

Structural Estimation of Dynamic Stochastic Optimizing Models of Intertemporal Choice For Dummies!

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<http://www.econ2.jhu.edu/people/ccarroll/SolvingMicroDSOPs-Slides.pdf>

- Efficient Solution Methods for Canonical C problem
 - CRRA utility
 - Plausible (microeconomically calibrated) uncertainty
 - Life cycle or infinite horizon
- How To Add a Second Choice Variable
- Method of Simulated Moments Estimation of Parameters

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1. *Journal of Management Studies*, 1991, 28, 1.

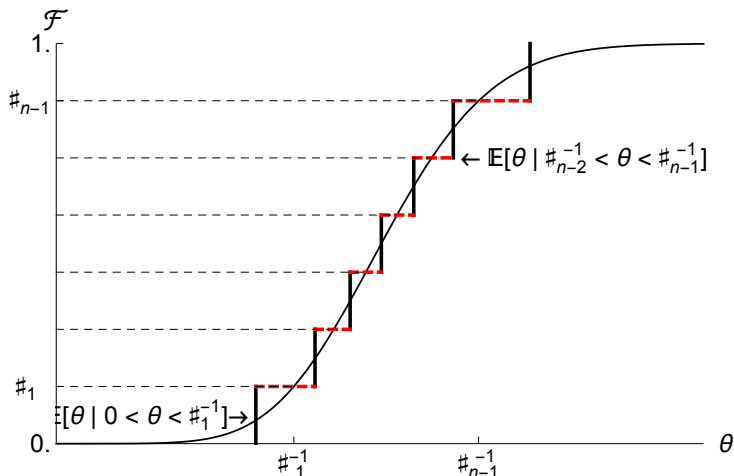
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- Non-Friedman (transitory/permanent) income process
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When Doesn't Normalization Work?

- Non-CRRA utility
- Non-Friedman (transitory/permanent) income process
 - e.g., AR(1)
 - But micro evidence is consistent with Friedman

Trick: Discretize the Risks

E.g. use an equiprobable 7-point distribution:



$$\mathbf{v}'_t(a_t) = \beta R \Gamma_{t+1}^{-\rho} \left(\frac{1}{n} \right) \sum_{i=1}^n u'(\mathbf{c}_{t+1}(\mathcal{R}_{t+1} a_t + \theta_i)) \quad (12)$$

Trick: Interpolate a Consumption Rule

- 1 Define a grid of points \vec{m} (indexed $m[i]$)
- 2 Use numerical rootfinder to solve $u'(c) = v'_t(m[i] - c)$
 - The c that solves this becomes $c[i]$
- 3 Construct interpolating function \hat{c} by linear interpolation
 - 'Connect-the-dots'

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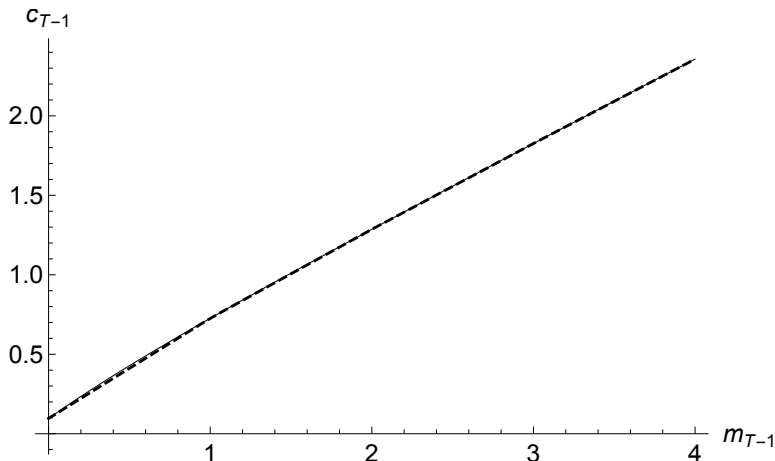
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Example: $\vec{m}_{T-1} = \{0., 1., 2., 3., 4.\}$ (solid is 'correct' soln)



Problem: Numerical Rootfinding is *Slow*

Numerical search for values of c_{T-1} satisfying $u'(c) = v'_t(m[i] - c)$ at, say, 6 gridpoints of \vec{m}_{T-1} may require hundreds or even thousands of evaluations of

$$v'_{T-1}(\overbrace{m_{T-1} - c_{T-1}}^{a_{T-1}}) = \beta_T \Gamma_T^{1-\rho} \left(\frac{1}{n} \right) \sum_{i=1}^n (\mathcal{R}_T a_{T-1} + \theta_i)^{-\rho}$$

Solution: The Method of Endogenous Gridpoints

- Define vector of *end-of-period* asset values \vec{a}
- For each $a[j]$ compute $v'_t(a[j])$

Each of these $v'_t[j]$ corresponds to a unique $c[j]$ via FOC:

$$c[j]^{-\rho} = v'_t(a[j]) \quad (14)$$

$$c[j] = (v'_t(a[j]))^{-1/\rho} \quad (15)$$

But the DBC says

$$a_t = m_t - c_t \quad (16)$$

$$m[j] = a[j] + c[j] \quad (17)$$

So computing v'_t at a vector of \vec{a} values has produced for us the corresponding \vec{c} and \vec{m} values at virtually no cost!

From these we can interpolate as before to construct $\hat{c}_t(m)$.

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Principles of Approximation

- Hard to approximate things that approach ∞ for relevant m
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Approximate Something That Would Be Linear in PF Case

Perfect Foresight Theory:

$$c_t(m) = (m + h_t)\underline{\kappa}_t \quad (18)$$

for market resources m and end-of-period human wealth h .

This is why it's a good idea to approximate c_t

Bonus: Easy to debug programs by setting $\sigma^2 = 0$ and testing whether numerical solution matches analytical!

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But What if You *Need* the Value Function?

Perfect foresight value function:

$$\bar{v}_t(m_t) = u(\bar{c}_t)\mathbb{C}_t^T \quad (19)$$

$$= u(\bar{c}_t)\underline{\kappa}_t^{-1} \quad (20)$$

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where the second line uses the fact demonstrated in Carroll (Forthcoming) that $\mathbb{C}_t = \kappa_t^{-1}$.

This can be transformed as

$$\begin{aligned} \bar{\lambda}_t &\equiv ((1 - \rho)\bar{v}_t)^{1/(1-\rho)} \\ &= c_t(\mathbb{C}_t^T)^{1/(1-\rho)} \\ &= (\blacktriangle m_t + \blacktriangle h_t)\underline{\kappa}_t^{-\rho/(1-\rho)} \end{aligned}$$

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Approximate Slope Too

Carroll (Forthcoming) shows that c_t^m exists everywhere.

Define *consumed* function and its derivative as

$$c_t(a) = (v'_t(a))^{-1/\rho} \quad (25)$$

$$c_t^a(a) = -(1/\rho) (v'_t(a))^{-1-1/\rho} v''_t(a) \quad (26)$$

and using chain rule it is easy to show that

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To Implement: Modify Prior Procedures in Two Ways

- 1 Construct \vec{c}_t^m along with \vec{c}_t in EGM algorithm
- 2 Approximate $c_t(m)$ using piecewise Hermite polynomial
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Problem: \hat{c} Below Bottom m Gridpoint and Extrapolation

Consider what happens as a_{T-1} approaches $\underline{a}_{T-1} \equiv -\underline{\theta}\mathcal{R}_T^{-1}$,

$$\begin{aligned} \lim_{a \downarrow \underline{a}_{T-1}} v'_{T-1}(a) &= \lim_{a \downarrow \underline{a}_{T-1}} \beta R \Gamma_T^{-\rho} \left(\frac{1}{n} \right) \sum_{i=1}^n (a \mathcal{R}_T + \theta_i)^{-\rho} \\ &= \infty \end{aligned}$$

This means our lowest value in \vec{a}_{T-1} should be $> \underline{a}_{T-1}$.

Suppose we construct \hat{c} by linear interpolation:

$$\hat{c}_{T-1}(m) = \hat{c}_{T-1}(\vec{m}_{T-1}[1]) + \hat{c}'_{T-1}(\vec{m}_{T-1}[1])(m - \vec{m}_{T-1}[1])$$

True c is strictly concave $\Rightarrow \exists m^- > \underline{m}_{T-1}$ for which

$$m^- - \hat{c}_{T-1}(m^-) < \underline{a}_{T-1}$$

Solution: Hard-Code the Bottom Point

Theory says that

$$\lim_{m \downarrow \underline{m}_{T-1}} c_{T-1}(m) = 0 \quad (28)$$

$$\lim_{m \downarrow \underline{m}_{T-1}} c_{T-1}^m(m) = \bar{\kappa}_{T-1} \quad (29)$$

- ① Redefine \vec{a} *relative* to \underline{a}_{T-1}
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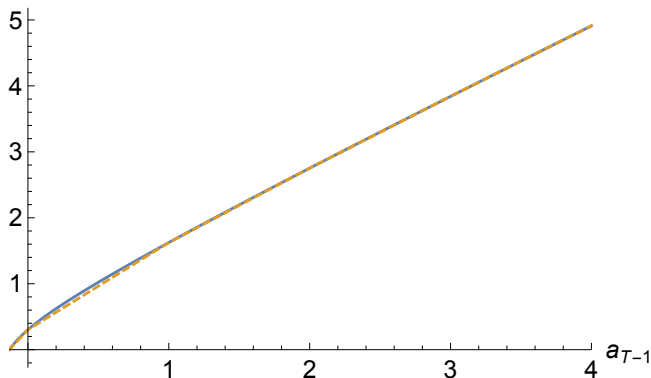
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Trick: Improving the a Grid

Grid Spacing: Uniform

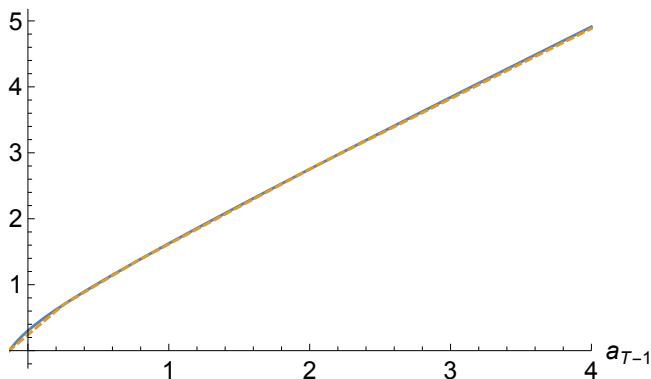
$$(u'_{T-1}(a_{T-1}))^{-1/\rho}, \dot{c}_{T-1}(a_{T-1})$$



Trick: Improving the a Grid

Grid Spacing: Same $\{\underline{a}, \bar{a}\}$ But Triple Exponential $e^{e^{\dots}}$ Growth

$$(u'_{T-1}(a_{T-1}))^{-1/\rho}, \dot{c}_{T-1}(a_{T-1})$$



The Method of Moderation

- Further improves speed and accuracy of solution
- See my talk at the conference!

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Imposing 'Artificial' Borrowing Constraints

$$\begin{aligned}
 v_{T-1}(m_{T-1}) &= \max_{c_{T-1}} u(c_{T-1}) + \mathbb{E}_{T-1}[\beta \Gamma_T^{1-\rho} v_T(m_T)] \\
 &\text{s.t.} \\
 a_{T-1} &= m_{T-1} - c_{T-1} \\
 m_T &= \mathcal{R}_T a_{T-1} + \theta_T \\
 a_{T-1} &\geq 0.
 \end{aligned}$$

Define \hat{c}_t^* as soln to unconstrained problem. Then

$$\hat{c}_{T-1}(m_{T-1}) = \min[m_{T-1}, \hat{c}_{T-1}^*(m_{T-1})]. \quad (30)$$

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A l l o c i t a t e →

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Procedure is very easy:

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- Above $m_{T-1}^{\#}$, $\hat{c}_{T-1}(m)$ obtained as before
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- Below $m_{T-1}^{\#}$, $\hat{c}_{T-1}(m) = m$

1. $\frac{1}{2}$ 2. $\frac{1}{3}$ 3. $\frac{1}{4}$ 4. $\frac{1}{5}$ 5. $\frac{1}{6}$ 6. $\frac{1}{7}$ 7. $\frac{1}{8}$ 8. $\frac{1}{9}$ 9. $\frac{1}{10}$ 10. $\frac{1}{11}$ 11. $\frac{1}{12}$ 12. $\frac{1}{13}$ 13. $\frac{1}{14}$ 14. $\frac{1}{15}$ 15. $\frac{1}{16}$ 16. $\frac{1}{17}$ 17. $\frac{1}{18}$ 18. $\frac{1}{19}$ 19. $\frac{1}{20}$ 20. $\frac{1}{21}$ 21. $\frac{1}{22}$ 22. $\frac{1}{23}$ 23. $\frac{1}{24}$ 24. $\frac{1}{25}$ 25. $\frac{1}{26}$ 26. $\frac{1}{27}$ 27. $\frac{1}{28}$ 28. $\frac{1}{29}$ 29. $\frac{1}{30}$ 30. $\frac{1}{31}$ 31. $\frac{1}{32}$ 32. $\frac{1}{33}$ 33. $\frac{1}{34}$ 34. $\frac{1}{35}$ 35. $\frac{1}{36}$ 36. $\frac{1}{37}$ 37. $\frac{1}{38}$ 38. $\frac{1}{39}$ 39. $\frac{1}{40}$ 40. $\frac{1}{41}$ 41. $\frac{1}{42}$ 42. $\frac{1}{43}$ 43. $\frac{1}{44}$ 44. $\frac{1}{45}$ 45. $\frac{1}{46}$ 46. $\frac{1}{47}$ 47. $\frac{1}{48}$ 48. $\frac{1}{49}$ 49. $\frac{1}{50}$ 50. $\frac{1}{51}$ 51. $\frac{1}{52}$ 52. $\frac{1}{53}$ 53. $\frac{1}{54}$ 54. $\frac{1}{55}$ 55. $\frac{1}{56}$ 56. $\frac{1}{57}$ 57. $\frac{1}{58}$ 58. $\frac{1}{59}$ 59. $\frac{1}{60}$ 60. $\frac{1}{61}$ 61. $\frac{1}{62}$ 62. $\frac{1}{63}$ 63. $\frac{1}{64}$ 64. $\frac{1}{65}$ 65. $\frac{1}{66}$ 66. $\frac{1}{67}$ 67. $\frac{1}{68}$ 68. $\frac{1}{69}$ 69. $\frac{1}{70}$ 70. $\frac{1}{71}$ 71. $\frac{1}{72}$ 72. $\frac{1}{73}$ 73. $\frac{1}{74}$ 74. $\frac{1}{75}$ 75. $\frac{1}{76}$ 76. $\frac{1}{77}$ 77. $\frac{1}{78}$ 78. $\frac{1}{79}$ 79. $\frac{1}{80}$ 80. $\frac{1}{81}$ 81. $\frac{1}{82}$ 82. $\frac{1}{83}$ 83. $\frac{1}{84}$ 84. $\frac{1}{85}$ 85. $\frac{1}{86}$ 86. $\frac{1}{87}$ 87. $\frac{1}{88}$ 88. $\frac{1}{89}$ 89. $\frac{1}{90}$ 90. $\frac{1}{91}$ 91. $\frac{1}{92}$ 92. $\frac{1}{93}$ 93. $\frac{1}{94}$ 94. $\frac{1}{95}$ 95. $\frac{1}{96}$ 96. $\frac{1}{97}$ 97. $\frac{1}{98}$ 98. $\frac{1}{99}$ 99. $\frac{1}{100}$ 100. $\frac{1}{101}$ 101. $\frac{1}{102}$ 102. $\frac{1}{103}$ 103. $\frac{1}{104}$ 104. $\frac{1}{105}$ 105. $\frac{1}{106}$ 106. $\frac{1}{107}$ 107. $\frac{1}{108}$ 108. $\frac{1}{109}$ 109. $\frac{1}{110}$ 110. $\frac{1}{111}$ 111. $\frac{1}{112}$ 112. $\frac{1}{113}$ 113. $\frac{1}{114}$ 114. $\frac{1}{115}$ 115. $\frac{1}{116}$ 116. $\frac{1}{117}$ 117. $\frac{1}{118}$ 118. $\frac{1}{119}$ 119. $\frac{1}{120}$ 120. $\frac{1}{121}$ 121. $\frac{1}{122}$ 122. $\frac{1}{123}$ 123. $\frac{1}{124}$ 124. $\frac{1}{125}$ 125. $\frac{1}{126}$ 126. $\frac{1}{127}$ 127. $\frac{1}{128}$ 128. $\frac{1}{129}$ 129. $\frac{1}{130}$ 130. $\frac{1}{131}$ 131. $\frac{1}{132}$ 132. $\frac{1}{133}$ 133. $\frac{1}{134}$ 134. $\frac{1}{135}$ 135. $\frac{1}{136}$ 136. $\frac{1}{137}$ 137. $\frac{1}{138}$ 138. $\frac{1}{139}$ 139. $\frac{1}{140}$ 140. $\frac{1}{141}$ 141. $\frac{1}{142}$ 142. $\frac{1}{143}$ 143. $\frac{1}{144}$ 144. $\frac{1}{145}$ 145. $\frac{1}{146}$ 146. $\frac{1}{147}$ 147. $\frac{1}{148}$ 148. $\frac{1}{149}$ 149. $\frac{1}{150}$ 150. $\frac{1}{151}$ 151. $\frac{1}{152}$ 152. $\frac{1}{153}$ 153. $\frac{1}{154}$ 154. $\frac{1}{155}$ 155. $\frac{1}{156}$ 156. $\frac{1}{157}$ 157. $\frac{1}{158}$ 158. $\frac{1}{159}$ 159. $\frac{1}{160}$ 160. $\frac{1}{161}$ 161. $\frac{1}{162}$ 162. $\frac{1}{163}$ 163. $\frac{1}{164}$ 164. $\frac{1}{165}$ 165. $\frac{1}{166}$ 166. $\frac{1}{167}$ 167. $\frac{1}{168}$ 168. $\frac{1}{169}$ 169. $\frac{1}{170}$ 170. $\frac{1}{171}$ 171. $\frac{1}{172}$ 172. $\frac{1}{173}$ 173. $\frac{1}{174}$ 174. $\frac{1}{175}$ 175. $\frac{1}{176}$ 176. $\frac{1}{177}$ 177. $\frac{1}{178}$ 178. $\frac{1}{179}$ 179. $\frac{1}{180}$ 180. $\frac{1}{181}$ 181. $\frac{1}{182}$ 182. $\frac{1}{183}$ 183. $\frac{1}{184}$ 184. $\frac{1}{185}$ 185. $\frac{1}{186}$ 186. $\frac{1}{187}$ 187. $\frac{1}{188}$ 188. $\frac{1}{189}$ 189. $\frac{1}{190}$ 190. $\frac{1}{191}$ 191. $\frac{1}{192}$ 192. $\frac{1}{193}$ 193. $\frac{1}{194}$ 194. $\frac{1}{195}$ 195. $\frac{1}{196}$ 196. $\frac{1}{197}$ 197. $\frac{1}{198}$ 198. $\frac{1}{199}$ 199. $\frac{1}{200}$ 200. $\frac{1}{201}$ 201. $\frac{1}{202}$ 202. $\frac{1}{203}$ 203. $\frac{1}{204}$ 204. $\frac{1}{205}$ 205. $\frac{1}{206}$ 206. $\frac{1}{207}$ 207. $\frac{1}{208}$ 208. $\frac{1}{209}$ 209. $\frac{1}{210}$ 210. $\frac{1}{211}$ 211. $\frac{1}{212}$ 212. $\frac{1}{213}$ 213. $\frac{1}{214}$ 214. $\frac{1}{215}$ 215. $\frac{1}{216}$ 216. $\frac{1}{217}$ 217. $\frac{1}{218}$ 218. $\frac{1}{219}$ 219. $\frac{1}{220}$ 220. $\frac{1}{221}$ 221. $\frac{1}{222}$ 222. $\frac{1}{223}$ 223. $\frac{1}{224}$ 224. $\frac{1}{225}$ 225. $\frac{1}{226}$ 226. $\frac{1}{227}$ 227. $\frac{1}{228}$ 228. $\frac{1}{229}$ 229. $\frac{1}{230}$ 230. $\frac{1}{231}$ 231. $\frac{1}{232}$ 232. $\frac{1}{233}$ 233. $\frac{1}{234}$ 234. $\frac{1}{235}$ 235. $\frac{1}{236}$ 236. $\frac{1}{237}$ 237. $\frac{1}{238}$ 238. $\frac{1}{239}$ 239. $\frac{1}{240}$ 240.

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$$\mathbf{c}_{t,i} = (\mathbf{v}'_t(\mathbf{a}_{t,i}))^{-1/\rho}, \quad (31)$$

$$= \left(\beta \mathbb{E}_t \left[\mathbf{R} \Gamma_{t+1}^{-\rho} (\dot{\mathbf{c}}_{t+1}(\mathcal{R}_{t+1} \mathbf{a}_{t,i} + \theta_{t+1}))^{-\rho} \right] \right)^{-1/\rho} (32)$$

- 2 Call the result \vec{c}_t and generate the corresponding $\vec{m}_t = \vec{c}_t + \vec{a}_t$
- 3 Interpolate to create $\hat{c}_t(m)$

Consumption Rules \dot{c}_{T-n} Converge

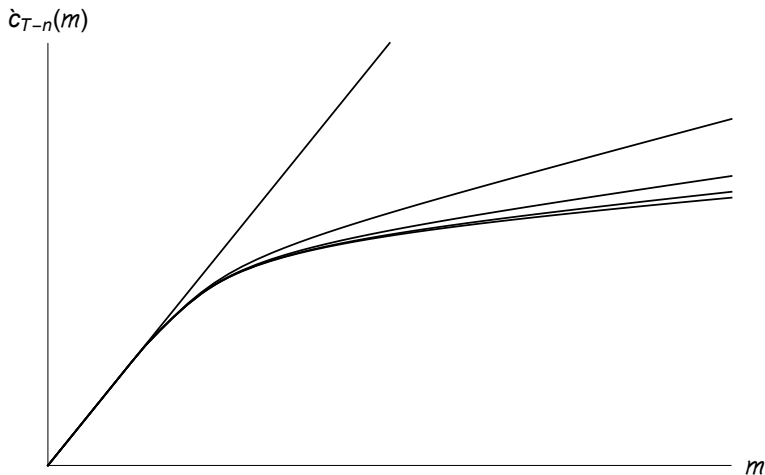


Figure: Converging $\dot{c}_{T-n}(m)$ Functions for $n = \{1, 5, 10, 15, 20\}$

Portfolio Choice

Now the consumer has a choice between a risky and a safe asset.

The portfolio return is

s.t.

$$R_{t+1} = R + (R_{t+1} - R)\zeta_t$$

$$m_{t+1} = (m_t - c_t)\mathbb{R}_{t+1} + \theta_{t+1}$$

$$0 \leq S_t \leq 1.$$

Portfolio Choice

Now the consumer has a choice between a risky and a safe asset.
The portfolio return is

$$\mathbb{R}_{t+1} = R(1 - \varsigma_t) + R_{t+1}\varsigma_t \quad (33)$$

$$= R + (R_{t+1} - R)\zeta_t \quad (34)$$

so (setting $\Gamma = 1$) the maximization problem is

s.t.

$$R_{t+1} = R + (R_{t+1} - R)\zeta_t$$

$$m_{t+1} = (m_t - c_t)\mathbb{R}_{t+1} + \theta_{t+1}$$

$$0 \leq S_t \leq 1,$$

$$u'(c_t) = \mathbb{E}_t[\beta R_{t+1} u'(c_{t+1})]. \quad (35)$$

Convergence

When the problem satisfies certain conditions (Carroll (Forthcoming)), it defines a 'converged' consumption rule with a 'target' ratio \check{m} that satisfies:

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m} \quad (37)$$

Define the target m implied by the consumption rule c_t as \check{m}_t .

Then a plausible metric for convergence is to define some value ϵ and to declare the solution to have converged when

$$|\check{m}_{t+1} - \check{m}_t| < \epsilon \quad (38)$$

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Trick: Coarse then Fine θ

- 1 Start with coarse grid for θ (say, 3 points)
- 2 Solve to convergence; call period of convergence n
- 3 Construct finer grid for θ (say, 7 points)
- 4 Solve for period $T - n - 1$ assuming \hat{c}_{T-n}
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Life Cycle Maximization Problem

$$v_t(m_t) = \max_{c_t} \left\{ u(c_t) + \beta_{t+1} \mathbb{E}_t[(\psi_{t+1} \Gamma_{t+1})^{1-\rho} v_{t+1}(m_{t+1})] \right\}$$

s.t.

$$a_t = m_t - c_t$$

$$m_{t+1} = a_t \underbrace{\left(\frac{R}{\psi_{t+1} \Gamma_{t+1}} \right)}_{\equiv \mathcal{R}_{t+1}} + \theta_{t+1}$$

β_s : probability alive (not dead) until age s given alive at age $s - 1$

$\hat{\beta}_s$: time-varying discount factor between age $s - 1$ and s

ψ_s : mean-one shock to permanent income

β : time-invariant discount factor

Details follow Cagetti (2003)

- Parameterization of Uncertainty
- Probability of Death
- Demographic Adjustments to β

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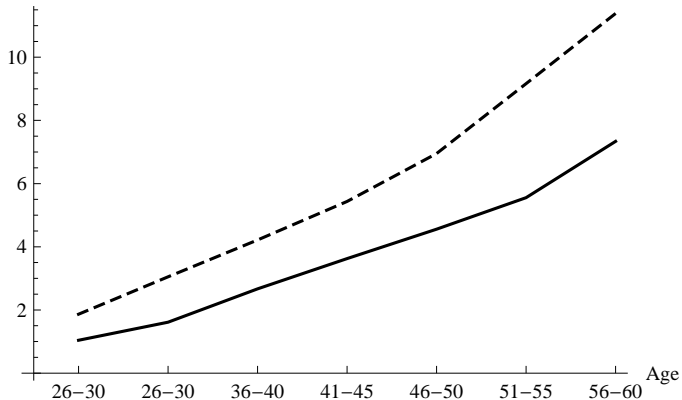


Figure: m from SCF (means (dashed) and medians (solid))

Simulated Moments

Given a set of parameter values $\{\rho, \Xi\}$:

- Start at age 25 with empirical m data
- Draw shocks using calibrated $\sigma_\psi^2, \sigma_\theta^2$
- Consume according to solved c_t

$\Rightarrow m$ distribution by age

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$\Rightarrow m$ distribution by age

```

GapEmpiricalSimulatedMedians[ $\rho, \triangleright$ ]:=
[ ConstructcFuncLife[ $\rho, \triangleright$ ];
  Simulate;
   $\sum_i^N \omega_i |\varsigma_i^\tau - s^\tau(\xi)|$ 
];

```

$$\xi = \{\rho, \mathbb{I}\} \quad (39)$$

solve

$$\min_{\xi} \sum_i^N \omega_i |\varsigma_i^T - s^T(\xi)| \quad (40)$$

Bootstrap Standard Errors (Horowitz (2001))

Yields estimates of

Table: Estimation Results

ρ	γ
4.68	1.00
(0.13)	(0.00)



References I

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- HOROWITZ, JOEL L. (2001): "The Bootstrap," in Handbook of Econometrics, ed. by James J. Heckman, and Edward Leamer, vol. 5. Elsevier/North Holland.