

# Structural Estimation of Dynamic Stochastic Optimizing Models of Intertemporal Choice For Dummies!

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June 2012

<http://www.econ2.jhu.edu/people/ccarroll/SolvingMicroDSOPs-Slides.pdf>

- Efficient Solution Methods for Canonical  $C$  problem
  - CRRA utility
  - Plausible (microeconomically calibrated) uncertainty
  - Life cycle or infinite horizon
- How To Add a Second Choice Variable
- Method of Simulated Moments Estimation of Parameters

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$$\log \theta \sim N(\sigma^2/2, \sigma^2) \quad \text{lognormal transitory shocks } \forall n \geq 0$$



## Bellman Equation

$$\mathbf{v}_t(\mathbf{m}_t, \mathbf{p}_t) = \max_{\mathbf{c}_t} u(\mathbf{c}_t) + \gamma \mathbb{E}_t[\mathbf{v}_{t+1}(\mathbf{m}_{t+1}, \mathbf{p}_{t+1})] \quad (4)$$

$m$ — 'market resources' (net worth plus current income)

$p$ — permanent labor income





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  - e.g., AR(1)
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# Trick: View Everything from End of Period

Define

$$v_t(a_t) = \mathbb{E}_t[\beta \mathcal{G}_{t+1}^{1-\rho} v_{t+1}(\mathcal{R}_{t+1} a_t + \theta_{t+1})] \quad (8)$$

so

$$v_t(m_t) = \max_{c_t} u(c_t) + v_t(m_t - c_t) \quad (9)$$

with FOC

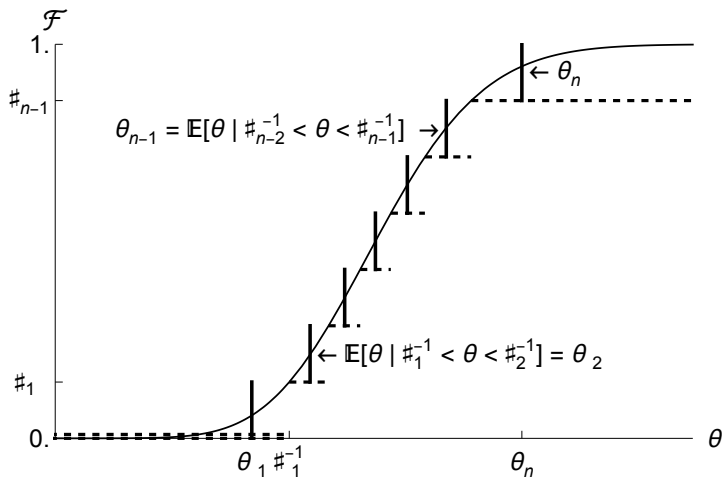
$$u'(c_t) = v'_t(m_t - c_t). \quad (10)$$

and Envelope relation

$$u'(c_t) = v'_t(m_t) \quad (11)$$

# Trick: Discretize the Risks

E.g. use an equiprobable 7-point distribution:





# Trick: Discretize the Risks

$$v'_t(a_t) = \beta R \mathcal{G}_{t+1}^{-\rho} \left( \frac{1}{n} \right) \sum_{i=1}^n u'(c_{t+1}(\mathcal{R}_{t+1}a_t + \theta_i)) \quad (12)$$

So for any particular  $m_{T-1}$  the corresponding  $c_{T-1}$  can be found using the FOC:

$$u'(c_t) = v'_t(m_t - c_t). \quad (13)$$

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[illegible]

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# Trick: Interpolate a Consumption Rule

- 1 Define a grid of points  $\vec{m}$  (indexed  $m[i]$ )
- 2 Use numerical rootfinder to solve  $u'(c) = v'_t(m[i] - c)$ 
  - The  $c$  that solves this becomes  $c[i]$
- 3 Construct interpolating function  $\hat{c}$  by linear interpolation
  - 'Connect-the-dots'

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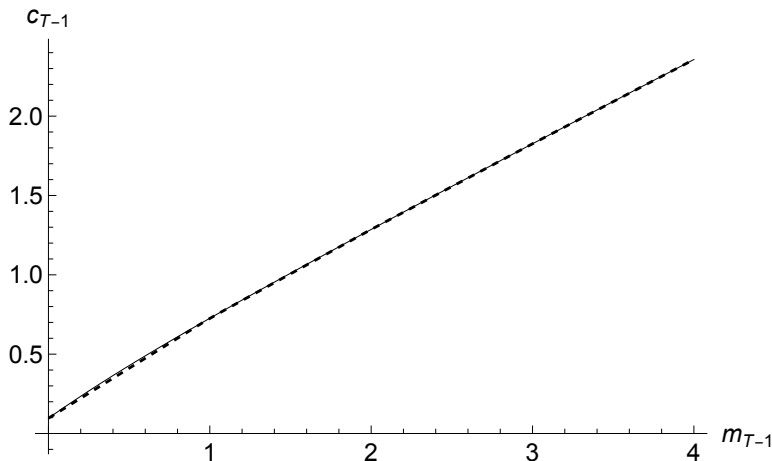
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# Trick: Interpolate a Consumption Rule

Example:  $\vec{m}_{T-1} = \{0., 1., 2., 3., 4.\}$  (solid is 'correct' soln)





# Problem: Numerical Rootfinding is *Slow*

Numerical search for values of  $c_{T-1}$  satisfying  $u'(c) = v'_t(m[i] - c)$  at, say, 6 gridpoints of  $\vec{m}_{T-1}$  may require hundreds or even thousands of evaluations of

$$v'_{T-1}(\overbrace{m_{T-1} - c_{T-1}}^{a_{T-1}}) = \beta_T \mathcal{G}_T^{1-\rho} \left( \frac{1}{n} \right) \sum_{i=1}^n (\mathcal{R}_T a_{T-1} + \theta_i)^{-\rho}$$

# Solution: The Method of Endogenous Gridpoints

- Define vector of *end-of-period* asset values  $\vec{a}$
- For each  $a[j]$  compute  $v'_t(a[j])$

Each of these  $v'_t[j]$  corresponds to a unique  $c[j]$  via FOC:

$$\begin{aligned} c[j]^{-\rho} &= v'_t(a[j]) \\ c[j] &= (v'_t(a[j]))^{-1/\rho} \end{aligned} \quad (14)$$

But the DBC says

$$\begin{aligned} a_t &= m_t - c_t \\ m[j] &= a[j] + c[j] \end{aligned} \quad (15)$$

So computing  $v'_t$  at a vector of  $\vec{a}$  values has produced for us the corresponding  $\vec{c}$  and  $\vec{m}$  values at virtually no cost!

From these we can interpolate as before to construct  $\hat{c}_t(m)$ .

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# Why Directly Approximating $v_t$ is a Bad Idea

## Principles of Approximation

- Hard to approximate things that approach  $\infty$  for relevant  $m$ 
  - Not a prob for Rep Agent models: 'relevant'  $m$ 's are  $\approx$  SS
- Hard to approximate things that are highly nonlinear

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# Approximate Something That Would Be Linear in PF Case

Perfect Foresight Theory:

$$c_t(m) = (m + h_t)\underline{\kappa}_t \quad (16)$$

for market resources  $m$  and end-of-period human wealth  $h$ .

This is why it's a good idea to approximate  $c_t$

Bonus: Easy to debug programs by setting  $\sigma^2 = 0$  and testing whether numerical solution matches analytical!

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# But What if You *Need* the Value Function?

Perfect foresight value function:

$$\begin{aligned}
 \bar{v}_t(m_t) &= u(\bar{c}_t) \mathbb{C}_t^T \\
 &= u(\bar{c}_t) \underline{\kappa}_t^{-1} \\
 &= u((\blacktriangle m_t + \blacktriangle h_t) \underline{\kappa}_t) \underline{\kappa}_t^{-1} \\
 &= u(\blacktriangle m_t + \blacktriangle h_t) \underline{\kappa}_t^{1-\rho} \underline{\kappa}_t^{-1} \\
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 \end{aligned} \tag{17}$$

where the second line uses the fact demonstrated in Carroll (2022) that  $\mathbb{C}_t = \kappa_t^{-1}$ .

This can be transformed as

$$\begin{aligned}
 \bar{\lambda}_t &\equiv ((1 - \rho) \bar{v}_t)^{1/(1-\rho)} \\
 &= c_t (\mathbb{C}_t^T)^{1/(1-\rho)} \\
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which is linear.

# Approximate Slope Too

Carroll (2022) shows that  $c_t^m$  exists everywhere.

Define *consumed* function and its derivative as

$$\begin{aligned} c_t(a) &= (v'_t(a))^{-1/\rho} \\ c_t^a(a) &= -(1/\rho) (v'_t(a))^{-1-1/\rho} v''_t(a) \end{aligned} \tag{19}$$

and using chain rule it is easy to show that

$$c_t^m = c_t^a / (1 + c_t^a) \tag{20}$$

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# To Implement: Modify Prior Procedures in Two Ways

- 1 Construct  $\vec{c}_t^m$  along with  $\vec{c}_t$  in EGM algorithm
- 2 Approximate  $c_t(m)$  using piecewise Hermite polynomial
  - Exact match to both level and derivative at set of points

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# Problem: $\hat{c}$ Below Bottom $m$ Gridpoint and Extrapolation

Consider what happens as  $a_{T-1}$  approaches  $\underline{a}_{T-1} \equiv -\underline{\theta}\mathcal{R}_T^{-1}$ ,

$$\lim_{a \downarrow \underline{a}_{T-1}} v'_{T-1}(a) = \lim_{a \downarrow \underline{a}_{T-1}} \beta R \mathcal{G}_T^{-\rho} \left( \frac{1}{n} \right) \sum_{i=1}^n (a \mathcal{R}_T + \theta_i)^{-\rho} \\ = \infty$$

This means our lowest value in  $\vec{a}_{T-1}$  should be  $> \underline{a}_{T-1}$ .

Suppose we construct  $\hat{c}$  by linear interpolation:

$$\hat{c}_{T-1}(m) = \hat{c}_{T-1}(\vec{m}_{T-1}[1]) + \hat{c}'_{T-1}(\vec{m}_{T-1}[1])(m - \vec{m}_{T-1}[1])$$

True  $c$  is strictly concave  $\Rightarrow \exists m^- > \underline{m}_{T-1}$  for which

$$m^- - \hat{c}_{T-1}(m^-) < \underline{a}_{T-1}$$



# Solution: Hard-Code the Bottom Point

Theory says that

$$\begin{aligned} \lim_{m \downarrow \underline{m}_{T-1}} c_{T-1}(m) &= 0 \\ \lim_{m \downarrow \underline{m}_{T-1}} c_{T-1}^m(m) &= \bar{\kappa}_{T-1} \end{aligned} \tag{21}$$

- ① Redefine  $\vec{a}$  *relative* to  $\underline{a}_{T-1}$
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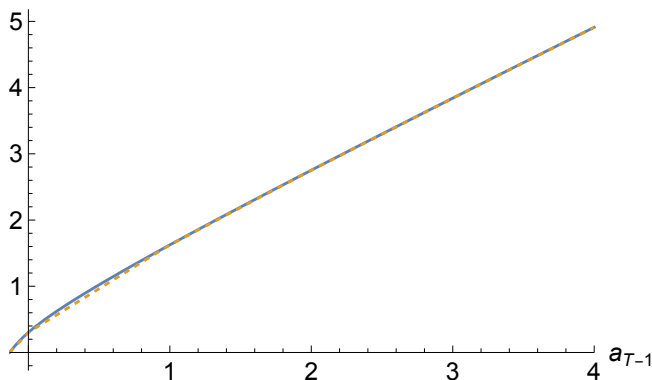
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# Trick: Improving the $a$ Grid

Grid Spacing: Uniform

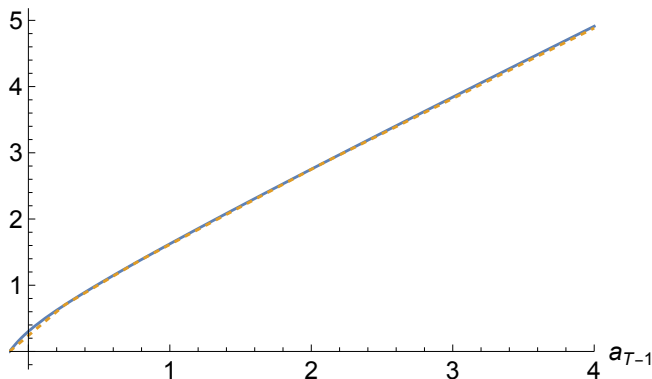
$$(u'_{T-1}(a_{T-1}))^{-1/\rho}, \dot{c}_{T-1}(a_{T-1})$$



# Trick: Improving the $a$ Grid

Grid Spacing: Same  $\{\underline{a}, \bar{a}\}$  But Triple Exponential  $e^{e^{\dots}}$  Growth

$$(v'_{T-1}(a_{T-1}))^{-1/\rho}, \dot{c}_{T-1}(a_{T-1})$$



# The Method of Moderation

- Further improves speed and accuracy of solution
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# Imposing 'Artificial' Borrowing Constraints

$$v_{T-1}(m_{T-1}) = \max_{c_{T-1}} u(c_{T-1}) + \mathbb{E}_{T-1}[\beta \mathcal{G}_T^{1-\rho} v_T(m_T)]$$

s.t.

$$a_{T-1} = m_{T-1} - c_{T-1}$$

$$m_T = \mathcal{R}_T a_{T-1} + \theta_T$$

$$a_{T-1} \geq 0.$$

Define  $\hat{c}_t^*$  as soln to unconstrained problem. Then

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Point where constraint makes transition from binding to not is

$$u'(m_{T-1}^{\#}) = v'_{T-1}(0.)$$

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Procedure is very easy:

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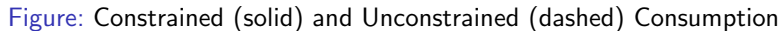
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③ Interpolate to create  $\hat{c}_t(m)$

# Recursion: Period $t$ Solution Given Period $t + 1$

## 1 Construct

$$\begin{aligned} c_{\bar{t},i} &= (v'_{\bar{t}}(a_{t,i}))^{-1/\rho}, \\ &= \left( \beta \mathbb{E}_t \left[ R \mathcal{G}_{t+1}^{-\rho} (\dot{c}_{t+1}(\mathcal{R}_{t+1} a_{t,i} + \theta_{t+1}))^{-\rho} \right] \right)^{-1/\rho}, \end{aligned} \quad (23)$$

- 2 Call the result  $\vec{c}_t$  and generate the corresponding  $\vec{m}_t = \vec{c}_t + \vec{a}_t$
- 3 Interpolate to create  $\dot{c}_t(m)$

# Consumption Rules $\dot{c}_{T-n}(m)$ Converge

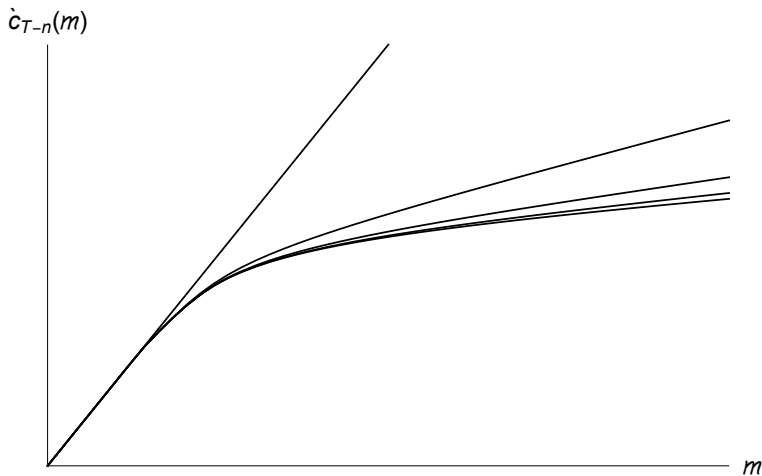


Figure: Converging  $\dot{c}_{T-n}(m)$  Functions for  $n = \{1, 5, 10, 15, 20\}$









# Portfolio Choice

Now the consumer has a choice between a risky and a safe asset.  
The portfolio return is

$$\begin{aligned}\mathbb{R}_{t+1} &= R(1 - \varsigma_t) + \mathbf{R}_{t+1}\varsigma_t \\ &= R + (\mathbf{R}_{t+1} - R)\varsigma_t\end{aligned}\tag{24}$$

so (setting  $\mathcal{G} = 1$ ) the maximization problem is

$$v_t(m_t) = \max_{\{c_t, \varsigma_t\}} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(m_{t+1})]$$

s.t.

$$\mathbb{R}_{t+1} = R + (\mathbf{R}_{t+1} - R)\varsigma_t$$

$$m_{t+1} = (m_t - c_t)\mathbb{R}_{t+1} + \theta_{t+1}$$

$$0 \leq \varsigma_t \leq 1,$$

$$u'(c_t) = \mathbb{E}_t[\beta \mathbb{R}_{t+1} u'(c_{t+1})]. \quad (25)$$

$$\begin{aligned} 0 &= \mathbb{E}_t[v'_{t+1}(m_{t+1})(\mathbf{R}_{t+1} - R)a_t] \\ &= a_t \mathbb{E}_t[u'(c_{t+1}(m_{t+1}))(\mathbf{R}_{t+1} - R)]. \end{aligned}$$



# Convergence

When the problem satisfies certain conditions (Carroll (2022)), it defines a ‘converged’ consumption rule with a ‘target’ ratio  $\check{m}$  that satisfies:

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m} \quad (26)$$

Define the target  $m$  implied by the consumption rule  $c_t$  as  $\check{m}_t$ .

Then a plausible metric for convergence is to define some value  $\epsilon$  and to declare the solution to have converged when

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- 1 Start with coarse grid for  $\theta$  (say, 3 points)
- 2 Solve to convergence; call period of convergence  $n$
- 3 Construct finer grid for  $\theta$  (say, 7 points)
- 4 Solve for period  $T - n - 1$  assuming  $\hat{c}_{T-n}$
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- Parameterization of Uncertainty
- Probability of Death
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# Empirical Wealth Profiles

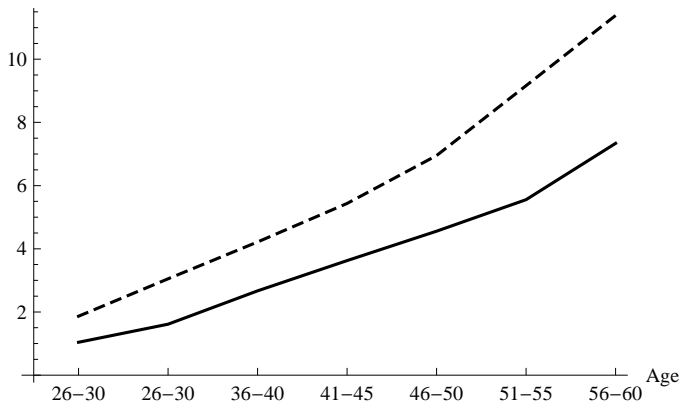


Figure:  $m$  from SCF (means (dashed) and medians (solid))

# Simulated Moments

Given a set of parameter values  $\{\rho, \Xi\}$ :

- Start at age 25 with empirical  $m$  data
- Draw shocks using calibrated  $\sigma_{\Psi}^2, \sigma_{\theta}^2$
- Consume according to solved  $c_t$

$\Rightarrow m$  distribution by age

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10

\_\_\_\_\_

```

GapEmpiricalSimulatedMedians[ $\rho, \beth$ ] :=
[
    ConstructcFuncLife[ $\rho, \beth$ ];
    Simulate;
    
$$\sum_i^N \omega_i |\varsigma_i^\tau - \mathbf{s}^\tau(\xi)|$$

];

```

---

$$\xi = \{\rho, \sqsupset\} \quad (28)$$

solve

$$\min_{\xi} \sum_i^N \omega_i |\varsigma_i^{\tau} - \mathbf{s}^{\tau}(\xi)| \quad (29)$$

## Yields estimates of

### Table: Estimation Results

| $\rho$ | $\beta$ |
|--------|---------|
| 4.68   | 1.00    |
| (0.13) | (0.00)  |

\_\_\_\_\_

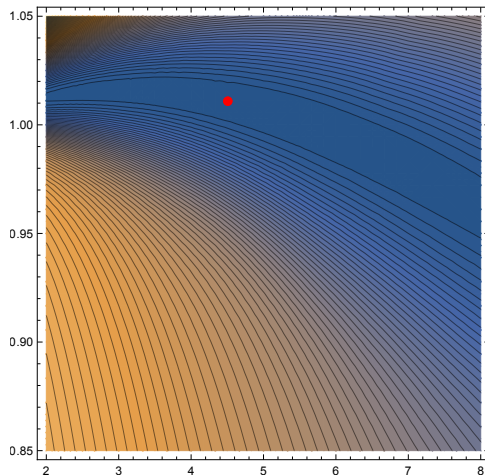


Figure: Point Estimate and Height of Minimized Function

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