OpenGamma Quantitative Research

Option Pricing with Fourier Methods

Richard White
Richard@opengamma.com

Abstract

We present details of how the price European calls and puts when the characteristic function of the terminal distribution is known in closed form. This includes a broad class of Lévy processes and the Heston model. Pricing is either via a single one-dimensional integral or the Fast Fourier transform (FFT). The latter gives prices (on a grid) for all strikes.

Contents

1 European Option Price

We take $t = 0$ as today. Given an exponential martingale process X_t, we assume that the price of an asset in a risk-neutral world[1] is given by

$$S_t = F_t e^{X_t} \tag{1}$$

where F_t is the forward price of the asset at time t and $E[e^{X_t}] = 1$ (by the exponential martingale property). If the probability density of X_t is given by $\rho_t(x)$[2] then the price of any European option[3] expiring at time T with payoff $g(S_T)$ is given by

$$V = D_T \int_{-\infty}^{\infty} g(F_T e^x) \rho_T(x) dx \tag{2}$$

where D_T is the discount factor. The price of a European call struck at K is given by

$$C(\kappa) = F_T D_T \int_{\kappa}^{\infty} (e^x - e^{\kappa}) \rho_T(x) dx \tag{3}$$

with $\kappa = \ln(\frac{K}{F_T})$. For any model that gives closed-form densities ρ_T, the price of European options can be found by simple one-dimensional integration.

1.1 Option Pricing using the Characteristic Function

By definition, the characteristic function $\phi_X(u)$ of a distribution, X, is the Fourier transform[4] (FT) of the probability density function (PDF) $\rho(x)$:

$$\phi_X(u) = E[e^{iuX}] = \int_{-\infty}^{+\infty} e^{iux} \rho(x) dx \tag{4}$$

Many models that do not offer a closed form for the terminal density do have closed forms for the characteristic function. Examples of such cases include a wide class of Levy processes and the Heston model[Hes93]. A large set of characteristic functions are listed in the appendix.

Obviously, one can recover the PDF from the characteristic function by taking the inverse Fourier transform. Option prices can then be calculated by integrating the product of the payoff

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[1]Specifically, in a measure where the zero-coupon bond $D(t, T)$ expiring at some horizon T is the numeraire.

[2]i.e. $P(X_t \in (x, x + dx)) = \rho_t(x) dx$

[3]The option can only be exercised on one fixed date and has a payoff that is a function of the value of the underlying asset at expiry only.

[4]Unfortunately there is no standard definition of a Fourier transform and its inverse. OpenGamma uses the definition common in physics (the angular frequency definition):

$$H(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} h(t) dt \qquad h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} H(\omega) d\omega$$

which is the form used to define a characteristic function (note: this is sometimes written with a factor of $\frac{1}{\sqrt{2\pi}}$ in front of both terms). The frequency definition is

$$H(f) = \int_{-\infty}^{\infty} e^{2\pi i f t} h(t) dt \qquad h(t) = \int_{-\infty}^{\infty} e^{-2\pi i f t} H(f) df$$

which has the virtue of the transform being identical apart from the sign in the exponent - most FFT implementations are based on this definition.

with the PDF - the prices are obtained via a two-dimensional integral. However, as pointed out in [Lew01], the inner product form of Parseval's theorem states that

$$\int_{-\infty}^{\infty} g(x)(h(x))^{\dagger}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}G(u)(H(u))^{\dagger}du \tag{5}$$

where $\dagger$ indicates complex conjugation and $G(\cdot)$ and $H(\cdot)$ are the FT of $g(\cdot)$ and $h(\cdot)$. This means that the price of any European option can be written as

$$V \quad = \quad D_T\int_{-\infty}^{\infty}[G(u)]^{\dagger}\phi_T(u)du \tag{6}$$

$$\text{where } G(u) \quad = \quad \int_{-\infty}^{+\infty}e^{iux}g(F_Te^x)dx \tag{7}$$

which is the integral of the product of the Fourier transform of the payoff and the characteristic function. For many payoffs, including calls and puts, the Fourier transform does not exist. However we can rewrite equation 3 as

$$C(\kappa) = F_TD_T\int_{-\infty}^{\infty}[e^{-(1+\alpha)x}(e^x - e^{\kappa})^+][e^{(1+\alpha)x}\,\rho_T(x)]dx \tag{8}$$

If $\alpha > 0$ and ρ_T has tails that fall off quicker than $e^{-(1+\alpha)|x|}$, then both terms in square brackets have well-defined Fourier transforms. In particular the FT of the second term is just $\phi(u-(1+\alpha)i)$ from the *frequency shifting* property [PTVF07].

The FT of the modified payoff is

$$\int_{-\infty}^{\infty}e^{iux}e^{-(1+\alpha)x}(e^x - e^{\kappa})dx \tag{9}$$

$$= \int_{\kappa}^{\infty}e^{(-\alpha+iu)x}dx - e^{\kappa}\int_{\kappa}^{\infty}e^{(-(1+\alpha)+iu)x}dx \tag{10}$$

$$= \frac{e^{\kappa(\alpha-ui)}}{\alpha - ui} - \frac{e^{\kappa(\alpha-ui)}}{(1+\alpha) - ui} \tag{11}$$

$$= \frac{e^{-\kappa\alpha}e^{iu\kappa}}{\alpha^2 + \alpha - u^2 - i(2\alpha + 1)u} \tag{12}$$

Putting this all together, the price of a European call option is given by

$$C(\kappa) = F_TD_T\frac{e^{-\alpha\kappa}}{2\pi}\int_{-\infty}^{\infty}e^{-iu\kappa}\frac{\phi(u - (1+\alpha)i)}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u}du \tag{13}$$

If we define a modified call price, $c(\kappa)$ as

$$c(\kappa) = \frac{C(\kappa)e^{\alpha\kappa}}{F_TD_T} \tag{14}$$

then equation 13 can be written as

$$c(\kappa) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iu\kappa}\hat{C}(u)du \tag{15}$$

$$\text{where} \quad \hat{C}(u) = \frac{\phi(u - (1+\alpha)i)}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u} \tag{16}$$

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so $c(\kappa)$ and $\hat{C}(u)$ are a transform pair. Equation 13 was first derived by Carr[CMS99] by directly considering the FT of the (modified) call price.

Since the modified call price is real, we must have $\hat{C}(-u) = [\hat{C}(u)]^{\dagger}$, so equation 15 can be rewritten as

$$c(\kappa) = \frac{1}{\pi} \int_0^{\infty} \Re(e^{-iu\kappa}\hat{C}(u))du \qquad (17)$$

and therefore $c(\kappa)$ can be calculated using standard numerical integration methods once suitable limits are established. Finally, the modified call price can be written as

$$c(\kappa) = -\frac{1}{2\pi} \int_{-\infty-(1+\alpha)i}^{\infty-(1+\alpha)i} e^{-i\nu\kappa} \frac{\hat{\phi}(\nu)}{\nu^2 + \nu i} d\nu \qquad (18)$$

and from this it is clear that we are integrating along a contour parallel to the real axis and shifted down by $1+\alpha$. It is possible to perform the above Fourier integral with $\alpha < 0$: First note that there are poles at $\nu = 0$ and $\nu = -i$; one can shift the contour across the poles and pick up the residues [Lew01]; for $-1 < \alpha < 0$, one must add $F_T D_T$ (i.e. we are pricing a covered call), while for $\alpha < -1$, equation 13 prices a put[5].

1.2 Integral Cut-Off

How does $\hat{C}(u)$ behave as $u \to \infty$? Consider the case of a Brownian process, with

$$\hat{C}(u) = e^{\frac{\sigma^2}{2}\alpha(1+\alpha)} \frac{e^{\frac{\sigma^2}{2}(2\alpha+1)ui}e^{-\frac{\sigma^2}{2}u^2}}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u} \qquad (19)$$

The numerator consists of a constant term, an oscillatory term, and a Gaussian tail term. At $u = 0$ this gives

$$\hat{C}(0) = \frac{e^{\frac{\sigma^2}{2}\alpha(1+\alpha)}}{\alpha^2 + \alpha} \qquad (20)$$

which is real (as it must be). A cut-off can be established by considering how the ratio $\frac{|\hat{C}(u)|}{\hat{C}(0)}$ behaves as $u \to \infty$. For the Gaussian process it is

$$\frac{|\hat{C}(u)|}{\hat{C}(0)} = \alpha(\alpha+1)\frac{e^{-\frac{\sigma^2}{2}u^2}}{u^2} \quad \text{as} \quad u \to \infty \qquad (21)$$

The cut-off can be established by finding a value R such that $\frac{|\hat{C}(R)|}{\hat{C}(0)} < \epsilon$. This analysis would indicate that choosing a value of α close to 0 or -1 will give a smaller cut-off. This is indeed true: however, the function $\hat{C}(u)$ will be very strongly peaked at $u = 0$ for α close to 0 or -1.

1.2.1 Finding the Cut-Off for Other Models

Consider the function

$$f(u) = \ln\left(|\hat{C}(u)|\right) - \ln\left(\epsilon\hat{C}(0)\right) \qquad (22)$$

For the Gaussian process we know this has exactly one root for $u > 0$ which is our required value R. We make the reasonable assumption that for all models this also holds. Equation 22

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[5]The same result can be obtained by repeating the Parseval argument with different payoffs.

can then be solved using a root finding algorithm. As we are approximating an integral cut-off, the accuracy of the root finder does not have to be high, so the answer can be obtained in a few iterations.

1.3 The Choice of α

As already mentioned, equation 8 implies that ρ_T must have tails that fall off quicker than $e^{-(1+\alpha)|x|}$. In particular we require that

$$0 < \int_{-\infty}^{\infty} e^{(1+\alpha)x} \rho_T(x) dx < \infty$$
$$\to 0 < \Re\left(\phi[-(1+\alpha)i]\right) < \infty \qquad \Im\left(\phi[-(1+\alpha)i]\right) = 0 \tag{23}$$

This can put a constraint on the allowed values of α. By direct substitution in the characteristic functions we can establish that there is no constraint for a Brownian process, and the following for these popular Levy processes:

$$-(1+G) < \alpha < M-1 \qquad \text{VG and CGMY} \tag{24}$$
$$-(a+b+1) < \alpha < a-b-1 \qquad \text{NIG} \tag{25}$$
$$-\frac{(\pi+a+b)}{a} < \alpha < \frac{\pi-a-b}{a} \qquad \text{Meixner} \tag{26}$$

For the worse case of the allowed parameters of the Levy processes, we have $-1 < \alpha < 0$, which is a good reason for making the choice $\alpha = -0.5$. In the Heston model we have

$$\alpha_\pm = \frac{2\nu\rho^2 - \nu - 2a\rho \pm \sqrt{\nu^2 - 4a\rho\nu + 4a^2}}{2\nu(1-\rho^2)} \tag{27}$$

where the $\pm$ indicates the upper and lower bounds on α. When $\rho = 0$ this reduces to

$$\alpha_\pm = -\frac{1}{2} \pm \sqrt{1 + \frac{4a^2}{\nu^2}} \tag{28}$$

so $\alpha = -0.5$ is always valid in this case. For $\rho = \pm 1$ we have

$$\alpha < \frac{a^2}{\nu(\pm 2a - \nu)} \tag{29}$$

which could exclude $\alpha = -0.5$.

If a value of α not in these ranges is used, the result will be spurious. Section 2.3 discusses optimising α to minimise computational time for a fixed accuracy.

1.4 Drift Correction

The exponential Martingale condition $E[e^{X_t}] = 1$ is not necessarily met by the characteristic functions listed in the appendix. For a single horizon, T, we can write

$$S_T = F_T e^{X_T - \omega_T} \tag{30}$$

and then set

$$e^{\omega_T} = E[e^{X_T}] = \phi_{X_T}(-i) \quad \rightarrow \quad \omega_T = \ln[\phi_{X_T}(-i)] = \psi_{X_T}(-i) \tag{31}$$

which returns the Martingale condition. We define $\hat{\phi}_{X_T}(u) = e^{-iu\omega_T}\phi_{X_T}(u)$ as the drift-corrected characteristic function, and use this whenever a characteristic function is required for pricing.[6]

2 Pricing Using the Fast Fourier Transform

Assume we have established some cut-off values for the integral, R, such that the call price

$$C^R(\kappa) = F_T D_T \frac{e^{-\alpha\kappa}}{2\pi} \int_{-R}^{R} e^{-i\kappa u}\hat{C}(u)du \tag{32}$$

satisfies the condition

$$\frac{|C^R(\kappa) - C(\kappa)|}{C(\kappa)} < \epsilon \tag{33}$$

where ϵ is the required relative error. We can then approximate the integral by sampling the integrand at spacing Δ to give

$$\frac{C(\kappa_m)}{F_T D_T} \approx \frac{e^{-\alpha\kappa_m}e^{-i\kappa_m R}}{2\pi}\Delta\sum_{j=0}^{M-1} e^{-i2\pi mj/M}\hat{C}_j$$

$$\text{where} \quad \hat{C}_j \equiv \hat{C}(u_j)$$

$$u_j = -R + j\Delta$$

$$M = \frac{2R}{\Delta} + 1 \tag{34}$$

$$\text{and} \quad \kappa_m = \begin{cases} \frac{2\pi m}{\Delta M} & m < M/2 \\ -\frac{2\pi(M-m)}{\Delta M} & m > M/2 \end{cases}$$

$$\text{for} \quad 0 \leq m < M$$

The limits of the integral are now $\pm R$, and κ has been discretised to κ_m. This is in the form of a Discrete Fourier Transform (DFT), allowing equation 34 to be written:

$$\frac{C_T(\kappa_m)}{F_T D_T} \approx \frac{e^{-\alpha\kappa_m}e^{-i\kappa_m R}}{2\pi}\Delta\left[DFT(\hat{C}_0....\hat{C}_{M-1})\right]_m \tag{35}$$

Discrete Fourier Transforms can be handled by Fast Fourier Transform (FFT) methods in $N\log_2 N$ time[PTVF07] and provide a whole spectrum of prices at different strikes - direct integration methods require a separate calculation for each required strike.

2.1 Sampling Frequency

The Nyquist critical frequency is $\omega_c = \frac{\pi}{\Delta}$ (or $f_c = \frac{1}{2\Delta}$ using the frequency definition of the FT). This means that if $H(\omega)$ is bandwidth-limited, i.e.

$$H(\pm(\omega_c + \eta)) = 0 \quad \forall \eta > 0 \tag{36}$$

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[6]We have used the *time shifting* property of FT[PTVF07]

then the function $h(\cdot)$ is completely determined by the samples. However, if $H(\cdot)$ has density outside this range, this is moved into the range giving the wrong answer (an effect known as aliasing). Since $H(\cdot)$ in our case is the modified option price, we can make an approximation of its bandwidth. If the modified option price is less than some tolerance for $|\kappa| > |\kappa_{max}|$, then the sampling spacing should be $\Delta = \frac{\pi}{\kappa_{max}}$

We proceed by considering the tail behaviour of Black-Scholes prices, and making the assumption the the value of κ_{max} established will not differ significantly from the true value for the model under consideration.

2.1.1 $\alpha > 0$

In this case, we are pricing calls. The price in a Black-Scholes world is

$$c(\kappa) = e^{\alpha\kappa}\left(N(d_1) - e^{\kappa}N(d_2)\right)$$
$$\text{where} \quad d_1 = \frac{-\kappa + \sigma^2 T/2}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T} \tag{37}$$

For $\kappa \ll 0$, $c(\kappa) \approx e^{\alpha\kappa}$ and for $\kappa \gg 0$, $c(\kappa) \approx ae^{-b\kappa^2}$. The right-hand side decays much faster and we need only consider the left.

For $\frac{c(\kappa)}{c(0)} < \epsilon$, we need

$$\kappa < \frac{1}{\alpha}\ln(\epsilon c(0)) \tag{38}$$

$$\rightarrow \kappa_{\max} \approx \left|\frac{\ln(\epsilon c(0))}{\alpha}\right| \tag{39}$$

$$\rightarrow \Delta \approx -\pi\frac{\alpha}{\ln\left(\epsilon(2N(\frac{\sigma\sqrt{T}}{2}) - 1)\right)} \tag{40}$$

Therefore, in this approximation there is a dependence on an unknown volatility level. However, if the model is being fitted to market data, an approximate volatility level to use is known. Otherwise, a conservative choice is to use a low value for the volatility.

2.1.2 $\alpha < -1$

In this case, we are pricing puts. In a Black-Scholes world the modified put price is

$$p(\kappa) = e^{\alpha\kappa}\left(e^{\kappa}(N(-d_2) - N(-d_1))\right) \tag{41}$$

We need only consider $\kappa \gg 0$, for which $p(\kappa) \approx e^{(1+\alpha)\kappa}$. It then follows that

$$\kappa_{\max} \approx \left|\frac{\ln(\epsilon p(0))}{1+\alpha}\right| \quad \rightarrow \quad \Delta \approx \pi\frac{1+\alpha}{\ln\left(\epsilon(2N(\frac{\sigma\sqrt{T}}{2}) - 1)\right)} \tag{42}$$

2.1.3 $-1 < \alpha < 0$

If α is in this range of values, we are pricing a call minus the forward multiplied by the discount factor. Let the modified form[7] of this be $x(\kappa)$

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[7]i.e. the price multiplied by $\frac{e^{\alpha\kappa}}{D_T F_T}$

$$x(\kappa) = e^{\alpha\kappa} \left(N(d_1) - e^{\kappa} N(d_2) - 1 \right)$$

$$\rightarrow \quad x(\kappa) = \begin{cases} 2 \left(N\left(\frac{\sigma\sqrt{T}}{2} \right) - 1 \right) & \kappa = 0 \\ -e^{(1+\alpha)\kappa} & \kappa \ll 0 \\ -e^{\alpha\kappa} & \kappa \gg 0 \end{cases} \tag{43}$$

After considering both signs of κ we arrive at the result

$$\kappa_{max} \approx \max \left(\left| \frac{\ln(-x(0)\epsilon)}{1 + \alpha} \right|, \left| \frac{\ln(-x(0)\epsilon)}{\alpha} \right| \right) \tag{44}$$

$$\rightarrow \quad \Delta \approx -\pi \frac{\min(1 + \alpha, -\alpha)}{\ln \left(2\epsilon \left(1 - N(\frac{\sigma\sqrt{T}}{2}) \right) \right)} \tag{45}$$

2.1.4 Short-Dated Options

As $T \rightarrow 0$, the price of the at-the-money puts and calls go to zero, implying that $\Delta \rightarrow 0$, and we are forced to take increasingly fine samples of $\hat{C}$. There is no such problem for $-1 < \alpha < 0$. In this case, Δ is maximised for $\alpha = -0.5$ to give $\Delta = \frac{\pi}{2\ln(\epsilon)}$. There are other reasons why $\alpha = -0.5$ is a good choice, which are discussed in section 1.3.

2.2 Sample Number

The spacing in κ-space, δ, is related to Δ by

$$\delta = \frac{2\pi}{N\Delta} \tag{46}$$

where N is the number of samples. A small Δ implies a large δ, which in turn means widely spaced strikes unless N is also increased, which adds to the computational burden. Having approximated the bandwidth, κ_{max}, and the function cut-off, R, we have the inequalities

$$N\delta \geq 2\kappa_{max} \tag{47}$$

$$N\Delta \geq 2R \tag{48}$$

$$\rightarrow N = \left\lfloor \frac{2R\kappa_{max}}{\pi} \right\rfloor \qquad \Delta = \frac{\pi}{\kappa_{max}} \qquad \delta = \frac{\pi}{R} \tag{49}$$

However, this gives no control over the size of δ or the value of N[8].

The standard trick to gain control over the size of δ is to zero-pad the DFT. In particular, if our function falls to zero after M samples (so the value R is exceeded), then one can stop sampling from the function and set the values to zero, giving a total of N samples. Furthermore, we only need sample the positive values of u which give the inequality $N \geq 2M - 1$ (generally, we will have $N \gg M$). Equation 48 is changed to read $M\Delta \geq R$, and combining with equation 46 gives the following inequality for N:

$$2M - 1 \leq N \leq \frac{2\pi}{\delta R} M \tag{50}$$

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[8]which is optimally a power of two

which has no solution unless $\delta < \frac{\pi}{R}$. We are free to choose any value of δ meeting this condition. Given that δ is the spacing of log-strike, we will generally want much smaller value than the upper limit. The procedure to determine M, N, and Δ is as follows:

1. Choose $\delta < \frac{\pi}{R}$

2. Set $N = f\left(\frac{2\kappa_{max}}{\delta}\right)$ where $f(\cdot)$ is a function that returns the next number greater than its argument that is a power of two.

3. Set $\Delta = \frac{2\pi}{N\delta}$

4. Set $M = \left\lfloor \frac{R}{\Delta} \right\rfloor = \left\lfloor \frac{R\delta N}{2\pi} \right\rfloor$

With this choice the Nyquist critical frequency, κ_c, is

$$\kappa_c = \frac{\pi}{\Delta} = \frac{N\delta}{2} > \kappa_{max} \tag{51}$$

If κ_{max} is well chosen, we are guaranteed not to have aliasing issues.

2.3 Optimising the Choice of α

The computational burden is dominated by the size of M, so a procedure to find the value of α that minimises M is desirable. One could proceed as follows: For a chosen value of δ:

1. Choose an allowed value of α in one of the three regions

2. Calculate κ_{max} and R

3. Calculate M, N, and Δ

4. Use the value of M to make a new guess of α (subject to it being an allowed value)

5. Repeat until M is minimised in that region

6. Repeat for the other two regions of α

We have not yet investigated how strongly M varies with α and thus whether the above optimisation routine is worth performing (there is no point optimising α only to find that the total computation time has increased).

A Characteristic Function

Some properties of characteristic functions that follow immediately from their definition are

$$\phi_X(0) = 1 \tag{52}$$
$$\phi_X(-i) = E[e^X] \tag{53}$$

and by differentiating with respect to u we have

$$\frac{d^n \phi_X(0)}{du^n} = (i)^n E[X^n] \tag{54}$$

so the nth derivative of the characteristic function at zero gives the nth moment (multiplied by some power of i).

The cumulant characteristic function, or characteristic exponent, is defined as $\psi(u) = \ln(\phi(u))$ and has the property that

$$\frac{d^n \psi_X(0)}{du^n} = (i)^n E[(X - E(X))^n] \tag{55}$$

so the nth derivative of the characteristic exponent at zero gives the nth central moment.

B Levy Processes

B.1 Normal Process

The best known Levy process is Brownian motion, which is driven by normally-distributed increments.

$$
\begin{aligned}
\phi_X(u; \mu, \sigma) &= E[\exp(iuX)] & (56) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{iux} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) & (57) \\
&= \exp\left(iu\mu - \frac{1}{2}\sigma^2 u^2\right) & (58) \\
& & (59)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_{X_t}(u) &= [\phi_X(u)]^t & (60) \\
&= \exp\left(t\left\{iu\mu - \frac{\sigma^2 u^2}{2}\right\}\right) & (61)
\end{aligned}
$$

B.2 Poisson Process

A Poisson process has up jumps of unit size at random times such that the number of jumps N_t in a interval of size t, is distributed as:

$$P(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \tag{62}$$

The time between jumps follows a exponential distribution with mean λ^{-1}. The characteristic function is:

$$
\begin{aligned}
\phi_{N_t}(u; \lambda) &= E[\exp(iuN_t)] & (63) \\
&= \sum_{n=0}^{\infty} e^{iun} \frac{e^{-\lambda t}(\lambda t)^n}{n!} & (64) \\
&= \exp(\lambda t(e^{iu} - 1)) & (65)
\end{aligned}
$$

B.2.1 Compound Poisson Process

This process has random jump sizes independently drawn from a distribution with characteristic function $\phi_Z(u)$.

$$X_t = \sum_{i=1}^{N_t} Z_i, \quad t \geq 0 \tag{66}$$

then

$$\phi_{X_t}(u; \lambda) = \exp(\lambda t(\phi_Z(u) - 1)) \tag{67}$$

B.3 The Variance Gamma Process

The characteristic function of VG is given by

$$\phi(u; \sigma, \nu, \theta) = \left(1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2\right)^{-1/\nu} \tag{68}$$

The distribution is symmetric for $\theta = 0$ and negative θ gives a negative skew. ν controls the excess kurtosis. For ease of comparison with other model three new variables are defined

$$C = 1/\nu \tag{69}$$

$$G = \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} - \frac{1}{2}\theta\nu\right)^{-1} \tag{70}$$

$$M = \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} + \frac{1}{2}\theta\nu\right)^{-1} \tag{71}$$

With this parameterisation the characteristic function becomes

$$\phi(u; C, G, M) = \left(\frac{GM}{GM + (M - G)iu + u^2}\right)^C \tag{72}$$

For sensible pricing we require that $E[e^X]$ is finite (see equation 30). This implies that $C > 0$, $G > 0$ and $M > 1$.

B.4 The Normal Inverse Gaussian Process

This has a characteristic function given by [Sch03]

$$\Phi_{NIG}(u; a, b, \delta) = \exp\left[-\delta(\sqrt{a^2 - (b + iu)^2} - \sqrt{a^2 - b^2})\right] \tag{73}$$

where $a > 0$, $a > |b|$, $a > |b + 1|$ and $\delta > 0$

B.5 The CGMY Process

The CGMY distribution has a characteristic function [CW04]

$$\phi_{CGMY}(u; C, G, M, Y) = \exp\left\{C\Gamma(-Y)[(M - iu)^Y \right.$$
$$\left. -M^Y + (G + iu)^Y - G^Y]\right\} \tag{74}$$

where $C > 0$, $G > 0$, $M > 1$ and $Y < 2$. This process is pure jump (i.e. no Brownian part). If $Y < 0$ the paths have a finite number of jumps, otherwise there are infinitely many jumps in any time interval. If $Y > 1$ the process is of infinite variation. The VG distribution is a special case of CGMY with $Y = 0$.

B.6 The Meixner Process

The characteristic function of the Meixner distribution is given by

$$\phi(u; a, b, \delta) = \left(\frac{\cos(b/2)}{\cosh(au - ib/2)} \right)^{2\delta} \tag{75}$$

where $a, \delta > 0$ and $-\pi < b < \pi - 2a$.

C Time Change Processes

Over some periods market activity (trading volumes, arrival of information etc.) will be greater than during other periods. This can be modelled by *business time* in which time moves forward relative to calendar time at a stochastic rate. If one drives the Levy processes by business time rather than calendar time, stochastic changes are introduced to the model parameters. A number of different stochastic clocks, for example integrated CIR, can be used.

The Time Changed Levy Process (TCLP) can be written as X_{Y_t} where Y_t is the business time. The characteristic function is given by [CGMY03]:

$$\begin{aligned} \phi_{TCLP}(u, t) &= E\left[e^{iuX_{Y_t}} \right] = E\left[e^{Y_t \psi_X(u)} \right] \\ &= E\left[e^{i\{-i\psi_X(u)\}Y_t} \right] = \phi_{Y_t}(-i\psi_X(u), t) \end{aligned} \tag{76}$$

where $\psi_X(u)$ is the cumulant characteristic function of the Levy process.

This stochastic clock has the effect of making one of the Levy model parameters stochastic[9]; which in turn makes the volatility, skew and kurtosis stochastic.

C.1 Integrated CIR Time Change

The model of Cox-Ingersoll-Ross is a mean-reverting positive process (used to model instantaneous interest rates)[10]

$$dy_t = \kappa(\eta - y_t)dt + \lambda\sqrt{y_t}dW_t \tag{77}$$

The *business time* elapsed over a calendar period t is the integrated CIR process:

$$Y_t = \int_0^t y_s ds \tag{78}$$

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[9]C in the case of VG and CGMY and δ in the case of NIG and Meixner.

[10]The Feller condition $2\kappa\eta > \lambda^2$ ensures that the process y_t cannot reach zero. However there is no reason to exclude $y_t = 0$ as the mean-reverting term will make it positive again, and it can be interpreted as a period of zero volatility.

Let y_0 be the initial rate of business time. The characteristic function is [Sch03]:

$$\phi(u, t; \kappa, \eta, \lambda, y_0) = E[e^{iuY_t}|y_0]$$
$$= \frac{e^{\kappa^2 \eta t/\lambda^2} \exp(2y_0 iu/\{\kappa + \gamma \coth(\gamma t/2)\})}{\left(\cosh(\gamma t/2) + \frac{\kappa}{\gamma} \sinh(\gamma t/2)\right)^{2\kappa\eta/\lambda^2}} \quad (79)$$

where $\quad \gamma = \sqrt{\kappa^2 - 2\lambda^2 iu}$

C.2 Integrated Ornstein-Uhlenbeck Processes

A Ornstein-Uhlenbeck (OU) process is defined by the SDE

$$dy_t = -\lambda y_t dt + dZ_t, \quad \lambda, y_0 > 0, \quad (80)$$

where Z is a subordinator (a Levy process with nonnegative drift, no Brownian part and only positive jumps called the background driving Levy process). The elapsed business time, Y_t, is given by equation 78.

C.2.1 The Gamma-OU Time Change

If the subordinator is driven by the Gamma distribution, the characteristic function is

$$\phi(u, t; \lambda, a, b, y_0) =$$
$$\exp\left(\frac{iuy_0}{\lambda}(1 - e^{-\lambda t}) - \frac{\lambda a}{iu - \lambda b}\left\{b \ln\left[\frac{b - iu\lambda^{-1}(1 - e^{-\lambda t})}{b}\right] + iut\right\}\right) \quad (81)$$

C.2.2 The Inverse Gaussian-OU Time Change

If the subordinator is driven by the Inverse Gaussian (IG) distribution, the characteristic function is

$$\phi(u, t; \lambda, a, b, y_0) = \exp\left(\frac{iuy_0}{\lambda}(1 - e^{-\lambda t}) + \frac{2aiu}{b\lambda}A(u, t)\right) \quad (82)$$

where

$$A(u, t) = \frac{1 - \sqrt{1 + \kappa[1 - \exp(-\lambda t)]}}{\kappa}$$
$$+ \frac{1}{\sqrt{1 + \kappa}}\left(\tanh^{-1}\left\{\frac{\sqrt{1 + \kappa[1 - \exp(-\lambda t)]}}{\sqrt{1 + \kappa}}\right\} - \tanh^{-1}\left\{\frac{1}{\sqrt{1 + \kappa}}\right\}\right)$$

and

$$\kappa = -\frac{2iu}{b^2\lambda}$$

D The Heston Model

In the Heston model [Hes93] the spot asset, S_t, follows a diffusion

$$dS_t = \mu(S_t, t)dt + \sqrt{V_t}S_t dW_t$$
$$dV_t = a[\theta - V_t]dt + \nu\sqrt{V_t}dZ_t \tag{83}$$
$$\langle dW_t, dZ_t \rangle = \rho dt$$

If $\rho = 0$ this can be viewed as an exponential Brownian process with a CIR stochastic clock. For $\rho \neq 0$ the characteristic function is given by [Gat06]

$$\phi(u) = e^{C(t,u) + D(t,u)V_0}$$
$$C(t,u) = \frac{a\theta}{\nu^2}\left([a - \rho\nu ui - d(u)]t - 2\ln\left[\frac{c(u) - e^{-d(u)t}}{c(u) - 1}\right]\right)$$
$$D(u,t) = \frac{a - \rho\nu ui + d(u)}{\nu^2}\left(\frac{1 - e^{-d(u)t}}{c(u) - e^{-d(u)t}}\right) \tag{84}$$
$$c(u) = \frac{a - \rho\nu ui + d(u)}{a - \rho\nu ui - d(u)} \qquad d(u) = \sqrt{(\rho\nu ui - a)^2 + iu\nu^2 + \nu^2 u^2}$$

In this form, the characteristic function does not suffer from any of the complex discontinuity problems discussed in [KJ06].

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