Normal Inverse Gaussian (NIG) Process

With Applications in Mathematical Finance

Cliff Kitchen

The Mathematical and Computational Finance Laboratory - Lunch at the Lab

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Stylized Empirical Facts

When modeling financial time series data even seemingly unrelated processes share stylized empirical facts, some of which are:

- Aggregational normality
- No incremental autocorrelation
- Bounded Quadratic Variation
- Asymmetric distribution of increments
- Heavy or semi-heavy tails
- Jumps in price trajectories

Gaussian vs Empirical

Compare the distribution of the log-returns to the normal

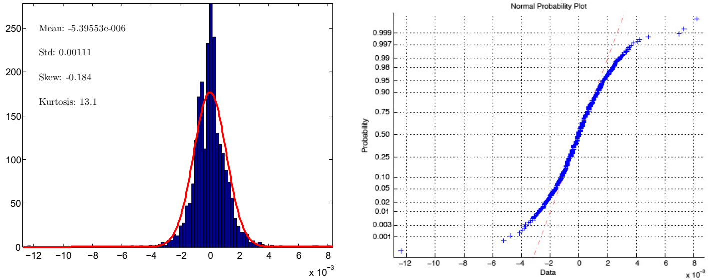

Figure: One-minute log-returns of DJX from Dec 10/08 - Dec 22/08, compared with the normal distribution

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We can see asymmetry and semi-heavy tails

Gaussian vs Empirical

Also compare the amplitude of the log-returns to the normal

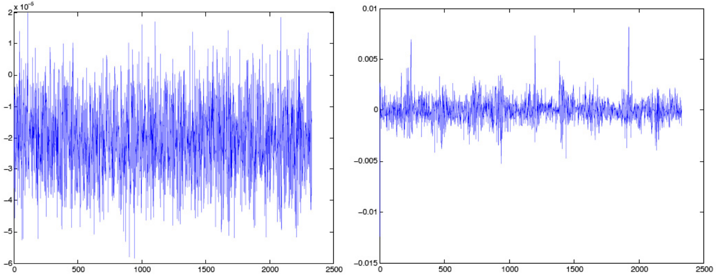

Figure: One-minute log-returns of DJX from Dec 10/08 - Dec 22/08, compared with log-return from the Black-Scholes model with the same annualized return and variance

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There is no representation of jumps in the Gaussian model

Lévy Processes

Given a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, a **Lévy process** $L = \{L_t, t \geq 0\}$ is an infinitely divisible continuous time stochastic process, $L_t : \Omega \to \mathbb{R}$, with stationary and independent increments.

Lévy processes are more versatile than Gaussian driven processes as they can model:

- Skewness
- Excess kurtosis
- Jumps

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Lévy Processes

Formal Definition

A càdlàg, adapted, real valued stochastic process $L = \{L_t, t \geq 0\}$ with $L_0 = 0$ a.s. is called a Lévy process if the following are satisfied:

- L has *independent increments*, i.e. $L_t - L_s$ is independent of $\mathcal{F}_s$ for any $0 \leq s < t \leq T$
- L has *stationary increments*, i.e. for any $s, t \geq 0$ the distribution of $L_{t+s} - L_t$ does not depend on t
- L is *stochastically continuous*, i.e. for all $t > 0$ and $\epsilon > 0$:

$$\lim_{s \to t} \mathbb{P}(|L_t - L_s| > \epsilon) = 0$$

Lévy Processes

The characteristic function of L_t describes the distribution of each independent increment is given by $\phi(u) = e^{t\,\eta(u)}$ ($t > 0$ and $u \in \mathbb{R}$), where $\eta(u)$ is the characteristic exponent of the process.

Lévy-Khintchine Formula

The characteristic exponent of L_t can be be expressed as

$$\eta(u) = \mathrm{i}\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} e^{\mathrm{i}ux} - 1 - \mathrm{i}ux\,\mathbf{1}_{\{|x|<1\}}\nu(dx)$$

Lévy processes are often represented by their Lévy triplet (γ, σ^2, ν)

Lévy Processes

The structure of the sample paths of L_t can be represented in an intuitive way

Lévy-Itô Decomposition

There exists $\gamma \in \mathbb{R}$, a Brownian motion B_{σ^2} with covariance matrix σ^2 and an independent Poisson random measure N such that, for each $t \geq 0$

$$L(t) = \gamma t + B_{\sigma^2}(t) + \int_{|x|<1} x\, \tilde{N}_t(dx) + \int_{|x|\geq 1} x\, N_t(dx)$$

History

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- It is a sub-class of the more general class of hyperbolic Lévy processes
- Shortly after its introduction Blaesild showed that the NIG distribution fit the log returns on German stock market data even better than the hyperbolic distribution, making this process one of great interest
- Barndorff-Neilsen originally introduced the process as an inverse Gaussian Lévy subordinated Brownian motion

Inverse Gaussian (IG) Process

The inverse Gaussian distribution is a two parameter continuous distribution that can be thought of as the first passage time of a Brownian motion to a fixed level $a > 0$.

Characteristic Function

$$\phi_{IG}(u; a, b) = \exp\left(-a(\sqrt{-2\mathrm{i}u + b^2} - b)\right)$$

The IG distribution is infinitely divisible we can define the IG process

$$X^{(IG)} = \{X_t^{(IG)}, t \geq 0\},$$

for $a, b > 0$, which starts at zero and has independent and stationary increments

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Inverse Gaussian (IG) Process

We use an inverse Gaussian Lévy subordinator by replacing the Brownian motion with a Gaussian process $X^{(G)} = \{X_t^{(G)}, t \geq 0\}$, where each $X_t^{(G)} = B_t + \gamma t$, and $\gamma \in \mathbb{R}$, where B_t is a standard Brownian motion. The inverse Gaussian subordinator is given by

$$T(t) = \inf\{s < 0; X_t^{(G)} = at\}, \qquad a > 0.$$

Each $T(t)$ has a density, and as a result the IG(a,b) law has density

Density Function

$$f_{T(t)}(x; a, b) = \frac{at}{\sqrt{2\pi}} \exp(atb) x^{-3/2} \exp\left(-\frac{1}{2}(a^2t^2x^{-1} + b^2x)\right)$$

Normal Inverse Gaussian (NIG) Process

Barndorff-Neilsen considered classes of normal variance-mean mixtures and defined the NIG distribution as the case when the mixing distribution is inverse Gaussian

Characteristic Function

$$\phi_{NIG}(u;\alpha,\beta,\delta,\mu) = \exp\left\{\delta\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+\mathrm{i}u)^2}\right)+\mathrm{i}\mu u\right\}$$

where

$$u \in \mathbb{R}, \quad \mu \in \mathbb{R}, \quad \delta > 0, \quad 0 \leq |\beta| \leq \alpha.$$

Normal Inverse Gaussian (NIG) Process

Each parameter in NIG($\alpha, \beta, \delta, \mu$) distributions can be interpreted as having a different effect on the shape of the distribution:

- α - tail heaviness of steepness
- β - symmetry
- δ - scale
- μ - location

Normal Inverse Gaussian (NIG) Process

Each parameter in NIG$(\alpha, \beta, \delta, \mu)$ distributions can be interpreted as having a different effect on the shape of the distribution:

- α - tail heaviness of steepness
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- μ - location

The NIG distribution is closed under convolution, in fact it is the only member of the family of general hyperbolic distributions to have the property

$$\mathrm{NIG}(\alpha, \beta, \delta_1, \mu_1) * \mathrm{NIG}(\alpha, \beta, \delta_2, \mu_2) = \mathrm{NIG}(\alpha, \beta, \delta_1 + \delta_2, \mu_1 + \mu_2)$$

Normal Inverse Gaussian (NIG) Process

Note that when using the NIG process for option pricing the location parameter of the distribution has no effect on the option value, so for convenience we will take $\mu = 0$.

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Again we have an infinitely divisible characteristic function and so we can define the NIG process $X^{(NIG)} = \{X_t^{(NIG)}, t \geq 0\}$, which again starts at zero and has independent and stationary increments each with an NIG(α, β, δ) distribution and the entire process has an NIG$(\alpha, \beta, \delta t)$ law.

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Note that $X_t^{(NIG)} = X_{T(t)}^{(G)}$ for each $t \geq 0$, where $T(t)$ is an inverse Gaussian subordinator which is independent of B_t with parameters $a = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$

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Normal Inverse Gaussian (NIG) Process

Lévy-Khintchine Triplet

The NIG process has no diffusion component making a pure jump process with Lévy triplet $(\gamma, 0, \nu_{\mathrm{NIG}}(dx))$, with

$$\begin{aligned} \gamma &= \frac{2\alpha\delta}{\pi}\int_0^1 \sinh(\beta x)\mathrm{K}_1(\alpha x)dx, \\ \nu_{\mathrm{NIG}}(dx) &= \frac{\alpha\delta}{\pi}\frac{\exp(\beta x)\mathrm{K}_1(\alpha|x|)}{|x|}dx, \end{aligned}$$

where $\mathrm{K}_\lambda(z)$ is the modified Bessel function of the third kind,

$$\mathrm{K}_\lambda(z) = \frac{1}{2}\int_0^\infty u^{\lambda-1}\exp\left(-\frac{1}{2}z(u+u^{-1})\right)du, \qquad x > 0$$

Normal Inverse Gaussian (NIG) Process

Lévy-Itô Decomposition

The NIG(α, β, δ) law can be represented in the form

$$X_t^{(NIG)} = \gamma t + \int_{|y|<1} y\tilde{\mathrm{N}}_t(dy) + \int_{|y|\geq 1} y\mathrm{N}_t(dy),$$

where N_t and $\tilde{\mathrm{N}}_t$ are Poisson and compensated Poisson measures respectively

Inverse Gaussian Random Variables

There are a variety of simulations codes available on-line but be warned that you must use the proper parameterization to simulate the IG process so I give an algorithm here

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IG(a, b) Random Number Generator

1. Generate a standard normal random number v.
2. Set $y = v^2$.
3. Set $x = (a/b) + y/(2b^2) + \sqrt{4aby + y^2}/(2b^2)$.
4. Generate a uniform random number u.
5. if $u \leq a/(a + xb)$, then return the number x as the IG(a, b) random number, else return $a^2/(b^2x)$ as the IG(a, b) random number.

Inverse Gaussian Random Variables

Below shows a frequency histogram of computing 5 simulations with 1000 samples of inverse Gaussian random variables using the above algorithm

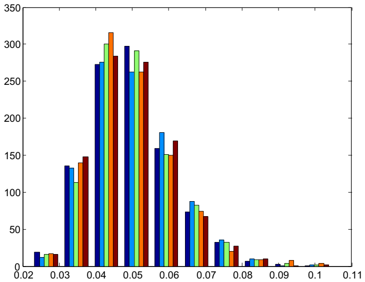

Figure: Frequency histogram of IG(1,20) variables

Inverse Gaussian Process

Simulation of an $X^{(IG)} = \{X_t^{(IG)}, t \geq 0\}$ process with law $\text{IG}(at, b)$ can easily be implemented once the above algorithm is available. To simulate the value of this process at time points $\{n\Delta t, n = 0, 1, \ldots\}$ use

Inverse Gaussian Process Simulation

1. Generate n independent $\text{IG}(a\Delta t, b)$ random numbers i_n, $n \geq 1$.
2. Set initial process value to zero, $X_0^{(IG)} = 0$.
3. Iterate path by $X_{n\Delta t}^{(IG)} = X_{(n-1)\Delta t}^{(IG)} + i_n$.

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Inverse Gaussian Process

Below shows 3 simulations each with 1000 partitions of the interval $T = 1$ of an inverse Gaussian process using the above algorithm

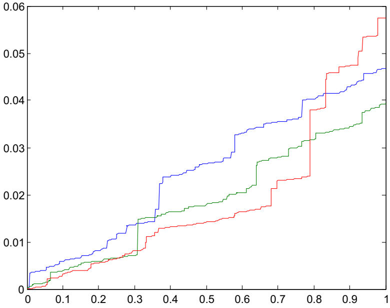

Figure: Sample paths from IG(1,20) process

Normal Inverse Gaussian Process

We can simulate an $X^{(NIG)} = \{X_t^{(NIG)}, t \geq 0\}$ process with law $\text{NIG}(\alpha, \beta, \delta t)$ as an inverse Gaussian time-changed Brownian motion with drift. To simulate the value of this process at time points $\{n\Delta t, n = 0, 1, \ldots\}$ use

Normal Inverse Gaussian Process Simulation

1. Simulate each state of an inverse Gaussian process $X^{(IG)} = \{X_t^{(IG)}, t \geq 0\}$ at time points $\{n\Delta t, n = 0, 1, \ldots\}$ using the algorithm above with $a = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$.
2. Difference each consecutive state of $X^{(IG)}$, $dt_{n\Delta t} = X_{n\Delta t}^{(IG)} - X_{(n-1)\Delta t}^{(IG)}$.

Normal Inverse Gaussian Process

Normal Inverse Gaussian Process Simulation

3. Simulate time change of a standard Brownian motion $W = \{W_t, t \geq 0\}$ by,
   - Simulate n independent standard normal random variables $\nu_n, \quad n > 0$.
   - Set $W_0 = W_{X_0^{(IG)}} = 0$.
   - $W_{n\Delta t} = W_{(n-1)\Delta t} + \sqrt{dt_{n\Delta t}}\nu_n$
4. Iterate path by $X_{n\Delta t}^{(NIG)} = \beta\delta^2 X_{n\Delta t}^{(IG)} + \delta W_{n\Delta t}$.

Normal Inverse Gaussian Process

Below shows 3 simulations each with 1000 partitions of the interval $T = 2$ of a NIG process using the above algorithm

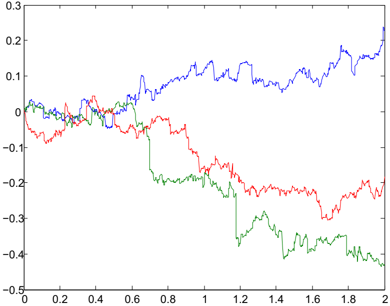

Figure: Sample paths of an NIG(50,-5,1) process

Method of Moments Framework

Method of moments (MOM) calibration technique does not require an explicit representation of the density function so it is very robust, however is it not as efficient as MLE

Given we know the characteristic function we can calculate the moment generating function, $M_X(u) = \phi_X(-\mathrm{i}u)$. Then the n^{th}-order moment can be calculated by taking the n^{th} derivative

$$\mathbb{E}\left[X^n\right] = M_X^{(n)}(0) = \frac{d^n M_X}{dt^n}(0)$$

Method of Moments Framework

We can convert the n^{th}-order moment to the central moment by

$$\mathbb{E}[(X-\mu)^n] = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \mathbb{E}[X^j] \mathbb{E}[X]^{n-j}$$

where $\mu = \mathbb{E}[X]$. Thus for $n = 2, 3, 4$ we have the population variance, skewness and kurtosis respectively and we can compare these to their sample counterparts.

Method of Moments Framework

Sample Moments

$$\text{Mean:} \quad m = \bar{x} = \frac{1}{N}\sum_{j=1}^{N} x_j$$

$$\text{Variance:} \quad v = \frac{1}{N-1}\sum_{j=1}^{N}(x_j - \bar{x})^2$$

$$\text{Skewness:} \quad s = \frac{1}{N}\sum_{j=1}^{N}\left(\frac{x_j-\bar{x}}{\bar{\sigma}}\right)^3$$

$$\text{Kurtosis:} \quad k = \left\{\frac{1}{N}\sum_{j=1}^{N}\left(\frac{x_j-\bar{x}}{\bar{\sigma}}\right)^4\right\} - 3$$

Moments of NIG Distribution

For the NIG distribution we know the moments

Population Moments

$$
\begin{aligned}
\mathbb{E}[X] &= \mu + \delta \frac{\beta/\alpha}{(1-(\beta/\alpha)^2)^{1/2}} \\
\mathrm{Var}[X] &= \delta^2 \alpha^{-1} \frac{\beta/\alpha}{(1-(\beta/\alpha)^2)^{3/2}} \\
\mathrm{Skew}[X] &= 3\alpha^{-1/4} \frac{\beta/\alpha}{(1-(\beta/\alpha)^2)^{1/4}} \\
\mathrm{Kurt}[X] &= 3\alpha^{-1/2} \frac{1+4(\beta/\alpha)^2}{(1-(\beta/\alpha)^2)^{1/2}}
\end{aligned}
$$

Maximum Likelihood Estimation Framework

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MLE has a higher probability of being close to the quantities being estimated than MOM, but this techniques relies on knowing the population density function. If the density is mis-specified MLE estimators will be inconsistent.

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MLE has a higher probability of being close to the quantities being estimated than MOM, but this techniques relies on knowing the population density function. If the density is mis-specified MLE estimators will be inconsistent.

For a distribution with no explicit density function a discrete Fourier transformation can be used to approximate the density but care must be given to ensure there is no mis-specification.

Maximum Likelihood Estimation Framework

The likelihood function of a sample $x_1, x_2, \ldots, x_n$ of n values from distribution can be computed with the density function associated with the sample as a function of θ, the distribution parameters, with $x_1, x_2, \ldots, x_n$ fixed.

$$l(\theta) = f_\theta(x_1, x_2, \ldots, x_n)$$

Maximum Likelihood Estimation Framework

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$$l(\theta) = f_\theta(x_1, x_2, \ldots, x_n)$$

Assuming the data is i.i.d. and since maxima are unaffected by monotone transformations, we need to maximize

$$\mathcal{L}(\theta) = \sum_{i=1}^{n} \log f_\theta(x_i)$$

This is done by simultaneously solving the corresponding partials w.r.t each parameter in θ

MLE for NIG Distribution

The density of the NIG distribution can be given explicitly

Density Function

$$f_{\mathrm{NIG}}(x;\alpha,\beta,\delta,\mu) = \frac{\alpha\delta K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\pi\sqrt{\delta^2+(x-\mu)^2}}\, e^{\delta\gamma+\beta(x-\mu)}$$

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The log-likelihood function, $\mathcal{L}_{\mathrm{NIG}}(\theta)$, is given by

$$\log\left(\frac{(\alpha^2-\beta^2)^{-1/4}}{\sqrt{2\pi}\alpha^{-1}\delta^{-1/2}\mathrm{K}_{-1/2}(\delta\sqrt{\alpha^2-\beta^2}}\right)-\frac{1}{2}\sum_{i=1}^{n}\log\left(\delta^2+(x_i-\mu)^2\right)$$
$$\sum_{i=1}^{n}\left[\log\,\mathrm{K}_1\left(\alpha\sqrt{\delta^2+(x_i-\mu)^2}\right)+\beta(x_i-\mu)\right]$$

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MLE for NIG Distribution

Now we take the corresponding partials and solve the system . . .

MLE for NIG Distribution

Now we take the corresponding partials and solve the system ...
Fortunately we can utilize the optimization toolbox in MATLAB without actually calculating these derivatives.

MATLAB Code

Define anonymous function:

```
f = @(param) -sum(log(nigpdf(R,param(1),param(2),param(3),param(4))));
```

Get inital estimates via MOM: `[ALPHA,BETA,DELTA,MU] = NIGmom(R);`

Assign values for optimization: `param = [ALPHA,BETA,DELTA,MU]`

Define optimization tolerance:

```
opt = optimset('diagnostics','on','display','iter','tolx',1e-12);
```

Run optimization: `est = fminunc(f,param,opt);`

Recapture Test Parameters

To test the calibration methods shown here I simulated and NIG$(50, -5, 5, 0)$ process over a different number of partitions of the interval [0,1]. Note that $\delta t = 5$, so δ will change proportionally to the number of intervals

Parameter	n	α	β	δ	μ
MOM	1,000	59.0564	-6.5816	0.0060	0.0002
MLE	1,000	47.3408	-6.5050	0.0050	0.0002
MOM	10,000	75.6475	-5.9593	0.0007	0.0000
MLE	10,000	54.2056	-5.9854	0.0005	0.0000
MOM	100,000	48.3898	-5.9496	0.0000	0.0000
MLE	100,000	48.3814	-5.9269	0.0001	0.0000
MOM	1,000,000	51.5604	-0.2797	0.0000	-0.0000
MLE	1,000,000	51.5604	-0.2797	0.0000	-0.0000

Framework

We consider an asset price model $S = \{S_t, t \geq 0\}$ that is an exponential of a Lévy process, specifically a NIG$(\alpha, \beta, \delta t)$ process $X^{(NIG)} = \{X_t^{(NIG)}, t \geq 0\}$. This process will evolve in the form

$$S_t = S_0 e^{X_t^{(NIG)}}, \qquad 0 \leq t \leq T$$

Overview

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Very efficient numerical evaluation for vanilla type options, however, a main drawback to this method is that this efficiency is not carried over when trying to evaluate exotic options

Required Assumptions

1. Assume that $\phi_{L_T}(z)$, the characteristic function of L_T, exists for all $z \in \mathbb{C}$ with $\Im(z) \in I_1 \subset [0,1]$.
2. Assume that P_{L_T}, the distribution of L_T, is absolutely continuous w.r.t. the Lebesgue measure λ with density ρ.
3. Consider an integrable, European-style, payoff function $g(S_T)$.
4. Assume that $x \to e^{-Rx}|g(e^{-x})|$ is bounded and integrable for all $R \in I_2 \subset \mathbb{R}$.
5. Assume that $I_1 \cap I_2 \neq \emptyset$.

Note that $R \in (-\infty, -1)$ is simply a dampening factor that is required for the integration below

General Raible Formula

Then by no arbitrage arguments the value of the option is equal to the expected payoff under the risk-neutral measure $\mathcal{Q}$, see Raible for details. The value of the option is given by

$$C_T(S,K) = \frac{e^{-rT-R\log(S_0)}}{2\pi}\int_{\mathbb{R}} e^{-\mathrm{i}u\log(S_0)}\mathcal{L}_\pi(R+\mathrm{i}u)\phi_{L_T}(\mathrm{i}R-u)du,$$

where, $\phi_{L_T}(z)$ is the characteristic function of the process under the risk-neutral measure and $\mathcal{L}_\pi(z)$ is the bilateral Laplace transformation for the payoff function at $z \in \mathbb{C}$, given by

$$\mathcal{L}_\pi(z) = \frac{K^{1+z}}{z(z+1)},$$

for the payoff of a European call given by $g(S_T) = (S_T - K)^+$

Raible Formula on an NIG Process

For the NIG$(\alpha, \beta, \delta t)$ process we have

Call Option Value on an NIG Process

$$C_T(S,K) = \frac{e^{-rT - R\log(S_0)}}{2\pi} \int_{\mathbb{R}} e^{-\mathrm{i}u\log(S_0)} \frac{K^{1+R+\mathrm{i}u}}{(R+\mathrm{i}u)(R+\mathrm{i}u+1)} * \exp\left\{ T\delta \left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2 - (\beta - (R+\mathrm{i}u))^2} \right) \right\}$$

Raible Formula on an NIG Process

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- R should have no effect in the option pricing formula, if it does, then there is an error in implementing the transformation
- We assume that we know the form of the characteristic function, under the risk-neutral measure $\mathcal{Q}$.

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Incomplete Markets

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When pricing using Lévy processes we have incomplete markets, their is no unique risk neutral measure. We must determine criteria or a method to pick the optimal measure with which to price our option with.

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The Esscher transform attempts to do this by choosing the equivalent martingale measure with minimal entropy by a utility-maximizing argument.

General Framework of Esscher Transform

Given our process X_t, let $f_t(x)$ be the density of our model under the physical measure $\mathcal{P}$. Then for some number $\theta \in \{\theta \in \mathbb{R} | \int_{\mathbb{R}} \exp(\theta y) f_t(y) dy < \infty\}$ we can define a new density

$$f_t^{(\theta)}(x) = \frac{\exp(\theta x) f_t(x)}{\int_{\mathbb{R}} \exp(\theta y) f_t(y) dy}$$

We need to choose θ so that the discounted stock price model is a martingale where the expectation is taken with respect to the law with density $f_t^{(\theta)}(t)$. For this, we need

$$\exp(r) = \frac{\phi(-\mathrm{i}(\theta + 1))}{\phi(-\mathrm{i}\theta)}$$

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Esscher Transform on NIG Process

The solution, θ^* is the Esscher transform martingale measure under $\mathcal{Q}$. If we are modeling the log-returns of a market under measure $\mathcal{P}$ by an NIG$(\alpha, \beta, \delta t)$ process, with no dividends, we have

$$\begin{aligned} e^r &= \frac{\exp\left\{\delta\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+\mathrm{i}(-\mathrm{i}(\theta+1)))^2}\right)\right\}}{\exp\left\{\delta\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+\mathrm{i}(-\mathrm{i}\theta))^2}\right)\right\}} \\ &= \frac{\exp\left\{\delta\left(-\sqrt{\alpha^2-(\beta+\theta+1)^2}\right)\right\}}{\exp\left\{\delta\left(-\sqrt{\alpha^2-(\beta+\theta)^2}\right)\right\}} \end{aligned}$$

which reduces to

$$r = \delta\left(\sqrt{\alpha^2-(\beta+\theta)^2}-\sqrt{\alpha^2-(\beta+\theta+1)^2}\right)$$

Esscher Transform on NIG Process

We now solve this for θ^* and we have an equivalent martingale measure $\mathcal{Q}$ which follows an NIG$(\alpha, \theta^* + \beta, \delta t)$ law. For convenience we write the adjusted parameter $\hat{\beta} = \theta^* + \beta$, where

$$\hat{\beta} = -1/2\,\frac{\delta^4 + \delta^2 r^2 + r\sqrt{-\delta^2\left(\delta^2 + r^2\right)\left(\delta^2 + r^2 - 4\,\delta^2\alpha^2\right)}}{\delta^2\left(\delta^2 + r^2\right)}$$

Apple Inc.

Apple Inc. (AAPL) is traded on NASDAQ. These data contains one-minute log-returns from Nov 28/08 - Dec 23/08

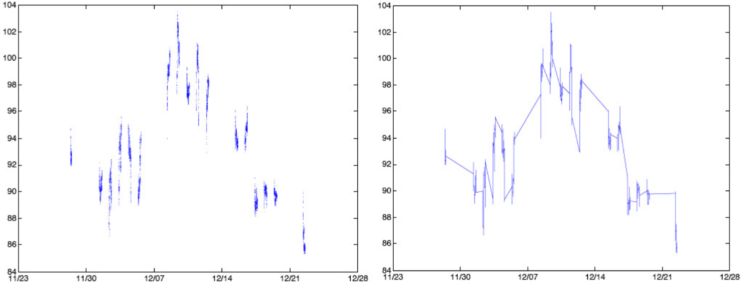

Figure: One-minute prices of AAPL from Nov 28/08 - Dec 23/08

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Apple Inc.

Although at first this doesn't look like any of the simulations we've created from an exponential NIG process, look more closely of the returns.

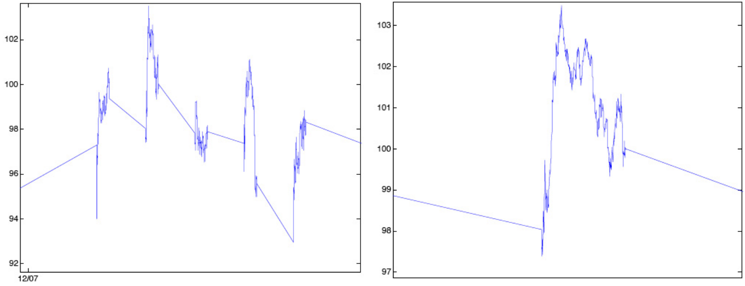

Figure: Magnification of AAPL stock data

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- The increments of the NIG process model the log-difference of the stock price $X_t^{(NIG)} - X_{t-1}^{(NIG)} = \log(S_t/S_{t-1})$
- Now calibrate the NIG(α, β, δ) parameters to the all the data expect the returns over the weekend, calibrate these returns separately
- Using statistical testing determine if the weekend returns follow the same distribution as all the other data, if not then discard them from your analysis
- Although it is not typically done the same analysis can be completed for the daily difference in closing and opening prices

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Apple Inc.

- We omit the weekend returns and calibrate our model by first using MOM and then MLE

$$\alpha = 174.0781 \quad \beta = -4.1078 \quad \delta = 0.0006 \quad \mu = 0.0000$$

- Now calculate the Esscher transform to get our equivalent martingale measure we set

$$\hat{\beta} = -.4999982018$$

- Given the log-returns follow an NIG$(\alpha, \hat{\beta}, \delta)$ distribution under the risk neutral measure $\mathcal{Q}$ we can compute the Raible pricing formula for a European call (assuming $r = 0.041$) with strike K $= 90$ and maturity on Jan 16/09

$$C_{24/260}(85.59, 90) = 4.59$$

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