Practice Problems 1a: More Logic

Math 228 **Solutions** Spring 2012

1. (a) Negation: The power goes off and the food does not spoil.
   Converse: If the food spoils, then the power went off.
   Contrapositive: If the food does not spoil, then the power did not go off.

   (b) Negation: The door is closed and the light is on.
   Converse: If the light is off then the door is closed.
   Contrapositive: If the light is on then the door is open.

   (c) Negation: $\exists x(x < 1 \wedge x^2 \geq 1)$
   Converse: $\forall x(x^2 < 1 \rightarrow x < 1)$
   Contrapositive: $\forall x(x^2 \geq 1 \rightarrow x \geq 1)$.

   (d) Negation: There is a natural number n which is prime but not solitary.
   Converse: For all natural numbers n, if n is solitary, then n is prime.
   Contrapositive: For all natural numbers n, if n is not solitary then n is not prime.

   (e) Negation: There is a function which is differentiable and not continuous.
   Converse: For all functions f, if f is continuous then f is differentiable.
   Contrapositive: For all functions f, if f is not continuous then f is not differentiable.

   (f) Negation: There are integers a and b for which $a \cdot b$ is even but a or b is odd.
   Converse: For all integers a and b, if a and b are even then ab is even.
   Contrapositive: For all integers a and b, if a or b is odd, then ab is odd.

   (g) Negation: There are integers x and y such that for every integer n, $x \leq 0$ and $nx \leq y$.
   Converse: For every integer x and every integer y there is an integer n such that if $nx > y$ then $x > 0$.
   Contrapositive: For every integer x and every integer y there is an integer n such that if $nx \leq y$ then $x \leq 0$.

   (h) Negation: There are real numbers x and y such that $xy = 0$ but $x \neq 0$ and $y \neq 0$.
   Converse: For all real numbers x and y, if $x = 0$ or $y = 0$ then $xy = 0$
   Contrapositive: For all real numbers x and y, if $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

   (i) Negation: There is at least one student in Math 228 who does not understand implications but will still pass the exam.
   Converse: For every student in Math 228, if they fail the exam, then they did not understand implications.
   Contrapositive: For every student in Math 228, if they pass the exam, then they understood implications.

2. (a) Direct proof.

   Proof. Let n be an integer. Assume n is even. Then $n = 2k$ for some integer k. Thus $8n = 16k = 2(8k)$. Therefore $8n$ is even. □

   (b) The converse is false. That is, there is an integer n such that $8n$ is even but n is odd. For example, consider $n = 3$. Then $8n = 24$ which is even but $n = 3$ is odd.

3. (a) Direct proof.

   Proof. Let n be an integer. Assume n is odd. So $n = 2k + 1$ for some integer k. Then

   $$7n = 7(2k + 1) = 14k + 7 = 2(7k + 3) + 1$$

   Since $7k + 3$ is an integer, we see that $7n$ is odd. □

(b) The converse is: for all integers n if $7n$ is odd, then n is odd. We will prove this by contrapositive.

Proof. Let n be an integer. Assume n is not odd. Then $n = 2k$ for some integer k. So $7n = 14k = 2(7k)$ which is to say $7n$ is even. Therefore $7n$ is not odd. □

4. (a) Direct proof.

Proof. Let a and b be integers. Assume a is even and b is a multiple of 3. Then $a = 2k$ and $b = 3j$ for some integers k and j. Now

$$ab = (2k)(3j) = 6(kj)$$

Since kj is an integer, we have that ab is a multiple of 6. □

(b) The converse is: for all integers a and b, if ab is a multiple of 6, then a is even and b is a multiple of 3. This is false. Consider $a = 3$ and $b = 10$. Then $ab = 30$ which is a multiple of 6, but a is not even and b is not divisible by 3.

5. We give a proof by contradiction.

Proof. Suppose, contrary to stipulation that $\log(7)$ is rational. Then $\log(7) = \frac{a}{b}$ with a and $b \neq 0$ integers. By properties of logarithms, this implies

$$7 = 10^{\frac{a}{b}}$$

Equivalently,

$$7^b = 10^a$$

But this is impossible as any power of 7 will be odd while any power of 10 will be even. □

6. Again, by contradiction.

Proof. Suppose there were integers x and y such that $x^2 = 4y + 3$. Now x^2 must be odd, since $4y + 3$ is odd. Since x^2 is odd, we know x must be odd as well. So $x = 2k + 1$ for some integer k. Then $x^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. Therefore we have,

$$4(k^2 + k) + 1 = 4y + 3$$

which implies

$$4(k^2 + k) = 4y + 2$$

and therefore

$$2(k^2 + k) = 2y + 1.$$

But this is a contradiction - the left hand side is even while the right hand side is odd. □