Practice Problems 5: Induction

Math 228 **Solutions** Spring 2012

1. *Proof.* We must prove that $1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$ for all $n \in \mathbb{N}$. Thus let $P(n)$ be the statement $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$. We will prove that $P(n)$ is true for all $n \in \mathbb{N}$.

   First we establish the base case, $P(0)$, which claims that $1 = 2^{0+1} - 1$. Since $2^1 - 1 = 2 - 1 = 1$, we see that $P(0)$ is true.

   Now for the inductive case. Assume that $P(k)$ is true for an arbitrary $k \in \mathbb{N}$. That is, $1+2+2^2+\cdots+2^k = 2^{k+1} - 1$. We must show that $P(k+1)$ is true (i.e., that $1 + 2 + 2^2 + \cdots + 2^{k+1} = 2^{k+2} - 1$). To do this, we start with the left hand side of $P(k+1)$ and work to the right hand side:

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} &= 2^{k+1} - 1 + 2^{k+1} && \text{by the inductive hypothesis} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1 \end{aligned}$$

   Thus $P(k+1)$ is true so by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

2. *Proof.* Let $P(n)$ be the statement "$7^n - 1$ is a multiple of 6." We will show $P(n)$ is true for all $n \in \mathbb{N}$.

   First we establish the base case, $P(0)$. Since $7^0 - 1 = 0$, and 0 is a multiple of 6, $P(0)$ is true.

   Now for the inductive case. Assume $P(k)$ holds for an arbitrary $k \in \mathbb{N}$. That is, $7^k - 1$ is a multiple of 6, or in other words, $7^k - 1 = 6j$ for some integer j. Now consider $7^{k+1} - 1$:

$$\begin{aligned} 7^{k+1} - 1 &= 7^{k+1} - 7 + 6 && \text{by cleverness: } -1 = -7 + 6 \\ &= 7(7^k - 1) + 6 && \text{factor out a 7 from the first two terms} \\ &= 7(6j) + 6 && \text{by the inductive hypothesis} \\ &= 6(7j + 1) && \text{factor out a 6} \end{aligned}$$

   Therefore $7^{k+1} - 1$ is a multiple of 6, or in other words, $P(k+1)$ is true. Therefore by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

3. *Proof.* Let $P(n)$ be the statement $1 + 3 + 5 + \cdots + (2n - 1) = n^2$. We will prove that $P(n)$ is true for all $n \geq 1$.

   First the base case, $P(1)$. We have $1 = 1^2$ which is true, so $P(1)$ is established.

   Now the inductive case. Assume that $P(k)$ is true for some fixed arbitrary $k \geq 1$. That is, $1+3+5+\cdots+(2k-1) = k^2$. We will now prove that $P(k+1)$ is also true (i.e., that $1+3+5+\cdots+(2k+1) = (k+1)^2$). We start with the left hand side of $P(k+1)$ and work to the right hand side:

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= k^2 + (2k + 1) && \text{by the induction hypothesis} \\ &= (k + 1)^2 && \text{by factoring} \end{aligned}$$

   Thus $P(k+1)$ holds, so by the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$. □

4. *Proof.* Let $P(n)$ be the statement $F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$. We will show that $P(n)$ is true for all $n \geq 0$. First the base case is easy because $F_0 = 0$ and $F_1 = 1$ so $F_0 = F_1 - 1$. Now consider the inductive case. Assume $P(k)$ is true, that is, assume $F_0 + F_2 + F_4 + \cdots + F_{2k} = F_{2k+1} - 1$. To establish $P(k+1)$ we work from left to right:

$$\begin{aligned} F_0 + F_2 + F_4 + \cdots + F_{2k} + F_{2k+2} &= F_{2k+1} - 1 + F_{2k+2} && \text{by the inductive hypothesis} \\ &= F_{2k+1} + F_{2k+2} - 1 \\ &= F_{2k+3} - 1 && \text{by the recursive definition of the Fibonacci numbers} \end{aligned}$$

   Therefore $F_0 + F_2 + F_4 + \cdots + F_{2k+2} = F_{2k+3} - 1$, which is to say $P(k+1)$ holds. Therefore by the principle of mathematical induction, $P(n)$ is true for all $n \geq 0$. □

5. *Proof.* Let $P(n)$ be the statement $2^n < n!$. We will show $P(n)$ is true for all $n \geq 4$. First, we check the base case and see that yes, $2^4 < 4!$ (as $16 < 24$) so $P(4)$ is true. Now for the inductive case. Assume $P(k)$ is true for an arbitrary $k \geq 4$. That is, $2^k < k!$. Now consider $P(k+1)$: $2^{k+1} < (k+1)!$. To prove this, we start with the left side and work to the right side.

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &< 2 \cdot k! && \text{by the inductive hypothesis} \\ &< (k+1) \cdot k! && \text{since } k+1 > 2 \\ &= (k+1)! \end{aligned}$$

Therefore $2^{k+1} < (k+1)!$ so we have established $P(k+1)$. Thus by the principle of mathematical induction $P(n)$ is true for all $n \geq 4$. □

6. The only problem is that we never established the base case. Of course, when $n = 0$, $0 + 3 \neq 0 + 7$.

7. *Proof.* Let $P(n)$ be the statement that $n + 3 < n + 7$. We will prove that $P(n)$ is true for all $n \in \mathbb{N}$. First, note that the base case holds: $0 + 3 < 0 + 7$. Now assume for induction that $P(k)$ is true. That is, $k + 3 < k + 7$. We must show that $P(k+1)$ is true. Now since $k + 3 < k + 7$, add 1 to both sides. This gives $k + 3 + 1 < k + 7 + 1$. Regrouping $(k+1) + 3 < (k+1) + 7$. But this is simply $P(k+1)$. Thus by the principle of mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$. □

8. The problem here is that while $P(0)$ is true, and while $P(k) \to P(k+1)$ for *some* values of k, there is at least one value of k (namely $k = 99$) when that implication fails. For a valid proof by induction, $P(k) \to P(k+1)$ must be true for all values of k greater than or equal to the base case.

9. *Proof.* Let $P(n)$ be the statement "there is a strictly increasing sequence $a_1, a_2, a_3, \ldots, a_n$ with $a_n < 100$." We will prove $P(n)$ is true for all $n \geq 1$. First we establish the base case: $P(1)$ says there is a single number a_1 with $a_1 < 100$. This is true - take $a_1 = 0$. Now for the inductive step, assume $P(k)$ is true. That is there exists a strictly increasing sequence $a_1, a_2, a_3, \ldots, a_k$ with $a_k < 100$. Now consider this sequence, plus one more term, a_{k+1} which is greater than a_k but less than 100. Such a number exists - for example, the average between a_k and 100. So then $P(k+1)$ is true, so we have shown that $P(k) \to P(k+1)$. Thus by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

10. We once again failed to establish the base case: when $n = 0$, $n^2 + n = 0$ which is even, not odd.

11. *Proof.* Let $P(n)$ be the statement "$n^2 + n$ is even." We will prove that $P(n)$ is true for all $n \in \mathbb{N}$. First the base case: when $n = 0$, we have $n^2 + n = 0$ which is even, so $P(0)$ is true. Now suppose for induction that $P(k)$ is true, that is, that $k^2 + k$ is even. Now consider the statement $P(k+1)$. Now $(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1 = k^2 + k + 2k + 2$. By the inductive hypothesis, $k^2 + k$ is even, and of course $2k + 2$ is even. An even plus an even is always even, so therefore $(k+1)^2 + (k+1)$ is even. Therefore by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

12. Hint: the idea is to define the sequence so that a_n is less than the distance between the previous partial sum and 2. That way when you add it into the next partial sum, the partial sum is still less than 2. You could do this ahead of time, or use a clever $P(n)$ in the induction proof. Let $P(n)$ be the statement, "there is a sequence of positive real numbers $a_1, a_2, a_3, \ldots, a_n$ such that $a_1 + a_2 + a_3 + \cdots + a_n < 2$." The base case should be easy (just pick $a_1 < 2$). For the inductive case, you know that $a_1 + a_2 + \cdots + a_k < 2$ so you just need to argue that you can find some a_{k+1} small enough to have $a_1 + a_2 + \cdots + a_k + a_{k+1} < 2$.