Non-archimedean geometry

Farbod Shokrieh

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1 Introduction

Conrad's note in [BCD+08] is a good overview. For a long term, [BGR84] was the main reference, but it is difficult to read. We will primarily use [Ber90] and [Bos14] as more technical references.

We'll partially follow [Ber90], which is freely available on the Cornell Library. The algebraic and analytic foundations for this course are developed in [BGR84].

The course will largely be self-contained, but we'll occasionally refer to outside theorems. The subject is currently very active; a lot of foundational material is not contained in books.

Non-archimedean geometry is related to many different fields. It uses algebraic geometry, analysis, algebra, and has applications in those fields, combinatorics, and number theory. Many of the rings that arise are non-noetherian, for example. If this course lived "over $\mathbf{C}$," it would be a combination of functional analysis, commutative algebra, and Riemann surfaces.

Non-archimedean geometry has many fantastic application (local Langlands, etc.) but sadly, we will not be able to say much about contemporary applications.

1.1 Motivation and background

First, what is the archimedean property? Fix a field K, and a "norm" (we'll give precise definitions later) $|\cdot| : K \to \mathbf{R}$. We say that K satisfies the *archimedean property* if for any $x \in K^\times$, there exists $n \in \mathbf{Z}$ such that $|nx| > 1$.

1.1.1 Example. The fields of real and complex numbers, with their classical absolute values, are archimedean.

According to a theorem of Gelfand and Mazur, $\mathbf{R}$ and $\mathbf{C}$ are the "only" examples of archimedean fields. Since there are many examples of non-archimedean fields, there is a sense in which $\mathbf{R}$ and $\mathbf{C}$ are exceptional.

1.1.2 Example. In classical (complex) algebraic geometry, one is interested in the zero sets of polynomials in $\mathbf{C}^n$. Call such sets *algebraic*. This gives us a topology on $\mathbf{C}^n$ and its subvarieties known as the *Zariski topology*. We can also give $\mathbf{C}^n$ the topology coming from the absolute value $|\cdot|$ on $\mathbf{C}$. This finer topology is called the *canonical topology*. It allows us to do analysis on complex varieties.

The canonical topology (and the analytic techniques accompanying it) allow for:

- Cauchy integrals
- holomorphic / meromorphic functions / differential forms
- Hodge theory
- Morse theory

and there are formal GAGA (for ***G****éométrie* ***A****lgébrique et* ***G****éométrie* ***A****nalytique*) corresponces after Serre's paper [Ser56]. Essentially, Serre's paper says that as far as sheaves and cohomology on proper varieties are concerned, algebraic geometry is the same as analytic geometry.

In modern algebraic geometry, we also care about solutions (to polynomial equations) in K^n, where K is for example $\mathbf{Q}$, $\mathbf{Q}_p$ (in number theory), or $\mathbf{C}((t))$ (in the deformation theory of complex varieties). All these fields have interesting absolute values which are *not* archimedean.

1.1.3 Example. Any field K can be given the "stupid" absolute value by

$$|x| = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Although it seems like a degenerate case, the theory of the "stupid" absolute value is highly interesting.

1.1.4 Example. The field $K = \mathbf{Q}$ has two classes of absolute values. One is the classical absolute value, which we denote by $|\cdot|_\infty$. There is of course the "stupid" absolute value, which we denote $|\cdot|_0$. Fix a prime p. There is an absolute value $|\cdot|_p$ on $\mathbf{Q}$, defined by

$$\left|\frac{a}{b}\right|_p = \left|p^e\frac{a'}{b'}\right|_p = p^{-\alpha},$$

where $\frac{a}{b} = p^e\frac{a'}{b'}$ is such that both a and b are relatively prime to p.

In a precise sense, Otrowski's theorem tells us that $\{|\cdot|_\infty, |\cdot|_0, |\cdot|_p : p \text{ prime}\}$ is essentially a complete list of the possible absolute values on $\mathbf{Q}$. Note that $|\cdot|_p$ and $|\cdot|_0$ are non-archimedean. Equivalently, the familiar triangle inequality $|x+y| \leqslant |x| + |y|$ can be strengthened to the inequality

$$|x+y| \leqslant \max\{|x|, |y|\}.$$

It is easy to prove that this "strong triangle inequality" holds for $|\cdot|_0$ and the $|\cdot|_p$.

It's a good idea to complete $\mathbf{Q}$ with respect to $|\cdot|_p$. We write $\mathbf{Q}_p$ for this completion, called the field of *p-adic numbers*. (Incidentally, it is easy to check that $\mathbf{Q}$ is not complete with respect to $|\cdot|_p$. Alternatively: general theorem about complete metric spaces...).

1.1.5 Example. The field $\mathbf{C}((t))$ is often given the *t-adic* absolute value:

$$\left|\sum_{i \geqslant j} a_i t^i\right|_t = e^{-j},$$

whenever $a_k \neq 0$. The algebraic closure of $\mathbf{C}((t))$ has an easy explicit description, and is called the field of *Puiseaux series*, first studied by Isaac Newton.

Recall how one constructs $\mathbf{R}$ from $\mathbf{Q}$. First form the ring of Cauchy sequences, then mod out by the (maximal) ideal of sequences converging to zero. The same procedure works for any topological field. However, when we completed $\mathbf{Q}$ to get $\mathbf{R}$, the operation of "filling in the holes" turned the totally disconnected $\mathbf{Q}$ into a connected space. This phenomenon is pathological! In the non-archimedean world, the completion is still totally disconnected. In fact, *all* non-archimedean fields are totally disconnected. This would seem to make any kind of analysis difficult (for example: how to discuss paths?).

It makes sense to talk about power series and their convergence over non-archimedean fields. Indeed, convergence of formal power series is far easier to check in the non-archimedean case than otherwise! (A series converges if and only if its terms tend to zero.) Recall that in complex analysis, a function is analytic if it can locally be written as a convergent power series. For X some "geometric object" over a non-archimedean field K, we can define a function $f\colon X \to K$ to be analytic if it is locally presentable by a convergent power series. This is a bad definition because the notion of "locally" behaves badly for totally disconnected spaces.

1.2 Approaches to non-archimedean geometry

Before the 1960's, things stood here; no satisfactory theory existed. Today a plethora of solutions to this problem exist:

1.2.1 Tate-Grothendieck, 1960's

John Tate realized that certain types of elliptic curves can be "formally uniformized" over arbitrary fields. He wrote to Grothendieck with this discovery, but Grothendieck was unimpressed. Despite this, Tate was able to use Grothendieck's mathematical machinery to construct the category of *rigid analytic spaces*, in which uniformization makes sense. Better theories exist today, but many foundational results are only written in the language of rigid spaces. The main difficulty here is that the "spaces" involved are not actually topological spaces; they only carry a Grothendieck topology. The introduction to [BCD+08] has a good historical overview. Affinoid algebras, the analytic substitute for polynomial rings, are the only aspect of the theory still actively used today.

1.2.2 Raynaud, 1970's

Raynaud, a former student of Grothendieck, developed an extremely powerful technique for approaching non-archimedean geometry using "formal models." A downside is that the theory is quite technical. It is analogous to writing down a variety $V_{/\mathbf{Q}}$ as the zero-set of polynomials with coefficients in $\mathbf{Z}$. In Raynaud's theory, the "model" for a rigid space is a formal scheme. A non-archimedean field K contains a valuation ring R, and one represents rigid K-varieties with formal R-schemes. Raynaud's theory is good for answering "algebraic" questions, e.g. of flatness, base change, fiber dimension, ...

1.2.3 Berkovich, early 1990's

Berkovich spaces, developed in [Ber90]. This turns Tate's rigid spaces into honest topological spaces by adding points. One ends up with a topological space together with a structure sheaf. The topological space is compact, Hausdorff, and locally path connected on connected components. In a precise sense, a Berkovich space is a "space of rank-one valuations."

1.2.4 Huber, late 1990's

Huber has defined in [Hub96] a category of adic spaces. In a precise sense, an adic space is the "limit of all formal models" (Riemann-Zariski space). Equivalently, an adic space is the "space of all valuations" on some reasonable ring. Recently, Scholze used adic spaces in [Sch12] to prove spectacular theorems.

Let K be a non-archimedean field. Berkovich's idea is to use seminorms to "add points" to K, to result in a $\mathbf{R}$-tree. This construction can be glued to turn each variety $X_{/K}$ into a topological space X^{an}. It is an extremely hard theorem that this "Berkovich space" deformation retracts onto a finite simplicial complex.

1.2.5 Theorem (Berkovich, Thuillier, Loeser-Hrushovski)**.** *Any analytic space (in the sense of Berkovich) has a strong deformation retract onto a finite simplicial complex.*

See [Ber99, Thu07] for partial results in this direction. So the "wild" spaces we will construct will always be homotopy equivalent to something managable. For curves, this simplicial complex will be a graph.

1.3 Tropicalization

Consider the curve $X\colon z_1 + z_2 = 1$. The idea of behind *amœbas* is that instead of looking at $X(\mathbf{C})$, which is hard to visualize, one should consider the set

$$\mathrm{trop}_t(X) = \{(-\log_t |z_1|, -\log_t |z_2|) \in \mathbf{R}^2 \colon (z_1, z_2) \in X(\mathbf{C})\}.$$

As a subset of $\mathbf{R}^2$, this can be easily plotted. If we let $t \to 0$, then the limiting set exists; it is a union of line segments. The better approach is to consider the following set:

$$\mathrm{trop}(X) = \{(v_t(z_1), v_t(z_2)) \colon (z_1, z_2) \in X(\mathbf{C}(\!(t)\!))\}.$$

More generally, let K be a non-archimedean field, $X_{/K}$ a variety. Given a rational map $g\colon X \dashrightarrow \mathbf{G}_{\mathrm{m}}^N$, let $f\colon X(K) \to \mathbf{R}^N$ be the map

$$f(x) = (-\log |g_1(x)|, \ldots, -\log |g_N(x)|)$$

and put $\mathrm{trop}(X) = \overline{\mathrm{im}(f)}$. Clearly this depends on the choice of rational map g. The tropicalization map factors through the analytification of X:

$$\begin{tikzcd} X \arrow[r] \arrow[dr, dotted] & X^{\mathrm{an}} \arrow[d] \\ & \mathrm{trop}(X) \end{tikzcd}$$

So $\mathrm{trop}(X)$ is a "snapshot" of X^{an}.

1.3.1 Theorem (Payne)**.** *X^{an} is the limit of all tropicalizations of X.*

See [Pay09] for a rigorous formulation and proof.

1.4 Applications

Non-archimedean geometry has many applications in diverse fields. These include:

Semistable reduction theorems in arithmetic geometry. Néron models, local heights, p-adic Hodge theory, Local Langlands. Motivic zeta functions, uniformizations, counting curves, arithmetic dynamics, complex dynamics, Bogomolov conjecture, minimal model program, toroidal embeddings / toric varieties. (Bruhat-Tits) buildings, resolution of singularities in characteristic p.

2 Absolute values and valuations

A good source for this section is the textbook [EP05]. Fix a field K for the remainder of the section.

2.1 Definitions and first properties

2.1.1 Definition. An *absolute value* (norm) on K is a function $|\cdot|\colon K \to \mathbf{R}$ with the following properties:

1. $|x| = 0$ if and only if $x = 0$
2. $|xy| = |x||y|$
3. $|x + y| \leqslant |x| + |y|$.

Eventually, we will relax many of these requirements. The obvious examples are $\mathbf{R}$, $\mathbf{C}$ with the standard absolute value. The definition has some immediate consequences.

2.1.2 Lemma. *Let $(K, |\cdot|)$ be a field with absolute value. Then*

1. $|1| = 1$
2. $|1/x| = 1/|x|$
3. *If $x \in \boldsymbol{\mu}(K)$, then $|x| = 1$*
4. $|-x| = |x|$
5. $|\cdot|\colon K^\times \to \mathbf{R}^{>0}$ *is a group homomorphism*

Given an absolute value, we can define a metric on K by

$$d(x, y) = |x - y|,$$

hence K becomes a topological space. We call the topology induced by $|\cdot|$ the *canonical* topology on K. To be pedantic, $U \subset K$ is open if for all $u \in U$, there exists $r > 0$ such that the "open disk"

$$\mathrm{D}^-(a, r) = \{x \in K : |x - u| < r\} \subset U.$$

2.1.3 Lemma. *The topology induced by $|\cdot|$ is discrete if and only if $|\cdot|$ is the "stupid" absolute value (also known as the trivial absolute value) given by*

$$|x| = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Proof. Indeed, $\Leftarrow$ is trivial. Showing $\Rightarrow$ is not trivial. If $|\cdot|$ is not trivial, then there exists $x \in K$ with $0 < |x| < 1$. The sequence $\{x^n : n \geqslant 1\}$ converges to zero because $|x^n| = |x|^n \to 0$. Since we're in a field, no $x^n = 0$, so $\{0\}$ is *not* open. $\square$

In fact, we get a topological field: addition, multiplication, and inversion are all continuous. Moreover, K is Hausdorff under the canonical topology. So far so good. Given an absolute value on a field, we get a Hausdorff topological field. However, much of the course will be centered around remedying various defects of the canonical topology.

2.1.4 Lemma. *If $|\cdot|$ is an absolute value on K and $0 < e \leqslant 1$, then $|\cdot|^e$ is also an absolute value on K.*

Proof. $|\cdot|^e$ is trivially multiplicative. Showing the triangle inequality is harder, and needs $e \leqslant 1$. It turns out that if $|\cdot|$ is non-archimedean, then $e > 1$ works as well. □

2.1.5 Theorem. *Let $|\cdot|, |\cdot|'$ be two absolute values on K. These induce the same topology on K if and only if there exists $e > 0$ such that $|\cdot|' = |\cdot|^e$.*

Proof. $\Leftarrow$ is obvious. For the converse, see [cite source]. □

Recall that for any field (commutative ring, even) K, there is a unique unital ring homomorphism $\mathbf{Z} \to K$, determined by $1 \mapsto 1$.

2.1.6 Definition. Let $|\cdot|\colon K \to \mathbf{R}$ be an absolute value. Let $f : \mathbf{Z} \to K$ be the unique ring map. If $\mathrm{im}(f)$ is bounded in $\mathbf{R}$, we say $|\cdot|$ is *non-archimedean.*

This is a topological condition. In other words, if $|\cdot|$ and $|\cdot|'$ are equivalent absolute values, then one is non-archimedean if and only if the other is.

2.1.7 Example. The trivial absolute value is non-archimedean.

2.1.8 Example. If K has positive characteristic, all absolute values on K are non-archimedean. Indeed, the image of $\mathbf{Z} \to K$ consists of $\{0\} \cup \mathbf{F}_p^\times$; the latter set consists of $(p-1)$-st roots of unity. So $|n| \leqslant 1$ for all $n \in \mathbf{Z}$.

2.1.9 Example. The field $\mathbf{Q}$ with its usual absolute value $|\cdot|_\infty$ is archimedean.

2.1.10 Theorem. *An absolute value $|\cdot|$ on K is non-archimedean if and only if for all $x, y \in K$,*

$$|x + y| \leqslant \max\{|x|, |y|\}. \tag{1}$$

Proof. $\Leftarrow$ is easy. Indeed, $|n| \leqslant 1$ by induction on n.

$\Rightarrow$ is more subtle. There exists some C such that $|n| \leqslant C$ for all $n \in \mathbf{Z}$. We use the binomial theorem:

$$\begin{aligned}
|(x+y)^n| &= \left|\sum \binom{n}{i} x^i y^{n-i}\right| \\
&\leqslant \sum \left|\binom{n}{i}\right| |x^i y^{n-i}| \\
&\leqslant C(n+1)\max\{|x|, |y|\}^n.
\end{aligned}$$

Taking n-th roots of both sides, we obtain

$$|x + y| \leqslant \sqrt[n]{C(n+1)}\max\{|x|, |y|\}.$$

Letting $n \to \infty$, we obtain the result. □

The inequality (1) is called the ultrametric inequality in older sources. Today, it is generally called the non-archimedean triangle inequality. Since the non-archimedean triangle inequality holds when raised to arbitrary positive powers, if $|\cdot|$ is non-archimedean, then $|\cdot|^e$ is an absolute value for all $e > 0$.

The following simple lemma is quite important.

2.1.11 Lemma. *In a non-archimedean field, if* $|x| \neq |y|$, *then* $|x+y| = \max\{|x|, |y|\}$.

Proof. We may assume $|x| < |y|$. We need to show $|x+y| = |y|$. If not, we have $|x+y| < |y|$. But

$$|y| = |y + x - x| \leqslant \max\{|y+x|, |x|\} < y,$$

a contradiction. □

2.2 Topological consequences

2.2.1 Center of a disk

Let $(K, |\cdot|)$ be a non-archimedean field. Let $a, a' \in K$ with $|a - a'| \leqslant r$ for some $r > 0$. Then $|x - a| \leqslant r$ if and only if $|x - a'| \leqslant r$. In other words: *all points in a disk of radius r are the center of the disk*! In other words, any point in the set

$$\mathrm{D}(a, r) = \{x \in K : |x - a| \leqslant r\},$$

is a “center” of $\mathrm{D}(a, r)$. To see this, suppose $|x - a| \leqslant r$. Then

$$\begin{aligned} |x - a'| &= |x - a + a - a'| \\ &\leqslant \max\{|x - a|, |a - a'|\} \\ &\leqslant r. \end{aligned}$$

2.2.2 Clopen sets

Clearly the “closed disk” $\mathrm{D}(a, r)$ is closed with respect to the canonical topology. (This works for any metric space.) The terrible problem with non-archimedean fields is that $\mathrm{D}(a, r)$ is also open! Indeed, let $x_0 \in \mathrm{D}(a, r)$ (so $|x_0 - a| \leqslant r$) and consider the “open ball”

$$\mathrm{D}^-(x_0, r) = \{x \in K : |x - x_0| < r\}.$$

We have $\mathrm{D}^-(x_0, r) \subset \mathrm{D}(a, r)$.

2.2.3 Boundaries

Contrary to intuition, in a non-archimedean field, the set $\{x \colon |x - a| = r\}$ is *not* the topological boundary (closure $\smallsetminus$ interior) of $\mathrm{D}(a, r)$. In fact, any small neighborhood of a point in this set is contained in $\mathrm{D}(a, r)$.

2.2.4 Disconnectedness

Putting everything together, the canonical topology on K is totally disconnected. In other words, the only non-empty connected components are singletons. Indeed, let $x_0 \in K$, and let A be the connected component containing x_0. Assume $x_1 \neq x_0$ is also in A. Let $|x_1 - x_0| > r > 0$. Then

$$A = (A \cap \mathrm{D}(r, x_0)) \sqcup (A \smallsetminus \mathrm{D}(r, x_0))$$

is a decomposition of A into a disjoint union of open nonempty subsets.

2.3 Ostrowski's theorem

Consider the field $\mathbf{Q}$. We already know of the following absolute values:

1. $|\cdot|_\infty^e$ for any $0 < e \leqslant 1$.
2. $|\cdot|_0$, the trivial absolute value.
3. For each prime p, we have an absolute value $|\cdot|_p^e$ for any $e > 0$. Recall that for any $\frac{a}{b} \in \mathbf{Q}$, write $\frac{a}{b} = p^\alpha \frac{a'}{b'}$ for a', b' not divisible by p. One has $|a/b|_p^e = (p^e)^{-\alpha}$. We generally normalize by requiring $|p|_p = 1/p$; there are good analytical reasons from this involving normalization of Haar measures.

The field $\mathbf{Q}$ is totally disconnected for all these topologies. For $|\cdot|_\infty$, this can be solved via completion. For the p-adic and trivial absolute values, we need Berkovich's theory.

2.3.1 Theorem (Ostrowski)**.** *All absolute values on* $\mathbf{Q}$ *are, up to topological equivalence, of the form* $|\cdot|_0$, $|\cdot|_\infty$, *or* $|\cdot|_p$ *for some* p.

Proof. Let $|\cdot|$ be an absolute value on $\mathbf{Q}$.

Case 1: $|\cdot|$ is non-archimedean. Then $|n| \leqslant 1$ for all $n \in \mathbf{Z}$. If $|p| = 1$ for all primes p, then by unique factorization, $|\cdot|$ is the trivial absolute value. If there exists a prime p for which $|p| < 1$, there is no $l \neq p$ prime such that $|l| < 1$. For, since l and p are coprime, there is $x, y \in \mathbf{Z}$ for which $px + ly = 1$. But then

$$1 = |1| = |px + ly| \leqslant \max\{|px|, |py|\} < 1,$$

a contradiction. So $|p| < 1$ for a unique p. After possibly replacing $|\cdot|$ by an equivalent absolute value, we may assume $|p| = 1/p$. Unique factorization, combined with $|l| = 1$ ($l \neq p$) tells us that $|\cdot| = |\cdot|_p$.

Case 2: $|\cdot|$ is archimedean. There exists $n \in \mathbf{Z}^{>0}$ such that $|n| > 1$; let n_0 be the smallest such n. Write $|n_0| = n_0^\alpha$. Given $n \in \mathbf{Z}^{>0}$, we need to show $|n| = |n|_\infty^\alpha$. Write n in base n_0:

$$n = a_0 + a_1 n_0 + \cdots + a_s n_0^s, \tag{2}$$

where each $0 \leqslant a_0 < n_0$ and $a_s \neq 0$. First we show $|n| \leqslant |n|_\infty^\alpha$. Put $C = \sum_{i \geqslant 0} n_0^{-\alpha i}$;

then

$$\begin{aligned}|n| &= \left|\sum_{i=0}^{s} a_i n_0^i\right| \\ &\leqslant \sum_{i=0}^{s} |a_i||n_0|^i \\ &\leqslant \sum_{i=0}^{s} n_0^{\alpha i} \\ &\leqslant n_0^{s\alpha}(1 + n_0^{-\alpha} + n_0^{-2\alpha} + \cdots) \\ &\leqslant C n^\alpha.\end{aligned}$$

So $|n^N| \leqslant C n^{\alpha N}$ for all N. Thus $|n| \leqslant \sqrt[N]{C} n^\alpha$; letting $N \to \infty$ yields $|n| \leqslant n^\alpha$. Now we prove $|n| \geqslant n^\alpha$. Once again write n in base n_0 as in (2). We get

$$\begin{aligned}|n_0^{s+1}| &\leqslant |n| + |n_0^{s+1} - n| \\ &\leqslant |n| + (n_0^{s+1} - n)^\alpha \\ &\leqslant |n| + (n_0^{s+1} - n_0^s)^\alpha.\end{aligned}$$

It follows that

$$|n| \geqslant n_0^{\alpha(s+1)} \left(1 - \left(1 - \frac{1}{n_0}\right)^\alpha\right) \geqslant C' n^\alpha,$$

where as above C' does not depend on n. The same trick (raise to N, let $N \to \infty$) yields the result. □

There are versions of Ostrowski's theorem for arbitrary Dedekind domains, or function fields like $\mathbf{F}_p(t)$.

2.4 Completion

Here, will discuss "completion" in the sense of metric spaces. One can also "complete" topological spaces; there is a more general notion of completion that works for "uniform spaces," which are topological spaces together with a uniform structure. We will focus on metrics coming from absolute values.

2.4.1 Definition. A metric space is *complete* if every Cauchy sequence converges.

Let K be a field with absolute value $|\cdot|$. Note that $(K, |\cdot|)$ is complete if and only if $(K, |\cdot|^e)$ is complete, so completeness of a field is a topological property.

2.4.2 Example. The field $\mathbf{Q}$ with $|\cdot|_0$ is complete (Cauchy sequences stabilize).

2.4.3 Example. The field $\mathbf{Q}$ with $|\cdot|_\infty$ is *not* complete. It's completion is denoted $\mathbf{R}$; this is as nice as we could hope for (as a topological space).

2.4.4 Example. The field $\mathbf{Q}$ with $|\cdot|_p$ is also not complete. For example, for any integer $n \geqslant 1$, the sequence

$$n, n^p, n^{p^2}, n^{p^3}, \ldots$$

is Cauchy, and (if p is odd) it does not converge in $\mathbf{Q}$ when we set $n = p - 1$. Alternatively, it is known that any complete metric space without isolated points is uncountable; since $\mathbf{Q}$ is countable, it cannot be complete in any non-discrete topology.

2.4.5 Definition. Let $(K, |\cdot|)$ be a field with absolute value. A *completion* of $(K, |\cdot|)$ is tuple $(K', |\cdot|', i)$, where

1. $(K', |\cdot|')$ is a complete field,
2. $i\colon K \hookrightarrow K'$ is an isometry (i.e. $|x| = |i(x)|'$ for all $x \in K$), and
3. for any isometry $j\colon K \hookrightarrow F$ from K into a complete field $(F, |\cdot|'')$, there exists a unique isometry $i'\colon K' \hookrightarrow F$ such that the following diagram commutes:

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[dr, "j"'] & K' \arrow[d, dotted, "i'"] \\ & F. \end{tikzcd}$$

So a completion is an embedding of K into a complete field that is "initial" among such embeddings. Standard arguments show that the completion of K is uniquely determined by the universal property.

2.4.6 Theorem. *Let K be a field with absolute value $|\cdot|$. Then:*

1. *A completion $(K', |\cdot|', i)$ of K exists.*
2. *$i(K)$ is dense in K'.*
3. *$|\cdot|$ is non-archimedean if and only if $|\cdot|'$ is non-archimedean.*
4. *If $|\cdot|$ is non-archimedean, then*

$$|K^\times| = \{|x| \colon x \in K^\times\} = |K'^\times|'.$$

Proof. 1, 2 are classical. Construct K' as the quotient of the ring of Cauchy sequences by the (maximal) ideal of nullsequences.

3. Obvious corollary of Theorem 2.1.10.

4. Roughly: suppose $x \in K'$ is written $x = \lim x_n$, for $\{x_n\}$ a Cauchy sequence in K. Then $|x|' = \lim |x_n|$. For $m, n \gg 0$, we have $|x_n - x_m| < |x|/2 < |x_n|$. By Lemma 2.1.11, this can only happen if $|x_n| = |x_m|$. So the sequence $\{|x_n|\}$ stabilizes. □

It is true (though we will not prove) that if $K \hookrightarrow K'$ is *any* isometry from K to a complete field with dense image, then K' is a completion of K.

Part 4 clearly fails when passing from $\mathbf{Q}$ to $\mathbf{R}$. On the other hand, now we know that $\mathbf{Q}_p$ has "no new absolute values."

2.5 The p-adics

For each prime p, we write $\mathbf{Q}_p$ for any completion of $\mathbf{Q}$ with respect to $|\cdot|_p$; we call $\mathbf{Q}_p$ the *field of p-adic numbers.* Define

$$\begin{aligned}\mathbf{Q}_p^\circ &= \mathbf{Z}_p = \{x \in \mathbf{Q}_p : |x| \leqslant 1\}\\ \mathbf{Q}_p^{\circ\circ} &= \mathfrak{m} = \{x \in \mathbf{Q}_p : |x| < 1\}.\end{aligned}$$

One calls $\mathbf{Z}_p$ the ring of *p-adic integers.* One has $\mathbf{Z}_p/\mathfrak{m} \simeq \mathbf{F}_p$. Choose a set $S = \{\alpha_0, \dots, \alpha_{p-1}\}$, where α_i is a p-adic integer such that $\alpha_i \equiv i \pmod p$. We could use $S = \{0, \dots, p-1\}$, but there are better choices (e.g. Teichmüller representatives). Then some elementary work shows that every $\alpha \in \mathbf{Q}_p$ has exactly one representative Cauchy sequence $\{a_i\}$, where

$$a_i = b_0 p^{-m} + b_1 p^{-(m-1)} + \cdots + b_{i-1} p^{i-1-m},$$

where the $b_j \in S$.

2.5.1 Example. Work in $\mathbf{Q}_3$ and let $S = \{0, 1, 2\}$. One such sequence is:

$$\begin{aligned}a_1 &= 1\\ a_2 &= 1 + 0 \cdot 3\\ a_3 &= 1 + 0 \cdot 3 + 2 \cdot 3^2\\ a_4 &= 1 + 0 \cdot 3 + 2 \cdot 3^2 + 1 \cdot 3^3\\ &\ \vdots\end{aligned}$$

Given a representation

$$a = b_0 p^{-m} + b_1 p^{-(m-1)} + \cdots,$$

where $b_0 \neq 0$, one can check that $|a|_p = p^m$.

The field $\mathbf{Q}_p$ is complete, totally disconnected, but not algebraically closed. An abstract algebraic closure $\overline{\mathbf{Q}_p}$ exists. However, unlike the passage from $\mathbf{R}$ to $\mathbf{C}$, the extension $\overline{\mathbf{Q}_p}/\mathbf{Q}_p$ is *not* finite. Moreover, $\overline{\mathbf{Q}_p}$ is not complete with respect to the unique absolute value extending the one on $\mathbf{Q}_p$. Let $\mathbf{C}_p = \widehat{\overline{\mathbf{Q}_p}}$; here we we are lucky and $\mathbf{C}_p$ is algebraically closed. The field $\mathbf{C}_p$ is the p-adic analogue of $\mathbf{C}$.

3 Absolute values and field extensions

A field extension is naturally a vector space over the base field, so our first results will be in the more general context of vector spaces over normed fields.

3.1 Normed vector spaces

3.1.1 Definition. Let K be a field with absolute value $|\cdot|$, V be a vector space over K. A *(K-vector space) norm* on V is a map $\|\cdot\| : V \to \mathbf{R}$, satisfying:

1. $\|v\| \geqslant 0$ for all $v \in V$, with equality if and only if $v = 0$.

2. $\|cv\| = |c|\|v\|$ for all $c \in K$, $v \in V$.

3. $\|v + w\| \leqslant \|v\| + \|w\|$ for all $v, w \in V$.

We get in the natural way a metric (hence topology) on V. Thus concepts like completeness and completion make sense. The whole machinery of subsection 2.4 carries through.

Two norms $\|\cdot\|_1$, $\|\cdot\|_2$ induce some topology if and only if

$$c_2\|\cdot\|_2 \leqslant \|\cdot\|_1 \leqslant c_1\|\cdot\|_2,$$

for some $c_1, c_2 > 0$. We call such norms *equivalent*. In classical linear algebra, one often rescales vectors to make them have norm one. Here, if $v \in V$, we might not be able to rescale v to get $\|cv\| = 1$. This is because $\{|c| \colon c \in K^\times\}$ can be a proper subgroup of $\mathbf{R}^{>0}$.

3.1.2 Example. Fix a basis $B = \{v_1, \ldots, v_d\}$ of V. Given $x = \sum a_i v_i$, put

$$\|x\|_{\mathrm{sup},B} = \|x\|_{\mathrm{max},B} = \max_{1 \leqslant i \leqslant d} |a_i|.$$

It is easy to show that this actually is a norm. If K is complete with respect to $|\cdot|$, then $(V, \|\cdot\|_{\mathrm{max}})$ is also complete. If $|\cdot|$ is non-archimedean, then

$$\|v + w\|_{\mathrm{max}} \leqslant \max\{\|v\|_{\mathrm{max}}, \|w\|_{\mathrm{max}}\}.$$

Finally, the max norm with respect to different bases are equivalent.

3.1.3 Theorem. *If K is complete with respect to $|\cdot|$ and V is finite-dimensional over K, then all K-vector space norms on V are equivalent. Moreover, V is also complete, all vector subspaces are closed, and if $|\cdot|$ is non-archimedean, then $\|\cdot\|$ is also non-archimedean, in the weaker sense that*

$$\|v + w\| \leqslant C \max\{\|v\|, \|w\|\},$$

for some fixed C.

Proof. Fix a basis $B = \{v_1, \ldots, v_d\}$ of V. Given any norm $\|\cdot\|$ on V, we show it's equivalent to $\|\cdot\|_{\mathrm{max},B}$. One direction doesn't use completeness:

$$\begin{aligned} \|x\| &= \left\|\sum a_i v_i\right\| \\ &\leqslant \sum |a_i|\|v_i\| \\ &\leqslant d \max_i \|v_i\| \max_i i|a_i|. \end{aligned}$$

Put $C_1 = d\max\{v_i\}$; this yields $|\cdot| \leqslant C_1\|\cdot\|_{\mathrm{max},B}$. To show that $\|\cdot\|_{\mathrm{max}} \leqslant C_2\|\cdot\|$, we induct on $d = \dim(V)$. For $d = 1$, the result is immediate: just observe $\|v\| = |a_1|\|v_1\|$. If $d > 1$, we may assume that all proper subspaces of V are complete. General nonsense tells us that subspaces are closed. Let $H_i = \sum_{j \neq i} K v_j$. Let $\pi_i \colon V \to V/H_i = L_i$ be the projection map. Define a new norm on L_i by

$$\|u\|_i = \inf\{\|v\| \colon \pi_i(v) = u\}.$$

Since H_i is closed, $\|u_i\| = 0$ implies $u \in H_i$. For $v = \sum a_i v_i$, we have $\pi_i(v) = a_i \pi_v(v_i)$. Note that $\|\pi_i(v)\|_i = |a_i| \|\pi_i(v_i)\|_i$, which implies

$$\frac{\|\pi_i(v)\|_i}{\|\pi_i(v_i)\|_i} \leqslant \frac{\|v\|}{\|\pi_i(v_i)\|_i} \leqslant B_i \|v\|,$$

where $B_i = \|\pi_i(v_i)\|_i^{-1}$. Put $C_2 = \max\{B_i\}$. □

The moral of the story is that finite-dimensional vector spaces over a complete field carry a unique topology.

3.2 Existence of extensions

3.2.1 Theorem. *Let K be a field complete with respect to $|\cdot|$. Let L/K be an extension.*

1. *There is exactly one extension of $|\cdot|$ to an absolute value $|\cdot|'$ on L. Moreover, $|\cdot|'$ is non-archimedean if and only if $|\cdot|$ is non-archimedean.*
2. *If $[L\colon K] < \infty$, then $(L, |\cdot|')$ is complete. If K is non-archimedean, then for all $x \in L$,*

$$|x|' = \left|\mathrm{N}_{L/K}(x)\right|^{\frac{1}{[L\colon K]}}. \tag{3}$$

Proof. 1. That there is *at most* one extension follows almost immediately from Theorem 3.1.3. The field L is a direct limit of finite extensions, so we may as well assume L is a finite extension of K. Since L is a finite-dimensional K-vector space, it has a unique topology induced by vector space norm. Since K is complete, L is also complete. Let $|\cdot|_1$ and $|\cdot|_2$ be two norms on L. If the topology induced by $|\cdot|_1$, then everything in sight is discrete. If $|\cdot|_1$ induces a non-discrete topology, then by Theorem 2.1.5, $|\cdot|_1 = |\cdot|_2^e$ for some e. Restricting to K, we see that $e = 1$.

2. Recall that for $x \in L$, the map "multiply by x" is a K-linear map $(x\cdot)\colon L \to L$. The *norm* of x is by definition $\mathrm{N}_{L/K}(x) = \det(x\cdot)$. It's not easy to show that (3) satisfies the triangle inequality. (One uses Hensel's lemma to show that $\mathrm{N}_{L/K}(x) \in K^\circ$ if and only if $x \in L^\circ$.) □

It is easy to show that (3) is the only possible extension of a norm, using the Galois definition of the norm. Abstract nonsense reduces us to the case where L/K is finite Galois. In this case, we know that

$$\mathrm{N}_{L/K}(x) = \prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma(x).$$

It follows that

$$\begin{aligned} |\,\mathrm{N}_{L/K}(x)| &= |\,\mathrm{N}_{L/K}(x)|' \\ &= \prod_{\sigma \in \mathrm{Gal}(L/K)} |\sigma(x)|' \\ &= {|x|'}^{[L\colon K]}. \end{aligned}$$

The last equality holds because $|\sigma(x)|' = |x|'$ for all σ.

3.2.2 Example. Let $\mathbf{Q}_p$ be the completion of $\mathbf{Q}$ with respect to the p-adic absolute value $|\cdot|_p$. By Theorem 3.2.1, $\overline{\mathbf{Q}_p}$ carries a unique extension of $|\cdot|_p$; the set $|\overline{\mathbf{Q}_p}^\times| \subset \mathbf{R}^{>0}$ is dense!

3.3 Archimedean fields

In general, if K is a complete non-archimedean field, then its algebraic closure $\overline{K}$ will not be complete. The only exception is when $\overline{K} = \mathbf{C}$.

3.3.1 Theorem (Gelfand, Mazur)**.** *The only complete archimedean fields are* $(\mathbf{R}, |\cdot|_\infty^e)$ *and* $(\mathbf{C}, |\cdot|_\infty^e)$ *for* $0 \leqslant e < 1$*.*

Proof. This is [EP05, 1.2.4]. If $(K, |\cdot|)$ is a complete archimedean field, then K has characteristic zero, so $\mathbf{Q} \hookrightarrow K$. The absolute value $|\cdot|$ restricted to $\mathbf{Q}$ is archimedean, so by Theorem 2.3.1, $|\cdot| = |\cdot|_\infty^e$ on $\mathbf{Q}$ for some $0 < e \leqslant 1$. By the universal property of (metric) completion, we get a continuous embedding $\mathbf{R} \hookrightarrow K$. By Corollary 3.3.4, $K = \mathbf{R}$ or $K = \mathbf{C}$. ☐

3.3.2 Corollary. *If* $(K, |\cdot|)$ *is an archimedean field, then* K *admits a dense isometry into either* $\mathbf{R}$ *or* $\mathbf{C}$*.*

3.3.3 Lemma. *Let* A *be a commutative Banach* $\mathbf{R}$*-algebra containing* $\mathbf{C}$*. For every* $a \in A$*, the set* $\{c \in \mathbf{C} \colon a - c \notin A^\times\}$ *is compact and nonempty.*

Proof. To show that the set, known as the *spectrum* of a, is nonempty, one uses the map $\mathbf{C} \to A$ given by $c \mapsto (a-c)^{-1}$. For details, see [Rud87, 18.6]. ☐

3.3.4 Corollary. *Let* $(K, |\cdot|)$ *be an extension of* $(\mathbf{R}, |\cdot|_\infty)$*. Then either* $K = \mathbf{R}$ *or* $K = \mathbf{C}$*.*

Proof. If K contains $\mathbf{C}$, then applying Lemma 3.3.3 shows that $K = \mathbf{C}$. If K does not contain any such j, the field $K(\sqrt{-1})$ must be $\mathbf{C}$, hence $K = \mathbf{R}$. ☐

If $(K, |\cdot|)$ is a non-archimedean field which is *not* complete, and L/K is an algebraic extension, how can we extend $|\cdot|$ to L?

3.3.5 Theorem. *Let* $(K, |\cdot|)$ *be a non-archimedean field,* L/K *an algebraic extension. Then* $|\cdot|$ *admits an extension to* L*.*

Proof. Let $\widehat{K}$ be the completion of K; it carries a unique extension of $|\cdot|$. Let $\overline{\widehat{K}}$ be an algebraic closure of $\widehat{K}$; this also carries a unique extension of $|\cdot|$. Since L/K is algebraic, we have an embedding $i : L \hookrightarrow \overline{\widehat{K}}$. The pullback via i of the absolute value on $\overline{\widehat{K}}$ is an extension of $|\cdot|$ to L. ☐

3.4 Krasner's lemma

3.4.1 Theorem (Krasner)**.** *Let* K *be an algebraically closed field with absolute value* $|\cdot|$*. Then* $\widehat{K}$ *is also algebraically closed.*

Proof. This works via "continuity of roots." Let L be an algebraic closure of $\widehat{K}$. We want to prove that $L = \widehat{K}$. Let $f \in \widehat{K}[x]$; write $f = \sum_{i=0}^n c_i x^i$ with $c_n = 1$. Let $\alpha \in L$ be a root of f, i.e. $f(\alpha) = 0$. It suffices to show that α can be approximated by elements $\beta_i \in K$ (so $\alpha = \lim \beta_i \in \widehat{K}$). Fix $\epsilon > 0$. We may choose $d_i \in K$ approximating the coefficients $c_i \in \widehat{K}$, so that $g = \sum d_i x^i \in K[x]$ is monic and satisfies $|g(\alpha)| \leqslant \epsilon^n$. Since K is algebraically closed, we have a factorization $g = \prod(x - \beta_i)$, so

$$|g(\alpha)| = \prod_{i=1}^{n} |\alpha - \beta_i| \leqslant \epsilon^n.$$

So at least one β_i satisfies $|\alpha - \beta_i| \leqslant \epsilon$. □

We mentioned that earlier that $(\mathbf{Q}, |\cdot|_p)$ is not complete. There is an explicit, algorithmic way to show this. One shows that if $(\mathbf{Q}, |\cdot|_p)$ were complete, then all $(p-1)$-st roots of unity would lie in $\mathbf{Q}$, which is not the case, at least if $p \geqslant 5$. The general technique relies on the isomorphism $\mathbf{Z}_p = \varprojlim \mathbf{Z}/p^i$, and uses Hensel's Lemma to construct compatible sequences $\{a_i \mod p^i\}$. This is done very nicely in [Gou97, 3.2.3].

For any non-archimedean field K, we have a sequence of embeddings:

$$K \hookrightarrow \widehat{K} \hookrightarrow \overline{\widehat{K}} \hookrightarrow \widehat{\overline{\widehat{K}}}. \tag{4}$$

Each field has a canonical (unique) extension of the absolute value $|\cdot|$ of K. We denote these absolute values by $|\cdot|$ as well. One might wonder if the process of passing to algebraic closure and them completion can be repeated indefinitely. By Theorem 3.4.1, the sequence (4) is is as far as we need to go: the field $\widehat{\overline{\widehat{K}}}$ is as far as we need to go. One thing to be careful of: the value group stays the same in the first and third embeddings (4), but $|\overline{\widehat{K}}^\times|$ may be strictly larger than $|\widehat{K}^\times|$.

3.4.2 Example. For $(\mathbf{Q}, |\cdot|_p)$, we get the following chain of embeddings:

$$\mathbf{Q} \hookrightarrow \mathbf{Q}_p \hookrightarrow \overline{\mathbf{Q}_p} \hookrightarrow \widehat{\overline{\mathbf{Q}_p}} = \mathbf{C}_p.$$

The field $\mathbf{C}_p$ is the "p-adic analogue of $\mathbf{C}$." See the book [Kob84] for a careful explanation of each step.

3.4.3 Definition. Let K be any field. The *field of Puiseaux series* over K, is

$$K\{\!\{t\}\!\} = \left\{ \sum_{k=k_0}^{\infty} a_i t^{k/n} : n \in \mathbf{Z}^{>0} \right\}.$$

That is, elements consist of formal sums $\sum_{r \in \mathbf{Q}} a_r t^r$ such that the set $\{r : a_r \neq 0\}$ is bounded below and an element of some $\mathbf{Z}[\frac{1}{n}]$. Alternatively,

$$K\{\!\{t\}\!\} = \varinjlim_n K(\!(t^{1/n})\!).$$

3.4.4 Theorem (Newton)**.** *If K is algebraically closed of characteristic zero, then $K\{\!\{t\}\!\} = \overline{K(\!(t)\!)}$.*

Newton proved this theorem by creating an early version of what is now called *Newton's method.* Thus we have a sequence of embeddings

$$K(t) \hookrightarrow K(\!(t)\!) \hookrightarrow K\{\!\{t\}\!\} \hookrightarrow \widehat{K\{\!\{t\}\!\}},$$

in which the last field is known as the field of *formal Puiseaux series.* It is sometimes written $K(\!(t^{\mathbf{Q}})\!)$. For a proof of Newton's theorem, see [Eis95, Cor. 13.15].

3.5 Immediate extensions and spherical closure

3.5.1 Definition. Let $(K, |\cdot|)$ be a non-archimedean field. Put

$$\begin{aligned} K^\circ &= \{x \in K\colon |x| \leqslant 1\} && \text{“valuation ring”} \\ K^{\circ\circ} &= \{x \in K\colon |x| < 1\} && \text{“(unique) maximal ideal”} \\ K^\natural &= K^\circ / K^{\circ\circ} && \text{“residue (class) field”.} \end{aligned}$$

Note that the non-archimedean triangle inequality implies that K° actually is a ring. One often puts $R = K^\circ$; this is a local ring with maximal ideal $\mathfrak{m} = K^{\circ\circ}$ and residue field $K^\natural = R/\mathfrak{m}$. All the assignments $(-)^\circ$, $(-)^{\circ\circ}$, and $(-)^\natural$ are functorial.

3.5.2 Lemma. *Let $(K, |\cdot|)$ be a non-archimedean field. Then the embedding $K \hookrightarrow \widehat{K}$ induces an isomorphism $K^\natural \xrightarrow{\sim} \widehat{K}^\natural$.*

Proof. It is trivial to check that:

$$K^\circ = \widehat{K}^\circ \cap K$$
$$K^{\circ\circ} = \widehat{K}^{\circ\circ} \cap K.$$

This gives us a well-defined map $K^\circ \to \widehat{K}^\natural$, namely $a \mapsto a + \widehat{K}^{\circ\circ}$. It has kernel $K^{\circ\circ}$, so we get an embedding $K^\natural \hookrightarrow \widehat{K}^\natural$. To see that this map is surjective, note that for any $x \in \widehat{K}^\circ$, the set $x + \widehat{K}^{\circ\circ}$ is an open set in $\widehat{K}$. Since K is dense in $\widehat{K}$, there is some $y \in (x + \widehat{K}^{\circ\circ}) \cap K$; then $y \mapsto x + \widehat{K}^{\circ\circ}$. $\square$

3.5.3 Definition. Let L/K be an arbitrary extension of fields with absolute value. If $|K^\times| = |L^\times|$ and $K^\natural \xrightarrow{\sim} L^\natural$, we say L is an *immediate extension* of K.

3.5.4 Example. If K is any non-archimedean field, then $\widehat{K}$ is an immediate extension of K.

3.5.5 Definition. We say a field $(K, |\cdot|)$ is *maximally complete* if it admits no proper immediate extensions.

3.5.6 Definition. Let (X, d) be a metric space. We say X is *spherically complete* if, whenever $\{D_i\}_{i \in I}$ is a nested set of closed disks (i.e. each $D_i = \mathrm{D}(a_i, r_i) = \{x \in X : d(x, a_i) \leqslant r_i\}$ for some a_i, r_i, and for any i, j, either $D_i \subset D_j$ or $D_j \subset D_i$), then $\bigcap_i D_i \neq \varnothing$.

It turns out that spherical completeness is important for functional analysis. For example, the proof of the Hahn-Banach theorem requires spherical completeness. As an easy exercise, show that a spherically complete metric space is complete.

3.5.7 Example. The fields $(\mathbf{R}, |\cdot|_\infty)$ and $(\mathbf{C}, |\cdot|_\infty)$ are spherically complete.

3.5.8 Example. The field $(\mathbf{C}_p, |\cdot|_p)$ is *not* spherically complete. This is non-trivial, and highly frustrating. See Chapter 3 of [Rob00] for an explicit example of a sequence of nested disks in $\mathbf{C}_p$ with empty intersection.

3.5.9 Theorem. *A non-archimedean field is spherically complete if and only if it is maximally complete.*

For a good general discussion of spherically complete fields, see Chapter 2 of [NBB71]. This theorem is a combination of results on p. 34 and 43 of that book.

3.5.10 Theorem (Krull)**.** *All valued fields have a maximally complete extension.*

This proved on p. 54 of [NBB71].

3.5.11 Theorem. *All non-archimedean fields can be embedded into a spherically complete extension with the same value group and residue field.*

3.5.12 Example. Let's apply the above results to $\mathbf{C}_p$. We get an algebraically closed, spherically complete extension of $\mathbf{Q}_p$. Denote this by $\mathbf{Q}_p^{\odot}$

We would like some notion of uniqueness for maximally complete extensions. Sadly, this doesn't work in general.

3.5.13 Theorem (Kaplansky)**.** *If $K^\natural$ is characteristic zero, there is a unique maximal extension.*

So this can't be applied to $\mathbf{Q}_p$. See Chapter 3 of [Rob00] for a construction using ultraproducts. He ends up with a field Ω_p which contains $\mathbf{C}_p$, is spherically complete and algebraically closed. Moreover, $|\Omega_p| = \mathbf{R}^{\geqslant 0}$, as opposed to $|\mathbf{C}_p^\times| = p^{\mathbf{Q}}$. As a field, $\mathbf{C} \simeq \mathbf{C}_p$. This is a special case of the general theorem that the isomorphism of an uncountable algebraically closed field depends only on its cardinality (the theory of such fields is *categorical*). See §3.5 of [Rob00].

There are more algebraic constructions due to Poonen [Poo93] and Kedlaya [Ked01], which rely on Mal'cev-Neumann rings (or fields of Hahn series) and their p-adic analogue. We will return to this in a later section on arbitrary valuations.

Let K be a field. Then the extension $\overline{K}/K$ is unique up to (possibly non-unique) isomorphism. That is, $\overline{K}$ is a unique isomorphism class of extensions of K. This fails for spherical completions.

The notion of "immediate extension" does not make sense for archimedean fields. All the notions of K°, $K^{\circ\circ}$, $K^\natural$, immediate extension, maximally complete, etc. only make sense for non-archimedean $|\cdot|$.

Krull's proof (that any field can be embedded into a maximally complete field) in fact shows that the (maximally complete) field is maximal (in a precise sense) among immediate extensions. So every field admits a maximally complete immediate extension.

Krasner's theorem tells us that the completion of an algebraically closed field is algebraically closed. It is not clear (and possibly not true) that every field admits a dense embedding into a spherically complete field. Chapter 3 of [Rob00] constructs a non-immediate algebraically closed spherically complete field. Conclusion: Krasner's lemma doesn't (necessarily) hold for spherical completion.

3.6 Hensel's lemma

This is the main reason for why $\mathbf{Q}_p$ is "easy." It reduces problems from K to (purely algebraic) problems in $K^\natural$. Since $\mathbf{Q}_p^\natural = \mathbf{F}_p$, this generally makes a problem effectively computable.

Let $(K, |\cdot|)$ be a complete non-archimedean field. We have a natural map $K^\circ[x] \to K^\natural[x]$ denoted $f \mapsto \bar{f}$, "reduce modulo $K^{\circ\circ}$."

3.6.1 Theorem (Hensel). *Assume $g \in K^\circ[x]$; assume $\tilde{g} \in K^\natural[x]$ is nonzero. Assume $\bar{g} = pq$ in $K^\natural[x]$, where*

- *p is monic,*
- $\gcd(p,q) = 1$ *in $K^\natural[x]$.*

Then $g = PQ$ in $K^\circ[x]$, where

- *P is monic,*
- *$\bar{P} = p$, $\bar{Q} = q$.*

See [Bos14] for a proof.

The usual proof uses completeness / denseness of K by successively approximating P and Q. But in fact, completeness / denseness is not the core reason for the truth of this theorem! There is a general notion of completion for arbitrary (higher rank) valuation rings, and in this context the standard Hensel's lemma can fail. It is replaced by something called the "henselization" of a ring. We will see more about this in subsection 4.3.

3.6.2 Corollary. *Let L/K be an algebraic extension. Then $\alpha \in L$ is integral over K° if and only if $\mathrm{N}_{K(\alpha)/K}(\alpha) \in K^\circ$.*

The proof is in [Bos14]. In this generality, one needs Hensel's lemma for the proof. We can now prove that the extension of a complete absolute value $|\cdot|$ on K to L via

$$|\alpha|' = |\,\mathrm{N}_{K(\alpha)/K}(\alpha)|^{\frac{1}{[K(\alpha):\,K]}},$$

satisfies the non-archimedean triangle inequality:

$$|\alpha+\beta|' \leqslant \max\{|\alpha|', |\beta|'\}.$$

We may assume $|\beta|' \geqslant |\alpha|'$. Dividing by β, it comes down to showing that $|1+\gamma|' \leqslant 1$ whenever $|\gamma| \leqslant 1$. In light of Corollary 3.6.2, this is now obvious.

3.6.3 Corollary. *Let $(K, |\cdot|)$ be a complete non-archimedean field, $f \in K^\circ[x]$ nonzero. Assume there is $\tilde{a} \in K^\natural[x]$ such that $f(\tilde{a}) = 0$, but $f'(\tilde{a}) \neq 0$. Then there exists $a \in K^\circ$ such that $a \equiv \tilde{a} \mod K^{\circ\circ}$ and $f(a) = 0$.*

For $K = \mathbf{Q}_p$, this means that given $f \in \mathbf{Z}_p[x]$ with a non-repeated root a modulo p, then f has a root in $\mathbf{Z}_p$. A direct proof works via a p-adic version of Newton's algorithm (with guaranteed solution). This is done very nicely in [Gou97].

3.7 Newton's method

This is really zero-dimensional tropical geometry. Let $(K, |\cdot|)$ be a complete non-archimedean field. Let $f = \sum a_n x^n \in K[x^{\pm 1}]$ be a Laurent polynomial of degree d. Let $\lambda_1, \ldots, \lambda_d \in \overline{K}$ be the roots of f. (Formally, we're looking at the scheme $\mathrm{Spec}(k[x^{\pm 1}]/f)$.) Put $\mathrm{val}(\cdot) = -\log|\cdot|$. The set

$$\{\mathrm{val}(\lambda_1), \ldots, \mathrm{val}(\lambda_d)\}$$

can be described combinatorially. Write $\mathrm{NP}(f)$ for the lower convex hull of the set of points $\{n, \mathrm{val}(a_n)\} \subset \mathbf{R}^2$. This is the *Newton Polygon* of f.

3.7.1 Theorem. *Let f be as above. Then*

1. *$-r$ is a slope in $\mathrm{NP}(f)$ if and only if f has a root λ with $\mathrm{val}(\lambda) = r$.*

2. *The number of λ with $\mathrm{val}(\lambda) = r$ is the length of the projection onto the x-axis of the line segment with slope $-r$.*

Proof. Use the fact that each a_i is an elementary symmetric function of the λ_js,together with the non-archimedean triangle inequality. For example, when $x^2 + bx + c = (x - \lambda_1)(x - \lambda_2)$, we have

$$\begin{aligned} b &= -(\lambda_1 + \lambda_2) \\ c &= \lambda_1 \lambda_2. \end{aligned}$$

□

3.7.2 Example. Consider the polynomial $f(x) = x^2 - (p+1)x + p \in \mathbf{Q}_p[x]$. Normalize our valuation by $\mathrm{val}(p) = 1$ and $|p|_p = 1/p$. Then $\mathrm{NP}(f)$ is the lower convex hull of $\{(0,1),(1,0),(2,0)\}$. This has slopes -1 and 0, each of length 1, so f has one root of valuation 0 and 1, respectively. Of course, the roots are 1 and p, so we already knew this.

As an exercise, prove Eisenstein's criterion for irreducibility using Newton's Method.

Newton's method can be generalized to convergent power series in more than one variable. This has seen many applications, e.g. to abelian varieties.

4 Arbitrary valuations

This theory was initiated by Krull in the 1930s. As motivation: if $(K, |\cdot|)$ is a non-archimedean field, the absolute value took values in $\mathbf{R}^{\geqslant 0}$. More precisely, $|\cdot| \colon K^\times \to \mathbf{R}^{>0}$ is a group homomorphism. We could just as well have used $\mathrm{val}(\cdot) = -\log_a |\cdot|$; the topology does not depend on the choice of base a.

4.1 Definitions

So we have a function $\mathrm{val}\colon K \to \mathbf{R} \cup \{\infty\}$ such that

1. $\mathrm{val}(x) = \infty$ if and only if $x = 0$,

2. $\mathrm{val}(xy) = \mathrm{val}(x) + \mathrm{val}(y)$, and

3. $\mathrm{val}(x+y) \geqslant \min\{\mathrm{val}(x), \mathrm{val}(y)\}$.

We have the *valuation ring* given by

$$K^\circ = \{x \in K \colon \ \mathrm{val}(x) \geqslant 0\},$$

with maximal ideal $\{x \in K \colon \ \mathrm{val}(x) > 0\}$ and residue field $K^\natural = K^\circ / K^{\circ\circ}$. Krull's idea was that there is nothing special about $\mathbf{R}$, all we need is to be able to "add valuations" and "compare valuations." He replaced $(\mathbf{R}, +, \leqslant)$ with an arbitrary totally ordered abelian group $(\Gamma, +, \leqslant)$. (The order $\leqslant$ is required to be total and compatible with $+$ in the sense that $\alpha \leqslant \beta \Rightarrow \gamma + \alpha \leqslant \gamma + \beta$.)

4.1.1 Definition. Let K be a field. A *valuation* on K (with value group Γ) is a function $\mathrm{val}\colon K \twoheadrightarrow \Gamma \cup \{+\infty\}$ such that

1. $\mathrm{val}(x) = +\infty$ if and only if $x = 0$,
2. $\mathrm{val}(xy) = \mathrm{val}(x) + \mathrm{val}(y)$, and
3. $\mathrm{val}(x + y) \geqslant \min\{\mathrm{val}(x), \mathrm{val}(y)\}$.

4.1.2 Example. Let $(K, |\cdot|)$ be a non-archimedean field. Then $\mathrm{val}(\cdot) = -\log|\cdot|$ gives K a valuation with Γ a subgroup of $\mathbf{R}$.

As an exercise, compute $\mathbf{Q}^\circ$ for $|\cdot|_p$.

4.1.3 Example. Let $\Gamma = \mathbf{Z} \times \mathbf{Z}$ with the lexicographic order. Put a valuation on $k(\!(x, y)\!)$ by setting

$$\mathrm{val}\left(\sum a_{m,n} x^m y^n\right) = \min\{(m, n) \in \Gamma\colon a_{m,n} \neq 0\}.$$

It is a good exercise to check that this actually is a valuation.

4.2 Topology and completion

Any field with valuation can be given a Hausdorff topology in a functorial way. We can't do this in the naive way via metrics, because a valued field (K, val) doesn't have an absolute value. But formally, the construction works the same way. Briefly, we give K the topology whose basis consists of open sets of the form

$$U_\gamma(a) = \{b \in K\colon \mathrm{val}(b - a) > \gamma\}.$$

It's easy to check that $\{U_\gamma(a)\colon a \in K, \gamma \in \Gamma\}$ form the basis of a Hausdorff topology. The group $\Gamma = 0$ if and only if all $U_\gamma(a) = \{a\}$, which is equivalent to the topology being discrete.

We call two valuations $\mathrm{val}_1, \mathrm{val}_2$ on K *equivalent* if there is an isomorphism of value groups $\phi\colon \Gamma_1 \xrightarrow{\sim} \Gamma_2$ such that $\mathrm{val}_2 = \phi \circ \mathrm{val}_1$. Clearly equivalent valuations induce the same topology on K.

There is a weaker notion of "equivalence," namely "inducing the same topology." Unfortunately, it does not coincide with equivalence. Rather, we have the following

4.2.1 Theorem. *Let* $\mathrm{val}_1, \mathrm{val}_2$ *be valuations on* K *with valuation rings* $K^{\circ,1}$ *and* $K^{\circ,2}$*. Then* val_1 *and* val_2 *induce the same topology on* K *if and only if the subring of* K *generated by* $K^{\circ,1}$ *and* $K^{\circ,2}$ *is strictly smaller than* K*.*

Proof. This is [EP05, 2.3.4]. □

Alternatively, val_1 and val_2 induce different topologies if and only if $K^{\circ,1}K^{\circ,2} = K$. It turns out that all "reasonable" topologies on a field come from valuations, in the following sense.

4.2.2 Definition. Let K be a field. A *v-topology* on K is a topology τ on K which makes K into a Hausdorff topological field (i.e., the field operations are continuous) such that for all open $U \ni 0$, there exists an open $V \ni 0$ such that whenever $xy \in V$, then either $x \in U$ or $y \in U$.

Clearly, if (K, val) is a valued field, then the induced topology is a v-topology. Indeed, for $U \ni 0$, choose $\gamma \in \Gamma$ such that $\{\mathrm{val} > \gamma\} \subset U$; then $\{\mathrm{val} > 2\gamma\}$ works for V. The converse is a deep theorem.

4.2.3 Theorem. *Let (K, τ) be a field with a v-topology. Then τ is induced by either an archimedean absolute value or a valuation.*

Proof. This is [EP05, B.1]. □

Let (K, val) be a valued field; we give it the natural topology. It still makes sense to ask whether K is complete, except now completion is phrased in terms of uniform spaces and filters. Recall, following [Bou89, I §6.1] that a set $\mathfrak{F}$ of subsets of a set X is a *filter* if

1. (downward closed) $U \in \mathfrak{F}$ and $U \subset V$ implies $V \in \mathfrak{F}$,
2. (intersections) $U, V \in \mathfrak{F}$ imples $V \cap V \in \mathfrak{F}$,
3. (non-trivial) $\varnothing \notin \mathfrak{F}$.

Roughly, a filter on X is a set of "large subsets" of X. The example you should keep in mind is: for $x \in X$ an element of a topological space, the filters

$$\begin{aligned} \mathfrak{B}_x &= \{U \subset X \colon U \text{ contains an open neighborhood of } x\} \\ \mathfrak{F}_x &= \{U \subset X \colon x \in U\}. \end{aligned}$$

Alternatively, if $\{x_\alpha\}_{\alpha \in I}$ is a sequence in X indexed by a directed set I, put

$$\mathfrak{F}_{\{x_\alpha\}} = \{U \subset X \colon \text{ for some } \alpha_0 \in I,\ \alpha \geqslant \alpha_0 \Rightarrow x_\alpha \in U\}.$$

One says a filter $\mathfrak{F}$ *converges* to $x \in X$, written $\mathfrak{F} \to x$, if every open $U \ni x$ contains an element of $\mathfrak{F}$, i.e. $\mathfrak{B}_x \subset \mathfrak{F}$. So the filter $\mathfrak{F}_{\{x_\alpha\}} \to x$ if and only if for all $U \ni x$, there exists $\alpha_0 \in I$ such that $\alpha \geqslant \alpha_0 \Rightarrow x_\alpha \in U$.

If K is a topological field, a filter $\mathfrak{F}$ on K is *Cauchy* if for each open $U \ni 0$, there exists $a \in K$ such that $a + U \in \mathfrak{F}$. As an exercise, check that the filter $\mathfrak{F}_{\{x_\alpha\}}$ is Cauchy if and only if for each $U \ni 0$, there is $\alpha_0 \in I$ such that $x_\alpha - x_\beta \in U$ for all $\alpha, \beta \geqslant \alpha_0$. Say K is *complete* if every Cauchy filter converges to some $x \in K$.

Just like the classical case, the completion of a topological field makes sense. One can do this directly (take the ring of convergent nets modulo nets converging to zero as in [EP05, 2.4.3]), or equivalently apply the general machinery of "Hausdorff completion of a uniform space" [Bou89, II §3.7]. Namely, $(K, +)$ is a topological group, and as such it carries a canonical uniform structure [Bou89, III §3]. The Hausdorff completion of $(K, +)$, denoted $(\widehat{K}, +)$, is *a priori* just a topological group. But by [Bou89, III §6.8], the field structure extends uniquely to $\widehat{K}$. By [EP05, 2.4.4], the extension $K \hookrightarrow \widehat{K}$ is dense and immediate.

4.3 Henselization

All this seems great, but Hensel's lemma can fail for complete fields in this generality [EP05, 2.4.6]. The "standard" proof of Hensel's Lemma just uses denseness and completion. A better field than $\widehat{K}$ to do arithmetic in is the *henselization* of K, denoted K^{h}. It is a field containing K for which Hensel's Lemma holds, and that is in some sense minimal with respect to these properties.

4.3.1 Definition. A valued field (K, val) is *henselian* if the following equivalent conditions hold:

1. The valuation on K admits a unique extension to K^{s}.

2. The ring K° satisfies the conclusion of Theorem 3.6.1.

3. Each finite K°-algebra is a direct product of local K°-algebras.

The equivalence $1 \Leftrightarrow 2$ is [EP05, 4.1.3], while $2 \Leftrightarrow 3$ is [Ray70, I §1.5].

4.3.2 Definition. Let (K, val) be a valued field. The *henselization* of K is a henselian field K^{h} together with an valuation-preserving embedding $K \hookrightarrow K^{\mathrm{h}}$, such that for any other henselian field L and embedding $K \hookrightarrow L$, there is a unique valuation-preserving map $K^{\mathrm{h}} \to L$ making the following diagram commute:

$$\begin{tikzcd} K \arrow[r, hook] \arrow[dr, hook] & K^{\mathrm{h}} \arrow[d, dashed] \\ & L \end{tikzcd}$$

4.3.3 Proposition. *Let K be a valued field. Then*

1. *K^{h} exists.*

2. *K^{h}/K is an algebraic extension, and is unique up to valuation-preserving isomorphism.*

3. *The valuation on K admits a canonical extension to K^{h}.*

4. *The extension K^{h}/K is immediate.*

By [EP05, 5.2.2], henselizations always exist. Note that K is *not* necessarily dense in K^{h}.

4.4 Rank of a valuation

Every valuation has an invariant known as the *rank*. The rank only depends on the value group. So, fix a totally ordered abelian group $(\Gamma, +, \leqslant)$. Let $\Delta \subset \Gamma$ be a subgroup. We say Δ is *convex* if for any $\delta \in \Delta$, $\gamma \in \Gamma$ such that $0 \leqslant \gamma \leqslant \delta$, we have $\gamma \in \Delta$.

Let Σ be the collection of all convex subgroups of Γ. The set Σ carries a natural ordering via inclusion. It is easy to see that $(\Sigma, \subset)$ is totally ordered. The *rank* of Γ, written $\mathrm{rk}(\Gamma)$, is the order-type of $(\Sigma, \subset)$.

Recall from Cantor that two totally ordered sets have the same *order-type* if and only if there is an order-preserving bijection between them.

We will be concerned exclusively with finite order-types, namely those of the finite sets $\{0, 1, \ldots, n\}$. So $\mathrm{rk}(\Gamma) = n$ if and only if there are exactly n distinct proper convex subgroups of Γ.

So $\mathrm{rk}(\Gamma) = 1$ if and only if 0 is the only convex proper subgroup of Γ. Clearly, every subgroup of $\mathbf{R}$ (with the obvious ordering) has no nontrivial convex subgroup. This follows from the archimedean property of $\mathbf{R}$. The converse holds.

4.4.1 Theorem. *Let Γ be a totally ordered abelian group. Then* $\operatorname{rk}(\Gamma) \leqslant 1$ *if and only if* Γ *admits an order-preserving embedding into* $\mathbf{R}$.

The rank of $(\Gamma, +, \leqslant)$ depends strongly on the choice of ordering $\leqslant$. For example, $(\mathbf{Z}^2, \leqslant_{\mathrm{lex}})$ has rank two and is discrete, i.e. it has a minimal positive element. The group $\mathbf{Z}[\sqrt{2}] \subset \mathbf{R}$ is isomorphic to $\mathbf{Z}^2$ as an abstract group, but has rank one and is not discrete.

4.5 Value groups, valuation rings, and residue fields

A valuation $\operatorname{val}\colon K \to \Gamma \cup \{+\infty\}$ induces a group homomorphism $\operatorname{val}\colon K^\times \twoheadrightarrow \Gamma$. We call $\Gamma = \operatorname{val}(K^\times)$ the *value group* of (K, val).

4.5.1 Definition. Let $(K, \operatorname{val}\colon K \to \Gamma \cup \{\infty\})$ be a valuation field. We put $\operatorname{rk}(\operatorname{val}) = \operatorname{rk}(\Gamma)$.

The valuation ring $K^\circ = \{x \in K\colon \operatorname{val}(x) \geqslant 0\}$ depends on val, but we tacitly exclude this from the notation.

Note that

$$\begin{aligned}
\operatorname{val}(1) &= 0 \\
\operatorname{val}(x^{-1}) &= -\operatorname{val}(x) \\
\operatorname{val}(-x) &= \operatorname{val}(x) \\
\operatorname{val}(x) < \operatorname{val}(y) \Rightarrow \operatorname{val}(x+y) &= \operatorname{val}(x).
\end{aligned}$$

One has $(K^\circ)^\times = \{x \in K\colon \operatorname{val}(x) = 0\}$. So $K^{\circ\circ}$ is the set of non-units of K°. Thus $K^{\circ\circ} \subset K^\circ$ is the unique maximal ideal and K° has residue (class) field $K^\natural = K^\circ / K^{\circ\circ}$.

An integral domain R is called a *valuation ring* if for all $x \in R_{(0)} = K$, either $x \in R$ or $x^{-1} \in R$. That is, $K = R \cup R^{-1}$. Clearly our K° is a valuation ring in this new sense. Conversely, we have:

4.5.2 Lemma. *Let $R \subset K$ be a valuation ring (in the new sense). Then there exists a canonical valuation on K such that $R = K^\circ$ with respect to that valuation.*

Proof. Let $\Gamma = K^\times / R^\times$; this is an abelian group. Give Γ an ordering by setting $xR^\times \leqslant yX^\times$ if $y/x \in R$. Check that this makes Γ a totally ordered abelian group. We define a valuation $\operatorname{val}\colon K \to \Gamma \cup \{+\infty\}$ by

$$\begin{aligned}
\operatorname{val}(0) &= +\infty \\
\operatorname{val}(a) &= aR^\times \qquad (a \neq 0).
\end{aligned}$$

See the exercises for chapter 5 of [AM72] for a careful proof of this. Check that with this definition, $R = K^\circ$. □

4.5.3 Corollary. *Let (K, val) be a valued field. Then $\Gamma \simeq K^\times / K^{\circ\times}$.*

4.5.4 Example. Let R be a valuation ring. Then the following are equivalent:

1. R is noetherian.

2. R is a principal ideal ring.

3. R is a discrete valuation ring.
4. $\Gamma \simeq \mathbf{Z}$.

Thus valuation rings are very rarely noetherian.

4.5.5 Lemma. *Any additive subgroup of* $\mathbf{R}$ *is either cyclic or dense.*

Proof. If $G \subset \mathbf{R}$ is not dense, then there is some $\epsilon > 0$ such that $G \cap (0, \epsilon) = \varnothing$. By basic analysis, G has a smallest positive element; call it a. For any $x \in \mathbf{R}$, write $x = na + c$ with $0 \leqslant c < a$. If $x \in G$, we must have $c = 0$, whence $G = \mathbf{Z}a$. □

Thus, if a valuation takes values in $\mathbf{R}$, the valuation ring is either a PID or non-noetherian.

4.5.6 Example. The only valuations on $\mathbf{Q}$ are the p-adic ones. These are discrete.

4.5.7 Example. On the field $K(x)$, there are

1. p-adic valuations, for p an irreducible polynomial in $K[x]$
2. $\mathrm{val}_\infty(f/g) = \deg g - \deg f$, called the *valuation at infinity*. We write $K(\!(x)\!) = \widehat{K(x)}$; one calls the extension of val_∞ to $K(\!(x)\!)$ the *x-adic valuation.*

One can prove that these are the only valuations which are trivial on K.

Let (K, val) be a valued field. Let $\Gamma' \supset \Gamma$ be a larger totally ordered abelian group. Fix $\gamma \in \Gamma'$. Define $\mathrm{val}' \colon K(x) \to \Gamma' \cup \{\infty\}$ by

$$\begin{aligned} \mathrm{val}'(0) &= 0 \\ \mathrm{val}'(f/g) &= \mathrm{val}'(f) - \mathrm{val}'(g) \\ \mathrm{val}'\left(\sum a_i x^i\right) &= \min_i\{\mathrm{val}(a_i) + \gamma i\}. \end{aligned}$$

Standard Gröbner basis theory is the special case where $\Gamma = 0$ and $\Gamma' = \mathbf{R}$. It's easy to show that val' is indeed a valuation. The real content is showing that $\mathrm{val}(f+g) \geqslant \max\{\mathrm{val}(f), \mathrm{val}(g)\}$; this is easy. Showing that $\mathrm{val}'(fg) = \mathrm{val}'(f)\,\mathrm{val}'(g)$ is a bit more technical. There is a general theory of Gröbner bases for fields with valuation—see the preprint [CM].

The following gives a more algebraic definition of the rank of a valuation.

4.5.8 Theorem. *Let* (K, val) *be a valued field. There is a bijection*

$$\{\textit{convex subgroups of}\ \Gamma\} \leftrightarrow \mathrm{Spec}(K^\circ),$$

defined by

$$\begin{aligned} \Delta \subset \Gamma &\mapsto \{x \in K \colon\ \mathrm{val}(x) > \delta \textit{ for all } \delta \in \Delta\} \\ \mathfrak{p} \subset K^\circ &\mapsto \{\gamma \in \Gamma \colon\ \pm\gamma < \mathrm{val}(x) \textit{ for all } x \in \mathfrak{p}\}. \end{aligned}$$

The proof is an easy exercise.

4.5.9 Corollary. *If* (K, val) *is a finite-rank valued field, then* $\mathrm{rk}(\mathrm{val}) = \dim(K^\circ)$.

By $\dim(R)$, we mean the Krull dimension of R. Of course, for an arbitrary ring R, the "Krull dimension" of R is an order type—namely that of the longest increasing chain of prime ideals. With that definition of dimension, $\dim(K^\circ) = \mathrm{rk}(\mathrm{val})$ for all valuation fields. We also see that $\mathrm{rk} = 1$ if and only if K° is a maximal subring of K.

4.6 Extensions of arbitrary valuations

We worked out the story for rank-one valuations in section 3. In general, if L/K is an arbitrary field extension, we might ask whether a valuation on K extends to one on L.

4.6.1 Theorem (Chevalley). *Let K be a field, $R \subset K$ a ring, $\mathfrak{p} \in \operatorname{Spec}(R)$. Then there exists a valuation ring $K^\circ \subset K$ such that $R \subset K$ and $\mathfrak{p} = R \cap K^{\circ\circ}$.*

Proof. This is a straightforward application of Zorn's Lemma. □

4.6.2 Corollary. *Let L/K be an arbitrary field extension. For any valuation ring $K^\circ \subset K$, there exists a valuation ring $L^\circ \subset L$ extending K°.*

Proof. Apply Theorem 4.6.1 to $R = K^\circ \subset K$. □

Alternatively, we could have said that the valuation $\operatorname{val}\colon K \to \Gamma$ extends to some $\operatorname{val}'\colon L \to \Gamma' \supset \Gamma$.

4.6.3 Lemma. *Any valuation ring is integrally closed.*

Proof. Let $K^\circ \subset K$ be a valuation ring. Suppose $a_0 + \cdots + a_{n-1}x^{n-1} + x^n = 0$ for some $x \in K$ and $a_i \in K^\circ$. We want to show that $x \in K^\circ$. If not, then $x^{-1} \in K^\circ$; in fact $x^{-1} \in K^{\circ\circ}$. Basic algebra yields

$$-1 = a_0 x^{-n} + a_1 x^{-(n-1)} + \cdots + a_{n-1}x^{-1} \in K^{\circ\circ},$$

a contradiction. □

For the rest of this section, L/K is a valued extension. That is,

$$\begin{aligned}
(K, K^\circ) &\subset (L, L^\circ), \\
L^\circ \cap K &= K^\circ, \\
\operatorname{val}\colon K^\times &\twoheadrightarrow \Gamma = K^\times/K^{\circ\times}, \\
\operatorname{val}'\colon L^\times &\twoheadrightarrow \Gamma' = L^\times/L^{\circ\times} \text{ extends } \operatorname{val}.
\end{aligned}$$

We have $K^\times \hookrightarrow L^\times \xrightarrow{\operatorname{val}} \Gamma'$. The kernel of this map is $K^\times \cap L^{\circ\times} = K^{\circ\times}$. So we get an order-preserving map $\Gamma \hookrightarrow \Gamma'$. Similarly, the inclusion $K^\circ \hookrightarrow L^\circ$ induces a field extension $K^\natural \hookrightarrow L^\natural$.

4.6.4 Definition. With the above notation, we put

$$\begin{aligned}
e(L^\circ/K^\circ) &= [\Gamma' \colon \Gamma] && \text{“ramification index”}, \\
f(L^\circ/K^\circ) &= [L^\natural \colon K^\natural] && \text{“residue degree”}.
\end{aligned}$$

We are interested in bounding the ramification index and residue degree. For example, the extension L/K is immediate if and only if $e = f = 1$. We can prove the best results when L/K is algebraic.

4.6.5 Theorem. *Let L/K be as above.*

1. If $[L\colon K] = n < \infty$, then $ef \leqslant n$.

2. More generally, let $(L, L_1^\circ), \ldots, (L, L_t^\circ)$ be pairwise non-isomorphic valued extensions of (K, K°). Then

$$\sum_i e(L_i^\circ/K^\circ) f(L_i^\circ/K^\circ) \leqslant n. \tag{5}$$

3. If $\Gamma = K^\times/K^{\circ\times} \simeq \mathbf{Z}$ and L/K is separable, then (5), if the L_i° range over all *valued extensions of (K, K°) in L.*

The proof of part 1 (which is a special case of Abhyankar's inequality) of this theorem follows from the following result.

4.6.6 Lemma. *Let L/K be as above; assume $[L\colon K] < \infty$. Pick $\{\omega_1, \ldots, \omega_f\} \subset L^\circ$ such that $\{\overline{\omega_1}, \ldots, \overline{\omega_f}\} \subset L^\natural$ is linearly independent over $K^\natural$. Also, let $\{\pi_1, \ldots, \pi_e\} \subset L^\times$ be such that $\{\mathrm{val}'(\pi_1), \ldots, \mathrm{val}'(\pi_e)\}$ are distinct in Γ'/Γ. Then $\{\omega_i \pi_j\}$ is linearly independent over K.*

4.6.7 Corollary. *If L/K is algebraic, then*

1. *Γ'/Γ is a torsion group,*
2. *$L^\natural/K^\natural$ is algebraic,*
3. *Γ and Γ' have the same rank.*

Proof. Parts 1 and 2 follow from the corresponding fact for any *finite* extension L/K. Part 3 only uses the fact that Γ'/Γ is torsion. More generally, $\Delta' \mapsto \Delta \cap \Gamma$ is an order-preserving bijection between convex subgroups of Γ' and convex subgroups of Γ *whenever* $\Gamma \hookrightarrow \Gamma'$ is a torsion extension. ☐

4.6.8 Lemma. *If K is an algebraically closed valued field, then*

1. *$K^\natural$ is algebraically closed,*
2. *Γ is divisible.*

Proof. 1. Let $\bar{f} = \overline{a_0} + \overline{a_1}x + \cdots + x^n \in K^\natural[x]$. Lift the $\overline{a_i} \in K^\natural$ to $a_i \in K^\circ$; we get a polynomial $f = a_0 + a_1 x + \cdots + a_n x^n \in K^\circ[x]$. Since K is algebraically closed, f has a root α in K. Since K° is integrally closed, $\alpha \in K^\circ$. Then $\bar{f}(\overline{\alpha}) = 0$.

2. Recall that $\Gamma = K^\times/K^{\circ\times}$. We prove the stronger result that $K^\times$ is divisible. For any $a \in K^\times$ and $n \geqslant 1$, the equation $x^n - a = 0$ has a solution in $K^\times$, so we're done. ☐

There is another notion of rank of abelian groups. Let G be an abelian group, then it is a $\mathbf{Z}$-module. The *rational rank* of G is

$$\begin{aligned} \mathrm{rr}(G) &= \dim_{\mathbf{Q}}(G \otimes_{\mathbf{Z}} \mathbf{Q}) \\ &= \max\{\#S\colon S \subset G \text{ is linearly independent}\}. \end{aligned}$$

Note that $\mathrm{rr}(G)$ does not depend on any ordering on G. Clearly $\mathrm{rr}(G) = 0$ if and only if G is torsion. The group $G_{\mathbf{Q}} = G \otimes_{\mathbf{Z}} \mathbf{Q}$ is divisible, and the quotient $G_{\mathbf{Q}}/G$

is torsion. In fact, $G_{\mathbf{Q}}$ is the "smallest" divisible group H containing G such that H/G is torsion. One calls $G_{\mathbf{Q}}$ the "divisible hull" of G.

If G is totally ordered, we may extend the ordering to $G_{\mathbf{Q}}$ by:

$$g \otimes \frac{1}{n} \leqslant h \otimes \frac{1}{m} \qquad \Leftrightarrow \qquad mg \leqslant nh.$$

As an exercise, check that this gives $G_{\mathbf{Q}}$ the structure of a totally ordered abelian group. Let K be a valued field, $\overline{K}$ an algebraic closure of K. Let Γ (resp. Γ') be the value group of K (resp. $\overline{K}$). Namely,

$$\Gamma = K^\times / K^{\circ\times},$$
$$\Gamma' = \overline{K}^\times / \overline{K}^{\circ\times}.$$

Then $\Gamma' = \Gamma_{\mathbf{Q}}$. This recovers our earlier claims that Γ' is divisible, Γ'/Γ is torsion, and the minimality of Γ' with respect to these properties.

If G is a totally ordered abelian group, we already defined the rank $\mathrm{rk}(G)$. One has $\mathrm{rk}(G) = \mathrm{rk}(G_{\mathbf{Q}})$.

4.6.9 Lemma. *Let G be a totally ordered group. Then* $\mathrm{rk}(G) \leqslant \mathrm{rr}(G)$.

Proof. Induct on $r = \mathrm{rk}(G)$. For $r = 0$, there is nothing to prove. If $r > 0$, let $0 = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_r = G$ be a chain of convex subgroups. By induction, $\mathrm{rr}(G_{r-1}) \geqslant r - 1$. Since G_{r-1} is convex, G_r/G_{r-1} is ordered, hence torsion-free. It follows that $\mathrm{rr}(G/G_{r-1}) \geqslant 1$. Then:

$$\mathrm{rr}(G) = \mathrm{rr}(G/G_{r-1}) + \mathrm{rr}(G_{r-1}) \geqslant 1 + r - 1 = r.$$

□

4.6.10 Definition. The *rational rank* of a valuation is the rational rank of its value group.

4.7 Transcendental extensions

The following deep theorem is an analogue of the inequality $ef \leqslant n$ from Theorem 4.6.5.

4.7.1 Theorem (Abhyankar's dimension inequality)**.** *Let L/K be an arbitrary extension of valued fields with value groups $\Gamma' \supset \Gamma$. Then*

$$\mathrm{tr.deg}(L^\natural/K^\natural) + \mathrm{rr}(\Gamma'/\Gamma) \leqslant \mathrm{tr.deg}(L/K)$$

Proof. This follows from something formally very similar to the proof of Theorem 4.6.5, namely the following lemma. □

4.7.2 Lemma. *Let $\{x_1, \ldots, x_r\} \subset L^\circ$ be such that $\{\overline{x_1}, \ldots, \overline{x_r}\} \subset L^\natural$ are algebraically independent over $K^\natural$, $\{y_1, \ldots, y_s\} \subset L^\times$ be such that $\{\mathrm{val}'(y_1), \ldots, \mathrm{val}'(y_s)\}$ represent linearly independent elements of Γ'/Γ. Then $\{x_1, \ldots, x_r, y_1, \ldots, y_s\} \subset L$ are algebraically independent over K.*

5 Berkovich spectrum of a normed ring

This generalizes Gelfand's theory. Much of the theory even works for non-commutative algebras. We do not require the existence of a base field, so our theory applies to "arithmetic" rings like $\mathbf{Z}$ or $\mathbf{Z}_2[x]/(7x^3-5)$. We could phrase everything in terms of (rank-one) valuations and semivaluations. Let A be a commutative (unital) ring.

5.1 Norms and seminorms

5.1.1 Definition. A *seminorm* on A is a function $\|\cdot\|\colon A \to \mathbf{R}^{\geqslant 0}$ satisfying

1. $\|0\| = 0$, $\|1\| \leqslant 1$,
2. $\|f+g\| \leqslant \|f\| + \|g\|$,
3. $\|fg\| \leqslant \|f\|\|g\|$.

If we replace axiom 2 with $\|f+g\| \leqslant \max\{\|f\|, \|g\|\}$, we say the norm $\|\cdot\|$ is non-archimedean. We will focus primarily on non-archimedean norms. We allow $x \neq 0$ to have $\|x\| = 0$. Define

$$\ker(\|\cdot\|) = \{x \in A\colon \|x\| = 0\}.$$

(Huber calls this the *support* of $\|\cdot\|$.) This is a prime ideal in A. We call $\|\cdot\|$ a *norm* if $\ker(\|\cdot\|) = 0$. It turns out that our hypotheses imply that either $\|1\| = 0$ or $\|1\| = 1$. For, if $\|1\| \neq 0$, then $\|1\| \leqslant \|1\|\|1\|$, so $1 \leqslant \|1\|$. If $\|1\| = 0$, then $\|f\| = 0$ for all $f \in A$. We want to allow this possibility so that the zero ring can have a norm.

If $(A, \|\cdot\|)$ is a ring with seminorm, then $(A, +)$ is naturally a seminormed group, i.e. it satisfies

$$\begin{aligned} \|0\| &= 0 \\ \|f+g\| &\leqslant \|f\| + \|g\| \\ \|-f\| &= \|f\|. \end{aligned}$$

(Alternatively, replace the last two equations by $\|f-g\| \leqslant \|f\| + \|g\|$.) Any seminormed group carries a natural topology, so we get a topology on A. This topology is Hausdorff if and only if $\ker(\|\cdot\|) = 0$. The (Hausdorff) completion $\widehat{A}$ exists (it can be defined via Cauchy sequences as usual), but the obvious map $Ai\colon \to \widehat{A}$ is injective if and only if $\ker(\|\cdot\|) = 0$. In fact, $|ker(i) = \ker(\|\cdot\|)$.

If we replace the axiom $\|fg\| \leqslant \|f\|\|g\|$ with $\|fg\| = \|f\|\|g\|$ and also assume $\|\cdot\| \neq 0$, we say that $\|\cdot\|$ is *multiplicative*. If for all $f \in A$, $\|f^n\| = \|f\|^n$, we say $\|\cdot\|$ is *power-multiplicative*.

If N is a subgroup of a seminormed group $(A, +)$, then we get a "residue seminorm" on A/N, defined by

$$\|g + N\| = \inf\{\|f\|\colon f - g \in N\}.$$

This gives a norm on A/N if and only if N is closed.

Let $\varphi\colon M \to N$ be a homomorphism between seminormed groups. We call φ *bounded* if there exists some $C > 0$ such that $\|\varphi(f)\| \leqslant C\|f\|$ for all $f \in M$. Clearly bounded functions are continuous. The converse does not hold in this level of generality.

We say two seminorms $\|\cdot\|, \|\cdot\|'$ on a group are *equivalent* if there exists $C_1, C_2 > 0$ such that

$$C_1\|\cdot\| \leqslant \|\cdot\|' \leqslant C_2\|\cdot\|.$$

Equivalent seminorms induce the same topology, but the converse is false in general.

5.1.2 Example. Let K be any field; consider $M = k[x]$. Let $f = \sum_0^n a_i x^i \in M$, and pick some $0 < \alpha < 1 < \alpha'$. Define two seminorms on M by:

$$\|f\| = \max\{\alpha^j : \alpha_j \neq 0\}$$
$$\|f\|' = \max\{\alpha'^j : \alpha_j \neq 0\}.$$

Let $\varphi\colon (M, \|\cdot\|') \to (M, \|\cdot\|)$ be the identity on M. It is easy to check that φ is bounded. Indeed,

$$\frac{\|\varphi(f)\|}{\|f\|'} = \frac{\|f\|}{\|f\|'} = \frac{\alpha^j}{\alpha'^i} \leqslant 1.$$

Now φ^{-1} is *not* bounded (this is easy), but φ^{-1} is continuous. We conclude that continuity does not imply boundedness. Moreover, bounded homeomorphisms need not have bounded inverses.

Let $\varphi\colon M \to N$ be a linear map of seminormed groups. As groups, $M/\ker\varphi \simeq \operatorname{im}\varphi$, but this isomorphis might not respect the seminorms on the two groups.

5.1.3 Definition. A linear map $\varphi\colon M \to N$ of seminormed groups is *admissible* (or *strict*) if the residue seminorm on $M/\ker\varphi$ and the seminorm on $\operatorname{im}\varphi$ are equivalent via the natural isomorphism $M/\ker\varphi \simeq \operatorname{im}\varphi$.

5.1.4 Definition. A *Banach ring* is a normed ring which is complete (with respect to the norm).

5.2 Examples and first properties

5.2.1 Example. Any ring is with the trivial norm

$$\|f\| = \begin{cases} 1 & \text{if } f \neq 0 \\ 0 & \text{if } f = 0 \end{cases}$$

is a Banach ring.

5.2.2 Example. The ring $(\mathbf{Z}, \|\cdot\|_\infty)$ is a Banach ring. The induced topology is discrete, but $\|\cdot\|_\infty$ is very far from being equivalent to the trivial norm on $\mathbf{Z}$.

5.2.3 Example. Let A be a Banach ring, $\mathfrak{a} \subset A$ a closed ideal. Then $A/\mathfrak{a}$, with the residue norm, is a Banach ring.

5.2.4 Example. In the context of the above example, if $\mathfrak{m} \subset A$ is any maximal ideal, then $A/\mathfrak{m}$ is a Banach field. Here we implicitly take advantage of Lemma 5.2.5.

5.2.5 Lemma. *Let A be a Banach ring, $\mathfrak{m} \subset A$ a maximal ideal. Then $\mathfrak{m}$ is closed.*

Proof. If $\mathfrak{a} \subset A$ is any ideal, then its closure $\mathrm{cl}(\mathfrak{a})$ is also an ideal. Of course, if $\mathfrak{a}$ is a proper ideal, then $\mathfrak{a} \cap A^\times = \varnothing$. But $A^\times$ is open. Indeed, if $x \in A^\times$, then $\mathrm{D}^-(x, r) \subset A^\times$ for $r = \frac{1}{2}\|x^{-1}\|^{-1}$. To see this, let $x+h \in \mathrm{D}^-(x, r)$; we wish to show that $x + h \in A^\times$. We know that $\|h\| < \frac{1}{2}\|x^{-1}\|^{-1}$, so $\|hx^{-1}\| \leqslant \|h\|\|x^{-1}\| < \frac{1}{2}$. Write $x + h = x(1 + hx^{-1})$; the element $1 + hx^{-1}$ is invertible by Lemma 5.2.6 $\square$

5.2.6 Lemma. *Let A be a Banach ring. Then $\mathrm{D}^-(1, 1) \subset A^\times$.*

Proof. We need to show that if $\|x\| < 1$, then $1 - x$ is invertible. The sequence

$$a_n = 1 + x + \cdots + x^n,$$

is Cauchy because $\|x\| < 1$, so it converges. One can easily check that the limit is $(1 - x)^{-1}$. $\square$

5.2.7 Example. Let A be a Banach ring with norm $\|\cdot\|$. Choose $r \in \mathbf{R}^{>0}$. We define the ring $A\left\langle r^{-1}T\right\rangle \subset A[\![T]\!]$ to be the set of power series $f = \sum_{i \geqslant 0} a_i T^i$ such that $\sum \|a_i\| r^i < \infty$. If A is non-archimedean, we only need to require $\|a_i\| r^i \to 0$. We call $A\left\langle r^{-1}T\right\rangle$ a ring of *convergent power series.* We define a norm on $A\left\langle r^{-1}T\right\rangle$ by:

$$\|f\| = \sum \|a_i\| r^i \qquad \text{archimedean case}$$
$$\|f\| = \max\{\|a_i\| r^i\} \qquad \text{non-archimedean case.}$$

We could have used the first definition in both cases, but the latter is easier to handle. Claim: $(A\left\langle r^{-1}T\right\rangle, \|\cdot\|)$ is a Banach ring.

5.2.8 Example. Let $\{A_i\}_{i \in I}$ be an arbitrary collection of Banach rings. Then the direct product $A = \prod A_i$ can be "too large" to be a Banach ring. It contains a natural Banach ring, namely the *bounded direct product*

$$\prod^{\mathrm{b}} A_i = \{(f_i)_{i \in I} \colon \sup \|f_i\| < \infty\}.$$

This is a banach ring with respect to the norm $\|(f_i)\| = \sup \|f_i\|$. Also, there is the *restricted direct product*:

$$\prod^{\mathrm{c}} A_i = \{(f_i)_{i \in I} \colon \lim \|f_i\| \to 0\},$$

and the direct sum $\bigoplus_i A_i$. Note that the restricted direct product and direct sum will not be unital rings.

5.3 Spectrum: definition and first properties

Let $(A, \|\cdot\|)$ be a normed ring. A seminorm $|\cdot|$ on A is called *bounded* (with respect to $\|\cdot\|$) if there exists $c > 0$ such that $|f| \leqslant c\|f\|$ for all $f \in A$. One also calls bounded seminorms *admissible.* If $|\cdot|$ is bounded, then the induced map $|\cdot| \colon A \to \mathbf{R}, f \mapsto |f|$ is continuous, where $\mathbf{R}$ has its usual topology and A has the topology induced by $\|\cdot\|$.

5.3.1 Definition. Let A be a commutative unital Banach ring with fixed norm $\|\cdot\|$. The *spectrum* of A, denoted $\mathscr{M}(A)$, is the set of bounded multiplicative seminorms on A. For $x \in \mathscr{M}(A)$, let $|\cdot|_x$ be the corresponding seminorm. We give $\mathscr{M}(A)$ the weakest topology making all maps $x \mapsto |f|_x$ continuous.

5.3.2 Theorem. *Let A be a commutative Banach ring. Then*

1. *$\mathscr{M}(A) \neq \varnothing$.*

2. *$\mathscr{M}(A)$ is Hausdorff.*

3. *$\mathscr{M}(A)$ is compact.*

Proof. 1. This is the hardest part of the proof. Let $\mathfrak{m} \subset A$ be a maximal ideal; $\mathfrak{m}$ is closed by Lemma 5.2.5. The quotient $A/\mathfrak{m}$ is a field, complete with respect to the residue norm. If $\mathscr{M}(A/\mathfrak{m}) \neq \varnothing$, then $\mathscr{M}(A) \neq \varnothing$. Indeed, if $|\cdot|$ is a bounded multiplicative seminorm on $A/\mathfrak{m}$, then $|\cdot|\colon A \to \mathbf{R}$ given by $f \mapsto |f + \mathfrak{m}|$ is a bounded multiplicative seminorm on A. (This is a special case of functoriality: if $f\colon A \to B$ is a bounded homomorphism of Banach rings, there is a natural map $f^*\colon \mathscr{M}(B) \to \mathscr{M}(A)$.) Without loss of generality, we may assume that A is a field.

Let S be the set of all non-zero bounded seminorms on A. Given $|\cdot|, |\cdot|' \in S$, we put $|\cdot| \leqslant |\cdot|'$ if $|f| \leqslant |f|'$ for all $f \in A$. This gives S a partial order. Since $\|\cdot\| \in S$, the set S is non-empty. If $\{|\cdot|_i\}_{i\in I} \subset S$ is a chain, then $\inf_i |\cdot|_i \in S$ is a lower bound for $\{|\cdot|_i\}_{i\in I}$. Zorn's lemma gives us a minimal element $|\cdot| \in S$. We claim that $|\cdot|$ is multiplicative. To show this, it's enough to show that $|f^{-1}| = |f|^{-1}$ for all $f \in A^\times$. For, we already know that $|fg| \leqslant |f||g|$, so just compute:

$$|fg|^{-1} = |f^{-1}g^{-1}| \leqslant |f^{-1}||g^{-1}| = |f|^{-1}|g|^{-1}.$$

We always have $1 \leqslant |f||f^{-1}|$, so assume there exists f with $|f|^{-1} < |f^{-1}|$. Let $r = |f^{-1}|^{-1} < |f|$. Consider the Tate algebra (from Example 5.2.7):

$$B = A\left\langle r^{-1}T\right\rangle = \left\{\sum_{i\geqslant 0} a_i T^i\colon\ \sum |a_i| r^i < \infty\right\}.$$

Note that $|\cdot|$ extends to B by $|\sum a_i T^i| = \sum |a_i| r^i$. We claim that $f - T$ is not invertible in B. If it did have an inverse, this would be $\sum f^{-i}T^i$, which has norm $\sum |f^{-i}| r^i = \sum 1$, which diverges. (See the proof of [Ber90, 1.2.1] for details). Consider the map $A \to B/\langle f - T\rangle \neq \varnothing$. This induces a smaller bounded seminorm on A, which contradicts the minimality of $|\cdot|$. So $|\cdot|$ is multiplicative.

2. For $x \neq y \in \mathscr{M}(A)$, we want to show there are disjoint open neighborhoods of x and y. Since $x \neq y$, there exists some $f \in A$ such that $|f|_x \neq |f|_y$, say $|f|_x < |f|_y$. Choose $r \in \mathbf{R}$ with $|f|_x < r < |f|_y$. Then $\{z \in \mathscr{M}(A)\colon |f|_z < r\}$ and $\{z \in \mathscr{M}(A)\colon |f|_z > r\}$ are the desired sets.

3. Let $C = \prod_{f\in A}[0, \|f\|]$; this is compact by Tychonoff's theorem. Note that $\mathscr{M}(A) \hookrightarrow C$ via $x \mapsto (|f|_x)_{f\in A}$. Moreover, $\mathscr{M}(A) \subset C$ is closed because all the defining properties of $\mathscr{M}(A)$ are closed conditions. Thus $\mathscr{M}(A)$ is itself compact. ∎

If $\|\cdot\|$ is already multiplicative, then $\mathscr{M}(A) \neq \varnothing$ because $\|\cdot\| \in \mathscr{M}(A)$. This is much harder in general. We didn't actually need the completeness of A in the above definition. This is because any bounded seminorm on A has a unique extension to $\widehat{A}$. We could have replaced $\|\cdot\|$ with any equivalent seminorm and get the same $\mathscr{M}(A)$.

Let $|\cdot|\colon A \to \mathbf{R}$ be a power-multiplicative bounded seminorm. Then $|f|^n \leqslant c\|f\|^n$; take n-th roots and let $n \to \infty$ to realize that we may assume $c = 1$. So all $x \in \mathscr{M}(A)$ satisfy $|f|_x \leqslant \|f\|$ for all $f \in A$.

The topology of $\mathscr{M}(A)$ is very mysterious. In general, one needs model theory to prove some basic facts (e.g. that $\mathscr{M}(A)$ "looks like" a simplicial complex). Here are two equivalent ways to define the topology:

1. The topology on $X = \mathscr{M}(A)$ is generated by open sets of the form
$$\{x \in X\colon |f|_x < \alpha\}, \qquad \{x \in X\colon |f|_x > \alpha\},$$
for $f \in A$, $\alpha \in \mathbf{R}$.

2. A filter $\mathfrak{F}$ converges to $x \in X$ if and only if the filter
$$|f|_{\mathfrak{F}} = \{\{|f|_u\colon u \in U\}\colon U \in \mathfrak{F}\}$$
converges to $|f|_x$ for all $f \in A$.

Let $x \in \mathscr{M}(A)$. Then $\mathfrak{p}_x = \ker(|\cdot|_x)$ is a closed prime ideal. For $f \in A$, $|f|_x$ depends only on $\bar{f} \in A/\mathfrak{p}_x$. But $|\cdot|_x$ is a bounded multiplicative norm on $A/\mathfrak{p}_x$. The ring $A/\mathfrak{p}_x$ is an integral domain, so we can pass to its field of fractions $(A/\mathfrak{p}_x)_{(0)}$, which carries the norm induced by $|\cdot|_x$. Let $\mathscr{H}(x) = \widehat{(A/\mathfrak{p}_x)_{(0)}}$; this is the *completed residue field* of A at x. For $f \in A$, write $f(x)$ for the image of f under the composite map
$$A \twoheadrightarrow A/\mathfrak{p}_x \hookrightarrow (A/\mathfrak{p}_x)_{(0)} \hookrightarrow \widehat{(A/\mathfrak{p}_x)_{(0)}} = \mathscr{H}(x).$$
We will write $|\cdot|$ for the canonical extension of $|\cdot|_x$ to $\mathscr{H}(x)$. So $|f(x)| = |f(x)|_x = |f|_x$. The map $A \to \mathscr{H}(x)$ is a "character," in the following sense.

5.3.3 Definition. Let A be a Banach ring. A *character* on A is a nonzero bounded homomorphism to a field complete with respect to some absolute value.

5.3.4 Definition. Let A be a Banach ring. The *Gelfand transform* on A is the natural map
$$A \to \prod_{x\in\mathscr{M}(A)}^{\mathrm{b}} \mathscr{H}(x) \qquad f \mapsto (f(x))_{x\in\mathscr{M}(A)}.$$

Let $B = \prod^{\mathrm{b}}_{x\in\mathscr{M}(A)} \mathscr{H}(x)$. Then B is a Banach ring, and the Gelfand transform $A \to B$ is a bounded map. We get an induced surjective map $\mathscr{M}(B) \to \mathscr{M}(A)$.

5.3.5 Lemma. *Let A be a Banach ring. Then $f \in A$ is invertible if and only if $f(x) \neq 0$ for all $x \in \mathscr{M}(A)$.*

Proof. If $f \in A^\times$, then $|1|_x = |ff^{-1}|_x = 1$, so $|f|_x \neq 0$, whence $f(x) \neq 0$ in $\mathscr{H}(x)$ for all x. If $f \notin A^\times$, then there is some maximal (hence prime) ideal $\mathfrak{m} \ni f$. By Theorem 5.3.2, $\mathscr{M}(A/\mathfrak{m}) \neq \varnothing$. Choose $|\cdot| \in \mathscr{M}(A/\mathfrak{m})$, and let $|\cdot|_x \in \mathscr{M}(A)$ be its pullback. Then $|f|_x = |f + \mathfrak{m}| = |\mathfrak{m}| = 0$. □

Let $x \in \mathscr{M}(A)$. We have seen that there is a natural character $\chi_x \colon A \to \mathscr{H}(x)$. Conversely, given a character $\chi \colon A \to K$, we get a bounded multiplicative seminorm $|\cdot|_\chi \colon A \to \mathbf{R}$, defined by $|f|_\chi = |\chi(f)|$. At the moment, different characters might induce the same seminorm on A. We handle this by noting that two characters χ', χ'' give the same point in $\mathscr{M}(A)$ if and only if they are equivalent in the following sense: there exists a character $\chi \colon A \to K$ such that the following diagram commutes:

$$\begin{tikzcd} & A \arrow[dl, "\chi'"'] \arrow[d, "\chi"] \arrow[dr, "\chi''"] & \\ K' & K \arrow[l, hook'] \arrow[r, hook] & K'' \end{tikzcd} \tag{6}$$

5.4 Comparison with algebraic spectrum

Let A be a (commutative, unital) ring. Recall that $\mathrm{Spec}(A)$ is the set of prime ideals in A, i.e.

$$\mathrm{Spec}(A) = \{\mathfrak{p} \subset A \colon \mathfrak{p} \text{ is prime}\}.$$

The set $\mathrm{Spec}(A)$ has a topology with closed sets

$$V_{\mathfrak{a}} = \{\mathfrak{p} \in \mathrm{Spec}(A) \colon \mathfrak{p} \supset \mathfrak{a}\},$$

where $\mathfrak{a}$ ranges over ideals in A. An *algebraic character* on A is a ring homomorphism from A to a field, $\chi \colon A \to K$.

Given $\mathfrak{p} \in \mathrm{Spec}(A)$, we get a character

$$\chi_{\mathfrak{p}} \colon A \twoheadrightarrow A/\mathfrak{p} \hookrightarrow (A/\mathfrak{p})_{(0)}.$$

Conversely, given a character $\chi \colon A \to K$, the ideal $\ker(\chi) \in \mathrm{Spec}(A)$. Two characters χ', χ'' give the same point in $\mathrm{Spec}(A)$ if and only if a diagram (6) exists in the algebraic category.

There is a natural map $\ker \colon \mathscr{M}(A) \to \mathrm{Spec}(A)$ (the *kernel map*), given by $|\cdot| \mapsto \ker(|\cdot|)$. This should not be mistaken for the specialization map we will encounter later.

5.4.1 Lemma.

1. *The map* $\ker \colon \mathscr{M}(A) \to \mathrm{Spec}(A)$ *is continuous.*
2. *If* $\|\cdot\|$ *(the fixed norm on* A*) is trivial, then*
   (a) $\ker \colon \mathscr{M}(A) \to \mathrm{Spec}(A)$ *is surjective,*
   (b) *there is a canonical section* $\mathrm{Spec}(A) \to \mathscr{M}(A)$

Proof. 1. Let $\mathfrak{a} \subset A$ be an ideal. We claim that

$$\ker^{-1}(V_{\mathfrak{a}}) = \bigcap_{f \in \mathfrak{a}} \{x \in \mathscr{M}(A) \colon |f|_x = 0\}.$$

Indeed, if x lies in the right hand side, then for all $f \in \mathfrak{a}$, $|f|_x = 0$, whence $f \in \mathfrak{a} \Rightarrow f \in \mathfrak{p}_x$, so we get $\mathfrak{a} \subset \mathfrak{p}_x$, i.e. $\mathfrak{p}_x \in V_{\mathfrak{a}}$. This is equivalent to $\ker(|\cdot|_x) \subset V_{\mathfrak{a}}$, i.e. $x \in \ker^{-1}(V_{\mathfrak{a}})$. In fact, all the above implications are equivalences.

2. Given $\mathfrak{p} \in \operatorname{Spec}(A)$, define $|\cdot|_{\mathfrak{p}} \in \mathscr{M}(A)$ by

$$|f|_{\mathfrak{p}} = \begin{cases} 0 & f \in \mathfrak{p} \\ 1 & f \notin \mathfrak{p} \end{cases}.$$

Since $\|\cdot\|$ is trivial, $|\cdot|_{\mathfrak{p}}$ is a bounded multiplicative norm on A. The map $\mathfrak{p} \mapsto |\cdot|_{\mathfrak{p}}$ is the desired canonical section. □

5.5 Functoriality of Berkovich spectrum

We want to show that $A \mapsto \mathscr{M}(A)$ is functorial in the appropriate sense. Let $\varphi\colon (A, \|\cdot\|) \to (B, \|\cdot\|')$ be a bounded homomorphism of commutative Banach rings. (As always, we assume that $\varphi(1) = 1$.)

5.5.1 Theorem.

1. *The map $\varphi^*\colon \mathscr{M}(B) \to \mathscr{M}(A)$ given by $\varphi^*(|\cdot|') = |\cdot|$, $|f| = |\varphi(f)|'$, is well-defined and continuous.*

2. *Assume the set*
$$\left\{\frac{\varphi(f)}{\varphi(g)} : f, g \in A \text{ and } \varphi(g) \in B^\times\right\}$$
is dense in B. Then φ^* is injective.

3. *If φ^* is injective, then $\mathscr{M}(B)$ is homeomorphic to its image in $\mathscr{M}(A)$.*

Proof. 1. That $|\cdot| = \varphi^*(|\cdot|')$ is multiplicative is obvious. To see that it is bounded, note that for all $f \in A$, we have

$$|f| = |\varphi(f)|' \leqslant c_1\|\varphi(f)\|' \leqslant c_1 c_2\|f\|.$$

Continuity follows from the fact that

$$(\varphi^*)^{-1}\left(\{x \in \mathscr{M}(A) \colon |f(x)| > \alpha\}\right) = \{y \in \mathscr{M}(B) \colon |\varphi(f)(y)| > \alpha\}.$$

2. Let $|\cdot|' \neq |\cdot|'' \in \mathscr{M}(B)$ be such that $\varphi^*(|\cdot|') = \varphi^*(|\cdot|'')$. Then there exists $h \in B$ such that $|h|' \neq |h|''$, so we may as well assume $|h|' < |h|''$. Let $2\epsilon < |h|'' - |h|'$. By assumption, there exists $f, g \in A$ such that $\varphi(g) \in B^\times$ and $\left\|h - \frac{\varphi(f)}{\varphi(g)}\right\|' < \epsilon$. Thus $\left|h - \frac{\varphi(f)}{\varphi(g)}\right| < \epsilon$ and $\left|h - \frac{\varphi(f)}{\varphi(g)}\right|'' < \epsilon$. Apply the triangle inequality:

$$\begin{aligned}\left|\frac{\varphi(f)}{\varphi(g)}\right|' &= \left|\frac{\varphi(f)}{\varphi(g)} - h + h\right|' \\ &\leqslant \left|\frac{\varphi(f)}{\varphi(g)} - h\right|' + |h|' \\ &< \epsilon + |h|' \\ &< |h|' - \epsilon.\end{aligned}$$

Similarly

$$\begin{aligned} |h|'' &= \left| h - \frac{\varphi(f)}{\varphi(g)} + \frac{\varphi(f)}{\varphi(g)} \right|'' \\ &\leqslant \left| h - \frac{\varphi(f)}{\varphi(g)} \right|'' + \left| \frac{\varphi(f)}{\varphi(g)} \right|'' \\ &< \epsilon + \left| \frac{\varphi(f)}{\varphi(g)} \right|''. \end{aligned}$$

It follows that $\left|\frac{\varphi(f)}{\varphi(g)}\right|' < \left|\frac{\varphi(f)}{\varphi(g)}\right|''$, so $|\varphi(f)|'|\varphi(g)|'' < |\varphi(f)|''|\varphi(g)|'$. This implies $|f||g| < |f||g|$, a contradiction.

3. This follows from the general fact, proved in [Folland, p.129], that if X is quasi-compact, Y is Hausdorff, then any continuous bijection $f\colon X \to Y$ is a homeomorphism. □

Note that the recurring norm

$$|f|_{\mathfrak{p}} = \begin{cases} 1 & f \notin \mathfrak{p} \\ 0 & f \in \mathfrak{p} \end{cases},$$

comes from the trivial seminorm on $A/\mathfrak{p}$.

5.6 Boundedness versus continuity for K-algebras

Usually (e.g., in classical algebraic geometry) one cares about rings that are also K-vector spaces, for some field K.

5.6.1 Definition. Let R be a commutative ring. A (unital, associative) *R-algebra* is an abelian group A equipped with the structure of both a (associative, unital) ring and an R-module, such that ring multiplication is R-bilinear in the sense that $r \cdot (xy) = (r \cdot x)y = x(r \cdot y)$.

It is equivalent to specify a (unital, associative) ring A together with a ring homomorphism $R \to \mathrm{Z}(A)$, where $\mathrm{Z}(A) = \{a \in A\colon ab = ba \text{ for all } b \in A\}$ is the center of A. We are mainly interested in commutative R-algebras, but some aspects of the theory (especially over $\mathbf{C}$) work just as well for possibly non-commutative algebras.

Let K be a field with absolute value, A a K-algebra. A *norm* $|\cdot|$ on A is both a ring norm and a K-vector space norm. This in particular requires that $\|av\| = |a|\|v\|$ for all $a \in K$, $v \in A$. So any norm on A "remembers" the norm on K. More precisely, if A is a normed K-algebra, $\|\cdot\|_x\colon A \to \mathbf{R}$ a bounded multiplicative seminorm. Then $|\cdot|_x$ induces an absolute value on $\mathscr{H}(x)$. But $\mathscr{H}(x)$ is a complete extension of K. It turns out that $|\cdot|_x$ on $\mathscr{H}(x)$ must restrict to $|\cdot|$ on K. This is Exercise 4.3.1 in Conrad's notes in [BCD+08].

Let $\varphi\colon A \to B$ be a K-algebra homomorphism. We assume that A and B have been given (K-)norms $\|\cdot\|$ and $\|\cdot\|'$. Clearly, if φ is bounded (i.e. $\|\varphi(f)\|' \leqslant c\|f\|$ for all $f \in A$), then φ is continuous. We have seen that the converse does not always hold.

Let's try working towards proving the converse. Let $\varphi\colon A \to B$ be a continuous homomorphism. Then φ is continuous if and only if it is continuous at 0. So there exists a neighborhood $U \ni 0$ such that $\varphi(U) \subset \{b \in B\colon \|b\|' < 1\}$. There is $\delta > 0$ such that $\{a \in A\colon \|a\| \leqslant \delta\} \subset U$. So $\|\varphi(x)\|' \leqslant 1$ whenever $\|x\| \leqslant \delta$. If we were working over $\mathbf{C}$, we would not rescale. But this doesn't work if the norm on K is trivial. If the norm $|\cdot|$ on K is *not* trivial (i.e., $|K^\times| \neq 1$) then by rescaling we can say that $\|x\| \leqslant |a| \Rightarrow \|\varphi(x)\|' \leqslant \frac{|a|}{\delta}$. So generally, $\|\varphi(x)\|' \leqslant \frac{1}{\delta}\|x\|$, and φ is bounded.

If $|\cdot|$ is trivial on K, then continuity does not imply boundedness. If $|\cdot|$ is nontrivial, many other (but not all) basic facts of functional analysis are also true, e.g. Banach's open mapping theorem (a bijective bounded K-algebra homomorphism $A \to B$ has bounded inverse).

5.7 Gelfand's theory for $K = \mathbf{C}$

Much of this material can be found in [Rud91, Ch. 10-11]. Here is our motivating example.

5.7.1 Example. Let X be a nonempty compact Hausdorff space. Let $A = \mathscr{C}^0(X)$ be the space of continuous $\mathbf{C}$-valued functions on X. With pointwise multiplication, A is naturally a (commutative, unital) $\mathbf{C}$-algebra. We give A the supremum norm:

$$\|f\| = \sup_{x \in X} |f(x)|.$$

It's a standard fact that A is complete, i.e. it is a Banach $\mathbf{C}$-algebra. *Warning*: the norm on $\mathscr{C}^0(X)$ is *not* in general multiplicative, i.e. we only have $\|fg\| \leqslant \|f\|\|g\|$.

5.7.2 Example. Let X be a finite set with n points. Then $C^0(X) = \mathbf{C}^n$.

It is natural to ask: "given any commutative Banach $\mathbf{C}$-algebra A, can we embed A into $\mathscr{C}^0(X)$ for some compact Hausdorff X?"

5.7.3 Example. We give $\mathrm{L}^1(\mathbf{R}^n)$ the structure of a $\mathbf{C}$-algebra via convolution:

$$(f * g)(x) = \int_{\mathbf{R}^n} f(y)g(x-y)\,\mathrm{d}y.$$

This makes $\mathrm{L}^1(\mathbf{R}^n)$ a commutative, but non-unital algebra. We give $\mathrm{L}^1(\mathbf{R}^n)$ a unit as follows. Let $A = \mathrm{L}^1(\mathbf{R}^n) \oplus \mathbf{C}\delta$, where δ is the "Dirac delta" with multiplication as follows:

$$(f_1 + \lambda_1\delta) * (f_2 + \lambda_2\delta) = (f_1 * f_2 + \lambda_2 f_1 + \lambda_1 f_2) + \lambda_1\lambda_2\delta.$$

Give A a norm by

$$\|f + \lambda\delta\| = \|f\|_{\mathrm{L}^1} + |\lambda|.$$

5.7.4 Example. Let X be a Banach space over $\mathbf{C}$. Let $\mathscr{B}(X)$ be the algebra of bounded linear operators on X, with $\|\cdot\|$ the operator norm. For example, if $\dim(X) = n$, then $\mathscr{B}(X) \simeq M_n(\mathbf{C})$. Note that if $n > 1$, then $\mathscr{B}(X)$ is *not* commutative. The theory of Gelfand-Mazur (and some of Berkovich's) work for non-commutative algebras as well.

5.7.5 Definition. Let A be a $\mathbf{C}$-algebra. Put

$$\sigma(f) = \{\lambda \in \mathbf{C} \colon \lambda - f \notin A^\times\}.$$

(In functional analysis, one calls $\sigma(f)$ the *spectrum* of f, but we will avoid using this terminology. The complement $\mathbf{C} \setminus \sigma(f)$ is called the *resolvent* of f.)

For K non-archimedean, one can give a definition for $\sigma(f)$, but it's quite a bit more complicated, so we won't reproduce it here. See [Ber90, Ch. 7] for details.

5.7.6 Theorem. *Let $f \in A$. Then*

1. *$\sigma(f) \neq \varnothing$,*
2. *$\sigma(f)$ is compact.*

Proof. Part 1 is essentially Lemma 3.3.3. For part 2, it suffices to show that $\sigma(f)$ is closed and bounded. That $\sigma(f)$ is closed follows (essentially) from the fact that $A^\times$ is open. For boundedness of $\sigma(f)$, see below. □

5.7.7 Definition. For $f \in A$, put

$$\rho(f) = \sup\{|\lambda| \colon \lambda \in \sigma(f)\}.$$

5.7.8 Theorem. *$\rho(f) = \lim_{n\to\infty} \|f^n\|^{1/n}$.*

The proof will be given in a more general setting, after giving an equivalent definition of $\rho(f)$.

5.7.9 Definition. Let A be a Banach $\mathbf{C}$-algebra. As a set,

$$\mathcal{M}(A) = \{\chi \colon A \to \mathbf{C} \text{ a homomorphism of } \mathbf{C}\text{-algebras}\}.$$

5.7.10 Lemma.

1. *Any $\chi \in \mathcal{M}(A)$ is bounded.*
2. *$\mathcal{M}(A)$ can be identified with the space of maximal ideals in A via $\chi \leftrightarrow \ker(\chi)$.*
3. *$\mathcal{M}(A) = \mathscr{M}(A)$*
4. *$f \in A^\times$ if and only if $\chi(f) \neq 0$ for all $\chi \in \mathcal{M}(A)$.*
5. *$\lambda \in \sigma(f)$ if and only if $\chi(f) = \lambda$ for some $\chi \in \mathcal{M}(A)$.*

Proof. 1. Use the fact that $\chi \colon A \to \mathbf{C}$ respects the $\mathbf{C}$-algebra structure of A.
2. For $x \in \mathcal{M}(A)$, $\mathscr{H}(x)$ is a complete field containing $\mathbf{C}$, we have $\mathscr{H}(x) = \mathbf{C}$.
4. If $f \notin A^\times$, then $f \notin \mathfrak{m}$ for all $\mathfrak{m}$ maximal.
5. Apply part 4 to $\lambda - f$. □

Parts 2 and 3 essentially say that "Berkovich analytification" does *not* produce any new points when we endow $\mathbf{C}$ with the archimedean absolute value. When we endow $\mathbf{C}$ with the trivial absolute value, the Berkovich analytification carries a lot of extra information.

For $f \in A$, the *Gelfand transform* of f is the function $\hat{f} \colon \mathcal{M}(A) \to \mathbf{C}$ defined by

$$\hat{f}(\chi) = \chi(f).$$

The Gelfand transform is *a priori* a function $A \to \prod_{\mathcal{M}(A)} \mathbf{C}$. We have seen that $\rho(f) = \sup_{\chi \in \mathcal{M}(A)} |\hat{f}(\chi)|$. Give $\mathcal{M}(A)$ the weakest topology making each $\hat{f}$ continuous (this recovers the topology on $\mathscr{M}(A)$ we defined earlier). Then the Gelfand transform is a map $A \to \mathscr{C}^0(\mathcal{M}(A))$ and

$$\rho(f) = \|\hat{f}\|_{\mathscr{C}^0(\mathcal{M}(A))}.$$

Let $B = \{\hat{f} \colon f \in A\} \subset \mathscr{C}^0(\mathcal{M}(A))$. Clearly, the Gelfand transform gives us a $\mathbf{C}$-algebra homomorphism $A \twoheadrightarrow B \hookrightarrow \mathscr{C}^0(\mathcal{M}(A)), f \mapsto \hat{f}$. What is the kernel of $A \twoheadrightarrow B$? Suppose $\hat{f} = 0$, i.e. $\hat{f}(\chi) = 0$ for all χ, i.e. $\chi(f) = f \mod \ker(\chi) = 0$ for all χ. In other words, $\hat{f} = 0$ if and only if $f \in \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of A. So

$$\ker(\hat{\cdot}) = \bigcap_{\mathfrak{m} \in \mathrm{mSpec}(A)} \mathfrak{m} = \text{Jacobson radical of } A = \ker(\rho).$$

So $\hat{\cdot}$ is injective if and only if ρ is a norm (not just a seminorm) on A, if and only if A is Jacobson semisimple. (We say a algebra A is *Jacobson semisimple* if its Jacobson radical vanishes.)

The algebra A has a norm $\|\cdot\|$ already, while $B \subset \mathscr{C}^0(\mathcal{M}(A))$ carries the sup-norm. The map $\hat{\cdot} \colon A \to B$ is an sometry if and only if $\rho(\cdot) = \|\cdot\|$. Since ρ is power-multiplicative, clearly a necessary condition is for $\|\cdot\|$ to be power-multiplicative. We will see later that $\rho(\cdot) = \|\cdot\|$ if and only if $\|\cdot\|$ is power-multiplicative.

5.7.11 Definition (Old)**.** A $\mathbf{C}$-Banach algebra A is called *uniform* if there exists a compact Hausdorff space X such that there is a bounded $\mathbf{C}$-algebra homomorphism $A \hookrightarrow \mathscr{C}^0(X)$ with image a closed subspace containing the constant functions and separating points of X.

(Recall that $S \subset \mathscr{C}^0(X)$ *separates points* if for any $x \neq y \in X$, there is $s \in S$ such that $s(x) \neq s(y)$.) An algebra $(A, \|\cdot\|)$ is uniform if and only if $\rho(\cdot) = \|\cdot\|$, if and only if $\|\cdot\|$ is power-multiplicative.

Given a Banach $\mathbf{C}$-algebra $(A, \|\cdot\|)$, one can always find a related uniform algebra $(A^{\mathrm{u}}, \rho(\cdot))$. Namely, A^{u} is the completion of $A/\ker(\rho)$ with respect to the norm ρ.

5.7.12 Example. Let $A = \mathrm{L}^1(\mathbf{R}^n) \oplus \mathbf{C}\delta$. Given any $\omega \in \mathbf{R}^n$, we have a character $\chi_\omega \colon A \to \mathbf{C}$, defined by

$$\chi_\omega(f + \lambda \cdot \delta) = \hat{f}(\omega) + \lambda.$$

Here, $\hat{f}$ is the usual Fourier transform of f. Also, there is the character

$$\chi_\infty(f + \lambda \cdot \delta) = \lambda.$$

It is known that $\mathcal{M}(A) = \{\chi_\omega\}_{\omega \in \mathbf{R}^n} \cup \{\chi_\infty\}$. So $\mathcal{M}(A)$ is the "spectrum" space. Moreover, the (weak) topology on $\mathcal{M}(A)$ is the same as the one-point compactification of $\mathbf{R}^n$.

As an exercise, make $[0,1]$ out of $[0,1]\cap\mathbf{Q}$ by announcing what you would like to be your continuous functions on $[0,1]$. For a more formal discussion, see the introduction to [Ber90].

The theory of Gelfand-Mazur has another analytic generalization to C*-algebras and non-commutative geometry in the sense of A. Connes. The "spectrum" can be replaced by the *unitary dual*, which plays a large role in the representation theory of real reductive groups.

5.8 General Banach rings

Let A be a Banach ring, $f\in A$.

5.8.1 Definition. The *spectral radius* of f is

$$\rho(f)=\lim_{n\to\infty}\|f^n\|^{1/n}=\inf\{\|f^n\|^{1/n}\colon n\geqslant 1\}.$$

5.8.2 Lemma. *Definition 5.8.1 makes sense.*

Proof. This follows from Theorem 5.8.3 applied to $a_n=\log\|f^n\|$. The sequence is subadditive because $\|f^{n+m}\|\leqslant\|f^n\|\|f^m\|$. □

5.8.3 Theorem (Fekete)**.** *Let $\{a_n\}_{n\geqslant 1}\subset\mathbf{R}\cup\{-\infty\}$. If $\{a_n\}$ is subadditive ($a_{n+m}\leqslant a_n+a_m$), then $\lim_{n\to\infty}\frac{a_n}{n}$ exists and is equal to $\inf\{\frac{a_n}{n}\colon n\geqslant 1\}$.*

Proof. If $\inf\{\frac{a_n}{n}\colon n\geqslant 1\}=-\infty$, then it easily follows that $\lim\frac{a_n}{n}=-\infty$.

Suppose the infimum is finite, and let $a=\inf\{\frac{a_n}{n}\colon n\geqslant 1\}$. Fix $\epsilon>0$, and let $k\gg 0$ be such that $\left|\frac{a_k}{k}-a\right|<\epsilon/2$. Let $l\gg 0$ be such that $\frac{a_r}{kl}<\epsilon/2$ for $r<k$. If $n>kl$, write $n=kq+r$ with $r<k$. Then $q\geqslant l$, so

$$a\leqslant\frac{a_n}{n}\leqslant\frac{a_{kq}+a_r}{kq+r}\leqslant\frac{qa_k+kl\epsilon/2}{kq}=\frac{a_k}{k}+\frac{l\epsilon}{2q}<a+\epsilon.$$

□

Recall the Gelfand transform $\hat{\cdot}\colon A\twoheadrightarrow B\subset\prod^{\mathrm{b}}_{x\in\mathscr{M}(A)}\mathscr{H}(x)$, given by

$$\hat{f}(x)=f\mod\ker|\cdot|_x.$$

The algebra B inherits the supremum norm.

5.8.4 Theorem (Spectral radius vs. Berkovich spectrum)**.** *Let A be a commutative Banach ring. Then for all $f\in A$, $\rho(f)=\|\hat{f}\|$, i.e.*

$$\lim_{n\to\infty}\|f^n\|^{1/n}=\max_{x\in\mathscr{M}(A)}|f(x)|.$$

Proof. First we show that $\lim_{n\to\infty}\|f^n\|^{1/n}\geqslant\max_{x\in\mathscr{M}(A)}|f(x)|$. Let $f\in A$ and $x\in\mathscr{M}(A)$. Then

$$\|f^n\|\geqslant|f^n|_x=|f|_x^n=|f(x)|^n,$$

so $\rho(f)\geqslant|f(x)|$ and therefore $\rho(f)\geqslant\|\hat{f}\|$.

To show $\lim_{n\to\infty} \|f^n\|^{1/n} \leqslant \max_{x\in\mathscr{M}(A)} |f(x)|$, we will prove that for any $r \in \mathbf{R}^{\geqslant 0}$, $\|\hat{f}\| < r$ implies $\rho(f) < r$. Fix r, and assume $\|\hat{f}\| < r$. Then, by boundedness, $|f(x)| < r$ for all $x \in \mathscr{M}(A)$. Let

$$A' = \left\{ \sum_{i=0}^{\infty} a_i T^i : a_i \in A, \sum_{i=0}^{\infty} a_i r^{-i} < \infty \right\}$$

be the ring of convergent power series with coefficients in A having radius of convergence r. Note that $\sum_{i=0}^{\infty} a_i r^{-i}$ is a norm on A'.

We claim that $1 - fT$ is invertible in A'. Assuming this claim, we have

$$\begin{aligned}
1 - fT \text{ is invertible } &\Leftrightarrow \sum_{i=0}^{\infty} f^i T^i \in A' \\
&\Leftrightarrow \sum_{i=0}^{\infty} a_i r^{-i} < \infty \\
&\Rightarrow \|f^i\| r^{-i} < 1 \text{ for } i \text{ sufficiently large} \\
&\Rightarrow \|f^i\|^{\frac{1}{i}} < r \\
&\Rightarrow \rho(f) < r,
\end{aligned}$$

and we are done. To prove the claim, note that

$$\begin{aligned}
1 - fT \text{ is invertible in } A' &\Leftrightarrow (1 - fT)(x) \neq 0 \; \forall x \in M(A') \\
&\Leftrightarrow |1 - fT|_x \neq 0 \; \forall x \in M(A')
\end{aligned}$$

To show this it suffices to show $|fT|_x < 1$ because

$$1 = |1|_x = |1 - fT + fT|_x \leqslant |1 - fT|_x + |fT|_x$$

and if $|1 - fT|_x = 0$ then the above inequality implies $1 < 0 + 1$.

To compute $|fT|_x$, note that $\|T\| = r^{-1}$. By assumption, we have $|f|_x < r$ for all $x \in \mathscr{M}(A)$. But the map

$$\Phi : A \to A'$$

sending $f \mapsto f$ induces an isometry

$$\Phi^* : \mathscr{M}(A') \to \mathscr{M}(A),$$

so $|f|_x = |f|_{\Phi^*(x)} < r$ for all $x \in \mathscr{M}(A')$, and therefore

$$|fT|_x = |f|_x |T|_x < rr^{-1} = 1.$$

$\square$

The following theorem shows that $\rho(\cdot)$ is a canonical seminorm

5.8.5 Theorem. *Let $(A, \|\cdot\|)$ be a commutative Banach algebra. Then*

1. *ρ only depends on the equivalence class of $\|\cdot\|$.*

2. *$\rho : A \to \mathbf{R}$ is a seminorm.*

3. *ρ is always power-multiplicative.*

4. *$\rho(f) \leqslant \|f\| \ \forall f \in A$, moreover, $\rho(\cdot) = \|\cdot\|$ if and only if $\|\cdot\|$ is power-multiplicative.*

These results can be restated in terms of properties of the Gelfand transform

5.8.6 Theorem. *Let $\hat{\cdot} : A \to B$ be the Gelfand transform. Then*

(a) $\ker \hat{\cdot} = \ker \rho$.

(b) *$\hat{\cdot}$ is an isometry with respect to $(A, \|\cdot\|)$ and $(B, \rho(\cdot))$ if and only if $\|\cdot\|$ is power-multiplicative.*

The ideal $\ker \rho$ is called the *quasinilradical* of A. Its elements are the quasinilpotents of A, i.e. those elements f of A for which

$$f(x) = 0 \ \forall x \in \mathscr{M}(A).$$

In algebraic geometry, $f \in R$ is identically zero on $\operatorname{Spec}(R)$ if and only if

$$f(\mathfrak{p}) = f + \mathfrak{p} = 0 \quad \forall \mathfrak{p} \in \operatorname{Spec}(R).$$

which happens if and only if

$$f \in \bigcap_{\mathfrak{p} \in \operatorname{Spec}(R)} \mathfrak{p} = \operatorname{rad}(\langle 0 \rangle),$$

which is known as the nilradical of R.

Note that the nilradical is contained in the quasinilradical because of the power-multiplicativity of ρ. Recall that we have shown that for $\mathbf{C}$-Banach algebras, the quasinilradical is equal to the Jacobson radical of A.

5.8.7 Definition. A Banach algebra $(A, \|\cdot\|)$ is called *uniform* if $\|\cdot\|$ is power-multiplicative.

5.8.8 Lemma. *$\|\cdot\|$ is power-multiplicative if and only if for all $f \in A$, $\|f^2\| = \|f\|^2$.*

Proof. Fix f. Define $\Phi(n) = \log \|f^n\|$. By assumption, we have

$$\Phi(2) = 2\Phi(1)$$

so $\Phi(2^m) = 2^m \Phi(1)$. We want $\Phi(n) = n\Phi(1)$. Note: Φ is subadditive, so $\Phi(n) \leqslant n\Phi(1)$. If $\Phi(n) < n\Phi(1)$, then let k be such that $k + n = 2^m$. Then

$$2^m \Phi(1) = \Phi(2^m) = \Phi(k+n) \leqslant \Phi(k) + \Phi(n) < k\Phi(1) + n\Phi(1) = 2^m \Phi(1)$$

which is a contradiction. $\square$

Proof of Theorem 5.8.5. Let

$$\rho(f) = \lim_{n \to \infty} \|f^n\|^{1/n}) \tag{7}$$

$$\rho(f) = \max_{x \in \mathscr{M}(A)} |f(x)| \tag{8}$$

denote the two equivalent definitions of ρ.

1. This follows immediately from (7).

2. $\rho(0) = 0, \rho(1) = 1$ and $\rho(fg) \leqslant \rho(f)\rho(g)$ follow from (7), $\rho(f+g) \leqslant \rho(f)+\rho(g)$ follows easily from (8).

3. (7)

4. By (7), we have $\rho(f) = \lim_{n\to\infty} \|f^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \|f\|^{n\frac{1}{n}} = \|f\|$. If $\rho = \|\cdot\|$, then by 3, $\|\cdot\|$ is power-multiplicative, so $\rho(f) = \|f^n\|^{\frac{1}{n}} = \|f\|$.

$\square$

Proof of Theorem 5.8.6. (a) Follows from (7).

(b) Follows from Theorem 5.8.5, 4.

$\square$

Recall that the kernel map $\ker\colon \mathscr{M}(A) \to \operatorname{Spec}(A)$ sending $x \mapsto \ker|\cdot|_x$ is continuous with respect to the Berkovich topology on $\mathscr{M}(A)$ and the Zariski topology on $\operatorname{Spec}(A)$.

5.8.9 Definition. The Zariski topology on $\mathscr{M}(A)$ is the weakest topology making ker continuous.

As an exercise, show that $\mathscr{M}(A)$ is irreducible with respect to the Zariski topology if and only if $\ker\rho$ is a prime ideal.

5.9 Uniformization of Banach Rings

Let $(A, \|\cdot\|)$ be a Banach Ring. We would like to find a uniform Banach Ring associated to A. Motivated by Theorem 5.8.5, we would like to use ρ, but as seen in that theorem, ρ is in general only a seminorm, and $A/\ker\rho$ might not be complete. Therefore we define the uniformization A^{u} of A to be

$$A^{\mathrm{u}} = \widehat{A/\ker\rho}$$

where $\hat{\cdot}$ refers to completion with respect to ρ. Then (A^{u}, ρ) is a uniform Banach ring (we denote the residue norm of ρ by ρ again). Let $\pi\colon A \to A^{\mathrm{u}}$ be the quotient map.

5.9.1 Theorem.

1. *For any bounded morphism $\phi\colon A \to B$ with B uniform we have*

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[dr, "\phi"'] & A^{\mathrm{u}} \arrow[d, dotted, "\exists!"] \\ & B \end{tikzcd}$$

2. *We have $\mathscr{M}(A^{\mathrm{u}}) = \mathscr{M}(A)$.*

3. *$\mathscr{M}(A^{\mathrm{u}})$ has one point if and only if A^{u} is a field and ρ is an absolute value on A^{u}.*

4. ρ is non-archimedean on A^{u} if and only if for all $x \in \mathscr{M}(A^{\mathrm{u}})$, $(\mathscr{H}(x), |\cdot|_x)$ is non-archimedean.

Proof. 1. By a similar trick as before, using power-multiplicativity of $\|\cdot\|_B$, we may assume

$$\|\phi(f)\|_B \leqslant \rho(f).$$

So if $\rho(f) = 0$ then $\phi(f) = 0$ and $\phi\colon A \to B$ factors through $\pi\colon A \to A^{\mathrm{u}}$.

2. The map $\pi\colon A \to A^{\mathrm{u}}$ is bounded since it's a quotient map. Then

$$\pi^*\colon \mathscr{M}(A^{\mathrm{u}}) \to \mathscr{M}(A)$$

is injective, since $\pi(A)$ is dense in A^{u}. Both $\mathscr{M}(A^{\mathrm{u}})$ and $\mathscr{M}(A)$ are compact Hausdorff spaces, so injectivity of π implies

$$\mathscr{M}(A^{\mathrm{u}}) \simeq \pi\left(\mathscr{M}(A^{\mathrm{u}})\right),$$

so we just need to show that π^* is surjective. Let $|\cdot|_x \in \mathscr{M}(A)$. Note that it suffices to give such a $|\cdot|_y \in \mathscr{M}(A/\ker\rho)$, since then $|\cdot|_y$ will extend uniquely to the completion A^{u}. Define

$$|f + \ker\rho|_y = |f|_x = |f(x)|.$$

This is well-defined as $\rho(f) = \max_{x \in \mathscr{M}(A)} |f(x)| = 0$ implies $f(x) = 0$ for all x.

3. $\Leftarrow$. If $(K, |\cdot|)$ is a field with absolute value, then $\mathscr{M}(K)$ is a point, because for any $x \in \mathscr{M}(K)$, $\ker|\cdot| = 0$. Thus $\mathscr{H}(x) = K$, and the character $K \to \mathscr{H}(K)$ is the identity.

$\Rightarrow$. Suppose $\mathscr{M}(A^{\mathrm{u}}) = \{x\}$. First, we claim that $\rho(\cdot)$ is multiplicative, which follows from (8). Next, we know that $f \in (A^{\mathrm{u}})^\times$ if and only if $f(x) \neq 0$. Thus $x = \rho$, so $(A^{\mathrm{u}})^\times = A \smallsetminus \ker\rho = 0$. We have shown that A^{u} is a field.

4. Assume that for all $x \in \mathscr{M}(A^{\mathrm{u}})$, $(\mathscr{H}(x), |\cdot|_x)$ is non-archimedean. Then as

$$\rho(f) = \max_{x \in \mathscr{M}(A^{\mathrm{u}})} |f(x)| = \max_{x \in \mathscr{M}(A^{\mathrm{u}})} |f|_x,$$

ρ is non-archimedean.

For the converse, recall that it suffices to check the archimedean property for integers. Assume that ρ is non-archimedean. Then $\rho(n) \leqslant 1$ for all $n \in \mathbf{Z}$. Therefore for $x \in \mathscr{M}(A^{\mathrm{u}})$,

$$|n|_x \leqslant \rho(n) \leqslant 1,$$

so $|\cdot|_x$ is non-archimedean. □

5.10 Products

Let $(A, \|\cdot\|_A)$ be a normed ring.

5.10.1 Definition. A *seminorm* on an A-module M is a function

$$\|\cdot\| : M \to \mathbf{R}$$

such that

1. • $\|0\| = 0$,

- $\|m+n\| \leqslant \|m\| + \|n\|$,
- $\|m\| = \|-m\|$.

2. There exists some $c > 0$ such that for all $f \in A, m \in M$,

$$\|f \cdot \| \leqslant c\|f\|_A\|m\|.$$

Fact: We may assume $c = 1$ by replacing $\|\cdot\|$ by $\|\cdot\|'$, defined as

$$\|m\|' = \sup_{f \in A \setminus \{0\}} \frac{\|f \cdot m\|}{\|f\|_A}.$$

Then

$$\|m\| \leqslant \|m\|' \leqslant c\|m\|$$

so $\|\cdot\|$ is equivalent to $\|\cdot\|'$.

Let M and N be A-modules. On the tensor product $M \otimes_A N$ we have some natural choices for seminorms.

1. If A, M, N are all archimedean, then for $s \in M \otimes_A N$, let

$$\|s\| = \inf \left(\sum_{i \in I} \|m_i\| \|n_i\| \right), \tag{9}$$

where we are taking the infimum of the above for all presentations of s:

$$s = \sum_{i \in I} m_i \otimes n_i. \tag{10}$$

2. If A, M, N are all non-archimedean, we prefer the following choice

$$\|s\| = \inf \left(\mathrm{Max}_{i \in I} \|m_i\| \|n_i\| \right)$$

Some remarks about this definition:

1. Definition (9) also makes sense in the non-archimedean setting, but the two definitons are nonequivalent in general.

2. Definition (10) will not work in the archimedean case.

3. Even when the seminorms on M and N are norms, the seminorm on $M \otimes_A N$ might have a nontrivial kernel.

4. In general $M \otimes_A N$ is not complete.

To remedy this, we define

$$M \hat{\otimes}_A N = \widehat{M \otimes_A N}$$

where completion is taken with respect to the above-defined norm. Then $M \hat{\otimes}_A N$ is also an $\hat{A}$-module.

Note that the map

$$M \otimes_A N \to M \hat{\otimes}_A N$$

is not in general injective.

5.10.2 Theorem (Gruson)**.** *Let K be a non-archimedean complete field, M and N be K-Banach vector spaces. Then the natural map*

$$M \otimes_A N \to M \hat{\otimes}_A N$$

is injective.

The completed tensor product $M\hat{\otimes}_A N$ also has the following universal property: Any bounded bilinear map $M \times N \to L$ where L is a complete normed A-module factors through the canonical map

$$M \times N \to M\hat{\otimes}_A N.$$

Our main interest in tensor products is extension of scalars, i.e. if A is a K-Banach algebra and L/K is a field extension, we want to study

$$A' = A\hat{\otimes}_K L$$

6 Useful results

One motivation is: let A be a $\mathbf{Q}_p$-algebra. We would like to understand $X_{\mathbf{Q}_p} = \mathscr{M}(A)$ in terms of $X_{\mathbf{C}_p} = \mathscr{M}(A\hat{\otimes}_{\mathbf{Q}_p}\mathbf{C}_p)$. Another reason is that by extending the base field, we can enlarge the value group. For example, we can obtain value group $\mathbf{R}$, which makes some things easier.

Let A, B be Banach K-algebras, and put $X = \mathscr{M}(A)$, $Y = \mathscr{M}(B)$. We have a commutative diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A\hat{\otimes}_K B, \end{tikzcd}$$

which (by functoriality) gives us a commutative diagram of spectra:

$$\begin{tikzcd} \mathscr{M}(A\hat{\otimes}_K B) \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & \{*\}. \end{tikzcd}$$

We put $X \times_K Y = \mathscr{M}(A\hat{\otimes}_K B)$. There are projection maps $X \times_K Y \to X$ and $\to Y$, corresponding e.g. to the maps $A \to A\hat{\otimes}_K B$, $f \mapsto f \otimes 1$. Write $\pi\colon X \times_K Y \to X$. Given $x \in X$, what is the fiber $\pi^{-1}(x)$? It turns out that $\pi^{-1}(x) \simeq \mathscr{M}(\mathscr{H}(x)\hat{\otimes}_K B)$. That is, topological and "algebraic" fibers agree. So in particular, $\pi^{-1}(x) \neq \varnothing$ for all x, so $X \times Y \to X$ is surjective. The algebraic incarnation of $\pi^{-1}(x) \hookrightarrow X \times Y$ is the map $A\hat{\otimes}_K B \to \mathscr{H}(x) \otimes_K B$. Since $\pi^{-1}(x)$ is compact Hausdorff, the injection $\psi^*\colon \pi^{-1}(x) \hookrightarrow X \times Y$ is a homeomorphism onto its image.

Recall from [Har77, II.3] the notion of fibers of a morphism. If $f\colon X \to Y$ is a morphism of schemes, $y \in Y$ a point, then the fiber of f over y, written $f^{-1}(y)$, is by definition the fiber product $X \times_Y \operatorname{Spec}(k(y)) = X \times_Y y$.

Let K be a non-archimedean field, L/K a field extension, and A a Banach K-algebra. Write $X = \mathscr{M}(A)$ and $X_L = \mathscr{M}(A\hat{\otimes}_K L)$. From the above discussion, we have a surjective continuous map $X_L \to X$.

6.1 Interlude in Galois theory

Let K be a field; fix an algebraic closure $\overline{K}$. Let L/K be an algebraic (but possibly infinite) extension. We say that L/K is:

- *normal* if all K-embeddings $L \hookrightarrow \overline{K}$ have the same image. Let $\mathrm{Aut}(L/K)$ be the group of field automorphisms of L which are trivial on K. For example, $\mathbf{Q}(\sqrt[3]{2})/\mathbf{Q}$ is not normal, but $\mathbf{Q}(\sqrt[3]{2}, \zeta_3)/\mathbf{Q}$ is normal.
- *separable* if for all $\alpha \in L$, the minimal polynomial $f \in K[x]$ of α has distinct roots in $\overline{K}$. Equivalently, $L \otimes_K \overline{K}$ is reduced. Any field extension in characteristic zero is separable, as are all extensions of finite fields. More generally, any extension of a perfect field is separable. (A field K is *perfect* if either K is characteristic zero or all $x \in K$ are of the form y^p for some $y \in K$.) For p a prime, the extension $\mathbf{F}_p(\sqrt[p]{t})/\mathbf{F}_p(t)$ is not separable.
- *purely inseparable* if for any $\alpha \in L \smallsetminus K$, the minimal polynomial of α has repeated roots. Equivalently, if K has characteristic $p > 0$, the extension L/K is purely inseparable if and only if for any $\alpha \in L$, there exists $n \geqslant 0$ such that $\alpha^{p^n} \in K$.
- *Galois* if it is normal and separable. Alternatively, $\#\,\mathrm{Aut}(L/K) = [L \colon K]$, or $L^{\mathrm{Aut}(L/K)} = K$.

Artin proved a "primitive element theorem" for Galois extensions. One corollary is that if L/K is finite separable, then $L = K(\alpha)$ for some $\alpha \in L$. Any algebraic extension L/K has a distinguished subset

$$S = \{\alpha \in L \colon \text{the minimal polynomial of } \alpha \text{ is separable}\}.$$

One can check that S is a field. Moreover, S/K is separable, and L/S is purely inseparable.

Let L/K be an algebraic Galois extension with Galois group $G = \mathrm{Gal}(L/K)$. Let $\mathfrak{F} = \{K \subset K' \subset L \colon K'/K \text{ finite Galois}\}$. The group G naturally carries a topology (the *Krull topology*); a basis of neighborhoods of $1 \in G$ is given by

$$\{\mathrm{Gal}(L/K') \colon K' \in \mathfrak{F}\}.$$

Moreover,

$$\mathrm{Gal}(L/K) = \varprojlim_{\mathfrak{F}} \mathrm{Gal}(K'/K),$$

as topological groups, where we give each $\mathrm{Gal}(K'/K)$ the discrete topology. So $\mathrm{Gal}(L/K)$ is a profinite group. Given any tower $K_2/K_1/K$ of Galois extensions, we have a short exact sequence of profinite groups:

$$1 \to \mathrm{Gal}(K_2/K_1) \to \mathrm{Gal}(K_2/K) \to \mathrm{Gal}(K_1/K) \to 1.$$

If L/K is an algebraic extension that is not normal, the *normal closure* of L (over K) is the composite of all images of K-embeddings $L \hookrightarrow \overline{K}$. If we write L' for the normal closure of L, then L'/L is normal, and L' is the smallest normal extension of L which contains L. If $[L \colon K] < \infty$, then $[L' \colon K] < \infty$.

6.1.1 Lemma. *If L/K is normal and purely inseparable, then* $\mathrm{Aut}(L/K) = 1$.

Proof. Actually, purely inseparable extensions are automatically normal. We need to show that for all $\sigma \in \mathrm{Aut}(L/K)$ and $a \in L$, we have $\sigma(a) = a$. Assume L has characteristic $p > 0$. Given $a \in L$, there exists $n \geqslant 0$ such that $a^{p^n} \in K$. Then $\sigma(a)^{p^n} = \sigma(a^{p^n}) = a^{p^n}$. Thus $(\sigma(a) - a)^{p^n} = 0$, so $\sigma(a) = a$. $\square$

6.2 Completion of normal extensions

Let L/K be an (algebraic) normal extension. Suppose we have an absolute value $|\cdot|$ on K, for which K is complete. We have seen that there is a unique extenion of $|\cdot|$ to L. If $[L\colon K] < \infty$, then $(L, |\cdot|)$ is complete. If not, let $\widehat{L}$ denote its completion.

6.2.1 Lemma. *The extension $\widehat{L}/K$ is normal, and $L \hookrightarrow \widehat{L}$ induces an isomorphism $\mathrm{Aut}(\widehat{L}/K) \xrightarrow{\sim} \mathrm{Aut}(L/K)$.*

Proof. The archimedean case can be checked directly, so we suppose K (and hence L) is non-archimedean. Note that any $\sigma \in \mathrm{Aut}(L/K)$ is an isometry with respect to $|\cdot|$. This follows from the norm-formula:

$$|x| = \left|\mathrm{N}_{K(x)/K}(x)\right|^{[K(x)\colon K]^{-1}},$$

which is clearly $\mathrm{Aut}(L/K)$-invariant. Alternatively, use the fact that for any $\sigma \in \mathrm{Aut}(L/K)$, the pullback $\sigma^*|\cdot|$ is also an extension of $|\cdot|$ to L, so uniqueness tells us that $\sigma^*|\cdot| = |\cdot|$, i.e. $|\cdot|$ is σ-invariant. Thus any $\sigma \in \mathrm{Aut}(L/K)$ extends by continuity to an element of $\mathrm{Aut}(\widehat{L}/K)$. $\square$

6.3 Action of $\mathrm{Aut}(\widehat{L}/K)$ on Berkovich spectra

Let L/K be a normal extension of complete non-archimedean fields. Let A be a Banach K-algebra. Write $A_L = A\widehat{\otimes}_K L$; this is a Banach L-algebra. Let $G = \mathrm{Aut}(L/K)$, $X = \mathscr{M}(A)$ and $X_L = \mathscr{M}(A_L)$. We want to relate X to X_L. The group G acts on A_L and hence, by functoriality X_L in the following way. Let $\sigma \in G$. Then $\sigma\colon L \to L$ induces a K-bilinear map $A \times L \to A_L$, namely $(f, a) \mapsto f\widehat{\otimes}\sigma(a)$, which is bounded because

$$|f\widehat{\otimes}\sigma(a)| \leqslant \|f\||\sigma(a)| = \|f\||a|.$$

The universal property of tensor products gives us a map $\sigma\colon A_L \to A_L$ extending $f\widehat{\otimes}a \mapsto f\widehat{\otimes}\sigma(a)$. This is an isometry. By functoriality, we get a continuous map $\sigma^*\colon X_L \to X_L$; it makes the following diagram commute:

$$\begin{tikzcd} X_L \arrow[r, "\sigma^*"] \arrow[dr] & X_L \arrow[d] \\ & X. \end{tikzcd}$$

This gives us a (right-) action of G on X_L over X. Give the quotient set X_L/G the quotient topology, i.e. the strongest topology making the quotient map $X_L \twoheadrightarrow X_L/G$ continuous. Since X_L is compact, the quotient X_L/G is quasi-compact. Since the action of G respects the map $X_L \to X$, we have a continuous map $X_L/G \to X_K$.

6.3.1 Theorem. *Let L/K be a normal extension, $X = \mathscr{M}(A)$ and X_L as above. Then the natural map $X_L/G \to X_K$ is a homeomorphism.*

6.3.2 Theorem. *Let A be a commutative Banach K-ring, L/K a normal extension.*

1. *If $[L\colon K] < \infty$, then $X_L/G \simeq X_K$, i.e.:*
   (a) *If L/K is separable, then $X_L/\operatorname{Gal}(L/K) \simeq X_K$.*
   (b) *If L/K is purely inseparable, then $X_L = X_K$.*
   (c) *Let S/K be the largest separable subextension of L/K. Then $X_L \simeq X_S$ and $X_S/\operatorname{Gal}(S/K) \simeq X_K$.*
2. *If $[L\colon K] = \infty$, then $\mathscr{M}(A\widehat{\otimes}_K L) = \mathscr{M}(A\widehat{\otimes}_K \widehat{L})$, and $\mathscr{M}(A\widehat{\otimes}_K \widehat{L})/\operatorname{Gal}(S/K) = \mathscr{M}(A)$, where S is defined as above. In particular,*

$$\mathscr{M}\left(A\widehat{\otimes}_K \widehat{\overline{K}}\right) = \mathscr{M}(A)/\operatorname{Gal}(K^{\mathrm{sep}}/K).$$

Proof. Essentially, it all comes down to proving 1(a). The archimedean case is $\mathbf{C}/\mathbf{R}$. First, we may assume that A is a field, which comes from examining the map $X_L \to X_K$ fiberwise. Use the primitive element theorem to write $L = K(\alpha) = K[x]/p$, where $p \in K[x]$ is the minimal polynomial of α. Then $A \otimes_K L = A[x]/p$, which is a product of fields. Use [Ber90, 1.2.3].

To prove 1(b), note that for all $f \in A\widehat{\otimes}_K L$, there exists $n \geqslant 0$ such that $f^{p^n} \in A$. Thus, any $|\cdot|_x \in \mathscr{M}(A)$ has a unique extension of $|\cdot|$ to $\mathscr{M}(A\widehat{\otimes}_K L)$, given by

$$|f|_{A\widehat{\otimes}_K L} = \left|f^{p^n}\right|_x^{1/p^n}.$$

Part 1(c) follows from (a) and (b). Part 2 follows from part 1, and the profinite description of $\operatorname{Gal}(L/K)$. □

In light of this theorem, we will usually assume our base field K is complete and algebraically closed, e.g. $\mathbf{C}_p$. If K is an arbitrary non-archimedean field, then $\widehat{\overline{K}}$ is such a field. Sometimes this field is denoted $\mathbf{C}_K$.

If L/K is a Galois extension with automorphism group G, then the action of G on X_L respects not only the topology, but also more refined data like the "type" of points.

6.4 Reduction / specialization map

Let $(A, |\cdot|)$ be a commutative Banach algebra, where $|\cdot|$ is non-archimedean. Define

$$A^\circ = \{f \in A\colon \rho(f) \leqslant 1\}$$
$$A^{\circ\circ} = \{f \in A\colon \rho(f) < 1\}.$$

By Theorem 5.8.5, A° is a ring and $A^{\circ\circ}$ is an ideal in A°. So $A^\natural = A^\circ/A^{\circ\circ}$ is a ring (not necessarily a field, as $A^{\circ\circ}$ may not be maximal). We will define a *reduction map* $\mathscr{M}(A) \to \operatorname{Spec}(A^\natural)$.

6.4.1 Lemma. *Let $\varphi\colon A \to B$ be a bounded homomorphism of Banach rings. Then φ induces maps*

$$\varphi^\circ\colon A^\circ \to B^\circ$$
$$\varphi^{\circ\circ}\colon A^{\circ\circ} \to B^{\circ\circ}$$
$$\varphi^\natural\colon A^\natural \to B^\natural.$$

Proof. Clearly, it suffices to prove the first two parts. Better, it suffices to show that $\rho(\varphi(f)) \leqslant \rho(f)$ for all $f \in A$. We know that there exists $C > 0$ such that $\|\varphi(f)\|_B \leqslant C\|f\|_A$. The rest is an easy computation:

$$\begin{aligned}
\rho(\varphi(f)) &= \lim_{n\to\infty} \|\varphi(f)^n\|^{1/n} \\
&= \lim_{n\to\infty} \|\varphi(f^n)\|_B^{1/n} \\
&\leqslant \lim_{n\to\infty} \left(C\|f^n\|_A\right)^{1/n} \\
&= \lim_{n\to\infty} \|f^n\|_A^{1/n} = \rho(f).
\end{aligned}$$

□

Let $x \in \mathscr{M}(A)$. We have an associated bounded homomorphism $\chi_x\colon A \to \mathscr{H}(x)$. By the lemma, we get a homomorphism $\chi_x^\natural\colon A^\natural \to \mathscr{H}(x)^\natural$. The field $\mathscr{H}(x)^\natural$ is often written $\widetilde{\mathscr{H}(x)}$ and called the *double residue field* of $\mathscr{M}(A)$ at x. The ideal $\ker(\chi_x^\natural) \subset A^\natural$ is prime, so we have defined a map

$$\operatorname{red}\colon \mathscr{M}(A) \to \operatorname{Spec}(A^\natural).$$

If $\|\cdot\| = |\cdot|_0$ is the trivial norm on A, then

$$\rho(f) = \begin{cases} 1 & f \text{ is not nilpotent} \\ 0 & f \text{ is nilpotent} \end{cases}$$

So $\ker(\rho)$ is the nilradical of A. Thus $A^\circ = A$, $A^{\circ\circ} = \operatorname{rad}(0)$, and $A^\natural = A/\operatorname{rad}(0)$. In particular, if A is reduced, then $A^\natural = A$.

For the moment, assume A is reduced, and let $\|\cdot\|_A = |\cdot|_0$. We have a *reduction map* $\operatorname{red}\colon \mathscr{M}(A) \to \operatorname{Spec}(A)$, but this is *not* the same as the "kernel map" $\ker\colon \mathscr{M}(A) \to \operatorname{Spec}(A)$. Indeed,

$$\operatorname{red}(x) = \{f \in A\colon |f(x)| < 1\} \supset \{f \in A\colon |f(x)| = 0\} = \ker(x).$$

If K is a non-archimedean field and A is a Banach K-algebra, then $A^\natural$ is a $K^\natural$-algebra. So $\operatorname{Spec}(A^\natural)$ is a scheme over the residue field $K^\natural$. For example, if $K = \mathbf{Q}_p$, then $K^\natural = \mathbf{F}_p$ and the reduction map $\mathscr{M}(A) \to \operatorname{Spec}(A^\natural)_{/\mathbf{F}_p}$ takes values in a scheme over a finite field. This is why we call the map *reduction* or *specialization*. If $x \in \mathscr{M}(A)$, we have a point $x^\natural = \operatorname{red}(x) = \mathfrak{p}_{x^\natural} \in \operatorname{Spec}(A^\natural)$. The algebraic residue field $k(x^\natural)$, namely $(A^\natural/\mathfrak{p}_{x^\natural})_{(0)}$, is canonically a subfield of $\mathscr{H}(x)^\natural$.

6.4.2 Theorem. *Let A be a noetherian Banach ring. Then the reduction map* $\operatorname{red}\colon \mathscr{M}(A) \to \operatorname{Spec}(A^\natural)$ *is anticontinuous.*

Proof. Recall that a map $f\colon X \to Y$ of topological spaces is *anticontinuous* if $f^{-1}(\text{open}) = \text{closed}$ and $f^{-1}(\text{closed}) = \text{open}$.

Let $\mathfrak{a} \subset A^\natural$ an ideal. Recall that the induced closed set $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(A^\natural)\colon \mathfrak{a} \in \mathfrak{p}\}$. Such closed sets define a topology on $\operatorname{Spec}(A^\natural)$. It turns out that

$$\operatorname{red}^{-1}(V(\mathfrak{a})) = \{x \in \mathscr{M}(A)\colon |f(x)| < 1 \text{ for all } f + A^{\circ\circ} \in \mathfrak{a}\}.$$

If A is noetherian, the ideal $\mathfrak{a} + A^{\circ\circ}$ has a finite generating set $\langle f_1, \dots, f_n\rangle$, and

$$\operatorname{red}^{-1}(V(\mathfrak{a})) = \bigcap_i \{x \in \mathscr{M}(A)\colon |f_i(x)| < 1\}.$$

As a finite intersection of open sets, this is open. □

If A is a K-algebra, one could define the reduction map using the "valuative criterion of properness" applied to $\operatorname{Spec}(\mathscr{H}(x)) \to \mathfrak{X}$, where $\mathfrak{X}_{/K^\circ}$ is a proper flat model, assuming such a model exists.

6.5 Ring-theoretic properties of Tate algebras

By a *Tate algebra* over a non-archimedean field K, we mean a ring of the form

$$B_{\boldsymbol{r}} = K\{\boldsymbol{r}^{-1}\boldsymbol{T}\} = \left\{\sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}} \boldsymbol{T}^{\boldsymbol{\nu}}\colon |a_{\boldsymbol{\nu}}|\boldsymbol{r}^{\boldsymbol{\nu}} \to 0\right\}.$$

Here, $\boldsymbol{r} = (r_1, \dots, r_n)$ is a tuple of real numbers, and we write $\boldsymbol{T}^{\boldsymbol{\nu}} = T_1^{\nu_1} \cdots T_n^{\nu_n}$ for any multi-index $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n)$. Clearly, $K\{\boldsymbol{r}^{-1}\boldsymbol{T}\}$ is a commutative ring, and a proof similar to that of Theorem 7.3.1 tells us that

$$|f| = \max_{\boldsymbol{\nu}} |a_{\boldsymbol{\nu}}|\boldsymbol{r}^{\boldsymbol{\nu}}$$

is a multiplicative norm equal to the spectral norm. If $|\cdot|$ on K is trivial and the $r_i \geqslant 1$, then $B_{\boldsymbol{r}} = K[T_1, \dots, T_n]$. If $|\cdot|$ is trivial on K and all $r_i < 1$, then $B_{\boldsymbol{r}} = K[\![T_1, \dots, T_n]\!]$, the ring of formal power series. Just as above, there is a general "we may assume $\boldsymbol{r} = 1$" principle.

6.5.1 Theorem (Maximum modulus principle)**.** *Let K be a complete non-archimedean field, $f \in K\{T_1, \dots, T_n\}$. Then for any $x = (x_1 \dots, x_n) \in K^n$ with $|x_i| \leqslant 1$,*

$$|f(x)| \leqslant |f|,$$

and moreover $|f| = |f(x)|$ for some x as above.

Proof. Recall that for $f = \sum a_{\boldsymbol{\nu}} T^{\boldsymbol{\nu}}$, the Gauss norm $|f| = \max\{|a_{\boldsymbol{\nu}}|\}$. The fact that $\rho(f) = \max_{x \in \mathscr{M}(A)} |f(x)|$ is a vast generalization of this theorem. □

There is a non-archimedean Weierstrass theory. This theory is essential in understanding why convergent power series act very like polynomials. For this theory, we assume $\boldsymbol{r} = \boldsymbol{1}$. The books [BGR84, Bos14] are good references. Put $B = K\{T_1, \dots, T_n\}$; here $\rho(f) = \max |a_{\boldsymbol{\nu}}|$.

6.5.2 Definition. For $f \in K\{T_1, \ldots, T_n\}$, write $f = \sum_{i \geqslant 0} g_i T_n^i$, where the $g_i \in K\{T_1, \ldots, T_{n-1}\}$. We say that f is (T_n, s)-*distinguished* if

1. $g_s \in K\{T_1, \ldots, T_{n-1}\}^\times$,
2. $|g_s| = |f|$ and $|g_s| > |g_i|$ for $i > s$.

6.5.3 Theorem. *Let $f, g \in K\{T_1, \ldots, T_n\}$ with g (T_n, s)-distinguished. Then there exists a unique $q \in K\{T_1, \ldots, T_n\}$ and $r \in K\{T_1, \ldots, T_{n-1}\}[T_n]$ with $\deg_{T_n}(r) < s$ such that*

$$f = gq + r.$$

Moreover, $|f| = \max\{|g||q|, |r|\}$.

6.5.4 Theorem (Weierstrass preparation)**.** *Let $f \in K\{T_1, \ldots, T_n\}$ be (T_n, s)-distinguished. Then there exists a unique monic $\omega \in K\{T_1, \ldots, T_{n-1}\}[T_n]$ with $\deg_{T_n}(\omega) = s$ and a unique $e \in K\{T_1, \ldots, T_n\}^\times$ such that $f = \omega e$. Moreover, $|\omega| = 1$, and ω is (T_n, s)-distinguished.*

In complex analysis, if $f(z)$ is meromorphic near zero and $f \neq 0$, it can be written as $z^k h(z)$, where $k \in \mathbf{Z}$ is the order of vanishing of f at zero, and $h(0) \neq 0$. Weierstrass generalized this to functions of several variables. Complex-analytic Weierstrass theory is essential in proving relating analytic and algebraic geometry (i.e. in proving GAGA-type theorems).

There is also a Noether normalization for Tate algebras. Let $\mathfrak{a} \subset K\{T_1, \ldots, T_n\}$ be a closed proper ideal. Then there is an injective K-algebra homomorphism (for some d) $K\{s_1, \ldots, s_d\} \hookrightarrow K\{T_1, \ldots, T_n\}$ such that the composite $K\{s_1, \ldots, s_d\} \to K\{T_1, \ldots, T_n\}/\mathfrak{a}$ is (module-)finite. In fact, $d = \dim(K\{t_1, \ldots, t_d\}/\mathfrak{a})$, the (ring-theoretic) Krull dimension.

6.5.5 Theorem. *Let K be a complete non-archimedean field. Then $K\{T_1, \ldots, T_n\}$ is a noetherian unique factorization domain, which is a Jacobson ring.*

Recall that a ring A is *Jacobson* if for all ideals $\mathfrak{a} \subset A$, the radical $\sqrt{\mathfrak{a}}$ is the intersection of all *maximal* ideals containing $\mathfrak{a}$. If we replace K with K°, the rings may not be noetherian. For example, $\mathbf{C}_p^\circ$ is already non-noetherian.

7 Examples of Berkovich spaces

7.1 Ostrowski's theorem and $\mathscr{M}(\mathbf{Z})$

Here, our Banach ring is $(A, \|\cdot\|) = (\mathbf{Z}, |\cdot|_\infty)$. Clearly, this is complete. We'd like to understand $\mathscr{M}(\mathbf{Z})$. This is the set of bounded multiplicative seminorms on $\mathbf{Z}$. First, observe that $|\cdot|_\infty$ is maximal among seminorms on $\mathbf{Z}$, in the sense that $|n| \leqslant |n|_\infty$ for all seminorms $|\cdot|$ and $n \in \mathbf{Z}$. Indeed,

$$|n| = |1 + \cdots + 1| \leqslant |n|_\infty \cdot |1| \leqslant |n|_\infty.$$

So, as a set, $\mathscr{M}(\mathbf{Z})$ consists of *all* multiplicative seminorms on $\mathbf{Z}$. We already know all multiplicative norms on $\mathbf{Z}$ by Ostrowski, so it seems like we should have a pretty good handle on $\mathscr{M}(\mathbf{Z})$. If $|\cdot|_x \in \mathscr{M}(\mathbf{Z})$ has a non-trivial kernel, this kernel $\mathfrak{p}_x = \ker(|\cdot|_x)$ is a prime ideal in $\mathbf{Z}$, so $\mathfrak{p}_x = (p)$ for some rational prime p. The induced norm on $\mathbf{F}_p = \mathbf{Z}/(p)$ must be the trivial norm.

7.1.1 Theorem. *Any absolute value on a finite field is trivial.*

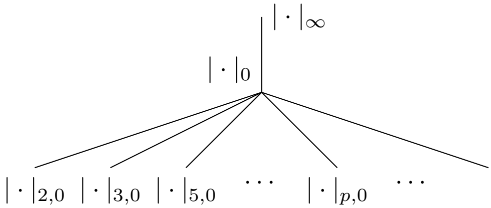
Proof. Let k be a finite field. We need to show that if $x \in k \setminus 0$, then $|x| = 1$. Let $q = \#k$; then $x^q = x$, so $|x|^q = |x|$, which forces $|x| = 1$. $\square$

So $\mathscr{M}(\mathbf{Z})$ contains the following types of points:

- $|\cdot|_0$, the trivial norm.
- $|\cdot|_{\infty,\varepsilon} := |\cdot|_\infty^\varepsilon$ for any $0 < \varepsilon \leqslant 1$.
- $|\cdot|_{p,\varepsilon}$, in which $|p|_{p,\varepsilon} = \varepsilon$, for all primes p and $0 < \varepsilon < 1$.
- $|\cdot|_{p,0}$ given by $|n|_{p,0} = \begin{cases} 1 & p \nmid n \\ 0 & p \mid n \end{cases}$.

In Berkovich's notation, we could write

$$\lim_{\varepsilon \to 0} |\cdot|_{\infty,\varepsilon} = |\cdot|_0 = \lim_{\varepsilon \to 1} |\cdot|_{p,\varepsilon}.$$



This picture is "true," but you need to be careful: the Berkovich topology is *not* the naive (metric) topology. Recall that $\mathscr{M}(\mathbf{Z})$ should carry the weakest topology making all maps $|\cdot| \mapsto |n|$ continuous. Since the norms in question are multiplicative, it suffices to ensure that the maps $|\cdot| \mapsto |p|$ are continuous for all primes p. In other words, the topology is generated by the sets

$$U_{p,t}^+ = \{|\cdot| : |p| > t\}$$
$$U_{p,t}^- = \{|\cdot| : |p| < t\}.$$

Define:

$$I_p = \{|\cdot|_{p,\varepsilon} : 0 \leqslant \varepsilon \leqslant 1\}$$
$$I_\infty = \{|\cdot|_{\infty,\varepsilon} : 0 \leqslant \varepsilon \leqslant 1\}.$$

7.1.2 Proposition.

1. *Any open set of* $\mathscr{M}(\mathbf{Z})$ *containing* $|\cdot|_0$ *contains all but finitely many* I_v *for* $v \in \{\textit{primes}\} \cup \{\infty\}$.
2. *The maps* $[0,1] \to \mathscr{M}(\mathbf{Z})$, $\varepsilon \mapsto |\cdot|_{v,\varepsilon}$ *are homeomorphisms onto the* I_v.

Proof. 1. Consider $U^+_{p,\varepsilon}$ for $t < 1$. Recall that if $q \neq p$ is a prime, then $|q|_{p,\varepsilon} = 1$. So $U^+_{p,t} \supset \bigcup_{v \neq p} I_v$. The similar statement holds for $U^-_{p,t}$. This (roughly) implies the first claim.

2. It suffices to show that the maps $f_v \colon [0,1] \to I_v$ are continuous, as bijectivity is easy. Let $U^+_{q,\varepsilon}(p) = U^+_{q,\varepsilon} \cap I_p$. Then

$$U^+_{q,\varepsilon}(p) = \begin{cases} I_p & t < 1 \text{ and } q \neq p \\ \varnothing & t > 1 \text{ and } q \neq p \\ \{|\cdot|_{p,\varepsilon} \colon \varepsilon > t\} & q = p \end{cases}$$

So $f_p^{-1}(U^+_{q,\varepsilon}(p))$ will be either $[0,1]$, $\varnothing$, or $\{\varepsilon > t\}$. Similarly one handles $U^-_{q,\varepsilon}(p)$, and the infinite places. $\square$

Recall the "kernel map" $\ker \colon \mathscr{M}(\mathbf{Z}) \to \operatorname{Spec}(\mathbf{Z})$. It sends everything to the generic point (0) *except* the $|\cdot|_{p,0}$, which map to $(p) \in \operatorname{Spec}(\mathbf{Z})$.

The whole picture generalizes to rings of integers in number fields.

Let A be a Dedekind domain (integrally closed domain of dimension one) and $|\cdot|_0$ the trivial norm on A. Recall that the map $\ker \colon \mathscr{M}(A) \to \operatorname{Spec}(A)$, $|\cdot| \mapsto \ker(|\cdot|)$, is continuous. We saw that there is a canonical section $\sigma \colon \operatorname{Spec}(A) \to \mathscr{M}(A)$, which sends $\mathfrak{p}$ to the pullback of the trivial norm on $A/\mathfrak{p}$. However, σ is not continuous, because $\mathscr{M}(A)$ is Hausdorff and $\operatorname{Spec}(A)$ is not.

7.2 Dedekind domains with trivial norm

7.2.1 Definition. A commutative ring A is called a *Dedekind domain* if it is an integrally closed, noetherian integral domain with Krull dimension 1.

This definition is "nice" in the sense that it is usually easy to check if a given ring is Dedekind, but it is not immediately clear why we should care about Dedekind rings.

7.2.2 Theorem. *A noetherian integral domain A is Dedekind if and only if any nonzero ideal $\mathfrak{a} \subset A$ factors uniquely as $\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_r$, where the $\mathfrak{p}_i$ are prime ideals.*

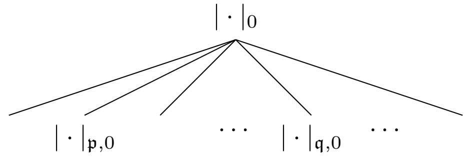
7.2.3 Example. If F is a number field, then the ring of integers O_F is Dedekind. For example, we could have $\mathbf{Z}$, $\mathbf{Z}[i]$, $\mathbf{Z}\left[\frac{\sqrt{-3}+1}{2}\right]$, etc.

7.2.4 Example. A curve C is nonsingular if and only if for all $U \subset C$ open, the ring $\mathrm{H}^0(U, \mathscr{O})$ is Dedekind. This is because one-dimensional varieties are normal if and only if they are smooth.

So proper algebraic curves are a kind of "geometrization" of the notion of a Dedekind domain.

7.2.5 Lemma. *A noetherian domain A is Dedekind if and only if $A_{\mathfrak{p}}$ is a discrete valuation ring for all nonzero primes $\mathfrak{p} \subset A$.*

Let A be a Dedekind domain with trivial norm $|\cdot|_0$. We'd like to understand $\mathscr{M}(A)$.



Fix $x \in \mathscr{M}(A)$, and let $\mathfrak{p}_x = \ker(x) = \{f \in A \colon |f(x)| = 0\}$. Let $\mathfrak{q}_x = \operatorname{red}(x) = \{f \in A \colon |f(x)| < 1\}$. Clearly $\mathfrak{p}_x \subset \mathfrak{q}_x$. If $\mathfrak{p}_x \neq 0$, then $\dim(A) = 1$ forces $\mathfrak{p}_x = \mathfrak{q}_x$. If $\mathfrak{p}_x = 0$, there are two possibilities: either $\mathfrak{q}_x = 0$ or $\mathfrak{q}_x \neq 0$.

Case 1: $\mathfrak{p}_x \neq 0$. Then

$$|f|_x = \begin{cases} 0 & f \in \mathfrak{p}_x \\ 1 & f \notin \mathfrak{p}_x \end{cases}$$

because $|f(x)| \leqslant |f|_0 = 1$ and $|f(x)| < 1$ if and only if $|f(x)| = 0$, if and only if $f \in \mathfrak{p}_x$. So $|\cdot|_x$ is the image of $\mathfrak{p}_x = \mathfrak{p}$ under the canonical section $\operatorname{Spec}(A) \to \mathscr{M}(A)$. Call this norm $|\cdot|_{\mathfrak{p},0}$.

Case 2: $0 = \mathfrak{p}_x = \mathfrak{q}_x$. One handles this exactly as in Case 1. We get that $|\cdot|_x = |\cdot|_0$.

Case 3: $0 = \mathfrak{p}_x \subsetneq \mathfrak{q}_x$. In the map $A \to A_{\mathfrak{q}_x}$, the target is a discrete valuation ring. The norm $|\cdot|_x$ extends to a multiplicative norm on $A_{\mathfrak{q}_x}$. Fix a generator π for the maximal ideal $\mathfrak{m} = (\mathfrak{q}_x)_{\mathfrak{q}_x}$ of $A_{\mathfrak{q}_x}$. The natural valuation on $A_{\mathfrak{q}_x}$ is

$$\operatorname{val}'(\varepsilon \pi^i) = i$$

whenever $\varepsilon \in A_{\mathfrak{q}_x}^\times$. If val_t is some other valuation with $\operatorname{val}(\pi) = t$, then $\operatorname{val}_t = t \operatorname{val}'$. So for $0 < t < \infty$, we have a multiplicative seminorm on A, restricted from one on $A_{\mathfrak{q}_x}$, given by

$$|\cdot|_{\mathfrak{q},t^{-1}} = e^{-\operatorname{val}_t}.$$

As $t \to 0^+$, $|\cdot|_{\mathfrak{q},t^{-1}} \to |\cdot|_0$, and as $t \to \infty$, $|\cdot|_{\mathfrak{q},t^{-1}} \to |\cdot|_{\mathfrak{q},0}$.

7.3 Berkovich closed disk

We consider convergent power series in one variable (i.e. analytic functions). The spectrum of this algebra will be the "Berkovich closed disk."

Fix a field K complete with respect to an absolute value $|\cdot|$. We do not assume K is non-archimedean, so we could have $K = \mathbf{R}$. Let $r > 0$, and define

$$A = K\left\langle r^{-1} T \right\rangle = \left\{ \sum_{i \geqslant 0} a_i T^i \colon \|f\| = \sum |a_i| r^i < \infty \right\}.$$

We want to think of $\mathscr{M}(A)$ as the closed analytic disk with radius r. The norm $\|\cdot\|$ is not even power-multiplicative (i.e., A is not a uniform algebra), which makes certain computations very annoying. What is A^{u}? In other words, what is the spectral radius norm $\rho(\cdot)$ on A, and the completion of A with respect to ρ? Since $\mathscr{M}(A) = \mathscr{M}(A^{\mathrm{u}})$, we still have the same analytic space.

7.3.1 Theorem. *Let* $f = \sum_{i \geqslant 0} a_i T^i \in K\left\langle r^{-1} T \right\rangle$.

1. *If K is non-archimedean, then*

$$\rho(f) = \max_{i \geqslant 0} |a_i| r^i$$
$$A^{\mathrm{u}} = K\{r^{-1}T\} = \left\{ f = \sum a_i T^i \colon \lim_{i \to \infty} |a_i| r^i = 0 \right\}.$$

2. *If $K = \mathbf{C}$, put $D_r = \{z \in \mathbf{C} \colon |z| \leqslant r\}$, $D_r^- = \{z \in \mathbf{C} \colon |z| < r\}$.*

$$\rho(f) = \max_{z \in D_r} |f(z)|$$
$$A^{\mathrm{u}} = \{f \in \mathscr{C}(D, \mathbf{C}) \colon f|_{D^-} \text{ is analytic}\}.$$

3. *If $K = \mathbf{R}$, then ρ is the restriction of the above ρ to real-analytic functions, and A^{u} is the subalgebra of $(A_{\mathbf{C}})^{\mathrm{u}}$ with real Taylor coefficients.*

4. *If K is non-archimedean, then $\rho(\cdot)$ multiplicative norm.*

Proof. We will ignore the archimedean case, as this essentially follows from the discussion on Gelfand's theory. So assume K is non-archimedean.

1. Let $|f| = \max |a_i| r^i$. First we show $\rho(f) \geqslant |f|$. Clearly $|f| \leqslant \|f\|$. We show later that $|\cdot|$ is power-multiplicative. Assuming this, we have

$$\rho(f) = \lim_{n \to \infty} \|f^n\|^{1/n} \geqslant \lim_{n \to \infty} |f^n|^{1/n} = |f|.$$

Now we show $\rho(f) \leqslant |f|$. First, assume $f \in K\left\langle r^{-1}T \right\rangle$ is a polynomial. Then $|f| = \max |a_i| r^i \geqslant |a_j| r^j$ for all $j \geqslant 0$. In other words, $|a_j| \leqslant |f| r^{-j}$. We now compute:

$$\|f\| = \sum_{j \leqslant \deg f} |a_j| r^j \leqslant \sum_{j \leqslant \deg f} |f| r^{-j} r^j = \sum_{j \leqslant \deg f} |f| = (1 + \deg f)|f|.$$

So in general,

$$\|f^n\|^{1/n} \leqslant (1 + n \deg f)^{1/n} |f^n|^{1/n},$$

whence $\rho(f) \leqslant |f|$. Polynomials are dense in $(K\left\langle r^{-1}T \right\rangle, \|\cdot\|)$. So for $f \in K\left\langle r^{-1}T \right\rangle$, there exists $\{p_n\} \subset K[T]$ such that $p_n \to f$. If $|f| < \rho(f)$, then since $|p_n| \geqslant \rho(p_n)$ for all n, we get a contradiction.

4. We prove that $|\cdot|$ is multiplicative. That $|fg| \leqslant |f||g|$ is clear from the definition. If the absolute value on K is trivial, the result is easy, so we assume it's not trivial. To show equality, we first assume $r = 1$. Put $B = K\{T\}$; then $B^\circ = K^\circ\{T\}$ and $B^{\circ\circ} = K^{\circ\circ}\{T\}$, so $B^\natural = K^\natural[T]$, which incidentally is not a field. Write $\pi \colon B \to B^\natural$ for the projection. Note that $|f| < 1$ if and only if $f^\natural = 0$.

We are trying to show that for $f, g \in K\{T\}$, $|fg| = |f||g|$. Clearly $|fg| \leqslant |f||g|$. If f or g are constant, there is nothing to prove. So assume $|f| = |g| = 1$. Then $f, g, fg \in B^\circ$, and $f^\natural, g^\natural \neq 0$. Since $B^\natural = K^\natural[T]$ is an integral domain, $(fg)^\natural \neq 0$, so $|fg| = 1$. For general f, g, since $|\cdot|$ is nontrivial, write $f = cf'$, $g = dg'$, where $c, d \in K$, and $|f'| = |g'| = 1$. We now compute:

$$|fg| = |cdf'g'| = |cd||f'g'| = |cd| = |f||g|.$$

We already have $A = K\left\langle r^{-1}T \right\rangle \subset B = K\{r^{-1}T\}$. To show that $B = A^{\mathrm{u}}$, we show that B is complete and A is dense in B. Density is easy ($K[T]$ is dense in both rings). Completeness is equivalent to the convergence of absolutely convergent series. Consider a series $\sum f_i$, where $f_i \in B$ are written $f_i = \sum_{j \geqslant 0} a_{ij} T^j$. Assume $|f_i| \to 0$; we will show that $\sum f_i$ converges. We know that $|a_{ij}| \leqslant |f_i|$ for all i, j. So $\lim_{i \to \infty} |a_{ij}| = 0$. Since K is complete non-archimedean, each $\sum_i a_{ij}$ is convergent; let b_j be the sum. One can easily check that $\sum_i f_i$ converges to $f = \sum_{j \geqslant 0} b_j T^j$. $\square$

By part 4, the spectral norm $\rho(\cdot) \in \mathscr{M}(A^{\mathrm{u}})$. One calls ρ the *Gauss norm* because the proof of its multiplicativity uses Gauss' lemma about the factorization of polynomials. In rigid-analytic geometry, $(K\{T\}, \rho)$ is called the *Tate algebra* in one variable. Since ρ is multiplicative, $K\{r^{-1}T\}$ is an integral domain. The proof of part 4 uses the same trick as is standard in proving Gauss' Lemma.

In the proof of Theorem 7.3.1, we assumed $r = 1$. The reasons for having this assumption are sometimes technical (as above), and sometimes historical, as early texts in rigid-analytic geometry typically worked with $r = 1$.

Consider $K\{r^{-1}T\}$. If $r \in |K^\times|$, then there exists $c \in K^\times$ such that $|c^{-1}| = r$, so we have a map $K\{T\} \to K\{r^{-1}T\}$, given by $T \mapsto cT$. This is a bounded K-algebra isomorphism. If $r \notin |K^\times|$, find a field $L \supset K$ such that $r \in |L^\times|$, base-change to L, and then use some kind of descent argument to "go back" to K.

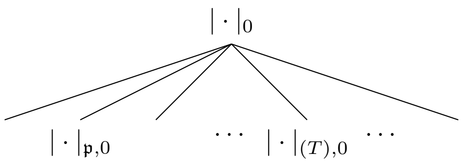
7.4 Constructing the open disk

Recall that $\mathscr{E}(0, r) = \mathscr{M}(K\{r^{-1}T\})$ is the Berkovich closed disk of radius r. If $0 < r < s$, then the inclusion $K\{s^{-1}T\} \hookrightarrow K\{r^{-1}T\}$ is bounded and injective. By functoriality, we have a continuous map $\mathscr{E}(0, r) \to \mathscr{E}(0, s)$, which induces a homeomorphism onto its image. (By Theorem 5.5.1, this follows from the fact that $K[T]$ is dense in both rings.)

Assume K is complete. Put $B = K\{r^{-1}T\}$; it carries the absolute value $|\sum a_i T^i| = \max\{|a_i| r^i\}$. If $|\cdot|$ is trivial on K, there are three possibilities:

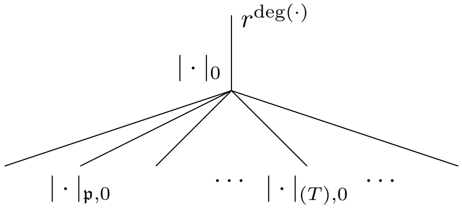
1. $r < 1$. Here $|f| = r^{\mathrm{val}_T(f)}$, and $B = K[\![T]\!]$. Let $|\cdot|_x \colon B \to \mathbf{R}$ be a bounded multiplicative seminorm. Let $t = |T|_x \geqslant 0$; then $|T|_x \leqslant |T| = r$. Note further that $|\cdot|_x$ is completely determined by $|T|_x$. This is because $|a_i T^i|_x = |T|_x^i = t^i$ if $a_i \neq 0$, and if $t \neq 0$, then since the t^i's are distinct, $|f| \max\{= t^{\mathrm{val}_T(f)}, |a_0|\}$ by the non-archimedean triangle inequality. So $\mathscr{E}(0, r) \simeq [0, r]$.

2. $r = 1$. Here $B = K[T]$ and $|f| = |f|_0 = \begin{cases} 1 & f \neq 0 \\ 0 & \text{else} \end{cases}$. This is a Dedekind domain with trivial valuation, so as we have already seen, its Berkovich spectrum looks like:

   where $\mathfrak{p} \subset K[T]$ ranges over the maximal ideals.

3. $r > 1$. We know that $\mathscr{E}(0,1) \hookrightarrow \mathscr{E}(0,r)$. In fact, $\mathscr{E}(0,1)$ consists of those seminorms with $|T|_x \leqslant 1$. Assume $|\cdot|_x \in \mathscr{E}(0,r) \smallsetminus \mathscr{E}(0,1)$. Then $r \geqslant |T|_x = t > 1$. As above, we get "$t$ determines $|\cdot|_x$," by $|f| = t^{\deg f}$. So $\mathscr{E}(0,r)$ looks like



If $|\cdot|$ is non-trivial, things are more complicated. For simplicity, we assume $K = \overline{K}$. It turns out that our classification of points is compatible with the Galois action, so we can quotient out by the Galois action to recover the picture over K. We want to find all possible $|\cdot|_x \in \mathscr{E}(0,r)$ when $r > 0$. There are some obvious points. First, recall that

$$\mathrm{D}(a,r) = \{x \in K \colon |x-a| \leqslant r\}$$
$$\mathrm{D}^-(a,r) = \{x \in K \colon |x-a| < r\}.$$

Let $a \in \mathrm{D}(0,r)$; then $f(a)$ makes sense for $f \in K\{r^{-1}T\}$. Then $f \mapsto |f(a)|$ is an element of $\mathscr{E}(0,r)$. We have defined an embedding $\mathrm{D}(0,r) \hookrightarrow \mathscr{E}(0,r)$; write it as $a \mapsto |\cdot|_{a,0}$. More generally, if $\mathrm{D}(a,s) \subset \mathrm{D}(0,r)$, then

$$|f|_{a,s} = \sup_{x \in \mathrm{D}(a,s)} |f(x)|$$

is a seminorm. By the maximum modulus principle, it is bounded and multiplicative. Indeed, write $f = \sum_{i \geqslant 0} bi(T-a)^i$; then

$$\max_{x \in \mathrm{D}(a,s)} |f(x)| = \max_{i \geqslant 0} |b_i| s^i,$$

which we have already shown is multiplicative.

Having $s \in |K^\times|$ is not important in the definition of $|\cdot|_{a,s}$, but it does make a (huge) difference when we look at $\mathscr{H}(x)$.

Even more generally, let $\mathfrak{F} = \{\mathrm{D}(a_i, r_i)\}_{i \in I}$ be a nested family of closed disks. Define $|\cdot|_{\mathfrak{F}} \colon K\{r^{-1}T\} \to \mathbf{R}$ by

$$|f|_{\mathfrak{F}} = \inf_{i \in I} |f|_{a_i, r_i}.$$

If $r_i \to 0$ and $\bigcap \mathrm{D}(a_i, r_i) \neq \varnothing$, then we recover an absolute value of the form $|\cdot|_a$. If $r_i \to r$ and $\bigcap \mathrm{D}(a_i, r_i) \neq \varnothing$, we get a seminorm of the form $|\cdot|_{a,s}$. But if K is not spherically complete (e.g. $\mathbf{C}_p$), then it might be the case that $\bigcap \mathrm{D}(a_i, r_i) = \varnothing$. By passing to a spherical completion of K, we can avoid these "funny points."

We have four types of points:

- "Type 1" (or classical). $|\cdot|_a$.

- "Type 2." $|\cdot|_{a,s}$ for $s \in |K^\times|$.
- "Type 3." $|\cdot|_{a,s}$ for $s \notin |K^\times|$.
- "Type 4." $|\cdot|_{\mathfrak{F}}$ for $\bigcap \mathfrak{F} = \varnothing$.

7.4.1 Theorem. *Let K be a complete, algebraically closed, non-archimedean field. If $|\cdot|_x \in \mathscr{E}(0,r)$, then $|\cdot|_x = \lim |\cdot|_{a_i,r_i}$ for some sequence of nested disks (in K) $\{\mathrm{D}(a_1,r_1), \mathrm{D}(a_2,r_2), \dots\}$.*

Proof. Given $|\cdot|_x$, consider the collection $\mathfrak{F}_x = \{\mathrm{D}(a, |T-a|_x)\}_{a\in \mathrm{D}(0,r)}$. We claim that $\subset$ gives $\mathfrak{F}_x$ a total ordering. Let $a, b \in \mathrm{D}(0,r)$; we can assume $|T-a|_x \geqslant |T-b|_x$. We show $\mathrm{D}(b, |T-b|_x) \subset \mathrm{D}(a, |T-a|_x)$. We compute:

$$|a-b| = |a-b|_x = |(T-b)-(T-a)|_x \leqslant \max(|T-a|_x, |T-b|_x) \leqslant |T-a|_x,$$

whence $b \in \mathrm{D}(a, |T-a|_x)$. Balls in non-archimedean fields are either disjoint or one contains the other, so we obtain the claim. Let

$$s = \inf_{a\in \mathrm{D}(0,r)} |T-a|_x,$$

and let $\{a_i\} \subset \mathrm{D}(0,r)$ be such that $s_i = |T-a_i|_x \to s$. The following claim finishes the proof:

Claim: for any $a \in \mathrm{D}(0,r)$, $|T-a|_x = \lim_{i\to\infty} |T-a|_{a_i,s_i}$. The claim suffices because of the Weierstrass preparation theorem: any $f \in K\{r^{-1}T\}$ can be written as $f = we$, where $|e|_x = 1$ and $w \in K[T]$. So $|f| = |w|_x$. Since K is algebraically closed, $w = \prod(T-\lambda)$, so $|w|_x = \prod |T-\lambda|_x$. In other words, $|\cdot|_x$ is completely determined by $\{|T-a|_x\}_{a\in \mathrm{D}(0,r)}$.

Now we prove the claim. Fix $a \in \mathrm{D}(0,r)$. By definition $|T-a|_x \geqslant s$. Suppose $|T-a|_x = s$. Then $|T-a|_x \leqslant |T-a_i|_x$. As above, we get $a \in \mathrm{D}(a_i, s_i)$. Note that:

$$|T-a|_{a_i,s_i} = \sup_{x\in \mathrm{D}(a_i,s_i)} |x-a| = \sup_{x\in \mathrm{D}(a,s_i)} |x-a| = s_i,$$

so all we need to prove is $s_i \to s$. Duh! The case $|T-a|_x > s$ is similar. □

We've completely described $\mathscr{E}(0,r)$ as a set. Now, we'd like to understand it topologically. Start by recalling that for a Banach algebra A and $x \in \mathscr{M}(A)$, $\mathscr{H}(x) = (A/\widehat{\ker|\cdot|_x})_{(0)}$. The Abhyankar inequality tells us that if L/K is an extension of valued fields, then $\operatorname{tr.deg}(L^\natural/K^\natural) + \operatorname{rr}(\Gamma'/\Gamma) \leqslant \operatorname{tr.deg}(L/K)$.

7.5 Classification of points in $K\{r^{-1}T\}$

A point $x \in \mathscr{E}(0,r)$ is of *type 1* (*classical*) if it is of the form $|\cdot|_{a,0}$ for $a \in \mathrm{D}(0,r)$.

7.5.1 Lemma. *$x \in \mathscr{E}(0,r)$ is type 1 if and only if $\mathscr{H}(x) = K$. In this case, $|\mathscr{H}(x)^\times| = |K^\times|$ and $\mathscr{H}(x)^\natural = K^\natural$.*

A point $x \in \mathscr{E}(0,r)$ is of *type 2* (*rational*) if it is of the form $|\cdot|_{a,r}$ for $r \in |K^\times|$.

7.5.2 Lemma. *$x \in \mathscr{E}(0,r)$ is type 2 if and only if $\mathscr{H}(x) \simeq \widehat{K(T)}$. Moreover, $|\mathscr{H}(x)^\times| = |K^\times|$ and $\mathscr{H}(x)^\natural = K^\natural(T)$.*

Roughly, this is the "$1+0 \leqslant 1$" case of Abhyankar's inequality.

A point $x \in \mathscr{E}(0,r)$ is *type 3* (*irrational*) if it is of the form $|\cdot|_{a,r}$ for $r \notin |K^\times|$.

7.5.3 Lemma. *$x \in \mathscr{E}(r,0)$ is type 3 if and only if $\mathscr{H}(x) = \widehat{K(T)}$. Moreover, $|\mathscr{H}(x)^\times| = \langle |K^\times|, r\rangle$, and $\mathscr{H}(x)^\natural = K^\natural$.*

A point $x \in \mathscr{E}(0,r)$ is *type 4* (*pathological*) if it is a limit of the previous types.

7.5.4 Lemma. *$x \in \mathscr{E}(0,r)$ is type 4 if and only if $\mathscr{H}(x) = \widehat{K(T)}$. Moreover, $|\mathscr{H}(x)^\times| = |K^\times|$ and $\mathscr{H}(x)^\natural = K^\natural$.*

For A a Banach K-algebra, $x \in \mathscr{M}(A)$, write $F_x = (A/\ker|\cdot|_x)_{(0)}$. Note that $\mathscr{H}(x) = \widehat{F_x}$.

7.5.5 Theorem. *Let K be a complete, algebraically closed, non-archimedean field, $r > 0$. Let $\mathscr{E}(0,r) = \mathscr{M}(K\{r^{-1}T\})$, and fix $x \in \mathscr{E}(0,r)$. Then:*

1. *The point x is classical if and only if $\mathscr{H}(x) = K$ (and therefore $|\mathscr{H}(x)^\times| = |K^\times|$, $\mathscr{H}(x)^\natural = K^\natural$). Moreover, $|\cdot|_x$ is a seminorm which is not a norm.*
2. *If x is not classical, then $\operatorname{tr.deg}(F_x/K) = 1$. Moreover, $|\cdot|_x$ is a norm. Further:*
   (a) x is rational if and only if:
   $$|\mathscr{H}(x)^\times| = |K^\times|$$
   $$\mathscr{H}(x)^\natural = K^\natural(t)$$
   (b) x is irrational if and only if, for $|\cdot|_x = |\cdot|_{a,s}$:
   $$|\mathscr{H}(x)^\times| = \langle |K^\times|, s\rangle$$
   $$\mathscr{H}(x)^\natural = K^\natural$$
   (c) x is pathological if and only if:
   $$|\mathscr{H}(x)^\times| = |K^\times|$$
   $$\mathscr{H}(x)^\natural = K^\natural.$$

Proof. It suffices to prove $\Rightarrow$ in all the above statements, because the possibilities for $(\mathscr{H}(x), |\mathscr{H}(x)^\times|, \mathscr{H}(x)^\natural)$ are mutually exclusive. Since $\mathscr{H}(x) = \widehat{F_x}$ is an immediate extension, we may work with $F_x = K\{r^{-1}T\}/\ker|\cdot|_x$.

1. $|\cdot|_x = |\cdot|_{a,0}$. Clearly $|T-a|_x = |a-a| = 0$, so $T-a \in \ker|\cdot|_x$. Thus F_x/K is algebraic. But K is algebraically closed already, so $K = F_x$.

2. If x is not rational, then $|\cdot|_x = |\cdot|_{a,s}$, or a limit of such seminorms. Clearly $\operatorname{tr.deg}(F_x/K) \leqslant 1$, but no nonzero polynomial (or power series, by Weierstrass' theorem) can vanish identically on $\mathrm{D}(a,s)$. So $\operatorname{tr.deg}(F_x/K) = 1$; indeed $F_x = K\{r^{-1}T\}_{(0)}$.

Suppose x is rational or irrational, i.e. $x = |\cdot|_{a,s}$. Then for $g \in K\{r^{-1}\}$, we have
$$|g|_x = \sup_{x \in \mathrm{D}(a,s)} |g(x)| = |g(p)|,$$

for some $p \in \mathrm{D}(a,s)$. Moreover,

$$|T-a|_x = \sup_{x\in \mathrm{D}(a,s)} |x-a| = s,$$

so $|\mathscr{H}(x)^\times|$ is as claimed. Suppose $|\cdot|_x$ is rational. Then $s \in |K^\times|$, so $|c| = s$ for some $c \in K^\times$. Let $\varphi\colon K\{r^{-1}T\} \to F_x$ be the canonical map; we claim that $\varphi(\frac{T-a}{c}) \in F_x^\circ$, for $|\frac{T-a}{c}|_{a,s} = 1$. Thus $\operatorname{tr.deg}(F_x^\natural/K^\natural) \geqslant 1$. Let t be the image of $\varphi(\frac{T-a}{c})$ in $F_x^\natural$. Then t is transcendental over $K^\natural$ and $F_x^\natural = K^\natural(t)$.

This trick won't work if x is irrational. That $F_x^\natural = K^\natural$ follows from Abhyankar. Alternatively, check that there is no $f \in K\{r^{-1}T\}$ such that $|\varphi(f)|_x = 1$. Let $f = \sum b_i(T-a)^i$; then $|f|_x = \max\{|b_i|s^i\} \neq 1$, otherwise $s^i = |b_i|$. But $|K^\times|$ is divisible because K is algebraically closed, so this would imply $s \in |K^\times|$. ☐

This classification of points in $\mathscr{E}(0,r)$ generalizes to $\mathbf{A}_K^{1,\mathrm{an}}$ and, more generally, to analytification of curves.

Note that we have actually constructed $\mathscr{E}(a,s)$ (the Berkovich closed disk centered at a with radius s) for all $a \in K$. For, any $\mathrm{D}(a,s) \subset \mathrm{D}(0,r)$ for some r. Then $\mathscr{E}(a,s) \hookrightarrow \mathscr{E}(0,r)$, where $|\cdot|_x \in \mathscr{E}(a,s)$ if and only if $|T-|_x \leqslant s$. We can also construct the open disk $\mathscr{E}^-(a,s)$. It embeds into some $\mathscr{E}(0,r)$, and consists of those $|\cdot|_x$ for which $|T-a|_x < s$.

7.5.6 Example. $\mathscr{E}^-(0,1) = \mathscr{E}(0,1) \smallsetminus \{\text{Gauss point}\}$.

7.6 R-trees

We want to show that $\mathscr{E}(0,r)$, with the Berkovich topology, is homeomorphic an $\mathbf{R}$-tree with the weak topology.

7.6.1 Definition. An $\mathbf{R}$-*tree* is a metric space such that any two points are connected by a unique path.

The metric topology on an $\mathbf{R}$-tree is called the *strong topology.*

7.6.2 Definition. Let (X,d) be an $\mathbf{R}$-tree. The *weak (or observer) topology* on X is the topology generated by all sets of the form $\mathrm{B}_x^-(\boldsymbol{v})$. Here, $\mathrm{B}_x^-(\boldsymbol{v})$ consists of those $y \in X \smallsetminus x$ such that the unique path from x to y "uses direction $\boldsymbol{v}$."

Intuitively, $\mathrm{B}_x^-(\boldsymbol{v})$ consists of all things that an observer sitting at point x sees when looking in the direction $\boldsymbol{v}$. It is a nice exercise to check that an $\mathbf{R}$-tree, with the observer topology, is compact Hausdorff. For $x \in X$, a fundamental system of open neighborhoods of x is given by the finite intersections $\bigcap_i \mathrm{B}_{x_i}^-(\boldsymbol{v}_i)$, such that $x \in \mathrm{B}_{x_i}^-(\boldsymbol{v}_i)$.

[Mention reference]

7.6.3 Lemma. *Let X be an $\mathbf{R}$-tree. Then:*

1. *X is Hausdorff with respect to the observer topology.*
2. *The strong (metric) topology is stronger than the weak (observer) topology. That is, the map $(X, \text{strong}) \to (X, \text{weak})$ given by $x \mapsto x$ is continuous.*

3. *X is connected in the strong topology if and only if it's path-connected, if and only if it's connected in the weak topology.*

4. *X is path-connected in the strong topology if and only if it's path-connected in the weak topology.*

Proof. Use the fact that a continuous injection $[0,1] \to X$ is continuous in the strong topology if and only if it's continuous in the weak topology. $\square$

7.7 $\mathscr{E}(0,r)$ as an R-tree

[picture: find reference]

Define a poset structure on $\mathscr{E}(0,r)$. Say $|\cdot|_x \leqslant |\cdot|_y$ if and only if $|f|_x \leqslant |f|_y$ for all $f \in K\{r^{-1}T\}$. The Gauss point is maximal, rational and pathological points are minimal, and containment of disks is detected via this relation. We'll use this ordering to give an interpretation of $\mathscr{E}(0,r)$ as a "Hasse diagram."

7.7.1 Lemma.

1. *$|\cdot|_{0,r} = |\cdot| = \rho(\cdot)$ is the unique maximal element of $\mathscr{E}(0,r)$ with respect to $\leqslant$.*

2. *$|\cdot|_{a,s} \leqslant |\cdot|_{a',s'}$ if and only if $\mathrm{D}(a,s) \subset \mathrm{D}(a',s')$.*

3. *If x is obtained by nested family $\{(\mathrm{D}(a_i,s_i)\}$ and y is obtained from a nested family $\{\mathrm{D}(b_i,t_i)\}$, then $x \leqslant y$ if and only if for all $k > 0$, there exists $m, n > k$ such that $\mathrm{D}(a_m,s_m) \subset \mathrm{D}(b_n,t_n)$.*

4. *Classical and pathological points (types 1 and 4) are minimal with respect to $\leqslant$.*

7.7.2 Definition. Given $x \in \mathscr{E}(0,r)$, we put

$$\mathrm{radius}(x) = \inf_{b \in \mathrm{D}(0,r)} |T - b|_x.$$

Note that if $|\cdot|_x = |\cdot|_{a,s}$, then $\mathrm{radius}(x) = s$. More generally, if x corresponds to $\{\mathrm{D}(a_i,s_i)\}_i$, then $\mathrm{radius}(x) = \lim s_i$. For,

$$|T - b|_{a,s} = \sup_{z \in \mathrm{D}(a,s)} |z - b| = \begin{cases} s & \text{if } b \in \mathrm{D}(a,s) \\ |a-b| & \text{if } b \notin \mathrm{D}(a,s) \end{cases}.$$

We can use this notion of radius to put a metric on $\mathscr{E}(0,r) \smallsetminus \mathrm{D}(0,r)$, i.e. on the set of non-classical points.

7.7.3 Definition. Let $x \leqslant y$ be in $\mathscr{E}(0,r) \smallsetminus \mathrm{D}(0,r)$. Put

$$d(x,y) = \log \frac{\mathrm{radius}(y)}{\mathrm{radius}(x)}.$$

This definition was known well before Berkovich spaces—Coleman used it to speak of "distance between annuli."

We will consider $\mathscr{E}(0,r)$ as an **R**-tree.

7.7.4 Theorem. *The Berkovich topology on $\mathscr{E}(0,r)$ is identical to the weak topology (for $\mathscr{E}(0,r)$ considered as an $\mathbf{R}$-tree via d).*

[picture: different depending on whether $r \in |K^\times|$.]

If $\mathrm{D}(a,s) \subset \mathrm{D}(0,r)$, then we get an embedding $\mathscr{E}(a,s) \hookrightarrow \mathscr{E}(0,r)$. Note that $|\cdot|_{a,s} \in \mathscr{E}(0,r)$. Then $\mathscr{E}(a,s) = \mathrm{B}_{|\cdot|_{a,s}}(|\cdot|_{a,s} \to |\cdot|_{a,0})$, with the obvious interpretation of "$|\cdot|_{a,s} \to |\cdot|_{a,0}$" as a direction. Here we write $\mathrm{B}_x(\boldsymbol{v}) = \mathrm{B}_x^-(\boldsymbol{v}) \cup \{x\}$.

The subset $\mathrm{D}(a,s) \subset \mathscr{E}(a,s)$ is dense (with respect to the Berkovich topology). So we have embedded $\mathrm{D}(a,s)$ as a dense subset in a compact, Hausdorff, locally connected space. Rational (type 2) points are also dense in $\mathscr{E}(a,s)$. This embedding gives a good explanation of the failure of "affinoid covers" to form an honest topology (instead of a Grothendieck topology) in rigid geometry.

7.8 Berkovich affine n-space

Let $\boldsymbol{r} = (r_1, \dots, r_n) \in \mathbf{R}^n$. Define the "polydisk" of radius $\boldsymbol{r}$ centered at $\mathbf{0}$ as

$$\mathscr{E}(\mathbf{0}, \boldsymbol{r}) = \mathscr{M}(K\langle \boldsymbol{r}^{-1}\boldsymbol{T}\rangle) \simeq \mathscr{M}(K\{\boldsymbol{r}^{-1}\boldsymbol{T}\}).$$

Only the case $n = 1$ is completely understood, in the sense of us having a classification theorem for points. Even for $n = 2$, we don't have a complete list of points.

What is $\mathbf{A}_K^{n,\mathrm{an}}$? This is not supposed to be compact. But this is a problem, because we only "know how" to construct compact spaces—namely the $\mathscr{M}(A)$ for A a Banach K-algebra. Morally, we would like

$$\mathbf{A}_K^{n,\mathrm{an}} = \bigcup_{\boldsymbol{r}>0} \mathscr{E}(\mathbf{0}, \boldsymbol{r}),$$

just like $\mathbf{C}^n$ is an increasing union of compact closed balls, in the classical theory.

If $f = \sum a_{\boldsymbol{\nu}} \boldsymbol{T}^{\boldsymbol{\nu}}$, we had the spectral norm $|f| = \max\{|a_{\boldsymbol{\nu}}| \boldsymbol{r}^{\boldsymbol{\nu}}\}$. But as $\boldsymbol{r} \to \infty$, these rings "converge to" a polynomial ring, in the sense that

$$\varprojlim_{\boldsymbol{r}\to\infty} K\{\boldsymbol{r}^{-1}\boldsymbol{T}\} = K[\boldsymbol{T}].$$

The only property of $|\cdot|$ on $K\{\boldsymbol{r}^{-1}\boldsymbol{T}\}$ that survives is the fact that it is compatible with the absolute value on K that we started out with. Since, even for $f \in K[\boldsymbol{T}]$, $\lim_{\boldsymbol{r}\to\infty} |f|_{\boldsymbol{r}} = \infty$, any multiplicative seminorm on $K[\boldsymbol{T}]$ is "bounded" in some sense. This suggests the following definition.

7.8.1 Definition. Let K be a complete non-archimedean field. Then $(\mathbf{A}_K^{n,\mathrm{an}})'$ is the set of all multiplicative seminorms on $K[T_1, \dots, T_n]$ that restrict to $|\cdot|$ on K. We give $(\mathbf{A}_K^{n,\mathrm{an}})'$ the weakest topology making all maps $|\cdot|_x \mapsto |f|_x$ continuous, as f ranges over $K[\boldsymbol{T}]$.

We will give $\mathbf{A}_K^{n,\mathrm{an}}$ the structure of a ringed space.

7.8.2 Theorem. $\mathbf{A}_K^{n,\mathrm{an}} = (\mathbf{A}_K^{n,\mathrm{an}})'$.

Proof. Exercise. □

If $K = R$ is a ring (not a field), then $|\cdot|_x$ on $R[T]$ needs to restrict to a *bounded* norm on R, i.e. $|\cdot| \in \mathscr{M}(R)$. Note that $R[T]$ is not itself a normed ring.

For K non-archimedean, complete, and algebraically closed, $\mathbf{A}_K^{1,\mathrm{an}}$ is an infinite ascending union of $\mathbf{R}$-trees. For general $n \geqslant 1$, any $x \in \mathbf{A}_K^{n,\mathrm{an}}$ has a residue field $\mathscr{H}(x)$, defined as before.

Let $X_{/K}$ be any affine, connected, smooth, n-dimensional variety. Write $K[X] = \Gamma(X, \mathscr{O}_X)$ be its coordinate ring. As a set, X^{an} consists of all multiplicative seminorms on $K[X]$ that restrict to $|\cdot|$ on K, with the weakest topology making all maps $|\cdot|_x \mapsto |f|_x$ continuous, for $f \in K[X]$. We would like to define X^{an} for general X via gluing, but this is tricky because the "basic objects" in Berkovich geometry are compact objects $\mathscr{E}(\mathbf{0}, \boldsymbol{r})$. These difficulties are not insurmountable, so one can (with some headache) define X^{an} for arbitrary varieties $X_{/K}$.

One can define $\mathbf{P}_K^{1,\mathrm{an}} = \mathbf{A}_K^{1,\mathrm{an}} \cup \{\infty\}$, or as $\mathscr{E}(0,1) \sqcup \mathscr{E}^-(0,1)$, just as in the classical case.

7.9 Equivalent definition of analytification

Recall that for $X_{/K}$ affine, we could define X^{an} as the set of multiplicative seminorms on $\Gamma(X, \mathscr{O})$ extending $|\cdot|$ on K. There is another approach. Let $X_{/K}$ be connected, smooth of dimension n. Here K is an algebraically closed complete non-archimedean field. As a set, we could let X^{an} be the set of pairs $(x, |\cdot|)$, where x is a (scheme-theoretic) point of X, and $|\cdot|\colon k(x) \to \mathbf{R}$ extends $|\cdot|$ on K. Recall that $k(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$ is the (algebraic) residue field of X at x.

There is an obvious forgetful map $i\colon X^{\mathrm{an}} \to X$, given by $(x, |\cdot|) \mapsto x$. Give X^{an} the weakest topology making:

1. i is continuous.

2. If $U \subset X$ is Zariski-open and $f \in \mathscr{O}_X(U)$ is a regular function, then the map $i^{-1}(U) \to \mathbf{R}$ given by $(x, |\cdot|) \mapsto |f(x)|$ is continuous.

As an exercise, check that this agrees with our previous definition for $\mathbf{A}_K^{1,\mathrm{an}}$. Better, analytification is a functor in the sense that a morphism $f\colon X \to Y$ induces a map $f_*\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$; this satisfies $(fg)_* = f_* g_*$.

7.10 Structure sheaves on Berkovich spaces

In many "geometric" theories (differential geometry, algebraic geometry, complex analysis, etc.) the objects in question carry a natural structure sheaf. In the case of affinoids ($\simeq \mathscr{M}(K\{\boldsymbol{r}^{-1}\boldsymbol{T}\}/\mathfrak{a})$) there is more concrete and rich theory. For $\mathbf{A}_K^{n,\mathrm{an}} = \bigcup_{\boldsymbol{r}} \mathscr{E}(0, \boldsymbol{r})$, it's true but nontrivial that we get the same theory.

The space $\mathbf{A}_R^{n,\mathrm{an}}$, when R is an integral domain, can be quite complicated. We are looking at multiplicative seminorms on the integral domain $R[T_1, \dots, T_n]$. Let $K_n = R[T_1, \dots, T_n]_{(0)} = R_{(0)}(T_1, \dots, T_n)$. Let $U \subset \mathbf{A}_R^{n,\mathrm{an}}$ be any subset, $f \in K_n$. We say f is defined on U if $f = g/h$ for some $h, g \in R[T_1, \dots, T_n]$ such that $h(x) \neq 0$ for all $x \in U$. (Recall $h(x) = h \mod |\cdot|_x$.) If f is defined on U, then $f(x) \in \mathscr{H}(x)$ makes sense for all $x \in U$.

7.10.1 Definition. Let U be any subset of $\mathbf{A}_R^{n,\mathrm{an}}$. The set of all analytic functions on U, denoted $\mathscr{O}_{\mathbf{A}_R^{n,\mathrm{an}}}(U)$, consists of those $f\colon U \to \prod_{x \in U} \mathscr{H}(x)$ which are "local limits" of rational functions defined on U.

We define the notion of *local limit*. A function $f\colon U \to \prod_{x\in U} \mathscr{H}(x)$ is a local limit of rational functions defined on U if for all $x \in U$, there exists a neighborhood $U \supset U' \ni x$ such that for all $\epsilon > 0$, there exists $g \in K_n$ defined on U' such that for all $x' \in U'$,

$$|f(x') - g(x')| < \epsilon.$$

We have given a rule, $U \mapsto \mathscr{O}_{\mathbf{A}_R^{n,\mathrm{an}}}(U)$.

7.10.2 Theorem. *The above definition of $\mathscr{O}_{\mathbf{A}_R^{n,\mathrm{an}}}$ gives a sheaf of local rings on $\mathbf{A}_R^{n,\mathrm{an}}$.*

That $\mathscr{O}_{\mathbf{A}_R^{n,\mathrm{an}}}$ is (presheaf) of rings takes a bit of work to show. Every $f \in \mathscr{O}_{\mathbf{A}_R^{n,\mathrm{an}}}(U)$ gives a continuous map $U \to \mathbf{R}$, $x \mapsto |f(x)|$. The fact that $\mathscr{O}_x = \varinjlim_{x\in U} \mathscr{O}(U)$ is local (with unique maximal deal the functions vanishing on x) isn't hard.

Recall $\mathscr{M}(\mathbf{Z})$. [picture.] By definition, $\mathscr{M}(\mathbf{Z}) = \mathbf{A}_{\mathbf{Z}}^{0,\mathrm{an}}$ and $K_0 = \mathbf{Q}$.

If x is "between" $|\cdot|_\infty$ and $|\cdot|_0$, then $\mathscr{O}_x = \mathbf{R}$. Same for open sets (strictly) within any "branch" $|\cdot|_0 \to |\cdot|_v$, namely you get $\mathbf{Q}_v$. $\mathscr{O}_{|\cdot|_{p,0}} = \mathbf{Z}_p$.

If U is "all branches but $p_1, \ldots, p_n$," then $\mathscr{O}(U) = \mathbf{Z}[\frac{1}{p_1\cdots p_n}]$. Also, $\mathscr{O}_{|\cdot|_0}$ is related to the adeles.

[mention source: PhD thesis on Berkovich affine line over $\mathbf{Z}$.]

7.11 Further topics

If we had another semester to learn about Berkovich spaces, here are some topics we'd cover:

1. Affinoid algebras and spaces, $\mathscr{M}(K\{\boldsymbol{r}^{-1}\boldsymbol{T}\}/\mathfrak{a})$.

2. "Models" versus Berkovich's skeletons for curves, abelian varieties, ...

3. Raynaud's theory of formal models.

4. Tropicalization. Let $X_{/K}$ be a scheme, where K is complete, algebraically closed. Suppose we have a rational map $X \to (K^\times)^N = T$. By functoriality we get a map $\varphi\colon X^{\mathrm{an}} \to T^{\mathrm{an}}$. There is a canonical map $T^{\mathrm{an}} \to N_{\mathbf{R}} \simeq \mathbf{R}^N$. ("moment map") Let M be the lattice of (algebraic) characters of T; note $M \simeq \mathbf{Z}^N$, and we put $N = \hom_{\mathbf{Z}}(M, \mathbf{Z})$. So $N_{\mathbf{R}} = \hom(M, \mathbf{R})$. Write $m \in M$ as χ^m. Our function $T^{\mathrm{an}} \to N_{\mathbf{R}}$ is

$$|\cdot|_x \mapsto -\log|\chi^m \circ \varphi|_x.$$

Let's look at type 1 (classical) points of T^{an}. Any $x = (x_1, \ldots, x_N)$ is mapped to $(\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n))$. The tropicalization of X is the image of $X^{\mathrm{an}} \to \mathbf{R}^N$.

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