23. $X_1, X_2, X_3, \ldots, X_k$ LET $Y = X_1 + X_2 + X_3 + \ldots + X_K$

IF $X_i \sim GEO(p)$ FIND MGF OF Y, WHAT IS DIST. OF Y

$f_X(x) = p \cdot q^{x-1} \quad x = 1, 2, \quad M_X(t) = \dfrac{pe^t}{1 - qe^t}$

$M_Y(t) = E(e^{tY})$

$M_Y(t) = E(e^{tX_1}, e^{tX_2}, \ldots, e^{tX_K})$

$M_Y(t) = M_{X_1}(t), M_{X_2}(t), \ldots, M_{X_K}(t)$

$M_Y(t) = \left(\dfrac{pe^t}{1-qe^t}\right)^K \Rightarrow M_Y(t) \sim NB(K, p)$

---

25. LET X_1, X_2, X_3 AND X_4 i RV

$X_2 = X_3 = X_4 \sim POI(5)$ NOW $Y = X_1 + X_2 + X_3 + X_4 \sim POI(2$

a) X_1

$Y = X_1 + 3\,POI(5) \sim POI(25)$

$Y = X_1 + POI(15) \Rightarrow X_1 \sim POI(25) - POI(15) \Rightarrow X_1 \sim POI(10)$

$X_1 \sim POI(10)$

b). $W = X_1 + X_2$

$W \sim POI(10) + POI(5)$

$W \sim POI(15)$

29) SAMPLE SIZE: n

pdf $f(x) = 1/x^2 \qquad 1 \leq x < \infty$

$\qquad\qquad 0 \qquad\qquad$ o.w.

RANGE SIZE → $\underbrace{\quad}_{k-1}$ $\quad 1 \quad$ $\underbrace{\quad}_{n-k}$

RV → $y_1 \; \ldots \; y_{k-1} \quad y_k \quad y_{k+1} \; \ldots \; y_n$

CDF/pdf → $F_Y(y_k) \qquad f_Y(y_k) \qquad 1 - F_Y(y_k)$

PERM. → $(k-1)! \qquad 1! \qquad (n-k)!$

SIMILAR TO A BINOMIAL WHERE $x = 1$

a) $f_Y(y_1, \ldots, y_n) = n! \, f_X(y_1) \times \ldots \times f_X(y_n)$

$$= n! \, \frac{1}{y_1^2} \times \cdots \times \frac{1}{y_n^2}$$

$$= \frac{n!}{(y_1 \cdot y_2 \cdots y_n)^2} \qquad 1 < y_1 < y_2 < \ldots < y_n < \infty$$

b) $f_{Y_1}(y_1) = \dfrac{n!}{(n-1)!} \cdot \left[1 - F_{Y_1}(y_1)\right]^{n-1} \cdot f_{Y_1}(y_1)$

$$= \frac{n \cdot (n-1)!}{(n-1)!} \cdot \left(1 - \int_1^{y_1} x^{-2}\, dx\right)^{n-1} \cdot \frac{1}{y_1^2} = n \left[1 - \left(-1 \cdot \frac{1}{x}\right)\Big|_1^{y_1}\right]^{n-1} \cdot \frac{1}{y_1^2}$$

$$= n \left[1 - \left(-\frac{1}{y_1} + 1\right)\right]^{n-1} \cdot \frac{1}{y_1^2} \Rightarrow f_{Y_1}(y_1) = n \left(1/y_1\right)^{n-1} \cdot 1/y_1^2 \qquad 1 < y_1 < \infty$$

c) $f_Y(y_n)$

$$f_{y_n} = \frac{n!}{(n-1)!}\left[F_{y_n}(y_n)\right]^{n-1} \cdot f_y(y_n)$$

$$= n\left[\int_1^{y_n} \frac{1}{x^2}\,dx\right]^{n-1} \cdot \frac{1}{y_n^2} = n \cdot \left(-\frac{1}{x}\Big|_1^{y_n}\right)^{n-1} \cdot \frac{1}{y_n^2}$$

$$f_{y_n} = n \cdot (1 - 1/y_n)^{n-1} \cdot 1/y_n^2 \qquad 1 < y_n < \infty$$

d) $R = Y_n - Y_1$ FOR $n = 2$

$$f_R(y_n, y_1) = \frac{n!}{(n-2)!} \cdot \frac{1}{y_1^2} \cdot \frac{1}{y_n^2} \cdot \left[\int_1^{y_n} \frac{1}{x^2}\,dx - \int_1^{y_1} \frac{1}{x^2}\,dx\right]^{n-2}$$

$$f_R(y_n, y_1) = \frac{n}{(n-2)!} \cdot \frac{1}{(y_1 y_n)^2} \cdot \left[-2(y_n-1)^2 + 2(y_1-1)\right]^{n-2}$$

$$f_R(y_n - y_1) = n.(n-1).(y_1 y_n)^{-2}.\,2(y_1 - y_n)^{n-2}$$

$s = y_1 \Rightarrow y_1 = s$

$r = y_n - s \quad y_n = r + s$

$$= f_{SR}(r,s) = n(n-1)(s.(r+s))^{-2}.\,2(s - r - s)^{n-2}$$

$$f_{RS}(r,s) = n(n-1)\,(sr + s^2)^{-2}.\,2r^{(n-2)} \qquad 1 < s < r = \infty$$

THE MARGINAL DENSITY OF R

$$f_R(r) = \int_1^r n(n-1)(sr + s^2)^{-2}.\,2r^{(n-2)}\,ds = \frac{-(n-1)nr^{n-5}\left(r\left(\frac{1}{r+s} + \frac{1}{s}\right) - 2\ln(r+s) + 2\ln\right.}{}$$

e) Y_R WHERE $r = \frac{(n+1)}{2}$

$$f_Y(y_R) = \frac{n!}{(R-1)!\,(n-R)!} \cdot F_Y(y_R)^{R-1}\; f_Y(y_R).\;(1 - F_Y(y_R))^{n-R}$$

$$= \frac{n!}{\left(\frac{n+1}{2} - 1\right)!\left(n - \frac{(n+1)}{2}\right)!} \cdot \left[\int_1^{y_R} \frac{1}{x^2}\,dx\right]^{R-1} \cdot \frac{1}{y_1^2} \cdot \left[1 - \int_1^{y_R} \frac{1}{x^2}\,dx\right]^{n-R}$$

$$= \frac{n!}{\left(\frac{n-1}{2}\right)!^2} \cdot (1 - 1/y_R)^{\frac{n-1}{2}} \cdot \frac{1}{y_R^2} \cdot \left(\frac{1}{y_R}\right)^{\frac{n-1}{2}} \qquad 1 < R < \infty$$

31). SIZE n $X_i \sim EXP(1)$ FIND pdf OF.

$f_X(x_i) = e^{-x_i}$ $x > 0$

a). Y_1

$$f(y_1) = \frac{n!}{(n-1)!} \cdot [1 - F(y_1)]^{n-1} \cdot f(y_1)$$

$$f(y_1) = n \cdot \left[1 - \int_0^{y_1} e^{-x} \, dx\right]^{n-1} \cdot e^{-y_1} = n \left[1 - \left(-e^{-x} \Big|_0^{y_1}\right)\right]^{n-1} \cdot e^{-y_1}$$

$$= n \cdot \left[1 - \left(1 - e^{-y_1}\right)\right]^{n-1} \cdot e^{-y_1} = n \cdot e^{-y_1(n-1)} \cdot e^{-y_1}$$

$$= n e^{-ny_1} \qquad y_1 > 0$$

b) Y_n

$$f(y_n) = \frac{n!}{(n-1)!} \cdot F(y_n)^{n-1} \cdot f(y_n)$$

$$f(y_n) = n \cdot \left[\int_0^{y_n} e^{-x} \, dx\right]^{n-1} \cdot e^{-y_n}$$

$$= n \left(-e^{-x} \Big|_0^{y_n}\right)^{n-1} \cdot e^{-y_n} = n \cdot \left(-e^{-y_n} + 1\right)^{n-1} \cdot e^{-y_n}$$

$$= n \left(1 - e^{-y_n}\right)^{n-1} \cdot e^{-y_n} \qquad y_n > 0$$

c). $R = Y_n - Y_1$

$$f_{1,n}(y_1, y_n) = \frac{n!}{(n-2)!} \cdot f(y_1) \cdot f(y_n) \cdot [F(y_n) - F(y_1)]^{n-2} \qquad 0 < y_1 < y_n < \infty$$

$$f_{x_1 x_n}(y_1, y_n) = \frac{n!}{(n-2)!} \cdot e^{-y_1} \cdot e^{-y_n} \cdot \left[\int_0^{y_n} e^{-x} dx - \int_0^{y_1} e^{-x} dx\right]^{n-2}$$

$$= n(n-1) \cdot e^{-(y_1 + y_n)} \cdot [(1 - e^{-y_n}) - (1 - e^{-y_1})]^{n-2}$$

$$= n \cdot (n-1) \cdot e^{-(y_1 + y_n)} \cdot (e^{-y_1} - e^{-y_n})^{n-2}$$

$s = y_1 \quad \rightarrow \quad y_1 = s$

$r = y_n - s \qquad y_n = r + s$

$$f_{RS}(r, s) = f_{x_1 x_n}(s, r+s) = n \cdot (n-1) \cdot e^{-(s + r + s)} \cdot (e^{-s} - e^{-(r+s)})^{n-2}$$

$$= n(n-1)\, e^{-(r+2s)} \left(e^{-s}(1 - e^{-r})\right)^{n-2}$$

$$= n(n-1)\, e^{-(r+2s)} \cdot e^{-(sn - 2s)} \cdot (1 - e^{-r})^{n-2} = n(n-1) \cdot e^{-(r + sn)} \cdot (1 - e^{-R})^{n}$$

FOR $0 < s < r < \infty$

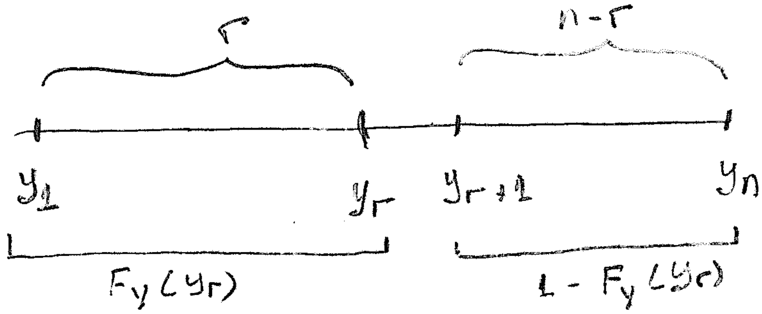
MARGINAL OF R

$$f_R(r) = \int_0^\infty n \cdot (n-1) \cdot e^{-r} \cdot (1 - e^{-r})^{n-2} \cdot e^{-sn}\, ds$$

$$= n(n-1) \cdot e^{-r} \cdot (1 - e^{-r})^{n-2} \cdot - \frac{e^{-sn}}{n} \Big|_0^\infty$$

$$f_R(r) = (n-1)\, e^{-r} \cdot (1 - e^{-r})^{n-2} \cdot 1 \qquad r > 0$$

d) FIRST r ORDER STATISTICS, $Y_1, \dots, Y_r$



$$f(y_1, \dots, y_r) = \frac{n!}{(n-r)!} \cdot e^{-(y_1 + \dots + y_r)} \cdot \left(e^{-y_r}\right)^{n-r}$$

35. X_1, X_2 $\quad X \sim EXP(\theta)$ $\quad Y = X_1 - X_2$

$f_X(x) = \frac{1}{\theta} e^{-x/\theta}$ $\quad M_X(t) = \frac{1}{1-\theta t}$

a) FIND MGF OF Y, DISTRIBUTION

$$M_Y(t_1, t_2) = E(e^{t_1 X_1 - t_2 X_2})$$

$$= M_{X_1}(t_1) \cdot M_{X_2}(t_2) = \frac{1}{1-\theta t_1} \cdot \frac{1}{1+\theta t_2}$$

$$= \frac{1}{1 + \theta t_2 - \theta t_1 - \theta^2 t_1 t_2} \Rightarrow \text{FOR } t_1 = t_2 \Rightarrow$$

$$M_Y(t) = \frac{1}{1-\theta^2 t^2} \Rightarrow Y \sim DE(\theta, 0)$$