EXERCISES

1). X RANDOM VARIABLE pdf $\begin{cases} f(x) = 4x^3 & 0 < x < 1 \\ 0 & \text{o.w} \end{cases}$

FIND THE pdf OF:

a) $Y = X^4$ $\quad F_Y(y) = P[Y \leq y] = P[X^4 \leq y]$

$= P[-y^{1/4} \leq X \leq y^{1/4}] = F_X(y^{1/4}) - F_X(-y^{1/4})$

$$f_Y(y) = \frac{d F_X(y^{1/4})}{dy} - \frac{d F_X(-y^{1/4})}{dy} =$$

$$f_Y(y) = f_X(y^{1/4}) \cdot \frac{d y^{1/4}}{dy} - f_X(-y^{1/4}) \cdot \frac{d -y^{1/4}}{dy} = f_X(y^{1/4}) \cdot \frac{y^{-3/4}}{4} - f_X(-y^{1/4}) \cdot -\frac{y^{-3/4}}{4}$$

$$f_Y(y) = \begin{cases} 0 & y \leq 0 \\ f_X(y^{1/4}) \cdot \frac{y^{-3/4}}{4} + f_X(-y^{1/4}) \frac{y^{-3/4}}{4} & 0 < y < 1 \end{cases}$$

$$f_Y(y) = \begin{cases} 0 & y \leq 0 \\ 4y^{3/4} \cdot \frac{1}{4y^{3/4}} + 0 & 0 < y < 1 \\ 0 & y > 1 \end{cases} \Rightarrow f_Y(y) = 1 \quad 0 \leq y \leq 1$$

$0 < y^{1/4} < 1$

$0 < y < 1$

b). $W = e^x$

$$F_W(w) = P[W \leq w]$$
$$= P[e^X \leq w]$$
$$= P[\ln e^X \leq \ln w]$$
$$= P[X \leq \ln w] = F_X(\ln w)$$

$$\frac{d\,F_W(w)}{dw} = \frac{d}{dw} F_X(\ln w) \cdot \frac{d \ln w}{dw} = f_X(\ln w) \cdot \frac{1}{w}$$

$$f_W(w) = \frac{4 \cdot (\ln w)^3}{w}$$

FOR

$$0 < \ln w < 1$$
$$e^0 < e^{\ln w} < e^1$$
$$1 < w < e$$

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c). $Z = \ln X$

$$F_Z(z) = P[Z \leq z]$$
$$= P[\ln X \leq z]$$
$$= P[e^{\ln X} \leq e^z] = P[X \leq e^z] = F_X(e^z)$$

$$\frac{d}{dz} F_Z(z) = \frac{d}{dz} F_X(e^z) = f_X(e^z) \cdot \frac{d e^z}{dz}$$

$$0 \leq e^z \leq 1$$
$$\ln 0 \leq \ln e^z < \ln 1$$
$$-\infty \leq z < 0$$

$$f_Z(z) = 4 \cdot e^{3z} \cdot e^z \Rightarrow 4e^{4z}$$

$$(x - 1/2)^2 = x^2 - \frac{2}{2}x + \frac{1}{4} = x^2 - x + \frac{1}{4}$$

d) $U = (X - 0.5)^2$

$$F_U(u) = P[U \leq u]$$
$$= P[(X - 1/2)^2 \leq u]$$
$$= P[|X - 1/2| \leq u^{1/2}]$$

$$X - 1/2 \leq u^{1/2} \Rightarrow \boxed{X \leq u^{1/2} + \frac{1}{2}}$$

OR

$$-(X - 1/2) \leq u^{1/2} \Rightarrow \boxed{X \geqslant -u^{1/2} + \frac{1}{2}}$$

$$F_U(u) = F_X(u^{1/2} + 1/2) - F_X(-u^{1/2} + 1/2)$$

$$= f_X(u^{1/2} + 1/2) \cdot \left(\frac{d\,u^{1/2}}{du} + \frac{d\,1/2}{du}\right) - \left[f_X\left(\frac{1}{2} - u^{1/2}\right)\left(\frac{d - u^{1/2}}{du} + \frac{d\,1/2}{du}\right)\right.$$

$$= f_X(u^{1/2} + 1/2) \cdot \frac{1}{2}u^{-1/2} + f_X\left(\frac{1}{2} - u^{1/2}\right) \cdot \frac{1}{2}u^{-1/2}$$

$$(a+b)^3 = \sum_{k=0}^{3} \binom{3}{k} b^{3-k} a^k = \binom{3}{0} b^3 \cdot 1 + \binom{3}{1} b^2 \cdot a + \binom{3}{2} b \cdot a^2 + \binom{3}{3} a^3$$

$$\binom{3}{0} = \frac{3!}{0!(3-0)!} = \frac{3\ 2\ 1}{1(3\ 2\ 1)} = 1$$

$$\binom{3}{1} = \frac{3!}{1!(2)!} = \frac{3\ 2\ 1}{1 \cdot 2 \cdot 1} = 3$$

$$\binom{3}{2} = \frac{3!}{2!(1)!} = \frac{3\ 2\ 1}{2\ 1} = 3$$

$$\binom{3}{3} = \frac{3!}{3!(0)!} = \frac{3\ 2\ 1}{3\ 2\ 1 \cdot 1} = 1$$

$$(a+b)^3 = b^3 + 3b^2a + 3ba^2 + a^3$$

$$\left(u^{1/2} + \frac{1}{2}\right)^3 = \frac{1}{8} + \frac{3 \cdot u}{2} + 3 \cdot u^{1/2} \cdot \frac{1}{4} + u^{3/2} = u^{3/2} + \frac{3u^{1/2}}{4} + \frac{3u}{2}$$

$$\left(u^{1/2} - \frac{1}{2}\right)^3 = -\frac{1}{8} - \frac{3 \cdot u}{2} + 3u^{1/2} \cdot \frac{1}{4} + u^{3/2} = u^{3/2} + \frac{3u^{1/2}}{4} - \frac{3u}{2}$$

$$= \frac{u^{-1/2}}{2}\left[4\left(u^{3/2} + \frac{3u^{1/2}}{4} + \frac{3u}{2} + \frac{1}{8}\right) - 4\left(u^{3/2} + \frac{3u^{1/2}}{4} - \frac{3u}{2} - \frac{1}{8}\right)\right]$$

$$= \frac{u^{-1/2}}{2}\left[4\left(u^{3/2} - u^{3/2} + \frac{3u^{1/2}}{4} - \frac{3u^{1/2}}{4} + \frac{3u}{2} + \frac{3u}{2} + \frac{1}{8} + \frac{1}{8}\right)\right]$$

$$f_U(u) = f_X(u^{1/2} + 1/2) \cdot \frac{1}{2} \frac{1}{u^{1/2}} + f_X(1/2 - u^{1/2}) \cdot \frac{1}{2} \frac{1}{u^{1/2}}$$

$$f_U(u) = \begin{cases} 0 & u \leq 0 \\ \dfrac{2(u^{1/2} + 1/2)^3}{u^{1/2}} + \dfrac{2(1/2 - u^{1/2})^3}{u^{1/2}} & 0 \leq u < 1/4 \\ 0 & u > 1/4 \end{cases}$$

$$u = (x - \tfrac{1}{2})^2$$

$$\sqrt{u} = x - \frac{1}{2}$$

$x_1 = \sqrt{u} + \frac{1}{2}$ $\quad 0 \leq u \leq 1/4 \quad$ IF $\quad 0 < x < 1$

$x_2 = \frac{1}{2} - \sqrt{u}$ $\quad 0 \leq u \leq 1/4 \quad$ IF $\quad 0 < x \leq 1$

A) $(u^{1/2} + 1/2)^3 = u^{3/2} + \dfrac{3u}{2} + \dfrac{3\sqrt{u}}{4} + \dfrac{1}{8}$

B) $(1/2 - u^{1/2})^3 = -u^{3/2} + \dfrac{3u}{2} - \dfrac{3\sqrt{u}}{4} + \dfrac{1}{8}$

$A + B = 0 + 3u - 0 + \dfrac{1}{4}$

$$f_U(u) = \begin{cases} 0 & u < 0 \\ 2(3u + 1/4) \cdot u^{-1/2} & \text{FOR } \quad 0 < u < 1/4 \\ 0 & u > 1/4 \end{cases}$$

2) LET X BE A RANDOM VARIABLE $X \sim UNIF(0,1)$ $f_X(x) = 1$

a) $Y = X^{1/4}$

$$F_Y(y) = P[Y \leq y]$$
$$= P[X^{1/4} \leq y]$$
$$= P[X \leq y^4]$$
$$= F_X(y^4)$$

$$f_Y(y) = \frac{d}{dy} F_X(y^4) = f_X(y^4) \cdot \left|\frac{d\,y^4}{dy}\right|$$

$$\boxed{f_Y(y) = 4y^3} \qquad 0 < y < 1$$

b) $W = e^{-X}$

$$F_W(w) = P[W \leq w]$$
$$= P[e^{-X} \leq w]$$
$$= P[\ln e^{-X} \leq \ln w]$$
$$= P[-X \leq \ln w] = P[X \geq -\ln w]$$
$$= 1 - F_X(-\ln w)$$

$$f_W(w) = -\frac{dF_X}{dw} \frac{d(-\ln w)}{dw}$$

$$= -f_X(-\ln w) \cdot \frac{-1}{w} \qquad \text{FOR} \quad e^{-1} < w < 1$$

$$\boxed{f_W(w) = \frac{1}{w}}$$

$w = e^{-x}$ $\quad -x = \ln w$ $\quad 0 < -\ln w < 1$ $\quad e^{-1} < w < 1$

$\ln(w) = \ln e^{-x}$ $\quad x = -\ln w$ $\quad -1 < \ln w < 0$

c) $Z = 1 - e^{-X}$

$$F_Z(z) = P[Z \leqslant z]$$
$$= P[1 - e^{-X} \leqslant z]$$
$$= P[-e^{-X} \leqslant z - 1] = P[e^{-X} \geqslant 1 - z]$$
$$= P[\ln e^{-X} \geqslant \ln(1-z)]$$
$$= P[-X \geqslant \ln(1-z)] = P[X \leqslant -\ln(1-z)]$$
$$= F_X(-\ln(1-z))$$

$$f_z(z) = \frac{d}{dz} -\ln(1-z) = -\left(\frac{-1}{1-z}\right) \Rightarrow f_Z(z) = \frac{1}{1-z} \quad \text{FOR } 0 < z < 1 - e^{-1}$$

$$\frac{d}{dz} \ln(1-z) \qquad u = 1 - z \Rightarrow \frac{d \ln(u)}{du} \cdot \frac{du}{dz}$$

$$\frac{du}{dz} = \frac{d}{dz}(1-z) = -1 \Rightarrow \frac{1}{u} \cdot -1 \Rightarrow \frac{-1}{1-z}$$

$$0 < x < 1$$
$$0 < -\ln(1-z) < 1$$
$$0 > \ln(1-z) > -1$$
$$1 > 1 - z > e^{-1}$$

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d) $U = X(1-X) \Rightarrow U = -X^2 + X \qquad -(X - \frac{1}{2})^2 = -(X^2 - X + \frac{1}{4})$

$$F_U(u) = P[U \leqslant u]$$
$$= P[X(1-X) \leqslant u]$$
$$= P[-(X - \tfrac{1}{2})^2 \leqslant u - \tfrac{1}{4}] = P[(X - \tfrac{1}{2})^2 \geqslant -u + 1/4]$$
$$= P[|X - \tfrac{1}{2}| \geqslant (-u + 1/4)^{1/2}]$$
$$= -(X - \tfrac{1}{2}) \leqslant (u - \tfrac{1}{4})^{1/2}$$

$x - \frac{1}{2} \geq (-u + 1/4)^{1/2} \Rightarrow x \geq (-u + 1/4)^{1/2} + 1/2$

$-(x - \frac{1}{2}) \leq -(-u + 1/4)^{1/2}$. $\quad -x \leq -(-u + 1/4)^{1/2} - 1/2 \Rightarrow x \geq (-u + 1/4)^{1/2}$

$F_U(u) = 2\left[1 - F_X(\sqrt{-u + 1/4} + \frac{1}{2})\right] = 2 - 2F_X(1/2 + \sqrt{1/4 - u})$

$f_U(u) = 2 f_X(1/2 + \sqrt{1/4 - u}) \cdot \frac{d}{du}(1/2 - \sqrt{1/4 - u})$

$= 2 \cdot \frac{d}{du} - \sqrt{1/4 - u} = 2 \cdot \frac{1}{2} \cdot (1/4 - u)^{-1/2} \Rightarrow \boxed{f_U(u) = (1/4 - u)^{-1/2}}$

$a = \frac{1}{4} - u$ $\qquad \frac{d}{da} - a^{1/2} \cdot -1 = \frac{1}{2} \cdot a^{-1/2}$

$\frac{da}{du} = \frac{d}{du}\left(\frac{1}{4}\right) - \frac{d}{du}u = -1$

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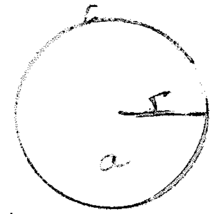
3)

$$\text{pdf} \begin{cases} f_R(r) = 6r(1-r) & 0 < r < 1 \\ 0 & \text{o.w.} \end{cases}$$



a). $C = 2\pi R \Rightarrow F_C(c) = P[C \leq c] = P[2\pi R \leq c]$

$$= P\left[R \leq \frac{c}{2\pi}\right] = F_R(c/2\pi)$$

$$f_C(c) = \frac{d\, F_R(c/2\pi)}{dc} = f_R(c/2\pi) \cdot \frac{d}{dc} c/2\pi = f_R(c/2\pi) \cdot \frac{1}{2\pi}$$

$$f_C(c) = \frac{6c}{2\pi}(1 - c/2\pi) \cdot \frac{1}{2\pi} = \frac{6c}{2\pi}\left(\frac{2\pi - c}{2\pi}\right) \cdot \frac{1}{2\pi} = \frac{6c(2\pi - c)}{(2\pi)^3}$$

$$\boxed{f_C(c) = \frac{6c(2\pi - c)}{(2\pi)^3} \qquad 0 < c < 2\pi}$$

b). $A = \pi R^2 \Rightarrow F_A(a) = P[A \leq a] = P[\pi R^2 \leq a]$

$$= P\left[R^2 \leq \frac{a}{\pi}\right] = P\left[-\left(\frac{a}{\pi}\right)^{1/2} \leq R \leq \left(\frac{a}{\pi}\right)^{1/2}\right] = F_R[(a/\pi)^{1/2}] - F_R[-(a/\pi)^{1/2}]$$

$$f_A(a) = \frac{d\, F_R[(a/\pi)^{1/2}]}{da} - \frac{d\, F_R[-(a/\pi)^{1/2}]}{da}$$

$$f_A(a) = f_R[(a/\pi)^{1/2}] \cdot \frac{d}{da}(a/\pi)^{1/2} - f_R[-(a/\pi)^{1/2}] \frac{d}{da} -(a/\pi)^{1/2}$$

$$f_A(a) = f_R[(a/\pi)^{1/2}] \cdot \frac{1}{\pi^{1/2}} \cdot \frac{1}{2} \frac{1}{a^{1/2}} - f_R[-(a/\pi)^{1/2}] \cdot \frac{1}{2\pi^{1/2}} \cdot \frac{1}{a^{1/2}}$$

SEE OTHER SIDE

$$f_A(a) = f_R\left[(a/\pi)^{1/2}\right] \frac{1}{2\pi^{1/2}} \cdot \frac{1}{a^{1/2}} - f_R\left[-(a/\pi)^{1/2}\right] \frac{-1}{2\pi^{1/2}} \cdot \frac{1}{a^{1/2}}$$

$$f_A(a) = \begin{cases} 0 & a < 0 \\ f_R(\sqrt{a/\pi}) \cdot \frac{1}{2} \frac{1}{\sqrt{\pi a}} + f_R(-\sqrt{a/\pi}) \cdot \frac{1}{2} \frac{1}{\sqrt{\pi a}} & 0 < a < \pi \end{cases}$$

$$f_A(a) = \begin{cases} 0 & a < 0 \\ 3 \cdot \frac{\sqrt{a}}{\sqrt{\pi}} \left(1 - \frac{\sqrt{a}}{\sqrt{\pi}}\right) \cdot \frac{1}{\sqrt{a} \cdot \sqrt{\pi}} + 0 & 0 < a < \pi \\ 0 & a > \pi \end{cases}$$

$$f_A(a) = \frac{3(\sqrt{\pi} - \sqrt{a})}{\pi^{3/2}} \qquad 0 < a < \pi$$

$0 < r < 1$

$0 < \sqrt{a/\pi} < 1$

$0 < a/\pi < 1$

$\boxed{0 < a < \pi}$

4). $X \sim WEI(\theta, \beta)$ FIND CDF AND pdf OF:

$$f_X(x) = \frac{\beta}{\theta^\beta} x^{\beta-1} e^{-(x/\theta)^\beta} \qquad \theta > 0 \qquad x > 0$$

$$F_X(x) = 1 - e^{-(x/\theta)^\beta} \qquad \beta > 0$$

a). $Y = (X/\theta)^\beta$

$$F_Y(y) = P[Y \leq y]$$

$$= P[(X/\theta)^\beta \leq y] = P[X/\theta \leq y^{1/\beta}] = P[X \leq \theta y^{1/\beta}]$$

$$= F_X(\theta y^{1/\beta})$$

$$F_Y(y) = 1 - e^{-\left(\frac{\theta y^{1/\beta}}{\theta}\right)^\beta} \Rightarrow \boxed{F_Y(y) = 1 - e^{-y}} \qquad y > 0$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= \frac{d}{dy} 1 - e^{-y} \Rightarrow$$

$$= -\frac{d}{dy} e^{-y} \Rightarrow \boxed{f_Y(y) = e^{-y}} \qquad y > 0$$

b) $W = \ln X$

$$F_W(w) = P[W \leq w]$$

$$= P[\ln X \leq w] = P[X \leq e^w] = F_X(e^w)$$

$$\boxed{F_W(w) = 1 - e^{-(e^w/\theta)^\beta}}$$

$$f_W(w) = \frac{d}{dw} 1 - e^{-(e^w/\theta)^\beta}$$

$$= \frac{d}{dw} - e^{-\left(\frac{e^w}{\theta}\right)^\beta}$$

$$f_W(w) = \frac{d}{dw} - e^{-\left(\frac{e^w}{\theta}\right)^\beta} \Rightarrow \boxed{f_W(w) = \beta \cdot e^{-\left(\frac{e^w}{\theta}\right)^\beta} \cdot \left(\frac{e^w}{\theta}\right)^\beta} \quad \text{FOR } w \text{ REAL}$$

$$u = -\left(\frac{e^w}{\theta}\right)^\beta \Rightarrow \frac{d}{du} - e^u \cdot \frac{du}{dw} = -e^u \cdot \beta \cdot -\frac{1}{\theta^\beta} \cdot e^{w\beta}$$

$$\frac{du}{dw} = \frac{d}{dw} - e^{w\beta} \cdot \frac{1}{\theta^\beta} = -\frac{1}{\theta^\beta} \cdot \frac{d}{dw} e^{w\beta} = -\frac{1}{\theta^\beta} \cdot e^{w\beta} \cdot \beta$$

$$a = w\beta \Rightarrow \frac{d e^a}{da} \cdot \frac{da}{dw} \Rightarrow e^a \cdot \beta \Rightarrow e^{w\beta} \cdot \beta$$

$$\frac{da}{dw} = \frac{d\, w\beta}{dw} = \beta$$

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c) $Z = (\ln X)^2$

$$F_Z(z) = P[Z \le z]$$

$$= P[(\ln x)^2 \le z] = P[|\ln x| \le z^{1/2}] = P[|e^{\ln x}| \le e^{z^{1/2}}]$$

$$= P[|X| \le e^{z^{1/2}}] \Rightarrow F_X(e^{z^{1/2}}) - F_X(-e^{z^{1/2}})$$

$$= 1 - e^{-\left(\frac{e^{z^{1/2}}}{\theta}\right)^\beta} - \left(1 - e^{-\left(-\frac{e^{z^{1/2}}}{\theta}\right)^\beta}\right) = e^{-\left(-\frac{e^{z^{1/2}}}{\theta}\right)^\beta} - e^{-\left(\frac{e^{z^{1/2}}}{\theta}\right)^\beta}$$

$$\boxed{F_Z(z) = e^{-\left(-\frac{e^{z^{1/2}}}{\theta}\right)^\beta} - e^{-\left(\frac{e^{z^{1/2}}}{\theta}\right)^\beta}} \quad \text{FOR } z > 0$$

8) - $X \sim$ pdf $f_X(x) = 4x^3 \quad 0 < x < 1$

a) - $Y = X^4$

$u(x) = x^4$

$u(y) = \pm y^{1/4} \Rightarrow f_Y(y) = \pm 4\, y^{3/4} \cdot \left| \frac{d\, y^{1/4}}{dy} \right| \quad 0 < y < 1$

$f_Y(y) = 4y^{3/4} \cdot \frac{1}{4} \cdot \frac{1}{y^{3/4}} + 4(-y^{3/4}) \cdot \frac{1}{4} \cdot \frac{1}{y^{3/4}} \Rightarrow f_Y(y) = 1$

b) $W = e^X$

$u(x) = e^x$

$z(w) = \ln(w) \Rightarrow f_W(w) = \ln(w) \cdot \left| \frac{d \ln(w)}{dw} \right|$

$f_W(w) = 4 \ln(w)^3 \cdot \frac{1}{w} \qquad 0 < \ln(w) < 1$

$e^0 < w < e^1$

$f_W(w) = \frac{4 \ln(w)^3}{w}$ FOR $1 < w < e$

c) $Z = \ln X$

$u(x) = \ln x$

$w(z) = e^z \Rightarrow f_Z(z) \cdot 4e^{3z} \left| \frac{d e^z}{dz} \right| \qquad 0 < e^z < 1$

$\ln 0 < \ln e^z < \ln(1)$

$f_Z(z) = 4\, e^{3z}\, e^z$

$f_Z(z) = 4e^{4z}$ FOR $-\infty < z < 0$

d) $U = (X - 0.5)^2$

$f(x) = (x - 0.5)^2$

$g(u) = \pm u^{1/2} + 0.5 \Rightarrow f_U(u) = 4.(\pm u^{1/2} + 0.5)^3 \cdot \left| \frac{d(u^{1/2} + 0.5)}{du} \right|$

$f_U(u) = 4(\pm u^{1/2} + 0.5)^3 \cdot \frac{1}{2u^{1/2}} \Rightarrow f_U(u) = \frac{2(u^{1/2} + 1/2)^3}{u^{1/2}} + \frac{2(1/2 - u^{1/2})^3}{u^{1/2}}$

$\boxed{f_U(u) = 2(3u + 1/4).\,u^{-1/2}}$ FOR $0 < u < 1/4$

$u = (x - 1/2)^2 \Rightarrow \pm\sqrt{u} = x - 1/2 \Rightarrow$

$$\begin{cases} x_1 = \sqrt{u} + \frac{1}{2} & 0 \leq u \leq 1/4 \quad \text{IF} \quad 0 \leq x \leq 1 \\ x_2 = \frac{1}{2} - \sqrt{u} & 0 \leq u \leq 1/4 \quad \text{IF} \quad 0 \leq x \leq 1 \end{cases}$$

$(a+b)^x = \sum_{y=0}^{x} \binom{x}{y} b^{x-y} a^y$ WHERE $\binom{x}{y} = \frac{x!}{y!(x-y)!}$

A) $(u^{1/2} + 1/2)^3 = u^{3/2} + \frac{3u}{2} + \frac{3\sqrt{u}}{4} + \frac{1}{8}$

B) $(1/2 - u^{1/2})^3 = -u^{3/2} + \frac{3u}{2} - \frac{3\sqrt{u}}{4} + \frac{1}{8}$

$A + B = 3u + \frac{1}{4}$

$2(3u + 1/4).\,u^{-1/2}$

9). LET $X \sim UNIF(0,1) \Rightarrow f_X(x) = 1$

a). $Y = X^{1/4}$

$$u(x) = x^{1/4}$$

$$w(y) = y^4$$

$$f_Y(y) = f_X(w(y)) \cdot \left| \frac{d\, w(y)}{dy} \right| = f_X(y^4) \cdot \left| \frac{d\, y^4}{dy} \right|$$

$$f_Y(y) = 1 \cdot 4 \cdot y^3 \Rightarrow f_Y(y) = 4y^3 \quad \text{FOR} \quad 0 < y < 1$$

$y = x^{1/4}$ $\qquad 0 < y^4 < 1$

$y^4 = x$ $\qquad 0 < y < 1$

b) $Y = e^{-X}$

$$u(x) = e^{-x}$$

$$\ln(y) = \ln e^{-x}$$

$$\ln(y) = -x$$

$$x = -\ln(y)$$

$$w(y) = -\ln(y)$$

$$f_Y(y) = f_X(-\ln y) \cdot \left| \frac{d -\ln y}{dy} \right| = 1 \cdot -\frac{1}{y} = f_Y(y) = -\frac{1}{y} \quad \text{FOR } e^{-1} < y < 1$$

$0 < -\ln(y) < 1$ $\qquad e^0 > y > e^{-1}$

$0 > \ln y > -1$ $\qquad e^{-1} < y < 1$

c). $Z = 1 - e^{-X}$

$u(x) = 1 - e^{-x}$

$y - 1 = -e^{-x}$

$e^{-x} = 1 - y$

$\ln e^{-x} = \ln(1-y)$

$x = -\ln(1-y)$

$w(y) = -\ln(1-y)$

$f_Y(y) = f_X(-\ln(1-y)) \left| \frac{d}{dy} -\ln(1-y) \right| = 1 \cdot \frac{1}{(1-y)}$ FOR $0 < y < 1 - e^{-1}$

$0 < -\ln(1-y) < 1$

$0 > \ln(1-y) > -1$

$1 > 1 - y > e^{-1}$

$0 > -y > e^{-1} - 1$

$0 < y < 1 - e^{-1}$

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d) $U = X(1-X)$

$U = -X^2 + X$

AS $-(X - \frac{1}{2})^2 = -X^2 + X - \frac{1}{4}$

$U - 1/4 = -(X - 1/2)^2$

$1/4 - U = (X - 1/2)^2$

$\pm(1/4 - U)^{1/2} = X - 1/2$

$\pm(\frac{1}{4} - U)^{1/2} = X - \frac{1}{2}$

$X_1 = \frac{1}{2} + (1/4 - U)^{1/2}$

$X_2 = \frac{1}{2} - (1/4 - U)^{1/2}$

$w(u) = \frac{1}{2} + (1/4 - u)^{1/2}$

$w(u) = \frac{1}{2} - (1/4 - u)^{1/2}$

$g(x) = -x^2 + x$

$$f_U(u) = f_X(w(u)) \cdot \left| \frac{d\, w(u)}{du} \right|$$

$$f_U(u) = f_X\left[\frac{1}{2} + (1/4 - u)^{1/2}\right] \cdot \left| \frac{d\, 1/2 + (1/4-u)^{1/2}}{du} \right| + f_X\left[1/2 - (1/4-u)^{1/2}\right] \cdot \left| \frac{d\, 1/2 - (1/4-u)^{1/2}}{du} \right|$$

$$f_U(u) = 1 \cdot \left| \frac{d\,(1/4-u)^{1/2}}{du} \right| + 1 \cdot \left| \frac{d - (1/4-u)^{1/2}}{du} \right| = 2 \cdot \frac{1}{2\sqrt{1/4-u}} = (1/4-u)^{-1/2}$$

$$f_U(u) = (1/4 - u)^{-1/2} \qquad 0 < u < 1/4$$

IF $u = 1/4$

$x_1 = \frac{1}{2} + (1/4 - \frac{1}{4})^{1/2} = 1/2$

$x_2 = \frac{1}{2} - (1/4 - 1/4)^{1/2} = 1/2$

IF $u = 0$

$x_1 = 1/2 + (1/4 - 0)^{1/2} = 1$

$x_2 = 1/2 - (1/4 - 0)^{1/2} = 0$

$a = 1/4 - u$

$\frac{da}{du} = \frac{d\,(1/4 - u)}{du}$

$\frac{da}{du} = -1$

$\frac{d\sqrt{a}}{da} \cdot \frac{da}{du}$

$\frac{1}{\sqrt{a}} \cdot -1$

$- \frac{1}{2\sqrt{1/4-u}}$

20). X pdf $f_X(x) = \frac{1}{2} e^{(-|x|)}$ $\quad -\infty < x < \infty$

a) FIND pdf OF $Y = |X|$

i) CDF METHOD

$$F_Y(y) = P[Y \le y] = P[|X| \le y]$$

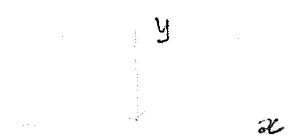
$$= P[-y \le X \le y] = F_X(y) - F_X(-y)$$

$$f_Y(y) = \frac{d F_X(y)}{dy} - \frac{d F_X(-y)}{dy}$$

$$f_Y(y) = f_X(y) \cdot \frac{dy}{dy} - \left(f_X(-y) \cdot -\frac{dy}{dy} \right)$$

$$f_Y(y) = \frac{1}{2} e^{(-y)} + \frac{1}{2} \cdot e^{(-y)} = e^{-y} \quad y > 0$$

ii) TRANSFORMATION METHOD



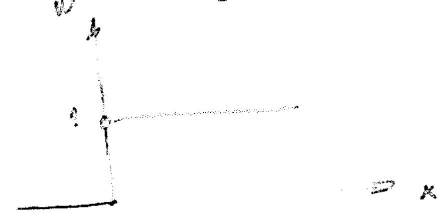
$$u(x) = |x|$$

$$w(y) = \pm y$$

$$f_Y(y) = f_X(y) \cdot \left|\frac{dy}{dy}\right| + f_X(-y) \left|\frac{d-y}{dy}\right| = \frac{1}{2} e^{-y} \cdot 1 + \frac{1}{2} e^{-y} \cdot 1$$

$$f_Y(y) = e^{-y} \quad y > 0$$

CDF OF W



b). $W = 0$ IF $X \leq 0$

$W = 1$ IF $X > 0$

FIND CDF OF W.

$$F_W(w) = P[W = 0] = 1/2$$

$$F_W(w) = P[W = 1] = 1/2$$

$$F_W(w) = P[W \leq w] = \begin{cases} 0 & \text{FOR } w < 0 \\ 1/2 & \text{FOR } 0 \leq w \leq 1 \\ 1 & \text{FOR } w \geq 1 \end{cases}$$

---

11). $X \sim BIN(n, p)$ FIND THE pdf OF $Y = n - X$

$$f_X(x) = \binom{n}{x} p^x q^{n-x} \quad \text{FOR } x = 0, 1, 2, \ldots, n$$

$$f_Y(y) = f_X(x(y))$$

$$f_Y(y) = f_X(n-y) = \binom{n}{n-y} p^{n-y} q^{n-(n-y)}$$

$$f_Y(y) = \frac{n!}{(n-y)!\,(n-y-n)!} \cdot q^y \cdot p^{n-y} = \frac{n!}{y!\,(n-y)!} \cdot (1-p)^y \cdot (1-q)^{n-y}$$

$$f_Y(y) = \binom{n}{y} \cdot (1-p)^y \cdot (1-q)^{n-y} = X \sim BIN(n, 1-p)$$

12) $X \sim NB(r, p)$ FIND pdf OF $Y = X - r$

$$f_X(x) = \binom{x-1}{r-1} p^r \; q^{x-r}$$

$$f_Y(y) = f_X(u(y))$$

$$f_Y(y) = f_X(y+r) = \binom{y+r-1}{r-1} \cdot p^{r-1} \cdot q^{y+r-1-(r-1)}$$

$$f_Y(y) = \frac{(y+r-1)!}{(r-1)!\,(y+r-1-r+1)!} \cdot p^{r-1} \cdot q^{y}$$

$$= \frac{(y+r-1)!}{y!\,(r-1)!} (1-p)^y \cdot (1-q)^{r-1} = \binom{y+r-1}{y} (1-p)^y \cdot (1-q)^{r-1}$$

$$f_Y(y) = Y \sim BIN(y+r-1,\ 1-p)$$

---

13) $X \sim$ pdf $f_X(x) = \frac{x^2}{24}$ $-2 < x < 4$ FIND pdf OF $Y = X^2$

i) CDF METHOD

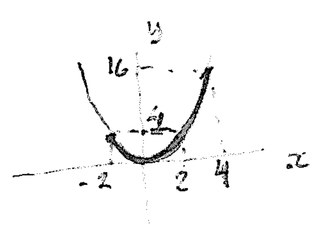
$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} < X < \sqrt{y})$$

$$F_Y(y) = F_X(y^{1/2}) - F_X(-y^{1/2})$$

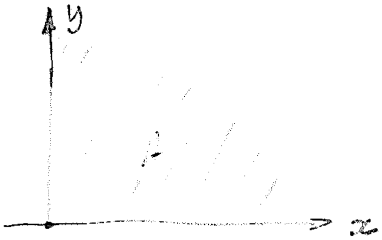
$$f_Y(y) = \frac{d\,F_X(y^{1/2})}{dy} - \frac{d\,F_X(-y^{1/2})}{dy} = f_X(y^{1/2}) \cdot \left|\frac{d\,y^{1/2}}{dy}\right| - f_X(-y^{1/2}) \cdot \left|\frac{d\,-y^{1/2}}{dy}\right|$$

$$f_Y(y) = \frac{1}{2} \cdot \frac{1}{y^{1/2}} \cdot f_X(y^{1/2}) + \frac{1}{2} \cdot \frac{1}{y^{1/2}} \cdot f_X(-y^{1/2}) = \frac{1}{2\sqrt{y}} \cdot \frac{y}{24} + \frac{1}{2\sqrt{y}} \cdot \frac{y}{24} = \frac{y}{24\sqrt{y}}$$

$$f_Y(y) = \begin{cases} \frac{\sqrt{y}}{24} & 0 < y < 4 \\ \frac{\sqrt{y}}{48} & 4 < y < 16 \end{cases}$$



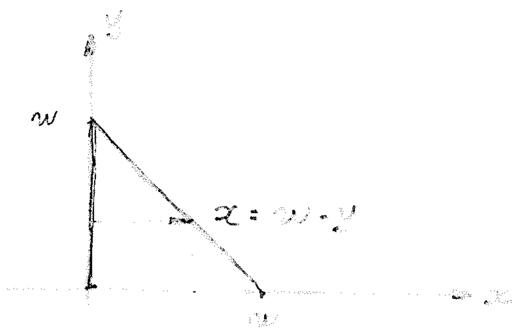
14. X AND Y pdf $f_{XY}(x,y) = 4e^{-2(x+y)}$ A $\begin{cases} 0 < x < \infty \\ 0 < y < \infty \end{cases}$



a) FIND THE CDF OF $W = X + Y$

$F_W(w) = P[W \le w] = P[X + Y \le w]$

- REGION W



$0 < y < w$

$0 < x < w - y$

$$F_W(w) = \int_0^w \int_0^{w-y} 4e^{-2(x+y)}\,dx\,dy = 1 - e^{-2w} - 2we^{-2w}$$

$w > 0$

① $\int_0^{w-y} 4e^{-2(x+y)}\,dx =$ $\int 4e^{u}\frac{-1}{2}\,du = -2e^{u}$

$u = -2x - 2y$

$\frac{du}{dx} = \frac{d(-2x)}{dx} - \frac{d(2y)}{dx} = -2$

$du = -2dx \Rightarrow dx = \frac{du}{-2}$

$-2e^{-2(x+y)}\Big|_0^{w-y} = -2\left[e^{-2(w-y+y)} - e^{-2(0+y)}\right]$

$= -2e^{-2w} + 2e^{-2y} = 2(e^{-2y} - e^{-2w})$

③ $\int_0^w 2(e^{-2y} - e^{-2w})\,dy = \int_0^w 2e^{-2y} - 2e^{-2w}\,dy$

$\int_0^w 2e^{u}\,dy - \int_0^w 2e^{-2w}\,dy$

$\int_0^w -e^{u}\,du - 2e^{-2w}y$

$-e^{-2y}\Big|_0^w - 2e^{-2w}y\Big|_0^w$

$-e^{-2w} + 1 - 2e^{-2w}w$

$= 1 - e^{-2w} - 2we^{-2w}$

$u = -2y$

$\frac{du}{dy} = \frac{d(-2y)}{dy}$

$du = -2\,dy \Rightarrow dy = \frac{du}{-2}$

b) $U = \dfrac{X}{Y}$ $\qquad V = X$

$v = x \Rightarrow x = v$

$u = \dfrac{x}{y} \qquad y = \dfrac{v}{u}$

$0 < v < \infty$

$0 < \dfrac{v}{u} < \infty \Rightarrow 0 < u < \infty$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial v}{\partial u} & \dfrac{\partial v u^{-1}}{\partial u} \\ \dfrac{\partial v}{\partial v} & \dfrac{\partial v u^{-1}}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & -v u^{-2} \\ 1 & u^{-1} \end{vmatrix} = v u^{-2}$$

$$f_{UV}(u,v) = f_{XY}(v, v/u) \cdot \frac{v}{u^2} = 4 e^{-2(v + v/u)} \cdot \frac{v}{u^2} =$$

$$f_{UV}(u,v) = \frac{4v}{u^2} \cdot e^{-2v(1+1/u)} \qquad v > 0 \text{ AND } u > 0$$

---

c) MARGINAL pdf OF U

$$f_U(u) = \int_0^\infty 4v \cdot \frac{1}{u^2} \cdot e^{-2v(1+1/u)}\, dv = \frac{4}{u^2} \int_0^\infty v \cdot e^{-v(2+2/u)}\, dv$$

$$= \frac{4}{u^2} \cdot \frac{1}{(2+2/u)^2} = \frac{1}{u^2} \cdot \frac{1}{(1+1/u)^2} = \frac{1}{(u+1)^2}$$

$$f_U(u) = \frac{1}{(u+1)^2} \qquad u > 0$$

15) X_1 AND X_2 $\quad n=2 \quad X_i \sim POI(\mu)$ ← DISCRETE $\quad f_X(x) = \dfrac{e^{-\mu}\mu^{x}}{x!}$

$x = 0, 1, 2$

FIND pdf OF $\quad Y = X_1 + X_2$

$$P_Y(Y=y) = f_Y(y) = \sum_{x=0}^{y} f_{X_1}(x)\, f_{X_2}(y-x)$$

$$f_Y(y) = \sum_{x=0}^{y} \frac{e^{-\mu_1}\mu_1^{x}}{x!} \cdot \frac{e^{-\mu_2}\mu_2^{y-x}}{(y-x)!} =$$

$$f_Y(y) = e^{(-\mu_1-\mu_2)} \sum_{x=0}^{y} \frac{\mu_1^{x}\cdot\mu_2^{y-x}}{x!\,(y-x)!}$$

THE BINOMIAL FORMULA

$$(a+b)^y = \sum_{x=0}^{y}\binom{y}{x} a^{y-x}\cdot b^{x} = (\mu_2+\mu_1)^y = \sum_{x=0}^{y}\binom{y}{x}\mu_2^{y-x}\mu_1^{x}$$

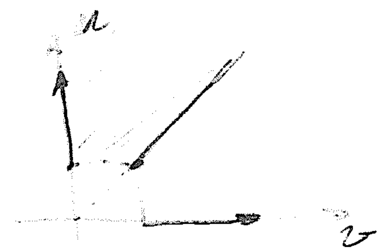
THEN

$$\sum_{x=0}^{y}\frac{\mu_1^{x}\cdot\mu_2^{y-x}}{x!\,(y-x)!}\cdot\frac{y!}{y!} = \frac{\sum_{x=0}^{y}\binom{y}{x}\mu_1^{x}\cdot\mu_2^{y-x}}{y!} = \frac{(\mu_2+\mu_1)^y}{y!}$$

SO ⇒ $f_Y(y) = \dfrac{e^{(-\mu_1-\mu_2)}\cdot(\mu_2+\mu_1)^y}{y!}$ ⇒ ASSUMING $\mu_1=\mu_2$ ⇒ $\dfrac{e^{-2\mu}\cdot 2\mu^{y}}{y!} = POI(2\mu)$

16) X_1 AND X_2 $\quad n=2 \quad$ pdf $= 1/x^3 \quad 1<x<\infty$

a) JOINT pdf $\quad U = X_1\ X_2$ AND $V = X_1$



$v = x_1 \quad \Rightarrow \quad x_1 = v \quad 1<v<\infty$

$u = x_1 . x_2 \quad x_2 = \dfrac{u}{v} \quad v<u<\infty$

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial v} & \dfrac{\partial x_2}{\partial v} \\ \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_2}{\partial u} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial v}{\partial v} & \dfrac{\partial u v^{-1}}{\partial v} \\ \dfrac{\partial v}{\partial u} & \dfrac{\partial u v^{-1}}{\partial u} \end{vmatrix} = \begin{vmatrix} 1 & -u.v^{-2} \\ 0 & \dfrac{1}{v} \end{vmatrix} = \frac{1}{v}$$

$$f_{UV}(u,v) = f_{X_1}(v)\ f_{X_2}(u/v)\ \cdot \frac{1}{v}$$

$$= \frac{1}{v^2} \cdot \frac{1}{u^2/v^2} \cdot \frac{1}{v} = \frac{1}{u^2 v}$$

$$f_{UV}(u,v) = \frac{1}{u^2 v} \qquad \text{FOR} \quad 1 < v < u < \infty$$

b). FIND MARGINAL pdf OF U

$$f_U(u) = \int_1^u \frac{1}{u^2 v}\, dv = \frac{1}{u^2} \cdot \int_1^u v^{-1}\, dv = \left.\frac{\ln(v)}{u^2}\right|_1^u = \frac{\ln(u)}{u^2} - 0$$

$$f_U(u) = \frac{\ln(u)}{u^2} \qquad u > 1$$

17) X_1 AND X_2 $\quad n=2 \quad X_i \sim GAM(\overset{\theta}{2}, \overset{\kappa}{1/2})$

$$f_X(x) = \frac{1}{\theta^K \Gamma(K)} x^{K-1} e^{-x/\theta} \qquad f_{X_i}(x_i) = \frac{1}{2^{1/2}\Gamma 1/2} \cdot x_i^{-1/2} e^{-x_i/2} =$$

a)

$$Y = \sqrt{X_1 + X_2} \implies Y^2 = X_1 + X_2$$

$$0 < y_1 < y_2 = \infty$$

$$y_1 = x_1^{1/2} \implies x_1 = y_1^2$$

$$y_2 = y_1 + x_2^{1/2} \implies x_2 = (y_2 - y_1)^2$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{\partial y_1^2}{\partial y_1} & \frac{\partial (y_2-y_1)^2}{\partial y_1} \\ \frac{\partial y_1^2}{\partial y_2} & \frac{\partial (y_2-y_1)^2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 2y_1 & -2y_2+2y_1 \\ 0 & 2y_2-2y_1 \end{vmatrix}$$

$$J = \left| 2y_1(2y_2 - 2y_1) \right| = \left| -4y_1^2 + 4y_2y_1 \right| = -4y_1^2 + 4y_2y_1$$

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1}(y_1^2) \cdot f_{X_2}(y_2-y_1)^2 \cdot (4y_1^2 + 4y_2y_1)$$

$$= \frac{1}{2^{1/2}\Gamma 1/2} \cdot (y_1^2)^{-1/2} \cdot e^{-y_1^2/2} \cdot \frac{1}{2^{1/2}\Gamma 1/2} \cdot (y_2-y_1)^{-2/2} \cdot e^{-\frac{(y_2-y_1)^2}{2}} \cdot 4(y_1^2+y_1y_2)$$

$$= \frac{1}{2 \cdot \pi^2} \cdot \frac{1}{y_1} \cdot e^{-y_1^2/2} \cdot \frac{1}{(y_2-y_1)} \cdot e^{-\frac{(y_2-y_1)^2}{2}} \cdot 4(-y_1^2 + y_1y_2)$$

$$= \frac{4}{2\pi^2} \cdot \frac{1}{(-y_1^2+y_1y_2)} \cdot \frac{(-y_1^2+y_1y_2)}{1} \cdot e^{-\frac{y_1^2}{2} + \frac{y_1^2}{2} + \frac{y_2^2}{2} - \frac{2y_1y_2}{2}}$$

$$= \frac{2}{\pi^2} \cdot e^{\frac{1}{2}(y_2^2 - y_1y_2)} \qquad y > 0$$

$$f_{Y_2}(y_2) = \int_0^{y_2} \frac{2}{\pi^2}\, e^{\frac{1}{2} y_2^2} - e^{\frac{1}{2}(y_1 y_2)} \, dy_1$$

$$= \frac{2}{\pi^2} \cdot e^{\frac{1}{2}(y_2^2 + y_2)} \,,\, \frac{\pi^{1/2}}{2^{1/2}} \cdot e^{-y_2^2/8} \Big|_0^{y_2}$$

$y_2 > 0$

18) X AND Y $\quad f_{X,Y}(x,y) = e^{-y} \quad 0<x<y<\infty$

a) $S = X + Y$ AND $T = X$ (JOINT pdf OF S AND T)

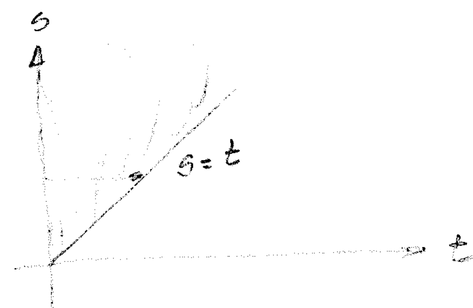
$t = x \qquad x = t$

$s = t + y \qquad y = s - t$

$$J = \begin{vmatrix} \frac{\partial s}{\partial x} & \frac{\partial t}{\partial x} \\ \frac{\partial s}{\partial y} & \frac{\partial t}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial (t+y)}{\partial x} & \frac{\partial x}{\partial x} \\ \frac{\partial (t+y)}{\partial y} & \frac{\partial x}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = |-1| = 1$$

$0 < t < s - t < \infty$

$0 < t < s/2 < \infty$



$f_{S,T}(s,t) = f_{X,Y}(t, s-t) \cdot 1$

$f_{S,T}(s,t) = e^{-(s-t)} \qquad 0 < t < s/2 < \infty$

b) $f_T(t) = \int_{2t}^{\infty} e^{(t-s)}\, ds = e^{t} \cdot -e^{-s}\Big|_{2t}^{\infty} \; e^{-\infty} + e^{-2t} = e^{-t} \qquad t > 0$

c) $f_S(s) = \int_{0}^{s/2} e^{(t-s)}\, dt = e^{-s} \cdot e^{t}\Big|_{0}^{s/2} = e^{-s}\left(e^{s/2} - 1\right) \qquad s > 0$

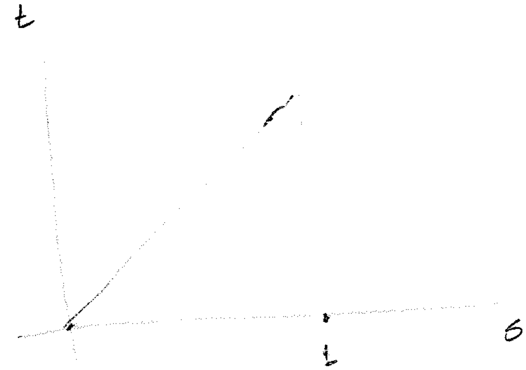
21). X AND Y CONTINUOS $f_{XY}(x,y) = 2(x+y)$

$0 < x < y < 1$

a) FIND $f_{ST}(s,t)$ WHERE $S = X$ AND $T = XY$

$s = x \qquad x = s$

$t = s.y \qquad y = t/s$



$$J = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} \quad \begin{vmatrix} 1 & -t/s^2 \\ 0 & 1/s \end{vmatrix} = \frac{1}{s}$$

$0 < s < t/s < 1 \qquad 0 < s^2 < t < s$

$$f_{ST}(s,t) = f_X(s) \cdot f_Y(t/s) \cdot \frac{1}{s}$$

$$= 2(s + t/s) \cdot \frac{1}{s} = 2\left(\frac{s^2+t}{s}\right) \cdot \frac{1}{s} = 2\left(1 + \frac{t}{s^2}\right)$$

$$f_S(s) = \int_{s^2}^{s} 2 + \frac{2t}{s^2} \; dt = \int 2\,dt + \int \frac{2t}{s^2}\,dt = 2\int dt + \frac{2}{s^2}\int t\,dt$$

$$= 2t + \frac{2}{s^2} \cdot \frac{t^2}{2} \Big|_{s^2}^{s} = 2s + 1 - 2s^2 - s^2 = -3s^2 + 2s + 1$$

$$f_T(t) = \int_0^{t^{1/2}} 2 + \frac{2t}{s^2} \; ds = \int 2\,ds + 2t\int s^{-2}\,ds =$$

$$= 2s + 2t \cdot \frac{-1}{2} \cdot s \Big|_0^{t^{1/2}} = 2\sqrt{t} - 2t \cdot \frac{\sqrt{t}}{2} = -t$$