(6pts) 1. For each of the following statements, write the converse, contrapositive, and the negation.

(a) $\exists x \forall y (x \geq 2y \rightarrow x > y + 1)$

Solution: Converse: $\exists x \forall y (x > y + 1 \rightarrow x \geq 2y)$ Contrapositive: $\exists x \forall y (x \leq y + 1 \rightarrow x < 2y)$ Negation: $\forall x \exists y (x \geq 2y \wedge x \leq y + 1)$

(b) For all integers a and b, if a is even and b is odd, then $a + b$ is odd.

Solution: Converse: For all integers a and b, if $a + b$ is odd, then a is even and b is odd. Contrapositive: For all integers a and b, if $a + b$ is not odd, then a is not even or b is not odd. Negation: There are integers a and b such that a is even and b is odd, but $a + b$ is not odd.

(5pts) 2. Prove the statement: For all integers n, if $5n$ is odd, then n is odd. Clearly state the style of proof you are using.

Solution: We will prove the contrapositive: if n is even, then $5n$ is even.

Proof. Let n be an arbitrary integer, and suppose n is even. Then $n = 2k$ for some integer k. Thus $5n = 5 \cdot 2k = 10k = 2(5k)$. Since $5k$ is an integer, we see that $5n$ must be even. This completes the proof. □

(5pts) 3. Prove the statement: For all integers a, b, and c, if $a^2 + b^2 = c^2$, then a or b is even. Hint: try a proof by contradiction.

Solution:

Proof. Suppose, contrary to stipulation, that there are integers a, b and c such that $a^2 + b^2 = c^2$ but a and b are both odd. Then $a = 2k + 1$ and $b = 2j + 1$ for some integers k and j. We then have

$$a^2 + b^2 = (2k + 1)^2 + (2j + 1)^2 = 4k^2 + 4k + 1 + 4j^2 + 4j + 1 = 4(k^2 + j^2 + k + j) + 2$$

So $c^2 = 4(k^2 + j^2 + k + j) + 2$. This means that c^2 is even, which can only happen if c is even. But then c^2 must be a multiple of 4. However, this is a contradiction because $4(k^2 + j^2 + k + j) + 2$ is not a multiple of 4. □

(4pts) 4. Order matters with quantifiers! Sometimes.

(a) Find a formula φ such that $\forall x \exists y \, \varphi$ is true but $\exists y \forall x \, \varphi$ is false.

Solution: There are many examples, but for instance φ could be $x < y$. It is true that for every x there is a y greater than it. However, there is not a y greater than every x

(b) Explain why you cannot find a formula ψ such that $\forall x \exists y \, \psi$ is false but $\exists y \forall x \, \psi$ is true.

Solution: Let's say $\exists y \forall x \psi$ is true. That means you can pick a y such that no matter what x is picked, ψ holds. Now say you pick that same y but keep it secret. Now you pick any x you like, and then reveal your previously selected y. Since you picked the same y as before, $\forall x \exists y \psi$ will also be true.