# Solutions to Bain and Engelhardt's Introduction to Probability and Mathematical Statistics

**06.01**

## Given: the pdf of x $f_x\left(x\right) = \begin{cases} 4x^3 & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases}$

## Find: PDF of a) $Y = X^4$

**Setup:** Use the CDF technique to get the CDF of Y in terms of a CDF of X
$F_Y(y) = P[Y \leq y] = P[X^4 \leq Y] = P[-y^{\frac{1}{4}} \leq X \leq y^{\frac{1}{4}}] = F_X(y^{\frac{1}{4}}) - F_X(-y^{\frac{1}{4}})$

**Steps: i)** Differentiate with respect to y to find an equation given in terms of the pdf of x:
$f_y(y) = \frac{d}{dy}F_X(y^{\frac{1}{4}}) - \frac{d}{dy}F_X(-y^{\frac{1}{4}}) = f_x(y^{\frac{1}{4}})\frac{d}{dy}y^{\frac{1}{4}} - f_x(-y^{\frac{1}{4}})\frac{d}{dy}-y^{\frac{1}{4}} = f_x(y^{\frac{1}{4}})\frac{y^{-\frac{3}{4}}}{4} - f_x(-y^{\frac{1}{4}})\frac{-y^{-\frac{3}{4}}}{4}$

**ii)** Plug in the original limits and function for the pdf of x, and compute the cdf for y

**Result:** $f_y(y) = \begin{cases} 4y^{\frac{3}{4}}\frac{1}{4y^{\frac{3}{4}}} & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases} = \begin{cases} 1 & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases}$

## Find: PDF of b) $W = e^X$

**Setup:** Use the CDF technique to get the CDF of W in terms of a CDF of X
$F_W(w) = P[W \leq w] = P[e^X \leq W] = P[X \leq lnW] = F_X(lnW)$

**Steps: i)** Differentiate with respect to w to find an equation given in terms of the pdf of x:
$f_w(w) = \frac{d}{dw}F_X(lnW)\frac{d}{dw}(lnw) = f_x(lnw)\frac{1}{w}$

**ii)** Plug in the original limits and function for the pdf of x, and compute the cdf for y

**Result:** $f_W(w) = \begin{cases} \frac{4(lnw)^3}{w} & , \quad 1 < w < e \\ 0 & , \quad o/w \end{cases}$

## Find: PDF of c) $Z = \ln x$

**Setup:** Use the CDF technique to get the CDF of Z in terms of a CDF of X
$F_Z(z) = P[Z \leq z] = P[lnx \leq z] = P[X \leq e^z] = F_X(e^z)$

**Steps: i)** Differentiate with respect to z to find an equation given in terms of the pdf of x:
$f_z(z) = \frac{d}{dz}F_X(e^z) = f_x(e^z)\frac{de^z}{dz}$

**ii)** Plug in the original limits and function for the pdf of x, and compute the cdf for y

**Result:** $f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \leq z < 0 \\ 0 & , \quad o/w \end{cases}$

## Find: PDF of d) $U = (X - 0.5)^2$

**Setup:** Use the CDF technique to get the CDF of U in terms of a CDF of X
$F_U(u) = P[U \leq u] = P[(X-0.5)^2 \leq u] = P[|X - 0.5| \leq u^{0.5}] = F_X(u^{1/2} + 1/2) - F_X(-u^{1/2} + 1/2)$

**Steps: i)** Differentiate with respect to u to find an equation given in terms of the pdf of x:
$f_U(u) = \frac{d}{du} F_X(u^{1/2} + 1/2) = f_x(u^{1/2} + 1/2)\frac{d}{du}(u^{1/2} + 1/2) - f_x(-u^{1/2} + 1/2)\frac{d}{du}(-u^{1/2} + 1/2)$
$f_x(u^{1/2} + 1/2)1/2u^{-1/2}) - f_x(-u^{1/2} + 1/2)1/2u^{-1/2}$

**ii)** INCOMPLETE

**Result:** $f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \leq z < 0 \\ 0 & , \quad o/w \end{cases}$

**06.02**

## Given: $X \sim Unif(0,1)$
## Find: a) PDF of $Y = X^{1/4}$

**Setup:** $F_Y(y) = P[Y \leq y] = P[X^{1/4} \leq y] = P[X \leq y^4] = F_X(y^4)$

**Steps: i)** find the pdf of x. Because X is a Uniform distribution with parameters 1 and 0, the pdf, which for Unif(a,b) is 1/b-a where $a < x < b$. Here, Unif(0,1) gives 1/1-0 =1

**ii)** Differentiate with respect to y to find an equation given in terms of the pdf of x:
$f_Y(y) = \frac{d}{dy} F_X(y^4) = 4y^3$

**Result:** $f_Y(y) = \begin{cases} 4y^3 & , \quad 0 < y < 1 \\ 0 & , \quad o/w \end{cases}$

## Find: b) PDF of $W = e^{-X}$

**Setup:** $F_W(z) = P[W \leq w] = P[e^{-X} \leq W] = P[-X \leq lnw] = P[X \geq -lnw] = 1 - F_x(-lnw)$

**Steps: i)** find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

**ii)** Differentiate with respect to w to find an equation given in terms of the pdf of x:
$f_W(w) = -\frac{d}{dw}F_X\frac{d}{dw}(-lnw) = -f_X(-lnw)\frac{-1}{w} \quad for \quad e^{-1} < w < 1 = -\frac{1}{w}$

**Result:** $f_W(w) = \begin{cases} \frac{1}{w} & , \quad e^{-1} < w < 1 \\ 0 & , \quad o/w \end{cases}$

# Find: c) PDF of $Z = 1 - e^{-X}$

**Setup:** $F_Z(z) = P[Z \leq z] = P[1 - e^{-X} \leq z] = P[-e^{-X} \leq z - 1] = P[e^{-X} \geq 1 - z] = P[-X \geq ln(1-z)] = P[X \leq -ln(1-z)] = F_x(-ln(1-z))$

**Steps: i)** find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

**ii)** Differentiate with respect to w to find an equation given in terms of the pdf of x:
$f_Z(z) = -ln(1-z) = -\frac{-1}{1-z} = \frac{1}{1-z} \quad for \quad 0 < z < e^{-1}$

**Result:** $f_W(w) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < e^{-1} \\ 0 & , \quad o/w \end{cases}$

# Find: d) PDF of $U = X(1 - X)$

**Setup:** $F_U(u) = P[U \leq u] = P[X(1-x) \leq u] = P[-X^2 + X \leq u] = P[-(X-1/2)^2 \leq u - 1/4] = P[(X-1/2)^2 \geq 1/4 - u] == P[|(X-1/2)| \geq (1/4-u)^{1/2}] =$

**Steps: i)** find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

**ii)** INCOMPLETE:
$f_Z(z) = -ln(1-z) = -\frac{-1}{1-z} = \frac{1}{1-z} \quad for \quad 0 < z < e^{-1}$

**Result:** $f_W(w) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < e^{-1} \\ 0 & , \quad o/w \end{cases}$

**06.03**

# Given: PDF $f_R(r) = \begin{cases} 6r(1-r) & , \quad 0 < r < 1 \\ 0 & , \quad o/w \end{cases}$

# Find: Distribution of the circumference

**Setup:** The circumference is $c = 2\pi r$ - we have the pdf in terms of x, so this is the transformation
$F_C(c) = P[C \leq c] = P[2\pi r \leq c] = P[r \leq c/2\pi] = F_x(c/2\pi)$

**Steps: i)** Differentiate with respect to c to find an equation given in terms of the pdf of x.
$f_C(c) = \frac{d}{dc}F_R(c/2\pi) = f_R(c/2\pi)\frac{d}{dc}(c/2\pi) = f_R(c/2\pi)(1/2\pi)$

**ii)** Plug the original pdf back into this new form:
$f_C(c) = \frac{6c}{2\pi}(1-(c/2\pi))(1/2\pi) = \frac{6c(2\pi - c}{2\pi^3} \quad if \quad 0 < c < 2\pi$

**Result:** $f_C(c) = \begin{cases} \frac{6c(2\pi-c}{2\pi^3} & , \quad 0 < c < 2\pi \\ 0 & , \quad o/w \end{cases}$

# Find: Distribution of the area

**Setup:** The area is $a = \pi r^2$ so the cdf $F_A(a) = P[A \le a] = P[\pi r^2 \le a] = P[r^2 \le a/\pi]$
$= P[|r| \le (a/\pi)^{1/2}] = P[-(a/\pi)^{1/2} \le c \le (a/\pi)^{1/2}] = F_r(a/\pi)^{1/2} - F_r - (a/\pi)^{1/2}$

**Steps: i)** Differentiate with respect to a to find an equation in terms of the pdf of x.
$f_A(a) = \frac{d}{da}F_R(a/\pi)^{1/2} - \frac{d}{da}F_R - (a/\pi)^{1/2} = f_R[(a/\pi)^{1/2}]\frac{d}{da}(a/\pi)^{1/2}f_R[-(a/\pi)^{1/2}]\frac{d}{da} - (a/\pi)^{1/2}$

**Result:** $f_A(a) = \begin{cases} \frac{3(\sqrt{\pi}-\sqrt{a}}{\pi^{3/2}} & , \quad 0 < a < \pi \\ 0 & , \quad o/w \end{cases}$

**06.10** Suppose $X$ has pdf $f_X(x) = \frac{1}{2}e^{-|x|}$ for all real $x$.

(a) Find the pdf of $Y = |X|$.
CDF Method
$F_Y(y) = P[Y \le y] = P[|x| \le y] = P[-y \le X \le y] = F_X(y) - F_X(-y)$
$f_Y(y) = \frac{dF_X(y)}{dy} - \frac{dF_X(-y)}{dy}$
$f_Y(y) = f_X(y)\frac{dy}{dx} - f_X(-y)(\frac{-dy}{dy})$
$f_y = \frac{1}{2}e^{-y} + \frac{1}{2}e^{-y} = e^{-y} \; y > 0$

(b) Let $W = 0$ if $X \le 0$ and $W = 1$ if $X > 0$. Find the CDF of $W$
$F_W(w) = P[W = 0] = \frac{1}{2}$
$F_W(w) = P[W = 1] = \frac{1}{2}$
$F_W(w) =$

$$\begin{cases} 0 & w \le 0 \\ \frac{1}{2} & 0 \le w \le 1 \\ 1 & w > 1 \end{cases}$$

**06.13** X has pdf

$$f(x) = \begin{cases} \frac{x^2}{24} & -2 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

We want pdf of the CDF $Y = X^2$ with regions: $(-2,0) \cup [0,4)$

$$[F_x(\sqrt{y}) - F_x(-\sqrt{y})] = \left[f_x(\sqrt{y})(\frac{1}{2}\sqrt{y}) - f_x(-\sqrt{y})(-\frac{1}{2}\sqrt{y})\right]$$

$$f_y(y) = \begin{cases} \frac{y}{48\sqrt{y}} + \frac{y}{48\sqrt{y}} & 0 < y < 4 \\ \frac{y}{48\sqrt{y}} & 4 \le y \le 16 \\ 0 & \text{otherwise} \end{cases}$$

$$f_y(y) = \begin{cases} \frac{\sqrt{y}}{24} & 0 < y < 4 \\ \frac{\sqrt{y}}{48} & 4 \le y \le 16 \\ 0 & \text{otherwise} \end{cases}$$

**06.14**

## Given: Joint PDF $f(x,y) = \begin{cases} 4e^{-2(x+y)} & , \quad 0 < x < \infty, 0 < y < \infty \\ 0 \quad , & o/w \end{cases}$

## Find: a) CDF of W=X+Y

**Setup:** $F_w(w) = P[W \le w] = P[X + Y \le w]$

**Steps:**

**i)** Express as a sum of probabilities, replace probabilities with binomials

**ii)** Simplify and Use Combinatorial Identity

**Result:** $\binom{n+m}{k}$ **06.15** This is a simplified version of example 6.4.5.

$X_1, X_2 \sim POI(\lambda)$ so the MGF of both is $e^{\lambda(e^t-1)}$. Thus by theorem 6.4.4

$$M_Y(t) = e^{\lambda(e^t-1)}e^{\lambda(e^t-1)} = e^{2\lambda(e^t-1)} \sim POI(2\lambda)$$

The pdf then of Y is

$$f_Y(y) = \begin{cases} \frac{e^{-2\lambda}(2\lambda)^y}{y!} & y = 0, 1, 2, ... \\ 0 & otherwise. \end{cases}$$

**06.16** Note: the pdf of $f_{x_1,x_2} = \frac{1}{x_1^2}\frac{1}{x_2^2}$

a) We need to find $f_{u,v} = f_{x_1,x_2}(x_1(u,v), x_2(u,v))|J|$ where J is our jacobian. First we let $u = x_1x_2$ and $v = x_1$ thus $x_1 = v$ and $x_2 = \frac{u}{v}$, now we can find J.

$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & 0 \end{vmatrix} = \frac{1}{v}$$

Finally, our pdf is:

$$\begin{aligned}
f_{U,V}(u,v) &= f_{x_1,x_2}(v, \frac{u}{v}) \left|\frac{1}{v}\right| \\
&= \frac{1}{v^2}\frac{1}{(\frac{u}{v})^2}\left|\frac{1}{v}\right| \\
&= \frac{1}{u^2 v}, 1 < v < u < \infty
\end{aligned}$$

**6.18** It is given that $X$ and $Y$ have a joint pdf given by

$$f(x,y) = e^{-y} \quad if \quad 0 < x < y < \infty. \tag{1}$$

**(a): Find the joint pdf of $S = X + Y$ and $T = X$.**

This can be done using the joint transformation method. By rearranging the above formulas we get $X = T$ and $Y = S - T$. Then it is easy to get the jacobian

$$J = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \tag{2}$$

whose determinant is clearly one. Note that the order in which you take partial derivatives is unimportant provided you are consistent - you will get the same determinant either way. Then we substitute in $X = T$ and $Y = S - T$ into the pdf and multiply by the determinant of the jacobian:

$$f_{S,T}(s,t) = f_{X,Y}(x(s,t), y(s,t)) \times 1 = \begin{cases} e^{t-s} & if \quad 0 < t < s/2 \\ 0 & otherwise \end{cases} . \tag{3}$$

The bounds of the function can be found in a few different ways. One way is to consider the bounds of the original function, $0 < x < y < \infty$. We can substitute in the new formulas for $X$ and $Y$ to get

$$0 < t < s - t < \infty. \tag{4}$$

Then it is apparent that

$$0 < 2t < s < \infty, \tag{5}$$

which then yields

$$0 < t < s/2, \tag{6}$$

the bounds of our new function.

**(b): Find the marginal pdf of T.**

The easiest way to do this is to "integrate out" S from the joint pdf we derived:

$$\begin{aligned}
f_T(t) &= \int_{-\infty}^{\infty} f_{S,T}(s,t)ds = \int_{2t}^{\infty} e^{t-s}ds \\
&= e^t \int_{2t}^{\infty} e^{-s}ds = e^t(-e^{-s}|_{2t}^{\infty}) \\
&= e^{-t} \quad if \quad t > 0.
\end{aligned} \tag{7}$$

**(c): Find the marginal pdf of S.**

This is just like part (b), except this time "integrate out" T:

$$\begin{aligned} f_S(s) &= \int_{-\infty}^{\infty} f_{S,T}(s,t)dt = \int_0^{s/2} e^{t-s}ds \\ &= e^{-s}\int_0^{s/2} e^t dt = e^{-s}(e^t|_0^{s/2}) \\ &= e^{-s}(e^{s/2}-1) \quad if \quad s>0. \end{aligned} \tag{8}$$

**06.23** We will use the property that independant identically distributed random variables has the form of 6.4.4, $M_Y(t) = [M_X(t)]^n$ where $Y = X_1+X_2+...+X_n$. then since $X_i \sim GEO(p)$

$$\begin{aligned} Mgf(Y) &= M_{X_1}(t)M_{X_2}(t)...M_{X_k}(t) \\ &= (M_X(t))^k \\ &= (\frac{pe^t}{1-qe^t})^k \sim NegativeBinomial(k,p) \end{aligned}$$

**06.25** First note, $X_1, X_2, X_3, X_4$ are all independant, but they are not IID as only $X_2, X_3, X_4 \sim POI(5)$ with $X_1$ not being listed. So formula 6.4.5 does not hold. 6.4.4 does though.

A)

$$\begin{aligned} Mgf(Y) &= M_{X_1}(t)M_{X_2+X_3+X_4}(t) \\ &= M_{X_1}(t)(M_{X_i}(t))^3 \end{aligned}$$

Since $X_2, X_3, X_4$ are iid 6.4.5 holds for moving to this mgf

$$\begin{aligned} &= M_{X_1}(t)(e^{\mu(e^t-1)})^3 \\ &= M_{X_1}(t)e^{3\mu(e^t-1)} \\ &= M_{X_1}(t)e^{15(e^t-1)} \\ e^{25(e^t-1)} &= M_{X_1}(t)e^{15(e^t-1)} \\ \frac{e^{25(e^t-1)}}{e^{15(e^t-1)}} &= M_{X_1}(t) \\ e^{10(e^t-1)} &= M_{X_1}(t) \sim POI(10) \end{aligned}$$

B) For $W = X_1 + X_2$ we have $X_1 \sim POI(10)$ and $X_2 \sim POI(5)$. So $POI(10+5) = POI(15)$

**06.29**

**Given: PDF** $f(x) = \begin{cases} \frac{1}{x^2} & , \quad 1 \leq x < \infty, 0 < y < \infty \\ 0 & , \quad o/w \end{cases}$

## Find: a) Joint PDF of the order statistics

**Setup:** $F_w(w) = P[W \leq w] = P[X + Y \leq w]$

**Steps: i)** Differentiate with respect to a to find an equation in terms of the pdf of x.
$f_A(a) = \frac{d}{da}F_R(a/\pi)^{1/2} - \frac{d}{da}F_R - (a/\pi)^{1/2} = f_R[(a/\pi)^{1/2}]\frac{d}{da}(a/\pi)^{1/2}f_R[-(a/\pi)^{1/2}]\frac{d}{da} - (a/\pi)^{1/2}$

**ii)** Simplify and Use Combinatorial Identity

**Result:** $\binom{n+m}{k}$

## Find: b) PDF of the smallest order statistic$Y_1$

**Setup:**

**Steps: i)**

**Result:**

## Find: c) PDF of the largest order statistic $Y_n$

**Setup:**

**Steps: i)**

**Result:**

## Find: d) PDF of the sample range $R = Y_n - Y_1, \quad forn = 2$

**Setup:** The area is $a = \pi r^2$ so the cdf $F_A(a) = P[A \leq a] = P[\pi r^2 \leq a] = P[r^2 \leq a/\pi]$
$= P[|r| \leq (a/\pi)^{1/2}] = P[-(a/\pi)^{1/2} \leq c \leq (a/\pi)^{1/2}] = F_r(a/\pi)^{1/2} - F_r - (a/\pi)^{1/2}$

**Steps: i)**

**Result:**

**Find: e) PDF of the sample median $R = Y_r - Y_1, \quad fornoddsothatr = (n+1)/2$**

**Setup:**

**Steps: i)**

**Result:** **06.35** Suppose $X_1, X_2$ are independent exponentially distributed random variables $X_i$ ~EXP($\theta$), and let $Y = X_1 - X_2$.
(a) Find the MGF of $Y$.
We can think of $Y = X_1 - X_2$ as $Y = X_1 + (-1)X_2$. Then using Theorem 6.4.1,

$$
\begin{aligned}
M_Y(t) &= (M_{X_1}(t))(M_{-X_2}(t)) \\
M_Y(t) &= (M_{X_1}(t))(M_{X_2}(-t)) \\
M_Y(t) &= \left(\frac{1}{1-\theta t}\right)\left(\frac{1}{1-\theta(-t)}\right) \\
M_Y(t) &= \left(\frac{1}{1-\theta t}\right)\left(\frac{1}{1+\theta t}\right) \\
M_Y(t) &= \frac{1}{1-\theta t+\theta t-\theta^2 t^2} \\
M_Y(t) &= \frac{1}{1-\theta^2 t^2}
\end{aligned}
$$

(b) What is the distribution of $Y$?
Since $\frac{1}{1-\theta^2 t^2}$ is the MGF of a double exponential, $Y \sim$ DE($\theta$,0). **07.11** a) First we need to know the $\mu$ and the $\sigma$. For a Uniform variable with $a = 0, b = 1$ we have $\mu = 1/2$ and $\sigma = 1/\sqrt{12}$ (Note: it is not $\sigma^2$). We also need to know that $n = 20$ from there we can use the CLT:

$$
\begin{aligned}
\Pr\left(\sum^{20} X_i < 12\right) &= \Pr\left(\frac{\sum X_i - 10}{\sqrt{20}\frac{1}{\sqrt{12}}} < \frac{12-10}{\sqrt{20}\frac{1}{\sqrt{12}}}\right) \\
&= \Phi\left(\frac{12-10}{\sqrt{20}\frac{1}{\sqrt{12}}}\right) \\
&\approx .9394
\end{aligned}
$$

b) **07.01** Consider a random sample of size n from a distribution with $CDFF(x) = 1 - \frac{1}{x} if 1 \leq x \leq \infty$

(a) Derive the CDF of the smallest order statistic, $X_{1:n}$

Solution: $G_1(y_1) = 1 - [1 - F_X(y_1)]^n = 1 - [1 - [1 - \frac{1}{y_1}]]^n = 1 - [\frac{1}{y_1}]^n$

$$
G_1(y_1) = \begin{cases} 1 - \frac{1}{[y_1]^n} & \text{if } 1 \leq y_1 \\ 0 & \text{if } 0 > y_1. \end{cases}
$$

(b) Find the limiting distribution of $X_{1:\,n}$ Solution:

$\lim_{n\to\infty} 1 - \frac{1}{{y_1}^n} = \begin{cases} 1 & \text{if } y_1 > 1 \\ 0 & \text{if } y_1 \leq 1 \end{cases}$

The limiting distribution of $X_{1:\,n}$ is degenerate at $y = 1$

(c) Find the limiting distribution of ${X^n}_{1:\,n}$

Solution:

$F_{{X^n}_{1:\,n}}(y) = P({X^n}_{1:\,n} \leq y) = P(X_{1:\,n} \leq y^{\frac{1}{n}}) = F_{X_{1:\,n}}(y^{\frac{1}{n}}) = 1 - \frac{1}{{y^{\frac{1}{n}}}^n} = 1 - \frac{1}{y_n}$

then, the limiting distribution of ${X^n}_{1:\,n} = \begin{cases} 1 - \frac{1}{y_n} & \text{if } y > 1 \\ 0 & \text{if } otherwise \end{cases}$