Solutions to Bain and Engelhardt's Introduction to Probability and Mathematical Statistics

06.01

Given: the pdf of x $f_x(x) = \begin{cases} 4x^3 & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases}$

Find: PDF of a) $Y = X^4$

Setup: Use the CDF technique to get the CDF of Y in terms of a CDF of X
$F_Y(y) = P[Y \leq y] = P[X^4 \leq Y] = P[-y^{\frac{1}{4}} \leq X \leq y^{\frac{1}{4}}] = F_X(y^{\frac{1}{4}}) - F_X(-y^{\frac{1}{4}})$

Steps: i) Differentiate with respect to y to find an equation given in terms of the pdf of x:
$f_y(y) = \frac{d}{dy}F_X(y^{\frac{1}{4}}) - \frac{d}{dy}F_X(-y^{\frac{1}{4}}) = f_x(y^{\frac{1}{4}})\frac{d}{dy}y^{\frac{1}{4}} - f_x(-y^{\frac{1}{4}})\frac{d}{dy}-y^{\frac{1}{4}} = f_x(y^{\frac{1}{4}})\frac{y^{-\frac{3}{4}}}{4} - f_x(-y^{\frac{1}{4}})\frac{-y^{-\frac{3}{4}}}{4}$

ii) Plug in the original limits and function for the pdf of x, and compute the cdf for y

Result: $f_y(y) = \begin{cases} 4y^{\frac{3}{4}}\frac{1}{4y^{\frac{3}{4}}} & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases} = \begin{cases} 1 & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases}$

Find: PDF of b) $W = e^X$

Setup: Use the CDF technique to get the CDF of W in terms of a CDF of X
$F_W(w) = P[W \leq w] = P[e^X \leq W] = P[X \leq lnW] = F_X(lnW)$

Steps: i) Differentiate with respect to w to find an equation given in terms of the pdf of x:
$f_w(w) = \frac{d}{dw}F_X(lnW)\frac{d}{dw}(lnw) = f_x(lnw)\frac{1}{w}$

ii) Plug in the original limits and function for the pdf of x, and compute the cdf for y

Result: $f_W(w) = \begin{cases} \frac{4(lnw)^3}{w} & , \quad 1 < w < e \\ 0 & , \quad o/w \end{cases}$

Find: PDF of c) $Z = \ln x$

Setup: Use the CDF technique to get the CDF of Z in terms of a CDF of X
$F_Z(z) = P[Z \leq z] = P[lnx \leq z] = P[X \leq e^z] = F_X(e^z)$

Steps: i) Differentiate with respect to z to find an equation given in terms of the pdf of x:
$f_z(z) = \frac{d}{dz}F_X(e^z) = f_x(e^z)\frac{de^z}{dz}$

ii) Plug in the original limits and function for the pdf of x, and compute the cdf for y

Result: $f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \leq z < 0 \\ 0 & , \quad o/w \end{cases}$

Find: PDF of d) $U = (X - 0.5)^2$

Setup: Use the CDF technique to get the CDF of U in terms of a CDF of X
$F_U(u) = P[U \leq u] = P[(X-0.5)^2 \leq u] = P[|X-0.5| \leq u^{0.5}] = F_X(u^{1/2} + 1/2) - F_X(-u^{1/2} + 1/2)$

Steps: i) Differentiate with respect to u to find an equation given in terms of the pdf of x:
$f_U(u) = \frac{d}{du} F_X(u^{1/2} + 1/2) = f_x(u^{1/2} + 1/2)\frac{d}{du}(u^{1/2} + 1/2) - f_x(-u^{1/2} + 1/2)\frac{d}{du}(-u^{1/2} + 1/2)$
$f_x(u^{1/2} + 1/2)1/2u^{-1/2}) - f_x(-u^{1/2} + 1/2)1/2u^{-1/2}$

ii) INCOMPLETE

Result: $f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \leq z < 0 \\ 0 & , \quad o/w \end{cases}$

06.02

Given: $X \sim Unif(0,1)$
Find: a) PDF of $Y = X^{1/4}$

Setup: $F_Y(y) = P[Y \leq y] = P[X^{1/4} \leq y] = P[X \leq y^4] = F_X(y^4)$

Steps: i) find the pdf of x. Because X is a Uniform distribution with parameters 1 and 0, the pdf, which for Unif(a,b) is 1/b-a where $a < x < b$. Here, Unif(0,1) gives 1/1-0 =1

ii) Differentiate with respect to y to find an equation given in terms of the pdf of x:
$f_Y(y) = \frac{d}{dy} F_X(y^4) = 4y^3$

Result: $f_Y(y) = \begin{cases} 4y^3 & , \quad 0 < y < 1 \\ 0 & , \quad o/w \end{cases}$

Find: b) PDF of $W = e^{-X}$

Setup: $F_W(z) = P[W \leq w] = P[e^{-X} \leq W] = P[-X \leq lnw] = P[X \geq -lnw] = 1 - F_x(-lnw)$

Steps: i) find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

ii) Differentiate with respect to w to find an equation given in terms of the pdf of x:
$f_W(w) = -\frac{d}{dw}F_X\frac{d}{dw}(-lnw) = -f_X(-lnw)\frac{-1}{w} \quad for \quad e^{-1} < w < 1 = -\frac{1}{w}$

Result: $f_W(w) = \begin{cases} \frac{1}{w} & , \quad e^{-1} < w < 1 \\ 0 & , \quad o/w \end{cases}$

Find: c) PDF of $Z = 1 - e^{-X}$

Setup: $F_Z(z) = P[Z \leq z] = P[1 - e^{-X} \leq z] = P[-e^{-X} \leq z - 1] = P[e^{-X} \geq 1 - z] = P[-X \geq ln(1-z)] = P[X \leq -ln(1-z)] = F_x(-ln(1-z))$

Steps: i) find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

ii) Differentiate with respect to w to find an equation given in terms of the pdf of x:
$f_Z(z) = -ln(1-z) = -\frac{-1}{1-z} = \frac{1}{1-z} \quad for \quad 0 < z < e^{-1}$

Result: $f_W(w) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < e^{-1} \\ 0 & , \quad o/w \end{cases}$

Find: d) PDF of $U = X(1-X)$

Setup: $F_U(u) = P[U \leq u] = P[X(1-x) \leq u] = P[-X^2 + X \leq u] = P[-(X-1/2)^2 \leq u - 1/4] = P[(X-1/2)^2 \geq 1/4 - u] == P[|(X-1/2)| \geq (1/4-u)^{1/2}] =$

Steps: i) find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

ii) INCOMPLETE:
$f_Z(z) = -ln(1-z) = -\frac{-1}{1-z} = \frac{1}{1-z} \quad for \quad 0 < z < e^{-1}$

Result: $f_W(w) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < e^{-1} \\ 0 & , \quad o/w \end{cases}$

06.03

Given: PDF $f_R(r) = \begin{cases} 6r(1-r) & , \quad 0 < r < 1 \\ 0 & , \quad o/w \end{cases}$

Find: Distribution of the circumference

Setup: The circumference is $c = 2\pi r$ - we have the pdf in terms of x, so this is the transformation
$F_C(c) = P[C \leq c] = P[2\pi r \leq c] = P[r \leq c/2\pi] = F_x(c/2\pi)$

Steps: i) Differentiate with respect to c to find an equation given in terms of the pdf of x.
$f_C(c) = \frac{d}{dc}F_R(c/2\pi) = f_R(c/2\pi)\frac{d}{dc}(c/2\pi) = f_R(c/2\pi)(1/2\pi)$

ii) Plug the original pdf back into this new form:
$f_C(c) = \frac{6c}{2\pi}(1-(c/2\pi))(1/2\pi) = \frac{6c(2\pi - c}{2\pi^3} \quad if \quad 0 < c < 2\pi$

Result: $f_C(c) = \begin{cases} \frac{6c(2\pi-c}{2\pi^3} & , \quad 0 < c < 2\pi \\ 0 & , \quad o/w \end{cases}$

Find: Distribution of the area

Setup: The area is $a = \pi r^2$ so the cdf $F_A(a) = P[A \le a] = P[\pi r^2 \le a] = P[r^2 \le a/\pi]$
$= P[|r| \le (a/\pi)^{1/2}] = P[-(a/\pi)^{1/2} \le c \le (a/\pi)^{1/2}] = F_r(a/\pi)^{1/2} - F_r - (a/\pi)^{1/2}$

Steps: i) Differentiate with respect to a to find an equation in terms of the pdf of x.
$f_A(a) = \frac{d}{da}F_R(a/\pi)^{1/2} - \frac{d}{da}F_R - (a/\pi)^{1/2} = f_R[(a/\pi)^{1/2}]\frac{d}{da}(a/\pi)^{1/2} f_R[-(a/\pi)^{1/2}]\frac{d}{da} - (a/\pi)^{1/2}$

Result: $f_A(a) = \begin{cases} \frac{3(\sqrt{\pi}-\sqrt{a}}{\pi^{3/2}} & , \quad 0 < a < \pi \\ 0 & , \quad o/w \end{cases}$

06.10 Suppose X has pdf $f_X(x) = \frac{1}{2}e^{-|x|}$ for all real x.

(a) Find the pdf of $Y = |X|$.
CDF Method
$F_Y(y) = P[Y \le y] = P[|x| \le y] = P[-y \le X \le y] = F_X(y) - F_X(-y)$
$f_Y(y) = \frac{dF_X(y)}{dy} - \frac{dF_X(-y)}{dy}$
$f_Y(y) = f_X(y)\frac{dy}{dx} - f_X(-y)(\frac{-dy}{dy})$
$f_y = \frac{1}{2}e^{-y} + \frac{1}{2}e^{-y} = e^{-y} \; y > 0$

(b) Let $W = 0$ if $X \le 0$ and $W = 1$ if $X > 0$. Find the CDF of W
$F_W(w) = P[W = 0] = \frac{1}{2}$
$F_W(w) = P[W = 1] = \frac{1}{2}$
$F_W(w) =$

$$\begin{cases} 0 & w \le 0 \\ \frac{1}{2} & 0 \le w \le 1 \\ 1 & w > 1 \end{cases}$$

06.13 X has pdf

$$f(x) = \begin{cases} \dfrac{x^2}{24} & -2 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

We want pdf of the CDF $Y = X^2$ with regions: $(-2,0) \cup [0,4)$

$$[F_x(\sqrt{y}) - F_x(-\sqrt{y})] = \left[f_x(\sqrt{y})(\frac{1}{2}\sqrt{y}) - f_x(-\sqrt{y})(-\frac{1}{2}\sqrt{y}) \right]$$

$$f_y(y) = \begin{cases} \dfrac{y}{48\sqrt{y}} + \dfrac{y}{48\sqrt{y}} & 0 < y < 4 \\ \dfrac{y}{48\sqrt{y}} & 4 \le y \le 16 \\ 0 & \text{otherwise} \end{cases}$$

$$f_y(y) = \begin{cases} \dfrac{\sqrt{y}}{24} & 0 < y < 4 \\ \dfrac{\sqrt{y}}{48} & 4 \le y \le 16 \\ 0 & \text{otherwise} \end{cases}$$

06.14

Given: Joint PDF $f(x,y) = \begin{cases} 4e^{-2(x+y)} & , \quad 0 < x < \infty, 0 < y < \infty \\ 0 \quad , & o/w \end{cases}$

Find: a) CDF of W=X+Y

Setup: $F_w(w) = P[W \le w] = P[X + Y \le w]$

Steps:

i) Express as a sum of probabilities, replace probabilities with binomials

ii) Simplify and Use Combinatorial Identity

Result: $\binom{n+m}{k}$

06.15 This is a simplified version of example 6.4.5.

$X_1, X_2 \sim POI(\lambda)$ so the MGF of both is $e^{\lambda(e^t-1)}$. Thus by theorem 6.4.4

$$M_Y(t) = e^{\lambda(e^t-1)}e^{\lambda(e^t-1)} = e^{2\lambda(e^t-1)} \sim POI(2\lambda)$$

The pdf then of Y is

$$f_Y(y) = \begin{cases} \frac{e^{-2\lambda}(2\lambda)^y}{y!} & y = 0, 1, 2, ... \\ 0 & otherwise. \end{cases}$$

06.16 Note: the pdf of $f_{x_1,x_2} = \frac{1}{x_1^2}\frac{1}{x_2^2}$

a) We need to find $f_{u,v} = f_{x_1,x_2}(x_1(u,v), x_2(u,v))|J|$ where J is our jacobian. First we let $u = x_1x_2$ and $v = x_1$ thus $x_1 = v$ and $x_2 = \frac{u}{v}$, now we can find J.

$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & 0 \end{vmatrix} = \frac{1}{v}$$

Finally, our pdf is:

$$\begin{aligned} f_{U,V}(u,v) &= f_{x_1,x_2}(v,\frac{u}{v})\left|\frac{1}{v}\right| \\ &= \frac{1}{v^2}\frac{1}{(\frac{u}{v})^2}\left|\frac{1}{v}\right| \\ &= \frac{1}{u^2v}, 1 < v < u < \infty \end{aligned}$$

6.18 It is given that X and Y have a joint pdf given by

$$f(x,y) = e^{-y} \quad if \quad 0 < x < y < \infty. \tag{1}$$

(a): Find the joint pdf of $S = X + Y$ and $T = X$.

This can be done using the joint transformation method. By rearranging the above formulas we get $X = T$ and $Y = S - T$. Then it is easy to get the jacobian

$$J = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \tag{2}$$

whose determinant is clearly one. Note that the order in which you take partial derivatives is unimportant provided you are consistent - you will get the same determinant either way. Then we substitute in $X = T$ and $Y = S - T$ into the pdf and multiply by the determinant of the jacobian:

$$f_{S,T}(s,t) = f_{X,Y}(x(s,t), y(s,t)) \times 1 = \begin{cases} e^{t-s} & if \quad 0 < t < s/2 \\ 0 & \qquad otherwise \end{cases}. \tag{3}$$

The bounds of the function can be found in a few different ways. One way is to consider the bounds of the original function, $0 < x < y < \infty$. We can substitute in the new formulas for X and Y to get

$$0 < t < s - t < \infty. \tag{4}$$

Then it is apparent that

$$0 < 2t < s < \infty, \tag{5}$$

which then yields

$$0 < t < s/2, \tag{6}$$

the bounds of our new function.

(b): Find the marginal pdf of T.

The easiest way to do this is to "integrate out" S from the joint pdf we derived:

$$\begin{aligned} f_T(t) &= \int_{-\infty}^{\infty} f_{S,T}(s,t)ds = \int_{2t}^{\infty} e^{t-s}ds \\ &= e^t\int_{2t}^{\infty} e^{-s}ds = e^t(-e^{-s}|_{2t}^{\infty}) \\ &= e^{-t} \quad if \quad t > 0. \end{aligned} \tag{7}$$

(c): Find the marginal pdf of S.

This is just like part (b), except this time "integrate out" T:

$$\begin{aligned} f_S(s) &= \int_{-\infty}^{\infty} f_{S,T}(s,t)dt = \int_0^{s/2} e^{t-s}ds \\ &= e^{-s}\int_0^{s/2} e^t dt = e^{-s}(e^t|_0^{s/2}) \\ &= e^{-s}(e^{s/2}-1) \quad if \quad s>0. \end{aligned} \tag{8}$$

6.21 Let X and Y be continuous random variables with a joint density function given by

$$f_{X,Y}(x,y) = 2(x+y) \quad if \quad 0<x<y<1 \quad and \quad 0 \quad otherwise. \tag{9}$$

(a) Find the joint density function of $S = X$ and $T = XY$.

We can solve for X and Y in terms of the new variables, to get $X = S$ and $Y = T/S$. Then the jacobian is given by

$$J = \begin{pmatrix} 1 & 0 \\ -T/S^2 & 1/S \end{pmatrix}. \tag{10}$$

Then the new pdf is given by

$$f_{T,S}(s,t) = f_{X,Y}(x(s,t),y(s,t)) \times |1/s| = \begin{cases} 2(s+t/s)\,|1/s|, & 0<s^2<t<s<1 \\ 0 & otherwise. \end{cases} \tag{11}$$

The bounds of this equation can be interpreted in the following way: the old triangular region in the xy plane got transformed to the region in the st plane between the lines $T = S$ and $T = S^2$.

(b) Find the marginal pdf of T.

To find the marginal of T, s needs to be "integrated out."

$$\begin{aligned} f_T(t) &= \int_{-\infty}^{\infty} f_{S,T}(s,t)ds \\ &= \int_t^{\sqrt{t}} 2(s+t/s)\,|1/s|\,ds = \int_t^{\sqrt{t}} 2(1+t/s^2)ds \\ &= 2\int_t^{\sqrt{t}} ds + 2t\int_t^{\sqrt{t}} 1/s^2 ds = 2(\sqrt{t}-t) + 2t(-1/s|_t^{\sqrt{t}} \\ &= 2\sqrt{t} - 2t + 2 - 2\sqrt{t} = \begin{cases} 2-2t & t \in (0,1) \\ 0 & otherwise. \end{cases} \end{aligned} \tag{12}$$

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06.23 We will use the property that independant identically distributed random variables has the form of 6.4.4, $M_Y(t) = [M_X(t)]^n$ where $Y = X_1+X_2+...+X_n$. then since $X_i \sim GEO(p)$

$$\begin{aligned} Mgf(Y) &= M_{X_1}(t)M_{X_2}(t)...M_{X_k}(t) \\ &= (M_X(t))^k \\ &= (\frac{pe^t}{1-qe^t})^k \sim NegativeBinomial(k,p) \end{aligned}$$

06.25 First note, X_1, X_2, X_3, X_4 are all independant, but they are not IID as only $X_2, X_3, X_4 \sim POI(5)$ with X_1 not being listed. So formula 6.4.5 does not hold. 6.4.4 does though.

A)

$$\begin{aligned} Mgf(Y) &= M_{X_1}(t)M_{X_2+X_3+X_4}(t) \\ &= M_{X_1}(t)(M_{X_i}(t))^3 \end{aligned}$$

Since X_2, X_3, X_4 are iid 6.4.5 holds for moving to this mgf

$$\begin{aligned} &= M_{X_1}(t)(e^{\mu(e^t-1)})^3 \\ &= M_{X_1}(t)e^{3\mu(e^t-1)} \\ &= M_{X_1}(t)e^{15(e^t-1)} \\ e^{25(e^t-1)} &= M_{X_1}(t)e^{15(e^t-1)} \\ \frac{e^{25(e^t-1)}}{e^{15(e^t-1)}} &= M_{X_1}(t) \\ e^{10(e^t-1)} &= M_{X_1}(t) \sim POI(10) \end{aligned}$$

B) For $W = X_1 + X_2$ we have $X_1 \sim POI(10)$ and $X_2 \sim POI(5)$. So $POI(10+5) = POI(15)$
06.29

Given: PDF $f(x) = \begin{cases} \frac{1}{x^2} & , \quad 1 \le x < \infty, 0 < y < \infty \\ 0 & , \quad o/w \end{cases}$

Find: a) Joint PDF of the order statistics

Setup: $F_w(w) = P[W \le w] = P[X + Y \le w]$

Steps: i) Differentiate with respect to a to find an equation in terms of the pdf of x.
$f_A(a) = \frac{d}{da}F_R(a/\pi)^{1/2} - \frac{d}{da}F_R - (a/\pi)^{1/2} = f_R[(a/\pi)^{1/2}]\frac{d}{da}(a/\pi)^{1/2}f_R[-(a/\pi)^{1/2}]\frac{d}{da} - (a/\pi)^{1/2}$

ii) Simplify and Use Combinatorial Identity

Result: $\binom{n+m}{k}$

Find: b) PDF of the smallest order statisticY_1

Setup:

Steps: i)

Result:

Find: c) PDF of the largest order statistic Y_n

Setup:

Steps: i)

Result:

Find: d) PDF of the sample range $R = Y_n - Y_1, \quad forn = 2$

Setup: The area is $a = \pi r^2$ so the cdf $F_A(a) = P[A \leq a] = P[\pi r^2 \leq a] = P[r^2 \leq a/\pi]$ $= P[|r| \leq (a/\pi)^{1/2}] = P[-(a/\pi)^{1/2} \leq c \leq (a/\pi)^{1/2}] = F_r(a/\pi)^{1/2} - F_r - (a/\pi)^{1/2}$

Steps: i)

Result:

Find: e) PDF of the sample median $R = Y_r - Y_1, \quad fornoddsothatr = (n+1)/2$

Setup:

Steps: i)

Result: **06.35** Suppose X_1, X_2 are independent exponentially distributed random variables $X_i \sim$EXP(θ), and let $Y = X_1 - X_2$.
(a) Find the MGF of Y.
We can think of $Y = X_1 - X_2$ as $Y = X_1 + (-1)X_2$. Then using Theorem 6.4.1,

$$\begin{aligned}
M_Y(t) &= (M_{X_1}(t))(M_{-X_2}(t)) \\
M_Y(t) &= (M_{X_1}(t))(M_{X_2}(-t)) \\
M_Y(t) &= \left(\frac{1}{1-\theta t}\right)\left(\frac{1}{1-\theta(-t)}\right) \\
M_Y(t) &= \left(\frac{1}{1-\theta t}\right)\left(\frac{1}{1+\theta t}\right) \\
M_Y(t) &= \frac{1}{1-\theta t+\theta t-\theta^2 t^2} \\
M_Y(t) &= \frac{1}{1-\theta^2 t^2}
\end{aligned}$$

(b) What is the distribution of Y?
Since $\frac{1}{1-\theta^2 t^2}$ is the MGF of a double exponential, $Y \sim$ DE(θ,0).

07.01 Consider a random sample of size n from a distribution with $CDFF(x) = 1 - \frac{1}{x} if 1 \leq x \leq \infty$

(a) Derive the CDF of the smallest order statistic, $X_{1:\,n}$

Solution: $G_1(y_1) = 1 - [1 - F_X(y_1)]^n = 1 - [1 - [1 - \frac{1}{y_1}]]^n = 1 - [\frac{1}{y_1}]^n$

$G_1(y_1) = \begin{cases} 1 - \frac{1}{[y_1]^n} & \text{if } 1 \leq y_1 \\ 0 & \text{if } 0 > y_1. \end{cases}$

(b) Find the limiting distribution of $X_{1:\,n}$ Solution:

$\lim_{n\to\infty} 1 - \frac{1}{{y_1}^n} = \begin{cases} 1 & \text{if } y_1 > 1 \\ 0 & \text{if } y_1 \leq 1 \end{cases}$

The limiting distribution of $X_{1:\,n}$ is degenerate at $y = 1$

(c) Find the limiting distribution of ${X^n}_{1:\,n}$

Solution:

$F_{{X^n}_{1:\,n}}(y) = P({X^n}_{1:\,n} \leq y) = P(X_{1:\,n} \leq y^{\frac{1}{n}}) = F_{X_{1:\,n}}(y^{\frac{1}{n}}) = 1 - \frac{1}{{y^{\frac{1}{n}}}^n} = 1 - \frac{1}{y_n}$

then, the limiting distribution of ${X^n}_{1:\,n} = \begin{cases} 1 - \frac{1}{y_n} & \text{if } y > 1 \\ 0 & \text{if } otherwise \end{cases}$

07.02

$$F(x) = \left\{ -\frac{1}{x^2}, \text{all real x} \right.$$

2a. $F_{X_{n:n}}(y) = (\frac{1}{1+e^{-y}})^n; \lim_{n\to\infty}(\frac{1}{1+e^{-y}})^n$ has no limiting distribution.

2b. $F_{X_{n:n}-\ln(n)}(y) = P[X_{n:n} - \ln(n) \leq y] = P[X_{n:n} \leq y + \ln(n)]$

$= F_{X_{n:n}}(y + \ln(n))^n = (\frac{1}{1+e^{-(y+\ln(n))}})^n = (\frac{1}{1+\frac{e^{-y}}{n}})^n;$

$\lim_{n\to\infty}(\frac{1}{1+\frac{e^{-y}}{n}})^n = e^{-e^{-y}}$

07.03 3a. $F(x) = \begin{cases} 1 - \dfrac{1}{x^2}, x > 1 \\ 0, x \leq 0 \end{cases}$

$F_{X_{1:n}}(y) = P[X_{1:n} \leq y] = 1 - P[X_{1:n} \geq y] = 1 - \frac{1}{y^{2n}}, y > 1$

$\lim_{n\to\infty}(1 - \frac{1}{y^{2n}}) = 1 - 0 = \begin{cases} 1, y > 1 \\ 0, y \leq 0 \end{cases}$

3b.
$F_{X_{n:n}}(y) = P[X_{n:n} \leq y] = 1 - P[X_{n:n} \geq y] = 1 - (1 - \frac{1}{y^2})^n = \frac{1}{y^{2n}}; lim?_{n\to\infty} \frac{1}{y^{2n}} = 0,$
Therefore $F_{X_{n:n}}(y)$ has no limiting distribution.

3c.
$F_{n^{-\frac{1}{2}}X_{n:n}}(y) = P[\frac{1}{\sqrt{n}}X_{n:n} \le y] = P[X_{n:n} \le \sqrt{n}y] = F_{X_{n:n}}(\sqrt{n}y) = (1-(\sqrt{n}y)^{-2})^n, for y > \frac{1}{\sqrt{n}}$

$\lim_{n\to\infty}(1-(\sqrt{n}y)^{-2})^n = (1-\frac{1}{ny^2})^n = \begin{cases} e^{-y^{-2}}, y > 0 \\ 0, y \le 0 \end{cases}$

07.07 The WEI $(1,2)$ distribution has pdf $f(x) = 2xe^{-x^2}$ for $x > 0$,mean $\mu = \Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$ and variance $\sigma^2 = \Gamma(2) - \Gamma(\frac{3}{2})^2 = 1 - \frac{\pi}{4}$
(a)According to the central limit theorem, this holds with $a = \mu - \frac{1.96\sigma}{\sqrt{n}}$ and $b = \mu + \frac{1.96\sigma}{\sqrt{n}}$, so if $n = 35$ we have $a = 0.7328$ and $b = 1.0397$
(b)For odd n, $X_{\frac{n+1}{2}:n}$ is approximately $N(x_{\frac{1}{2}}, \frac{c^2}{n})$, where $c^2 {=} \frac{1}{4f(x_{\frac{1}{2}})^2}$. Now $F(x_{\frac{1}{2}}) = \frac{1}{2}$, because $F(x) = 1 - e^{-x^2}$, it implies that $x_{\frac{1}{2}} = \sqrt{ln2}$. Also because $c^2 = \frac{1}{4ln2}$, we have $a = x_{\frac{1}{2}} - \frac{1.96c}{\sqrt{n}}$ and $b = x_{\frac{1}{2}} + \frac{1.96c}{\sqrt{n}}$, so when $n = 35$ we have $a = 0.6336$ and $b = 1.0315$
07.11 a) First we need to know the μ and the σ. For a Uniform variable with $a = 0, b = 1$ we have $\mu = 1/2$ and $\sigma = 1/\sqrt{12}$ (Note: it is not σ^2). We also need to know that $n = 20$ from there we can use the CLT:

$$\begin{aligned}
\Pr\left(\sum^{20} X_i < 12\right) &= \Pr\left(\frac{\sum X_i - 10}{\sqrt{20}\frac{1}{\sqrt{12}}} < \frac{12-10}{\sqrt{20}\frac{1}{\sqrt{12}}}\right) \\
&= \Phi\left(\frac{12-10}{\sqrt{20}\frac{1}{\sqrt{12}}}\right) \\
&\approx .9394
\end{aligned}$$

b) We let $Y = \sum^{20} X_i$, let Y' be our 90th percentile that we want to find. So we setup our probability as $\Pr(Y \le Y') = .9$, .9 as we are interested in the 90th percentile. Using μ, σ, and n from part (a) we solve with CLT:

$$\begin{aligned}
\Pr(Y \le Y') &= \Pr\left(\frac{Y - \mu n}{\sigma\sqrt{n}} \le \frac{Y' - \mu n}{\sigma\sqrt{n}}\right) \\
&= \Pr\left(Z \le \frac{Y' - 10}{\sqrt{20}\frac{1}{\sqrt{12}}}\right) \text{Note: Z is standard normal due to CLT} \\
.9 &= \Phi\left(\frac{Y' - 10}{\sqrt{20}\frac{1}{\sqrt{12}}}\right)
\end{aligned}$$

We now solve for Y'. We know (from a chart or list) that .9 from Φ is $z \approx 1.285$. So we set our final equation for finding out Y' with that in mind.

$$\begin{aligned}
\frac{Y' - 10}{1.291} &= 1.285 \\
Y' &\approx 11.658
\end{aligned}$$

07.12 a) First, an understanding that the wording here implies that X is actually "failures" of weapons. So the given p would normally be q in other contexts. So using the binomial theorem we would have $p = .05$ and $q = .95$. Knowing that we can use the Binomial theorem easily:

$$\begin{aligned}\Pr(X \geq 1) &= 1 - \Pr(X < 1)\\ &= 1 - \binom{n}{0}(.05)^0(.95)^n\end{aligned}$$

We now solve for n from the above equation knowing that the desired probability is .99

$$\begin{aligned}.99 &= 1 - (.95)^n\\ \ln .95^n &= \ln .01\\ n &= \frac{\ln .01}{\ln .95}\end{aligned}$$

So n, since it must be an integer, is rounded to 90.
b)
07.13 From the hint we know that $Y_n = \sum^n X_i$ where $X_i \sim Geo(p)$. So for $\sum^n X_i$ the $\mu = \frac{n}{p}$ and $\sigma^2 = \frac{nq}{p^2}$. Then by the CLT:

$$\begin{aligned}\Pr(Y_n \leq y) &= \Pr\left(\sum^n X_i \leq y\right)\\ &= \Pr\left(\frac{\sum^n X_i - \frac{n}{p}n}{\sqrt{n}\sqrt{\frac{nq}{p^2}}} \leq \frac{y - \frac{n}{p}n}{\sqrt{n}\sqrt{\frac{nq}{p^2}}}\right)\\ &= \Phi\left(\frac{y - \frac{n}{p}n}{\sqrt{n}\sqrt{\frac{nq}{p^2}}}\right)\end{aligned}$$

07.16 a) We need two things for this proof. First, we need to know μ and σ^2 of $\overline{X}$. We know this is $\mu = \mu$ and $\sigma^2 = \frac{\mu^2}{n}$ from facts of the sample mean distribution of $POI(\mu)$. Next the theorems from section 7.6, namely 7.6.2 and from 7.7, 7.7.2. These will let us prove the following:

$$\begin{aligned}\Pr\left[|\overline{X_n} - \mu| < \epsilon\right] &\geq 1 - \frac{\mu^2}{\epsilon^2 n}\\ \lim_{n\to\infty} \Pr\left[|\overline{X_n} - \mu| < \epsilon\right] &= 1\end{aligned}$$

From this we now know that $\overline{X} \xrightarrow{P} \mu$ from 7.6.3. For our goal, $e^{\overline{X_n}}$ we simply need to know 7.7.2. Since $\overline{X} \xrightarrow{P} \mu$ then $e^{\overline{X}} \xrightarrow{P} e^{\mu}$

b) It has been shown elsewhere in the text that any $\overline{X_n}$ will converge to $N(0,1)$ if standardized. The theorem we need to use then, is 7.7.6 which states that a function of an already convergent series also converges to an asymptotic normal distribution. (For an almost direct example see Example 7.7.3)

Our $g(y)$ here is $e^{-\overline{X_n}}$ where $g(y) = e^y$. So then $g'(y) = -e^{-y}$ and using 7.7.6 we can find our distribution if $\frac{d}{d\mu}e^{-\mu} = -e^{-\mu}$ then $N(e^{\mu}, \frac{-e^{-2\mu}\mu^2}{n})$

c) From parts (a) we know that $\overline{X_n} \xrightarrow{P} \mu$ and $e^{\overline{-X}} \xrightarrow{P} e^{-\mu}$. So we can use theorem 7.7.3 via section (2), which states that $X_nY_n \xrightarrow{P} cd$. In our case we have the prior two found distributions. So then by the theorem $\overline{X_n}e^{\overline{X_n}} \xrightarrow{P} \mu e^{-\mu}$

08.01 $X_i \sim N(101, 4)$ so then $\mu = 101$ and $\sigma^2 = 4$ or $\sigma = 2$. We just use the CLT to solve this:

$$\begin{aligned}
\Pr\left(20 \text{ bags will weigh at least 1 ton}\right) &= \Pr\left(\sum^{20} X_i \geq 2000\right) \\
&= \Pr\left(\frac{\sum^{20} X_i - 2020}{2\sqrt{20}} \geq \frac{2000-2020}{2\sqrt{20}}\right) \\
&= 1 - \Phi\left(-2.23\right) \\
&\approx 0.987
\end{aligned}$$

08.02 a) Since both S and B are normal variables we may transform them, via theorem 8.3.1 into a new normal variable. The values we need are μ and σ of both S and B. For S we have $\mu = 1$, $\sigma^2 = .0004$, for B $\mu = 1.01$ and $\sigma^2 = .0009$. For the question, we want the probability that $S > B$ so in other words $\Pr(S - B > 0)$. This means $S - B = Y$ is a new normal variable (by theorem), with values $\mu = -.01$ and $\sigma^2 = .0013$. We will use the CLT to solve the probability, so we need $\sigma = .036$. Using that we solve:

$$\begin{aligned}
\Pr\left(S - B > 0\right) &= \Pr\left(Y > 0\right) \\
&= \Pr\left(\frac{Y-(-.01)}{.036} > \frac{0-(-.01)}{.036}\right) \\
&= \Phi\left(\frac{.01}{.036}\right) \\
&\approx 0.39
\end{aligned}$$

b) We now assume that for S and B that σ^2 are identical for each, but unknown. We do know our desired probability, .95 so we will solve for that instead. Very similar in approach to part (a), we just solve for σ now. Important fact is that the $N(-.01, \sigma^2+\sigma^2) = N(-.01, 2\sigma^2)$ so $\sigma = \sigma\sqrt{2}$

$$\Phi\left(\frac{.01}{\sigma\sqrt{2}}\right) = .95$$

So we find the value in our table, 1.65 and solve for σ

$$\begin{aligned}\frac{.01}{\sigma\sqrt{2}} &= 1.65 \\ &\approx .00428\end{aligned}$$