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A Numerical Implementation of Spherical Object Collision Probability

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Abstract

Collision probability analysis for spherical objects exhibiting linear relative motion is accomplished by combining covariances and physical object dimensions at the point of closest approach. The resulting covariance ellipsoid and hardbody can be projected onto the plane perpendicular to relative velocity when the relative motion is assumed linear. Collision potential is determined from the object footprint on the projected, two-dimensional, covariance ellipse. The resulting double integral can be reduced to a single integral by various methods. This work addresses the numerical computation of this single integral using Simpson's one-third rule to achieve at least two significant figures of accuracy over a wide range of parameters.

Introduction

The assumptions involved in the probability formulation are well defined in references [1–3] and are summarized here for the reader's convenience. Space object collision probability analysis (COLA) is typically conducted with the objects modeled as spheres. At the point of closest approach, each object's positional uncertainty is combined and their radii summed. By assuming linear relative motion, the resultant is projected onto a plane perpendicular to the relative velocity where the collision probability is calculated. The combined covariance size, shape, and orientation are coupled with physical object sizes to determine collision potential. The projection results in a double integral that can be reduced to a single integral by using the error function or a contour integral.

Probabilistic collision dynamics requires computational techniques that are sufficiently efficient, robust, and accurate for decision makers. The probability integral can be numerically approximated in many ways. Although several computational methods have been developed, they remain proprietary and unavailable to the general public. For this effort the contour integral method was implemented and tested, but tended to be more computationally intensive than what follows. This work

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evaluates the error function integral in Cartesian space using Simpson's one-third rule [4] as originally developed by The Aerospace Corporation. It is currently available in the Satellite Tool Kit Advanced Conjunction Analysis Tool (STK/AdvCAT) feature [5] and is also used to generate the Satellite Orbital Conjunction Reports Assessing Threatening Encounters in Space (SOCRATES) [6]. The number of intermediate steps is adjusted based on combined standard deviations, object size, and miss distance. A slight modification is made at the upper and lower bounds of integration due to the tendency of Simpson's rule to underestimate in those regions. Testing shows the numerical method to be accurate to more than two significant figures for covariance aspect ratios ranging from 1 to 500 in the operational decision range. The decision range is defined in this paper as collision probability between 10^{-1} and 10^{-7}.

Formulation

This method addresses the practical aspects of the double integral's numerical approximation in the collision plane. The minor and major axes of the projected covariance are equivalent to the standard deviations σx and σy respectively. The other required parameters are the position of the secondary object's distance relative to the primary in the covariance frame (xm and ym) and the radius of the combined object (OBJ). For the reader's convenience, a nomenclature section is included at the end of this document. The two-dimensional probability equation for the combined spherical object is given as

$$P = \frac{1}{2\pi\sigma x\sigma y}\int_{-OBJ}^{OBJ}\int_{-\sqrt{OBJ^2-(x)^2}}^{\sqrt{OBJ^2-(x)^2}} \exp\left[\left(\frac{-1}{2}\right)\left[\left(\frac{x+xm}{\sigma x}\right)^2+\left(\frac{y+ym}{\sigma y}\right)^2\right]\right]\mathrm{d}y\,\mathrm{d}x \tag{1}$$

The above equation can be reduced to a single integral through the use of the error (erf) function [1] as

$$P = \frac{1}{\sqrt{8\pi}\sigma x}\left[\int_{-OBJ}^{OBJ}\left[\operatorname{erf}\left[\frac{\left(ym+\sqrt{OBJ^2-x^2}\right)}{\left(\sigma y\sqrt{2}\right)}\right] + \operatorname{erf}\left[\frac{\left(-ym+\sqrt{OBJ^2-x^2}\right)}{\left(\sigma y\sqrt{2}\right)}\right]\right]\exp\left[\frac{-(x+xm)^2}{2\sigma x^2}\right]\mathrm{d}x\right] \tag{2}$$

Numerically efficient error function routines are readily available in mathematical software packages such as Numerical Recipes [7]. The n-series expression for equation (2) can then be expressed as

$$P = \frac{OBJ2}{\sqrt{8\pi}\sigma xn}\sum_{i=0}^{n}\left[\left[\operatorname{erf}\left[\frac{\left[ym+\frac{2OBJ}{n}\sqrt{(n-i)i}\right]}{\left(\sigma y\sqrt{2}\right)}\right] + \operatorname{erf}\left[\frac{\left[-ym+\frac{2OBJ}{n}\sqrt{(n-i)i}\right]}{\left(\sigma y\sqrt{2}\right)}\right]\right]\right.$$

$$\exp\left[\frac{-\left[\frac{OBJ(2i-n)}{n}+xm\right]^2}{2\sigma x^2}\right]\Bigg] \tag{3}$$

where the integration variable is represented by

$$x=\frac{OBJ(2i-n)}{n} \tag{4}$$

and

$$\mathrm{d}x=\frac{OBJ2}{n} \tag{5}$$

There are many numerical integration methods that can adequately evaluate the above expression. For this work, Simpson's one-third rule [4] was chosen for its simplicity and computational efficiency. By taking advantage of symmetry while expanding the nth-order series and regrouping terms, an alternate m-series expression is created using the following values

$$\mathrm{d}x=\frac{OBJ}{2m} \tag{6}$$

$$y(x)=\sqrt{OBJ^2-x^2} \tag{7}$$

The odd-order m-series expression is

$$m\_odd=4\sum_{i=1}^{m}\Bigg[\left[\mathrm{erf}\left[\frac{(ym+y(x))}{(\sigma y\sqrt{2})}\right]-\mathrm{erf}\left[\frac{(ym-y(x))}{(\sigma y\sqrt{2})}\right]\right]$$
$$\left[\exp\left[\frac{-(xm+x)^2}{2\sigma x^2}\right]+\exp\left[\frac{-(xm-x)^2}{2\sigma x^2}\right]\right]\Bigg] \tag{8}$$

where x in the m_odd series is defined as

$$x=(2i-1)\,\mathrm{d}x-OBJ \tag{9}$$

The even-order m-series expression is

$$m\_even=2\sum_{i=1}^{m-1}\Bigg[\left[\mathrm{erf}\left[\frac{(ym+y(x))}{(\sigma y\sqrt{2})}\right]-\mathrm{erf}\left[\frac{(ym-y(x))}{(\sigma y\sqrt{2})}\right]\right]$$
$$\left[\exp\left[\frac{-(xm+x)^2}{2\sigma x^2}\right]+\exp\left[\frac{-(xm-x)^2}{2\sigma x^2}\right]\right]\Bigg]$$
$$+2\left[\mathrm{erf}\left[\frac{(ym+OBJ)}{(\sigma y\sqrt{2})}\right]-\mathrm{erf}\left[\frac{(ym-OBJ)}{(\sigma y\sqrt{2})}\right]\right]\left[\exp\left[\frac{-(xm)^2}{2\sigma x^2}\right]\right] \tag{10}$$

where x in the m_even series is defined as

$$x=(2i)\,\mathrm{d}x-OBJ \tag{11}$$

The zeroth term in the m-series is heuristically adjusted to compensate for the tendency of Simpson's rule to underestimate this particular kind of integral at the limits,

$$m\_0 = 2\left[\left[\operatorname{erf}\left[\frac{(ym + y(x))}{(\sigma y\sqrt{2})}\right] - \operatorname{erf}\left[\frac{(ym - y(x))}{(\sigma y\sqrt{2})}\right]\right]\right.$$
$$\left.\left[\exp\left[\frac{-(xm + x)^2}{2\sigma x^2}\right] + \exp\left[\frac{-(xm - x)^2}{2\sigma x^2}\right]\right]\right] \tag{12}$$

where x in the above expression is defined as

$$x = 0.015\,\mathrm{d}x - OBJ \tag{13}$$

The probability can then be computed from the m-series as

$$P = \frac{\mathrm{d}x}{3\sqrt{8\pi}\sigma x}(m\_0 + m\_even + m\_odd) \tag{14}$$

The accuracy of P is dependent on the value chosen for m. Numerical testing has shown that m can be adequately determined from

$$m = \operatorname{int}\left(\frac{5OBJ}{\min\left(\sigma x, \sigma y, \sqrt{xm^2 + ym^2}\right)}\right) \tag{15}$$

with a lower limit of 10 and an upper limit of 50.

The derived summation is sufficient for combined object radii up to 100 meters, miss distances up to 100 kilometers, and covariance aspect ratios up to 500.

Numerical Testing

Approximately 60,000 test cases were used to evaluate the numerical expression given by equation (14). These cases had all parameters normalized to a minor axis standard deviation of one. The object size varied from 10^{-3} to 10^{+3}, the miss distance varied from 10^{-4} to 10^{+3} with position ranging from 0^0 to 90^0 relative to the minor axis (due to axial symmetry of the covariance ellipse), and the aspect ratio varied from 1 to 500. In the representative plots that follow, a gray dashed line indicates the operational decision region of two significant figures for collision probability ranging from 10^{-1} and 10^{-7}. Ideally, the method should never produce results in this region, thus ensuring sufficient accuracy for decision-making. If an approximation does produce less than two significant figures in this region, then it should overestimate (erring on the conservative side). An overestimate is represented with a black plus sign and an underestimate is marked with a gray "x."

The reference ("truth") probability was computed with MATHCAD 11 set to the highest tolerance that would still allow convergence of the double integral in equation (1). For these cases the tolerance was set to 10^{-11}. The following figures are a representative sampling of test case results.

As the figures indicate, the method always produced results that were better than two significant figures in the operational decision region. If greater accuracy is desired, the parameter m should be increased.

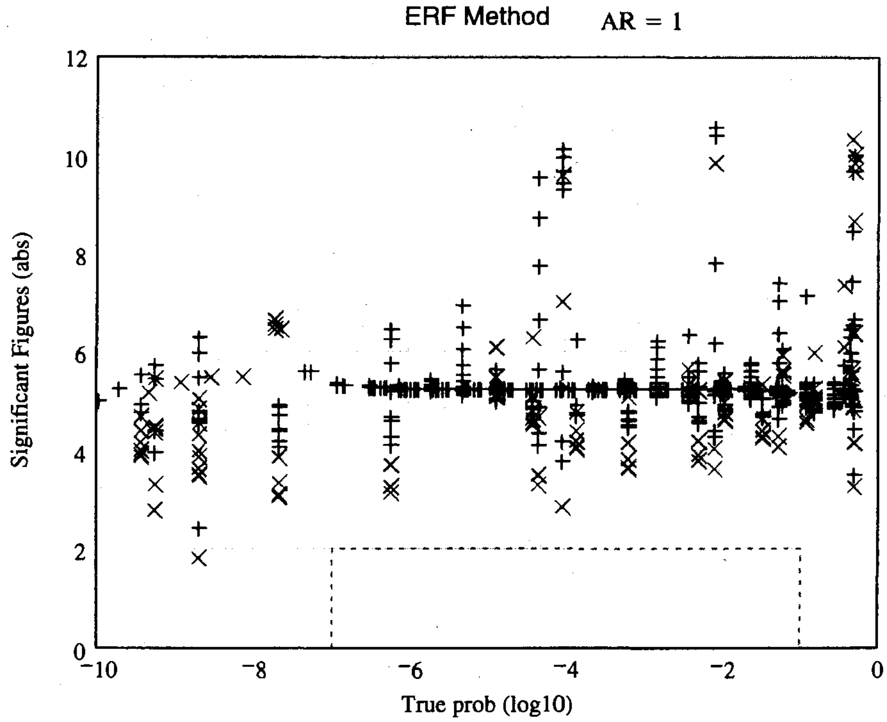

FIG. 1. Significant Figures Versus Probability for $OBJ \leq$ Miss Distance and $AR = 1$.

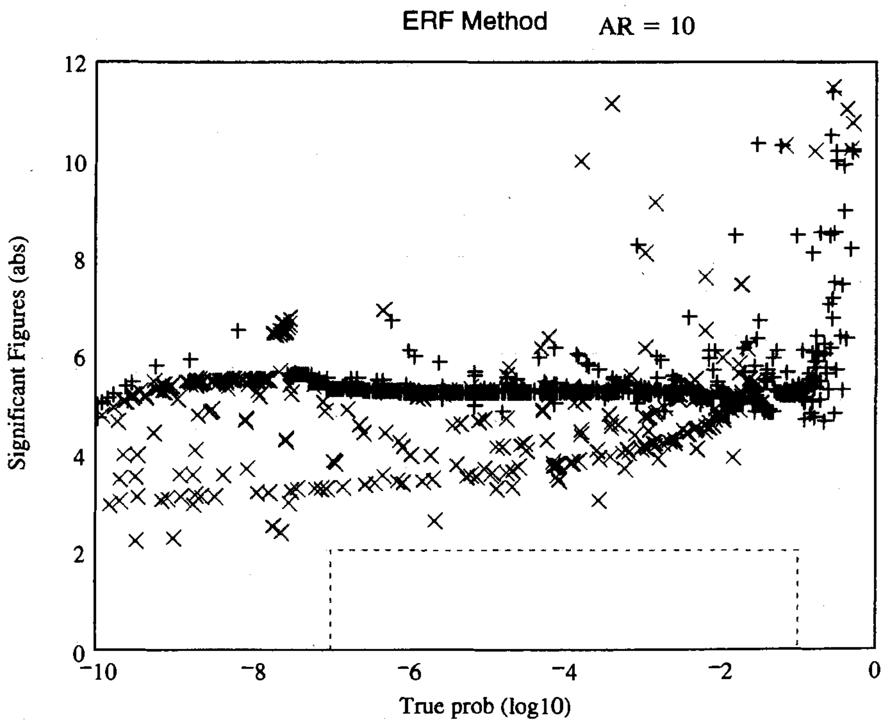

FIG. 2. Significant Figures Versus Probability for $OBJ \leq$ Miss Distance and $AR = 10$.

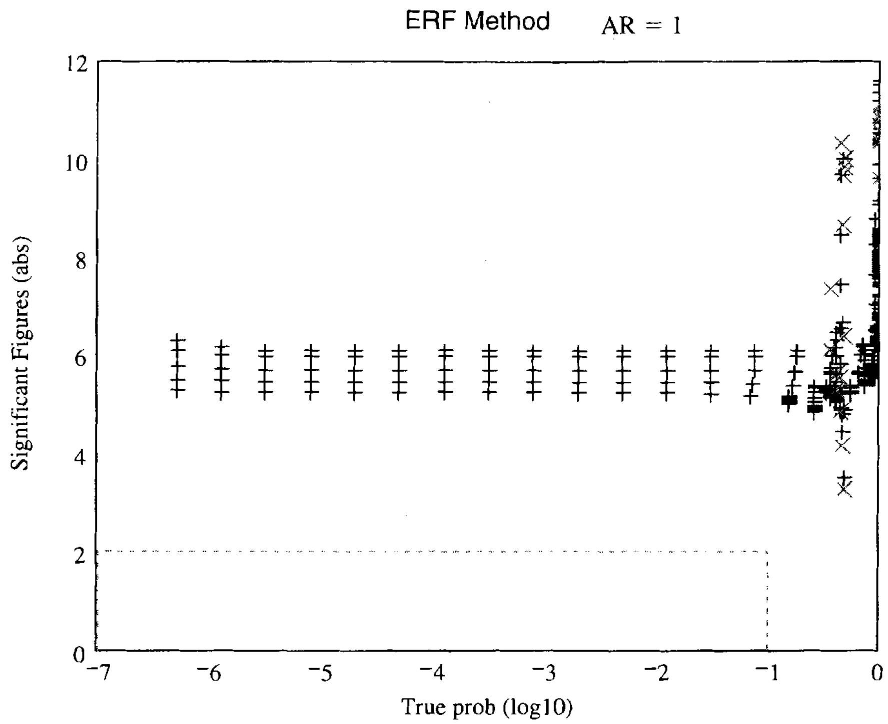

FIG. 3. Significant Figures Versus Probability for $OBJ >$ Miss Distance and $AR = 1$.

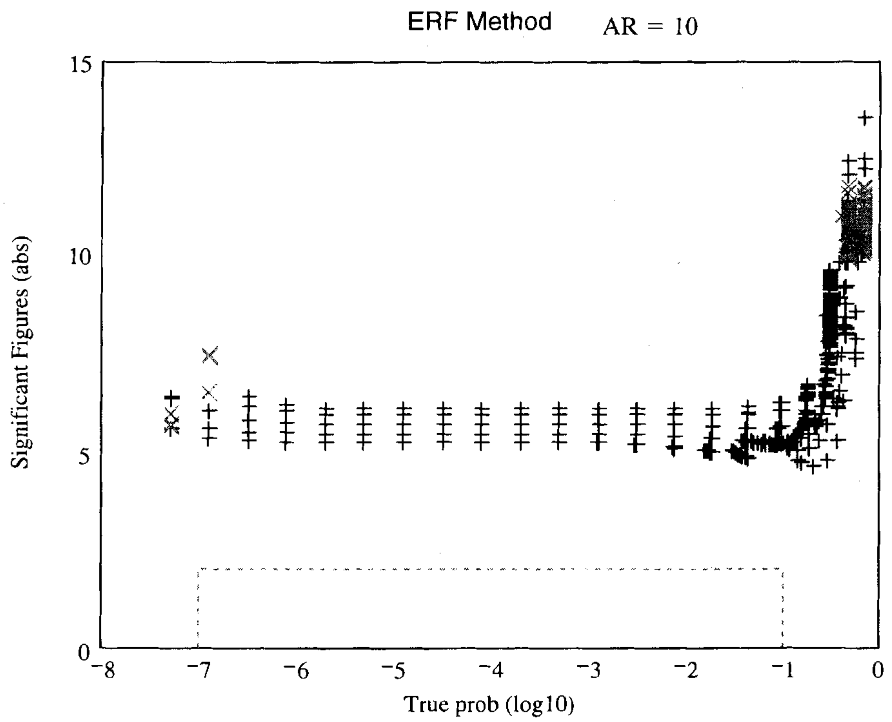

FIG. 4. Significant Figures Versus Probability for $OBJ >$ Miss Distance and $AR = 10$.

Conclusion

A numerical series approximation was presented that computes the collision probability of spherical objects. The double integral was reduced to a single integral by using the error function. The single integral was then analyzed using Simpson's one-third rule with a slight modification at the limits of integration. Numerical testing showed sufficient accuracy in the operational decision region over a wide range of parameters.

Nomenclature

AR	Aspect ratio (ratio of major-to-minor ellipse axes)
dx	Step size
erf	Error function
i	Counter
m	Number of terms in numerical series
m_0	Zeroth term for m summation
m_even	Even-numbered terms for m summation
m_odd	Odd-numbered terms for m summation
n	Number of terms in numerical series
OBJ	Radius of the combined object
P	Two-dimensional probability
x	Minor axis integration variable
xm	x-component of the projected miss distance
y	Major axis integration variable
y_abs	y-component intermediate distance
ym	y-component of the projected miss distance
z	Error function argument
σx	Minor axis standard deviation
σy	Major axis standard deviation

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