Math 228 **Practice Problems 5: Induction** Spring 2012

Instructions: The problems below are purely for you to practice. I will not collect these, but it is still a good idea to write out your solutions in full. Any of these problems or problems similar are fair game for quizzes and exams.

1. Use induction to prove for all $n \in \mathbb{N}$ that $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$.

2. Prove that $7^n - 1$ is a multiple of 6 for all $n \in \mathbb{N}$.

3. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all $n \geq 1$.

4. Prove that $F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$ where F_n is the nth Fibonacci number.

5. Prove that $2^n < n!$ for all $n \geq 4$. (Recall, $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$.)

6. What is wrong with the following "proof" of the "fact" that $n + 3 = n + 7$ for all values of n (besides of course that the thing it is claiming to prove is false)?

   Proof. Let $P(n)$ be the statement that $n + 3 = n + 7$. We will prove that $P(n)$ is true for all $n \in \mathbb{N}$. Assume, for induction that $P(k)$ is true. That is, $k + 3 = k + 7$. We must show that $P(k + 1)$ is true. Now since $k + 3 = k + 7$, add 1 to both sides. This gives $k + 3 + 1 = k + 7 + 1$. Regrouping $(k + 1) + 3 = (k + 1) + 7$. But this is simply $P(k + 1)$. Thus by the principle of mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

7. The proof in the previous problem does not work. But if we modify the "fact," we can get a working proof. Prove that $n + 3 < n + 7$ for all values of $n \in \mathbb{N}$. You can do this proof with algebra (without induction) the goal of this exercise is to write out a valid induction proof.

8. Find the flaw in the following "proof" of the "fact" that $n < 100$ for every $n \in \mathbb{N}$.

   Proof. Let $P(n)$ be the statement $n < 100$. We will prove $P(n)$ is true for all $n \in \mathbb{N}$. First we establish the base case: when $n = 0$, $P(n)$ is true, because $0 < 100$. Now for the inductive step, assume $P(k)$ is true. That is, $k < 100$. Now if $k < 100$, then k is some number, like 80. Of course $80 + 1 = 81$ which is still less than 100. So $k + 1 < 100$ as well. But this is what $P(k + 1)$ claims, so we have shown that $P(k) \rightarrow P(k + 1)$. Thus by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

9. While the above proof does not work (it better not - the statement it is trying to prove is false!) we can prove something similar. Prove that there is a strictly increasing sequence $a_1, a_2, a_3, \ldots$ of numbers (not necessarily integers) such that $a_n < 100$ for all $n \in \mathbb{N}$. (By *strictly increasing* we mean $a_n < a_{n+1}$ for all n - so each term must be larger than the last.)

10. What is wrong with the following "proof" of the "fact" that for all $n \in \mathbb{N}$, the number $n^2 + n$ is odd?

    Proof. Let $P(n)$ be the statement "$n^2 + n$ is odd." We will prove that $P(n)$ is true for all $n \in \mathbb{N}$. Suppose for induction that $P(k)$ is true, that is, that $k^2 + k$ is odd. Now consider the statement $P(k + 1)$. Now $(k + 1)^2 + (k + 1) = k^2 + 2k + 1 + k + 1 = k^2 + k + 2k + 2$. By the inductive hypothesis, $k^2 + k$ is odd, and of course $2k + 2$ is even. An odd plus an even is always odd, so therefore $(k + 1)^2 + (k + 1)$ is odd. Therefore by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

11. Now give a valid proof (by induction - even though you might be able to do so without using induction) of the statement, "for all $n \in \mathbb{N}$, the number $n^2 + n$ is even."

12. Prove that there is a sequence of positive real numbers $a_1, a_2, a_3, \ldots$ such that the partial sum $a_1 + a_2 + a_3 + \cdots + a_n$ is strictly less than 2 for all $n \in \mathbb{N}$. This is a bit tricky - think about how you could define what a_{k+1} is to make the induction argument work.