_i \leq c$ and zero otherwise. Find a statistic that is a function of $Y_1, Y_2, ...Y_n$ and also unbiased for $F_x(c) = \phi(\frac{c-\mu}{\sigma})$:

$$\begin{aligned}
P(Y_i = 1) &= P(X_i \leq c) \\
&= P(\frac{X_i - \mu}{\sigma} \leq \frac{c-\mu}{\sigma}) \\
&= P(N(0,1) \leq \frac{c-\mu}{\sigma}) \\
&= \phi(\frac{c-\mu}{\sigma}) \\
&= F_x(c)
\end{aligned}$$

$$
\begin{aligned}
E(Y_i) &= 1(P(Y_i = 1)) + 0(P(Y_i = 0)) \\
&= P(Y_i = 1) \\
&= \phi(\frac{c-\mu}{\sigma})
\end{aligned}
$$

$$
\begin{aligned}
E(\frac{1}{n}\sum_{i=1}^{n} Y_i) &= \frac{1}{n}\sum_{i=1}^{n} E(Y_i) \\
&= \frac{1}{n} n\phi(\frac{c-\mu}{\sigma}) \\
&= \phi(\frac{c-\mu}{\sigma})
\end{aligned}
$$

Then $\frac{1}{n}\sum\limits_{i=1}^{n} Y_i = \bar{Y}$ which is unbiased for $F_x(c) = \phi(\frac{c-\mu}{\sigma})$

08.05 a) We can find the distribution by use of moment generating functions. Since $\sum^{10} T$ then we have $M_{\sum^{10} T}(t)$. Evaluating this out will lead as follows:

$$
\begin{aligned}
M_{\sum^{10} T}(t) &= E\left(e^{t\sum^{10} T}\right) \\
&= \prod^{10} E\left(e^{tT_i}\right) \\
&= \left(\frac{1}{1-100t}\right)^{10}
\end{aligned}
$$

Which is known to be a gamma random variable with parameters $GAM(100, 10)$

b) Let $X \sim GAM(100, 10)$. We need to find $\Pr(X \geq 548)$ as the days in 1.5 years is 548. By using our hint we transform this probability into one in which we are able to find readily (by use of a table).

$$
\begin{aligned}
\Pr(X \geq 548) &= \Pr\left(\frac{2X}{100} \geq \frac{2 * 548}{100}\right) \\
&= \Pr\left(\chi^2 \geq \frac{548}{50}\right)
\end{aligned}
$$

Since our $\nu = 20$ we find that our probability is $1 - .05 = .95$

c) For two years, ie 730 days, we need to find $\Pr(X \geq 730) = .95$. All of our solutions rely on finding a needed quantity in a table. We are solving for the κ of our $GAM(100, \kappa)$ variable. So we get our percentile as $\frac{730}{50} = 14.6$ then using either table 4 or table 5, we find that for the .05 percent we need a $\nu \approx 24$. Actually a little less, but due to the fact we are parts, 24 is needed. Then our $\kappa = \frac{\nu}{2} = 12$

8.8 Suppose that $X \sim \chi^2(m)$ and $Y \sim \chi^2(n)$, and X and Y are independent. Is $X - Y \sim \chi^2$ if $n > m$? No, $X - Y$ can take on negative values and a random variable with a χ^2 distribution cannot.

8.9 Suppose that X is approximately $\chi^2(m)$ and $S = X + Y$ is approximately $\chi^2(m+n)$ and X and Y are independent. Use MGFs to show that $S - X$ is approximately $\chi^2(n)$:

First we have $M_x(t) = (1-2t)^{\frac{-m}{2}}$ and $M_s(t) = (1-2t)^{\frac{-(m+n)}{2}}$. With these then we can get our MGF of interest, so $M_s(t) = M_x(t)M_y(t)$, finally then:

$$\begin{aligned}
M_y(t) &= \frac{M_s(t)}{M_x(t)} \\
&= \frac{(1-2t)^{\frac{-(m+n)}{2}}}{(1-2t)^{\frac{-m}{2}}} \\
&= (1-2t)^{\frac{-n}{2}}
\end{aligned}$$

which is the MGF of $\chi^2(n)$ Thus $Y = S - X$ is approximately $\chi^2(n)$.

08.10 We have $n = 15$. With $\overline{X} = \frac{\sum_{i=1}^{15} Y_i}{15}$ where $Y \sim EXP(\theta)$ we have in fact $\frac{Z}{15}$ where $Z \sim GAM(\theta, 15)$. So to solve our probability, we proceed as follows:

$$\begin{aligned}
\Pr\left(\frac{cZ}{15} \leq \theta\right) &= \Pr\left(15cZ \leq 15\theta\right) \\
&= \Pr\left(\frac{2cZ}{\theta} \leq \frac{15\theta 2}{\theta}\right) \\
&= \Pr\left(\frac{2Z}{\theta} \leq \frac{30}{c}\right)
\end{aligned}$$

We now transform $\frac{2Z}{\theta} \sim \chi^2(30)$. For the probability of .95 we look in our table and find that for $\kappa = 30$ we need a $\gamma = 43.77$ so then $\frac{30}{c} = \gamma = 43.77$ and then $c = .685$

08.12 Since X_1 and X_2 are $\sim N(0, 25)$ we know that being squared they are "close" to the χ^2 distribution. So our goal in finding the probability will be to rework D into something involving χ^2. First, since $D = \sqrt{X_1^2 + X_2^2}$ we will get rid of the square root.

$$\Pr\left(\sqrt{X_1^2 + X_2^2} \leq 12.25\right) = \Pr\left(X_1^2 + X_2^2 \leq (12.25)^2\right)$$

Since $X^2 = (X-0)^2$ and since $\mu = 0$ we can rework the remaining distribution into a χ^2 as follows

$$\begin{aligned}
\Pr\left(X_1^2 + X_2^2 \leq (12.25)^2\right) &= \Pr\left((X-\mu)^2 + (X^2-\mu)^2 \leq (12.25)^2\right) \\
&= \Pr\left(\frac{(X-\mu)^2 + (X-\mu)^2}{\sigma^2} \leq \frac{(12.25)^2}{\sigma^2}\right) \\
&= \Pr\left(\sum^2 \frac{(X-\mu)^2}{\sigma^2} \leq \frac{(12.25)^2}{\sigma^2}\right) \\
&= \Pr\left(\sum^2 \frac{(X-0)^2}{25} \leq \frac{(12.25)^2}{25}\right) \\
&= \Pr\left(\chi^2(2) \leq \frac{(12.25)^2}{25}\right) \\
&= \Pr\left(\chi^2(2) \leq 6\right)
\end{aligned}$$

Using a table we find this probability to be $\approx .95$

08.13 a) We just convert the $\overline{X}$ to standard normal, and evaluate. $\Pr\left(\frac{\overline{X}-0}{\frac{1}{4}} \leq \frac{\frac{1}{2}-0}{\frac{1}{4}}\right)$ which we see is then $\Phi(2)$

b) Let $Y = Z_1 - Z_2$ then $Y \sim N(0,2)$. Thus $\Pr\left(\frac{Y}{\sqrt{2}} \leq \frac{2}{\sqrt{2}}\right)$ which is $\Phi(1.41) \approx .921$

c) By theorem, this is the same as part (b) as our $Y = X_1 + X_2$ is still $Y \sim N(0,2)$

d) By theorem corollary 8.3.2, $\sum^{16} Z^2 \sim \chi^2(16)$ as our $\mu = 0$ and $\sigma^2 = 1$. So then $\Pr\left(\chi^2(16) \leq 32\right) \approx .99$

e) We want to use theorem 8.3.6 part 3. First we divide both by $n-1$ to get S^2. Next we will multiply both sides by $n-1$ and divide by σ^2 which will give us a distribution of $\chi^2(15)$.

$$\begin{aligned}
\Pr\left(\sum^{16}(X-\overline{X})^2 \leq 25\right) &= \Pr\left(\frac{15\sum^{16}(X-\overline{X})^2}{(15)(1)} \leq \frac{15*25}{15*1}\right) \\
&= \Pr\left(\chi^2(15) \leq 25\right)
\end{aligned}$$

Which we find to be $\approx .95$

08.16 a) We make $\overline{X}$ into standard normal then calculate.

$$
\begin{aligned}
\Pr\left(3 < \overline{X} < 7\right) &= \Pr\left(\frac{3-6}{\frac{5}{3}} < \frac{\overline{X}-6}{\frac{5}{3}} < \frac{7-6}{\frac{5}{3}}\right) \\
&= \Pr\left(-1.8 < \Phi < .6\right) \\
&= \Phi(.6) - \Phi(-1.8) \\
&\approx .6898
\end{aligned}
$$

b) By theorem 8.4.3 we have $t(n-1) = t(8)$. So $\Pr(t(8) > 1.86) = 1 - \Pr(t(8) < 1.86) \approx .05$

c) By theorem 8.3.6 we multiply by $n-1$ and divide by σ^2 to give us a $\chi^2(8)$. Then $\Pr(\chi^2(8) \leq 10.22) = .75$

08.17 a) Simply use the table to find $Y(22.31) - Y(7.26)$

b) Using tables, this is found to be 27.14

c) We use some simply manipulations to get Y isolated.

$$
\begin{aligned}
\Pr\left(\frac{Y}{1+Y} \geq \frac{11}{16}\right) &= \Pr\left(\frac{1+Y}{Y} \leq \frac{16}{11}\right) \\
&= \Pr\left(\frac{1}{Y} \leq \frac{16}{11} - 1\right) \\
&= \Pr\left(Y \geq \frac{11}{5}\right) \\
&= 1 - .1 = .90
\end{aligned}
$$

d) Use of tables for $T(2.65) - T(.87)$

e) Again, this is found to be .265 using a table

f) The absolute value will put this as a $1 - T(c) + T(-c) = .02$ or $T(c) = .98 + T(-c)$. So we need $\Pr(T \leq c) = .99$ which found gives us a $c = 2.5$.

g) Use of tables to find the probability.

h) The probability is simplified to $\Pr(X \leq \frac{1}{.25})$ which we find to be .975

08.18 a) By theorem 8.3.4 we know that $V_1 + V_2 = \chi^2(14)$ Thus our result, via table, is $\approx .144$

b) By definition of the t distribution $\frac{Z}{\sqrt{\frac{V_1}{5}}} \sim t(5)$. By table the probability is then .95

c) We need to make the distribution a t distribution. We need to divide by our V_2 and multiply by $\sqrt{9}$ to get the form we need for the t. Thus:

$$
\begin{aligned}
\Pr\left(Z \geq .611\sqrt{V_2}\right) &= \Pr\left(\frac{Z}{\sqrt{\frac{V_2}{9}}} \geq 3 * .611\right) \\
&= \Pr\left(t(9) \geq 1.83\right) \\
&= 1 - .95 = .05
\end{aligned}
$$

d) If we multiply both sides by a $\frac{9}{5}$ we will get an F distribution which we can use a table to calculate the result. Thus, $\Pr(F(5,9) \leq 2.61) = .9$

e) We can convert this to an F distribution with a little work:

$$
\begin{aligned}
\Pr\left(\frac{V_1}{V_1+V_2} \leq b\right) &= \Pr\left(\frac{V_1+V_2}{V_1} \geq \frac{1}{b}\right) \\
&= \Pr\left(1 + \frac{V_2}{V_1} \geq \frac{1}{b}\right) \\
&= \Pr\left(\frac{V_2}{V_1} \geq \frac{1}{b} - 1\right) \\
&= \Pr\left(\frac{5}{9}\frac{V_2}{V_1} \geq \frac{5}{9}\left(\frac{1}{b} - 1\right)\right) \\
&= \Pr\left(F(9,5) \geq \frac{5}{9}\left(\frac{1}{b} - 1\right)\right) \\
&= \Pr\left(F(9,5) \geq \frac{5}{9}\left(\frac{1}{b} - 1\right)\right)
\end{aligned}
$$

We now find our value for an $F(9,5) = .90$ which turns out to be 2.61. We must take the inverse of this since we are $\geq$ and our table is in a format for $\leq$. We now set that equal to our percentile and solve for b.

$$
\begin{aligned}
\frac{1}{2.61} &= \left(\frac{5}{9}\left(\frac{1}{b} - 1\right)\right) \\
\frac{9}{5}\frac{1}{2.61} + 1 &= \frac{1}{b} \\
b &\approx 5.92
\end{aligned}
$$

9.01 Find the MMEs of θ based on a random sample $X_1, ..., X_n$ from each of the following pdf's:

a) $f(x;\theta) = \theta x^{\theta-1}; 0 < x < 1, 0 < \theta$

First set the sample moment to the population moment.

$$\begin{aligned}\overline{X} &= E[X] \\ &= \int_0^1 x\theta x^{\theta-1}dx \\ &= \int_0^1 \theta x^{\theta}dx \\ &= \frac{\theta x^{\theta+1}}{\theta+1}\Big|_0^1 \\ &= \frac{\theta}{\theta+1}\end{aligned}$$

Now solve for θ in terms of $\overline{X}$. $\frac{\theta}{\theta+1} = \overline{X} \longrightarrow \hat{\theta} = \frac{\overline{X}}{1-\overline{X}}$

b) $f(x;\theta) = (\theta+1)x^{-\theta-2}; 1 < x, 0 < \theta$
First set the sample moment to the population moment.

$$\begin{aligned}\overline{X} &= E[X] = \int_1^\infty x(\theta+1)x^{-\theta-2}dx \\ &= \int_1^\infty (\theta+1)x^{-\theta-1}dx \\ &= \frac{\theta+1}{-\theta}x^{-\theta}\Big|_1^\infty \\ &= \frac{\theta+1}{\theta}\end{aligned}$$

Now solve for θ in terms of $\overline{X}$. $\frac{\theta+1}{\theta} = \overline{X} \longrightarrow \hat{\theta} = \frac{1}{\overline{X}-1}$

c) $f(x;\theta) = \theta^2 x e^{-\theta x}; 0 < x, 0 < \theta$
First set the sample moment to the population moment.

$$\begin{aligned}\overline{X} &= E[X] \\ &= \int_0^\infty \theta^2 x^2 e^{-\theta x}dx \\ &= -\theta e^{-\theta x}x^2 - 2xe^{-\theta x} - \frac{2}{\theta}e^{-\theta x}\Big|_0^\infty \\ &= \frac{2}{\theta}\end{aligned}$$

Now solve for θ in terms of $\overline{X}$. $\frac{2}{\theta} = \overline{X} \longrightarrow \hat{\theta} = \frac{2}{\overline{X}}$

9.2 Find Methods of Moments Estimators based on a random sample of size n.

a) $X_i \sim NB(3,p)$
$\mu_1^{'} = \frac{r}{p} = \frac{3}{p} = \overline{X}$
$\hat{p} = \frac{3}{\overline{X}}$

b) $X_i \sim Gam(2, k)$
$\mu_1^{'} = k\theta = 2k = \overline{X}$
$\hat{k} = \frac{\overline{X}}{2}$

c) $X_i \sim Wei(\theta, 1/2)$
$\mu_1^{'} = \theta\Gamma(1 + \frac{1}{\beta}) = \theta\Gamma(1 + \frac{1}{1/2}) = \theta\Gamma(3) = \theta(3-1)! = 2\theta = \overline{X}$
$\hat{\theta} = \frac{\overline{X}}{2}$

d) $X_i \sim DE(\theta, \eta)$
$\mu_1^{'} = \eta = \overline{X}$
$\mu_2^{'} = \sigma^2 + \mu^2 = 2\theta^2 + \eta^2$
$\hat{\theta} = \sqrt{\frac{2\theta^2 + \eta^2 - \overline{X}^2}{2}} = \frac{\hat{\sigma}}{\sqrt{2}}$

e) $X_i \sim EV(\theta, \eta)$
$\mu_1^{'} = \eta - \gamma\theta = \overline{X}$
$\mu_2^{'} = \sigma^2 + \mu^2 = \frac{\pi^2\theta^2}{6} + (\eta - \gamma\theta)^2$
$\hat{\theta} = \frac{\sqrt{6}}{\pi}\hat{\sigma}$
$\hat{\eta} = \overline{X} + \gamma(\frac{\sqrt{6}}{\pi})\hat{\sigma}$

f) $X_i \sim Par(\theta, k)$
$\mu_1^{'} = \frac{\theta}{(k-1)} = \overline{X}$
$\mu_2^{'} = \sigma^2 + \mu^2 = \frac{\theta^2 k}{[(k-2)(k-1)^2]} + \frac{\theta^2}{(k-1)^2}$
$\frac{\hat{k}}{\hat{k}-2} = \frac{\hat{\sigma^2}}{\overline{X}^2} \rightarrow \frac{\frac{2\sigma^2}{\overline{X}^2}}{\frac{\sigma^2}{\overline{X}^2 - 1}}$
$\hat{\theta} = \overline{X}(\hat{k} - 1)$

9.3 Find MLEs for the following pdfs, based on a sample of $X_1, ..., X_n$ random variables. (a),(b), and (c) are all found in a similar manner. We will go over the steps in part (a) and the remaining will just show the work without comment.

a) $f(x;\theta) = \begin{cases} \theta x^{\theta-1} & 0 < x < 1 \\ 0 & otherwise \end{cases}$

$L = \theta^n (x_1 ... x_n)^{\theta-1}$	Set the likelyhood function equal to our joint pdf
$\ln L = n \ln\theta + (\theta - 1)(\ln x_1 ... \ln x_n)$	Now we take the natural log of this to get it in a workable state
$\frac{d}{d\theta} \ln L = \frac{n}{\theta} + (\ln x_1 ... \ln x_n) = 0$	We now set our equation equal to zero to find the estimator

Solving the last equation for θ will yield our estimator, which in this case is $\hat{\theta} = \frac{-n}{(\ln x_1 ... \ln x_n)}$.

b) $f(x;\theta) = \begin{cases} (\theta+1)x^{-\theta-2} & 1 < x \\ 0 & otherwise \end{cases}$

$$\begin{aligned} L &= (\theta+1)^n (x_1 ... x_n)^{-\theta-2} \\ \ln L &= n \ln(\theta+1) - (\theta+2)(\ln x_1 ... \ln x_n) \\ \frac{d}{d\theta} \ln L &= \frac{n}{(\theta+1)} - (\ln x_1 ... \ln x_n) = 0 \\ \hat{\theta} &= \frac{n}{(\ln x_1 ... \ln x_n)} - 1 \end{aligned}$$

c) $f(x;\theta) = \begin{cases} \theta^2 x e^{-\theta x} & 0 < x < 1 \\ 0 & otherwise \end{cases}$

$$\begin{aligned} L &= \theta^{2n} (x_1 ... x_n) e^{-\theta(x_1 ... x_n)} \\ \ln L &= 2n \ln \theta + \ln x_1 + ... + \ln x_n - \theta(x_1 + ... + x_n) \\ \frac{d}{d\theta} \ln L &= \frac{2n}{\theta} - (x_1 ... x_n) = 0 \\ \hat{\theta} &= \frac{2}{\overline{\overline{X}}} \end{aligned}$$

09.05 $f(x;\theta) = \frac{2\theta^2}{x^3}$, $x \geq \theta, \theta > 0$. $L(\theta) = \frac{2^n \theta^{2n}}{(x_1 ... x_n)^3}$ if $x_1 \geq \theta$, that is, if $x_{1:n} \geq \theta, ..., x_n \geq \theta$. Then $L(\theta)$ is maximized at $\hat{\theta} = X_{1:n}$.

9.6 Find MLEs based on a random sample $x_1 ... x_n$ for the following pdfs.

a) $f(x;\theta_1,\theta_2) = \begin{cases} \frac{1}{\theta_1 - \theta_2} & \theta_1 < x < \theta_2 \\ 0 & otherwise \end{cases}$

b) $f(x;\theta,\eta) = \begin{cases} \theta n^\theta x^{-\theta-1} & \eta \leq x, 0 < \theta, 0 < \eta < \infty \\ 0 & otherwise \end{cases}$

9.7 Let $x_1 ... x_n$ be a random sample from $X_i \sim Geo(p)$. Find MLEs for the following.

a) $E(X) = \frac{1}{p}$

b) $Var(X) = \frac{(1-p)}{p^2}$

c) $P[X > k] = (1-p)^k$

9.21 Consider a random sample of size n from a Bernoulli distribution X_i is approximately $BIN(1,p)$

a) find the CRLB for the variances of unbiased estimators of p.

$$f(x;p) = p^x (1-p)^{1-x}$$

$$ln(f(x;p)) = xln(p) + (1-x)ln(1-p)$$

$$(ln(f(x;p)))' = \frac{x}{p} - \frac{1-x}{1-p} = \frac{x-p}{p(1-p)}$$

$$E(((ln(f(x;p)))')^2) = E((\frac{x-p}{p(1-p)})^2) = \frac{1}{(p(1-p))^2}E((x-p)^2)$$

$$= \frac{E(x^2 - 2px + p^2)}{(p(1-p))^2} = \frac{p(1-p) + p^2 - 2p^2 + p^2}{(p(1-p))^2} =$$

$$\frac{1}{p(1-p)}$$

$$CLRB = \frac{(p')^2}{n(\frac{1}{p(1-p)})} = \frac{p(1-p)}{n}$$

b) Find the CRLB for the variances of unbiased estimators of p(1-p).

$$(p(1-p))' = 1 - 2p$$

$$E(((ln(f(x;p)))')^2) = \frac{1}{p(1-p)}$$

$$CLRB = \frac{(1-2p)^2}{n(\frac{1}{p(1-p)})} = \frac{p(1-p)(1-2p)^2}{n}$$

c) Find a UMVUE of p.

$$Var(\bar{x}) = \frac{p(1-p)}{n}$$

This is the same as the CRLB for p, thus, $\bar{x}$ is a UMVUE of p