JULES.jl

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1 Equation Set

The equation set is written in terms of five prognostic conservative variables: three momenta, mass density, and specific entropy. While defining and deriving these equations, it will prove convenient to define a 4-vector $u^\alpha = (1, u, v, w)$ for $\alpha = (t, x, y, z)$. This allows us to write a conservation law for a variable ϕ as

$$\partial_t \rho\phi + \partial_i \rho u_i \phi = \partial_\alpha \rho u^\alpha \phi = \text{ sources and sinks.}$$

Because mass conservation written in this form is just

$$\partial_\alpha \rho u^\alpha = 0, \tag{1}$$

this immediately provides some useful properties, namely that

$$\partial_\alpha \rho u^\alpha \phi = \rho u^\alpha \partial_a \phi$$

and

$$u^\alpha \partial_\alpha \rho = -\rho \partial_\alpha u^\alpha.$$

Written with this notation, and using $\tau^{(i)}$ to denote the stress tensor for component i, the momentum equations are

$$\begin{align}
\partial_\alpha \rho u^\alpha u &= -\partial_x p - \boldsymbol{\nabla} \cdot \tau^{(x)} \tag{2}\\
\partial_\alpha \rho u^\alpha v &= -\partial_y p - \boldsymbol{\nabla} \cdot \tau^{(y)} \tag{3}\\
\partial_\alpha \rho u^\alpha w &= -\partial_z p - \rho g - \boldsymbol{\nabla} \cdot \tau^{(z)} \tag{4}\\
& \tag{5}
\end{align}$$

To derive an equation for specific entropy

$$s = s_0 + c_v \ln\left(\frac{T}{T_0}\right) - R \ln\left(\frac{\rho}{\rho_0}\right),$$

we can write its conservation law as

$$\begin{aligned}
\partial_\alpha \rho u^\alpha s &= \partial_\alpha \rho u^\alpha c_v \ln\left(\frac{T}{T_0}\right) - \partial_\alpha \rho u^\alpha R \ln\left(\frac{\rho}{\rho_0}\right) \\
&= \frac{1}{T}\partial_\alpha \rho u^\alpha c_v T - \frac{R}{\rho}\partial_\alpha \rho u^\alpha \rho.
\end{aligned}$$

The first law of thermodynamics allows us to express $\partial_\alpha \rho u^\alpha c_v T$ (a conservation law for the internal energy) in terms of a heating rate $Q - \boldsymbol{\nabla} \cdot J + \epsilon$ (which includes contributions from diabatic heating Q, convergence of conductive heat fluxes J, and dissipation ϵ) and a work rate $-p\boldsymbol{\nabla} \cdot \mathbf{u}$. Substituting this back into the entropy equation gives

$$\begin{aligned}
\partial_\alpha \rho u^\alpha s &= \frac{1}{T}(Q - \boldsymbol{\nabla} \cdot J + \epsilon - p\boldsymbol{\nabla} \cdot \mathbf{u}) - \frac{R}{\rho}\partial_\alpha \rho u^\alpha \rho \\
&= \frac{1}{T}(Q - \boldsymbol{\nabla} \cdot J + \epsilon - p\boldsymbol{\nabla} \cdot \mathbf{u}) - R u^\alpha \partial_\alpha \rho \\
&= \frac{1}{T}(Q - \boldsymbol{\nabla} \cdot J + \epsilon - p\boldsymbol{\nabla} \cdot \mathbf{u}) + R\rho \partial_\alpha u^\alpha \\
&= \frac{1}{T}(Q - \boldsymbol{\nabla} \cdot J + \epsilon - p\boldsymbol{\nabla} \cdot \mathbf{u}) + \frac{p}{T}\boldsymbol{\nabla} \cdot \mathbf{u} \\
&= \frac{1}{T}(Q - \boldsymbol{\nabla} \cdot J + \epsilon).
\end{aligned}$$

We probably could have written this equation down without going through the derivation, but the derivation might be a useful starting point when we eventually try to derive equations for moist entropy (which will be much more complicated).

In summary, our equation set is

$$\partial_\alpha \rho u^\alpha = 0 \tag{6}$$

$$\partial_\alpha \rho u^\alpha u = -\partial_x p - \boldsymbol{\nabla} \cdot \tau^{(x)} \tag{7}$$

$$\partial_\alpha \rho u^\alpha v = -\partial_y p - \boldsymbol{\nabla} \cdot \tau^{(y)} \tag{8}$$

$$\partial_\alpha \rho u^\alpha w = -\partial_z p - \rho g - \boldsymbol{\nabla} \cdot \tau^{(z)} \tag{9}$$

$$\partial_\alpha \rho u^\alpha s = \frac{1}{T}(Q - \boldsymbol{\nabla} \cdot J + \epsilon) \tag{10}$$

In addition to these prognostic equations, we need a set of diagnostic equations that relate T, p, τ, Q, $\mathbf{J}$, and ϵ to prognostic fields. Diagnostic equations for p and T can be obtained from the definition of s and the ideal gas law, Q is typically provided by model "physics", and $\mathbf{J}$, τ, and ϵ are provided by a sub-grid-scale turbulence closure.

2 Time integration