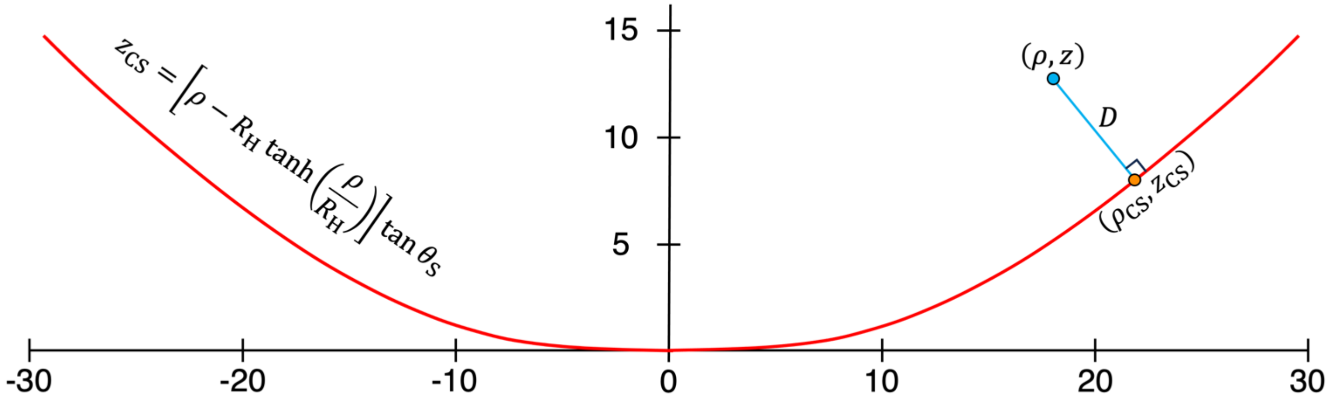
Problem: Given a point (ρ, z), what is the minimum distance to the current sheet given by the equation: $z_{\mathrm{CS}} = \left[\rho - R_{\mathrm{H}} \tanh\left(\frac{\rho}{R_{\mathrm{H}}}\right)\right] \tan\theta_{\mathrm{S}}$?



Solution: The squared distance between (ρ, z) and $(\rho_{\mathrm{CS}}, z_{\mathrm{CS}})$ is given by

$$D^2 = (\rho_{\mathrm{CS}} - \rho)^2 + (z_{\mathrm{CS}} - z)^2 = (\rho_{\mathrm{CS}} - \rho)^2 + \left(\left[\rho_{\mathrm{CS}} - R_{\mathrm{H}} \tanh\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right)\right] \tan\theta_{\mathrm{S}} - z\right)^2$$

Taking $\frac{\partial D^2}{\partial \rho_{\mathrm{CS}}}$:

$$\frac{\partial D^2}{\partial \rho_{\mathrm{CS}}} = 2(\rho_{\mathrm{CS}} - \rho) + 2\left(\left[\rho_{\mathrm{CS}} - R_{\mathrm{H}} \tanh\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right)\right] \tan\theta_{\mathrm{S}} - z\right)\left[1 - R_{\mathrm{H}} \frac{\partial}{\partial \rho_{\mathrm{CS}}}\left(\tanh\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right)\right)\right] \tan\theta_{\mathrm{S}},$$

where,

$$\frac{\partial}{\partial \rho_{\mathrm{CS}}}\left(\tanh\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right)\right) = \left(1 - \tanh^2\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right)\right)\frac{1}{R_{\mathrm{H}}},$$

yielding,

$$\frac{\partial D^2}{\partial \rho_{\mathrm{CS}}} = 2(\rho_{\mathrm{CS}} - \rho) + 2\left(\left[\rho_{\mathrm{CS}} - R_{\mathrm{H}} \tanh\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right)\right] \tan\theta_{\mathrm{S}} - z\right) \tanh^2\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right) \tan\theta_{\mathrm{S}}.$$

Setting $\frac{\partial D^2}{\partial \rho_{\mathrm{CS}}} = 0$ and solving for ρ_{CS} gives the minimum distance to the current sheet, but, no closed form solution exists. However, ρ_{CS} can be found numerically via the Newton-Raphson root-finding method. This requires knowledge of the second derivative $\frac{\partial^2(D^2)}{\partial {\rho_{\mathrm{CS}}}^2}$:

$$\frac{\partial^2(D^2)}{\partial {\rho_{\mathrm{CS}}}^2} = \frac{\partial}{\partial \rho_{\mathrm{CS}}}\left(2(\rho_{\mathrm{CS}} - \rho) + 2\left(\left[\rho_{CS} - R_{\mathrm{H}} \tanh\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right)\right] \tan\theta_{\mathrm{S}} - z\right) \tanh^2\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right) \tan\theta_{\mathrm{S}}\right)$$

$$\frac{\partial^2(D^2)}{\partial {\rho_{\mathrm{CS}}}^2} = 2 + 2\frac{\partial}{\partial \rho_{\mathrm{CS}}}\left[\left(\left(\rho_{\mathrm{CS}} - R_{\mathrm{H}} \tanh\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right)\right) \tan\theta_{\mathrm{S}} - z\right) \tanh^2\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right) \tan\theta_{\mathrm{S}}\right]$$

$$= 2 + 2\frac{\partial}{\partial \rho_{\mathrm{CS}}}\left(\left(\rho_{\mathrm{CS}} - R_{\mathrm{H}} \tanh\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right)\right) \tan\theta_{\mathrm{S}} - z\right) \tanh^2\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right) \tan\theta_{\mathrm{S}} + \left(\left(\rho_{\mathrm{CS}} - R_{\mathrm{H}} \tanh\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right)\right) \tan\theta_{\mathrm{S}} - z\right) \frac{\partial}{\partial \rho_{\mathrm{CS}}}\left(\tanh^2\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right) \tan\theta_{\mathrm{S}}\right),$$

where,

$$\frac{\partial}{\partial \rho_{\mathrm{CS}}}\left(\left(\rho_{\mathrm{CS}} - R_{\mathrm{H}} \tanh\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right)\right) \tan\theta_{\mathrm{S}} - z\right) = 1 - R_{\mathrm{H}}\left(1 - \tanh^2\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right)\right) \tan\theta_{\mathrm{S}} \frac{1}{R_{\mathrm{H}}} = \tanh^2\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right) \tan\theta_{\mathrm{S}},$$

and,

$$\frac{\partial}{\partial \rho_{\mathrm{CS}}}\left(\tanh^2\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right) \tan\theta_{\mathrm{S}}\right) = 2\tanh\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right)\left(1 - \tanh^2\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right)\right)\frac{1}{R_{\mathrm{H}}} \tan\theta_{\mathrm{S}},$$

therefore:

$$\frac{\partial^2(D^2)}{\partial {\rho_{\mathrm{CS}}}^2} = 2 + 2\tanh^4\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right) \tan^2\theta_{\mathrm{S}} + \frac{4}{R_{\mathrm{H}}}\left(\left(\rho_{\mathrm{CS}} - R_{\mathrm{H}} \tanh\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right)\right) \tan\theta_{\mathrm{S}} - z\right) \tanh\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right)\left(1 - \tanh^2\left(\frac{\rho_{\mathrm{CS}}}{R_{\mathrm{H}}}\right)\right) \tan\theta_{\mathrm{S}}.$$

The solutions for $\frac{\partial D^2}{\partial \rho_{\mathrm{CS}}}$ and $\frac{\partial^2(D^2)}{\partial {\rho_{\mathrm{CS}}}^2}$ can be used along with the initial guess $\rho_0 = \rho$ to find the point on the current sheet, $(\rho_{\mathrm{CS}}, z_{\mathrm{CS}})$, which is the minimum distance between (ρ, z) and $(\rho_{\mathrm{CS}}, z_{\mathrm{CS}})$.