Math 228 **Practice Problems 8: Binomial Identities** Spring 2012

**Instructions**: The problems below are purely for you to practice. I will not collect these, but it is still a good idea to write out your solutions in full. Any of these problems or problems similar are fair game for quizzes and exams.

1. Give a combinatorial proof for the identity $1 + 2 + 3 + \cdots + n = \binom{n+1}{2}$.

2. A woman is getting married. She has 15 best friends but can only select 6 of them to be her bride's maids, one of which needs to be her maid of honor. How many ways can she do this?
   (a) What if she first selects the 6 bride's maids, and then selects one of them to be the maid of honor?
   (b) What if she first selects her maid of honor, and then 5 other bride's maids?
   (c) Explain why $6\binom{15}{6} = 15\binom{14}{5}$.

3. Consider the bit strings in $\mathbf{B}_2^6$ (bit strings of length 6 and weight 2).
   (a) How many of those bit strings start with 1?
   (b) How many of those bit strings start with 01?
   (c) How many of those bit strings start with 001?
   (d) Are there any other strings we have not counted yet? Which ones, and how many are there?
   (e) How many bit strings are there total in $\mathbf{B}_2^6$?
   (f) What binomial identity have you just given a combinatorial proof for?

4. Let's count *ternary* digit strings - strings in which each digit can be 0, 1, or 2.
   (a) How many ternary digit strings contain exactly $n$ digits?
   (b) How many ternary digit strings contain exactly $n$ digits and $n$ 2's.
   (c) How many ternary digit strings contain exactly $n$ digits and $n-1$ 2's. (Hint: where can you put the non-2 digit, and then what could it be?)
   (d) How many ternary digit strings contain exactly $n$ digits and $n-2$ 2's. (Hint: see previous hint)
   (e) How many ternary digit strings contain exactly $n$ digits and $n-k$ 2's.
   (f) How many ternary digit strings contain exactly $n$ digits and no 2's. (Hint: what kind of a string is this?)
   (g) Use the above parts to give a combinatorial proof for the identity

$$\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + 2^3\binom{n}{3} + \cdots + 2^n\binom{n}{n} = 3^n$$

5. Give a combinatorial proof for the identity $P(n,k) = \binom{n}{k}k!$