1 Wealth In Utility Model

This appendix considers how to solve a model with a utility function that allows a role for financial balances distinct from the implications those balances might have for consumption expenditures: $\mathrm{u}(c, a)$.

For purposes of articulating the exact structure of the model it is useful to define a sequence for events that are in principle simultaneous. These correspond essentially to a set of steps that will be executed in order by the code solving (or simulating) the model. Notationally the sequence of steps can be indexed by time gaps of infinitesimal duration ϵ. We conceive of the sequence of events in the period as follows, where for example the notation 2ϵ means that the event is conceived of has happening two instants after the beginning of the period and because the period is of total duration 1, $1-1\epsilon$ means that the event is conceived of as happening an instant before the end of the period:

0ϵ: The period begins with the consumer knowing the ratio of end-of-last-period assets to permanent income. Call this $k_{t+0\epsilon}$:

$$k_{t+0\epsilon} = e_{t-1\epsilon} \tag{1}$$

- If this is the initial period, there is no $e_{t-1\epsilon}$ because there was no previous period
- In that case, some assumption must be made about the starting value of k

1ϵ: The consumer learns the transitory ($\boldsymbol{\xi}$) and permanent ($\boldsymbol{\Psi}$) shocks

- In simulation, the consumer's permanent income is updated according to

$$\mathbf{p}_t = \boldsymbol{\Phi}_t \mathbf{p}_{t-1} \boldsymbol{\Psi}_t \tag{2}$$

- $\boldsymbol{\Phi}$ is the predictable (say, life cycle) growth factor
- Because the problem is normalized by permanent income, we do not need to keep track of $\mathbf{p}$ during the solution stage
- Normalized income is therefore $y_{t+1\epsilon} = \boldsymbol{\theta}_{t+1\epsilon}$

2ϵ: The consumer calculates normalized bank balances

$$b_{t+2\epsilon} = k_t/(\boldsymbol{\Phi}_t \boldsymbol{\Psi}_t) \tag{3}$$

3ϵ: Market resources are determined

$$m_{t+3\epsilon} = b_{t+2\epsilon} + y_{t+1\epsilon} \tag{4}$$

4ϵ: The consumer decides how much to consume for the year

- We imagine that this amount is immediately deducted from market resources; concretely, imagine that the amount $c_{t+4\epsilon}$ is put into an untouchable escrow account
- The escrow account funds a constant *flow rate* of consumption that will be maintained throughout the interval from $t+4\epsilon$ to $t+1-\epsilon$

5ϵ: The consumption decision determines the amount of investable assets:

$$a_{t+5\epsilon} = m_{t+3\epsilon} - c_{t+4\epsilon} \tag{5}$$

6ϵ: The consumer makes a decision about the proportion of assets $a_{t+5\epsilon\epsilon}$ to invest in the risky asset, $\tilde{\varsigma}_{t+6\epsilon}$

$1-2\epsilon$: The rate of return on the risky asset $\tilde{\mathbf{R}}$ for the year is determined; combining this with the riskless (but potentially time-varying) return $\tilde{\mathsf{R}}_t$, this yields the portfolio return

$$\tilde{\mathbb{R}}_t = \varsigma_t \tilde{\mathbf{R}}_t + (1-\varsigma_t)\tilde{\mathsf{R}}_t \tag{6}$$

$1-1\epsilon$: End-of-period (December 31 11:59:59$\bar{9}$) financial balances are

$$e_{t+1-1\epsilon} = \tilde{\mathbb{R}}_{t+1-2\epsilon} a_{t+5\epsilon} \tag{7}$$

The model in the main text considers the problem at the point we have designated 4ϵ above: After the realization of all of the stochastic variables that determine m.

$$\mathrm{v}_{t+4\epsilon}(m_{t+3\epsilon}) = \max_{\{c_{t+4\epsilon},\tilde{\varsigma}_{t+6\epsilon}\}} \mathrm{u}(c_{t+4\epsilon}, m_{t+3\epsilon} - c_{t+4\epsilon}) + \mathbb{E}_{t+6\epsilon}[\beta_{t+1}\mathrm{v}_{t+1+4\epsilon}(m_{t+1+4\epsilon})]. \tag{8}$$

Using this notation we can now unambiguously define period-t post-all-decisions but pre-realization-of-returns expected value as being calculable immediately after the portfolio share has been chosen (and as in the main text we use a Gothic font for this v because the Goths flourished after the Romans):

$$\mathfrak{v}_{t+7\epsilon}(a_{t+4\epsilon}, \tilde{\varsigma}_{t+6\epsilon}) = \mathbb{E}_{t+7\epsilon}[\beta_{t+1}\mathrm{v}_{t+1+4\epsilon}(m_{t+1+4\epsilon})]. \tag{9}$$

Having now established this conceptual sequence, we can dispense with the ϵ timing conventions for all variables, and simply use a t subscript to denote the value of any variable determined at any point within the period, leaving the reader to remember the logic of the implicit timing above. For example, beginning-of-period-$(t+1)$ (January 01 12:00:00) financial capital is the same as end-of-period-t assets because an infinitesimal amount of time separates them:

$$k_{t+1} = e_t \tag{10}$$

since the value of e_t is in principle determined at the last instant of period t; that is, by e_t we expect the reader to understand us to mean what we wrote more elaborately as $e_{t+1-1\epsilon}$ above. Likewise, in the simpler notation, we can rewrite (9) more compactly as

$$\mathfrak{v}_t(a_t, \tilde{\varsigma}_t) = \mathbb{E}_t[\beta_{t+1} v_{t+1}(k_{t+1})]. \tag{11}$$

Now we can imagine inserting another step (in principle, between steps 5ϵ and 6ϵ; but now that our timing is clear, we will use the simpler notation) to calculate the optimal risky share as the share that maximizes expected value:

$$\tilde{\varsigma}_t^* = \arg\max_{\tilde{\varsigma}_t} \quad \mathfrak{v}_t(a_t, \tilde{\varsigma}_t) \tag{12}$$

which lets us construct a function $\overset{*}{\mathfrak{v}}_t(a_t)$ that calculates expected-value-given-optimal-portfolio-choice (with the asterisk accent indicating this is the maximum):

$$\overset{*}{\mathfrak{v}}_t(a_t) = \mathfrak{v}_t(a_t, \tilde{\varsigma}_t^*) \tag{13}$$

whose derivative is calculable as

$$\begin{aligned} \overset{*}{\mathfrak{v}}{}_t^a(a_t) &= \left(\frac{d}{da_t}\right) \mathfrak{v}_t(a_t, \tilde{\varsigma}_t^*) \\ &= \mathfrak{v}_t^a(a_t, \tilde{\varsigma}_t^*) + \underbrace{\mathfrak{v}_t^{\tilde{\varsigma}}(a_t, \tilde{\varsigma}_t^*)}_{=0 \texttt{ by FOC}} \left(\frac{d\tilde{\varsigma}_t^*}{da_t}\right). \end{aligned} \tag{14}$$

Collecting all of this, in our new notation the Roman-step problem is

$$\mathrm{v}_t(m_t) = \max_{c_t} \quad \mathrm{u}(c_t, m_t - c_t) + \overset{*}{\mathfrak{v}}_t(m_t - c_t) \tag{15}$$

with FOC

$$\mathrm{u}^c - \mathrm{u}^a = \overset{*}{\mathfrak{v}}{}^a(m_t - c_t). \tag{16}$$

and the Envelope theorem says

$$\mathrm{v}_t^m(m) = \mathrm{u}^a + \overset{*}{\mathfrak{v}}{}^a(a_t) \tag{17}$$

To make further progress, we now must specify the structure of the utility function. We consider two utility specifications, respectively called `CobbDouglas` and `CDC`. The CDC function is designed to capture the following:

1. Remain homothetic so that the problem scales
2. Allow different relative risk aversion for fluctuations in wealth ϱ versus in consumption ρ
3. Allow utility weights for consumption and wealth that are independently determined from their risk aversions

$$\begin{aligned} \texttt{CobbDouglas:} \qquad \mathrm{u}(c, a) &= \left(\frac{(c^{1-\delta}a^{\delta})^{1-\rho}}{1-\rho}\right) \\ &= \left(\frac{(c(a/c)^{\delta})^{1-\rho}}{1-\rho}\right) \\ &= \left(\frac{(c(c/a)^{-\delta})^{1-\rho}}{1-\rho}\right) \\ \texttt{CDC:} \qquad \mathrm{u}(c, a, \tilde{\varsigma}) &= (1-\delta)\acute{\mathrm{u}}(c) + \delta\acute{\mathrm{u}}(\mathbb{E}_t[e])\grave{\mathrm{u}}(e/\,\mathbb{E}_t[e]) \\ \acute{\mathrm{u}}(c) &= \left(\frac{c^{1-\rho}}{1-\rho}\right) \\ \grave{\mathrm{u}}(e) &= \left(\frac{e^{1-\varrho}}{1-\varrho}\right) \end{aligned}$$

$$\acute{\mathrm{u}}^c = c^{-\rho}$$
$$\grave{\mathrm{u}}^e = e^{-\varrho}$$

and the relationship between u^a and u^c allows us to write

$$\mathrm{u}^c - \mathrm{u}^a = \left(1 - (c/a)\left(\frac{\delta}{1-\delta}\right)\right)\mathrm{u}^c \tag{18}$$

In the `CobbDouglas` value function, relative risk aversion with respect to (proportional) fluctuations in a (for a fixed c) is given by $\delta\rho$, while relative risk aversion with respect to (proportional) fluctuations in c (for a fixed a) is given by $(1-\delta)\rho$. Suppose we calibrate δ to $1/3$, so that in the last period of life a consumer who faced no risk would choose to set $a = (1/2)c$. In such a case, when we consider introducing rate of return risk, the consumer's relative aversion to consumption risk will be twice as large as their relative aversion to fluctuations in financial balances.

1.1 Solution

Now, as in the main text, designate a matrix of values of end-of-period assets in $[\mathrm{a}]_{\mathrm{i}}$ for $\mathrm{i} \in \{0, ..., \mathrm{I}-1\}$ and for each element in i compute the corresponding matrix of values of Gothic maximized value (for the particular problem we have specified, these matrices will have only one dimension – they will be vectors):

$$[\overset{*}{\mathfrak{v}}{}^a_t] = \overset{*}{\mathfrak{v}}{}^a_t([\mathrm{a}]) \tag{19}$$

That is, $[\overset{*}{\mathfrak{v}}{}^a_t]_{\mathrm{i}} = \overset{*}{\mathfrak{v}}{}^a_t([\mathrm{a}]_{\mathrm{i}})\ \forall\ \mathrm{i}$.

1.2 CobbDouglas Model

Now suppose for convenience we define $\grave{\rho} = \rho(1-\delta) + \delta$ so that

$$\mathrm{u}^c(c,a) = c^{-\grave{\rho}} a^{\delta(1-\rho)}(1-\delta) \tag{20}$$
$$\mathrm{u}^a(c,a) = c^{-\grave{\rho}} a^{\delta(1-\rho)}\ \delta \tag{21}$$

and we define a pseudo-inverse function

$$\mu^{-1}(\bullet) = \left(\frac{\bullet}{1-\delta}\right)^{-1/\grave{\rho}} \tag{22}$$

Then

$$\mu^{-1}(\mathrm{u}^c_t - \mathrm{u}^a_t) = \mu^{-1}\left(1 - (c_t/a_t)(\delta/(1-\delta))\right)(a^{-\delta(1-\rho)/\grave{\rho}})c_t \tag{23}$$

Now we use the fact that for an optimizing consumer

$$\begin{aligned}\mu^{-1}(\mathrm{u}^c_t - \mathrm{u}^a_t)/a^{-\delta(1-\rho)/\grave{\rho}} &= \mu^{-1}(\overset{*}{\mathfrak{v}}{}^a_t(a))/a^{-\delta(1-\rho)/\grave{\rho}} \\ \mu^{-1}\left(1 - (c/a)(\delta/(1-\delta))\right)c &= \mu^{-1}(\overset{*}{\mathfrak{v}}{}^a_t(a))a^{\delta(1-\rho)/\grave{\rho}}.\end{aligned} \tag{24}$$

For any fixed $a > 0$ this is a nonlinear equation that can be solved for the unique c that satisfies it. (See below for discussion of options for solving the equation).

Define the 'consumed' function obtained in that manner as $\mathfrak{c}_t(a)$.

Now for convenience define a matrix of values of μ^{-1} calculated at the values in [a]:

$$[\overset{*}{\mu}_t^{-1}] = \mu^{-1}(\overset{*}{\mathfrak{v}}_t^a([\mathrm{a}])) \tag{25}$$

So for any given $[\mathrm{a}]_\mathrm{i}$ we must have

$$[\mathrm{m}]_\mathrm{i} = [\mathrm{a}]_\mathrm{i} + \mathfrak{c}_t([\mathrm{a}]_\mathrm{i}) \tag{26}$$

We can now construct a consumption function $\mathrm{c}_{t+3\epsilon}(m_{t+3\epsilon}) \equiv \mathrm{c}_t(m_t)$ that corresponds to the Roman period (by interpolating among using $\{[\mathrm{m}], [\mathrm{c}]\}$).

If income is nonstochastic, say at $y_t = 1$, we can (if we like) define $[\mathrm{b}] = [\mathrm{m}] - 1$ and construct the Greek version of the solution from

$$\upsilon_t(b_t) = \mathrm{v}_t(b_t + 1) \tag{27}$$

$$\chi_t(b_t) = \mathrm{c}_t(b_t + 1) \tag{28}$$

and so on.

1.3 CDC Utility Specification

If we define a pseudo-inverse function

$$\mu^{-1}(\bullet) = \bullet^{-1/\rho} \tag{29}$$

and a corresponding

$$\mathfrak{c}_t(a) = \overset{*}{\mu}^{-1} \tag{30}$$

as in the main text, we can construct a list of gridpoints and an interpolating consumption function as in the basic model in the main text.