Generating Functions

1 Generating Functions

We complete our discussion of sequences with a powerful tool of discrete mathematics. A method of manipulating sequences: the generating function. The idea is this: instead of an infinite sequence (for example: $2, 3, 5, 8, 12, \ldots$) we look at a single function which encodes the sequence. But not a function which gives the nth term as output. Instead, a function whose power series (like from calculus 2) "displays" the terms of the sequence. So for example, we would look at the power series $2+3x+5x^2+8x^3+12x^4+\cdots$ which displays (as coefficients) the sequence $2, 3, 5, 8, 12, \ldots$.

An infinite power series is simply an infinite sum of terms of the form $c_n x^n$ were c_n is some constant. So we might write a power series like this:

$$\sum_{k=0}^{\infty} c_k x^k$$

or expanded like this

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

When viewed in the context of generating functions, we call such a power series a *generating series*. The generating series generates the sequence

$$c_0, c_1, c_2, c_3, c_4, c_5, \ldots$$

that is, the sequence generated by a generating series is simply the sequence of *coefficients* of the infinite polynomial.

> **Example**: What sequence is represented by the generating series $3 + 8x^2 + x^3 + \frac{x^5}{7} + 100x^6 + \cdots$?
>
> *Solution*: We just read off the coefficients of each x^n term. So $a_0 = 3$ since the coefficient of x^0 is 3 ($x^0 = 1$ so this is the constant term). What is a_1? It is NOT 8, since 8 is the coefficient of x^2, so 8 is the term a_2 of the sequence. To find a_1 we need to look for the coefficient of x^1 which in this case is 0. So $a_1 = 0$. We have $a_3 = 1$, $a_4 = 0$, and $a_5 = \frac{1}{7}$. So we have the sequence
>
> $$3, 0, 8, 1, \frac{1}{7}, 100, \ldots$$
>
> Note, when discussing generating functions, we always start our sequence with a_0.

Now you might very naturally ask why we would do such a thing. One reason is that encoding a sequence with a power series helps us keep track of which term is which in the sequence. For example, if we write the sequence $1, 3, 4, 6, 9, \ldots, 24, 41, \ldots$ it is impossible to which term 24 is (even if we agreed that the first term was supposed to be a_0). However, if wrote the generating series instead we would have $1 + 3x + 4x^2 + 6x^3 + 9x^4 + \cdots + 24x^{17} + 41x^{18} + \cdots$. Now it is clear that 24 is the 17th term of the sequence (that is, $a_{17} = 24$). Of course to get this benefit we could have displayed our sequence in any number of ways - perhaps $\boxed{1}_0\boxed{3}_1\boxed{4}_2\boxed{6}_3\boxed{9}_4\cdots\boxed{24}_{17}\boxed{41}_{18}\cdots$ - but we do not do this. The reason is that the generating series looks like an ordinary power series (although we are interpreting it differently) so we can do things with it that we ordinarily do with power series - such as write down what it converges to.

For example, from calculus we know that the power series $1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots+\frac{x^n}{n!}+\cdots$ converges to the function e^x. So we can use e^x as a way of talking about the sequence of coefficients of the power series for e^x. When we write down a nice compact function which has an infinite power series that we view as a generating series, then we call that function a *generating function.* In this example we would say,

$$1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \ldots, \frac{1}{n!}, \ldots \text{ has generating function } e^x$$

1.1 Building Generating Functions

The e^x example is very specific - we have that one rather odd sequence, and the only reason we know its generating function is because we happen to know the Taylor series for e^x. Our goal now is to gather some tricks to build the generating function when given a particular sequence.

Let's see what the generating functions are for some very simple sequences. The simplest of all: 1, 1, 1, 1, 1, ... What does the *generating series* look like? It is simply $1 + x + x^2 + x^3 + x^4 + \cdots$. Now, can we find a closed formula for this power series? Yes! This particular series is really just a geometric series with common ratio x. So if we use our "multiply, shift and subtract" technique, we have

$$\begin{aligned} S &= 1 + x + x^2 + x^3 + \cdots \\ -xS &= \phantom{1 + {}} x + x^2 + x^3 + x^4 + \cdots \\ \hline (1-x)S &= 1 \end{aligned}$$

Therefore we see that

$$1 + x + x^2 + x^3 \cdots = \frac{1}{1-x}$$

You might remember from calculus that this is only true on the interval of convergence for the power series - when $|x| < 1$. That is true for us, but we don't care - we are never going to plug anything in for x, so as long as there is some value of x for which the generating function and generating series agree, we are happy. And in this case we are happy:

> The generating function for $1, 1, 1, 1, 1, 1, \ldots$ is $\dfrac{1}{1-x}$

Now let's use this basic generating function to find generating functions for more sequences. What if we replace x by $-x$. We get

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \text{ which generates } 1, -1, 1, -1, \ldots$$

If we replace x by $3x$ we get

$$\frac{1}{1-3x} = 1 + 3x + 9x^2 + 27x^3 + \cdots \text{ which generates } 1, 3, 9, 27, \ldots$$

So by replacing the x in $\frac{1}{1-x}$ we can get generating functions for a variety of sequences. But not all. For example, you cannot plug in anything for x to get the generating function for $2, 2, 2, 2, \ldots$. However, we are not lost yet. Notice that each term of $2, 2, 2, 2, \ldots$ is the result of multiplying the terms of $1, 1, 1, 1, \ldots$ by a constant (2). So let's multiply the generating function by 2 as well.

$$\frac{2}{1-x} = 2 + 2x + 2x^2 + 2x^3 + \cdots \text{ which generates } 2, 2, 2, 2, \ldots$$

Similarly, to find the generating function for the sequence $3, 9, 27, 81, \ldots$, we note that this sequence is the result of multiplying each term of $1, 3, 9, 27, \ldots$ by 3. Since we have the generating function for $1, 3, 9, 27, \ldots$ we can say

$$\frac{3}{1-3x} = 3 \cdot 1 + 3 \cdot 3x + 3 \cdot 9x^2 + 3 \cdot 27x^3 + \cdots \text{ which generates } 3, 9, 27, 81, \ldots$$

What about the sequence $2, 4, 10, 28, 82, \ldots$? Here the terms are always 1 more than powers of 3. We have added the sequences $1, 1, 1, 1, \ldots$ and $1, 3, 9, 27, \ldots$ term by term. Therefore we can get a generating function by adding the respective generating functions:

$$2 + 4x + 10x^2 + 28x^3 + \cdots = (1+1) + (1+3)x + (1+9)x^2 + (1+27)x^3 + \cdots = \frac{1}{1-x} + \frac{1}{1-3x}$$

The fun does not stop there: if we replace x in our original generating function by x^2 we get

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 \cdots \text{ which generates } 1, 0, 1, 0, 1, 0, \ldots$$

How could we get $0, 1, 0, 1, 0, 1, \ldots$? Start with the previous sequence and *shift* it over by 1. But how do you do this? To see how shifting works, let's first try to get the generating function for the sequence $0, 1, 3, 9, 27, \ldots$. We know that $\frac{1}{1-3x} = 1 + 3x + 9x^2 + 27x^3 + \cdots$. To get the zero out front, we need the generating series to look like $x + 3x^2 + 9x^3 + 27x^4 + \cdots$ (so there is no constant term). Multiplying by x has this effect. So the generating function for $0, 1, 3, 9, 27, \ldots$ is $\frac{x}{1-3x}$. This will also work to get the generating function for $0, 1, 0, 1, 0, 1, \ldots$:

$$\frac{x}{1-x^2} = x + x^3 + x^5 + \cdots \text{ which generates } 0, 1, 0, 1, 0, 1 \ldots$$

What would happen if we add (term by term) the sequences $1, 0, 1, 0, 1, 0, \ldots$ and $0, 1, 0, 1, 0, 1, \ldots$. We should get $1, 1, 1, 1, 1, 1 \ldots$. What happens when we add the generating functions? It works (try it)!

$$\frac{1}{1-x^2} + \frac{x}{1-x^2} = \frac{1}{1-x}$$

Here's a tricky one: what happens if you take the derivative of $\frac{1}{1-x}$? We simply get $\frac{1}{(1-x)^2}$. But if we differentiate term by term in the power series, we get $(1 + x + x^2 + x^3 + \cdots)' = 1 + 2x + 3x^2 + 4x^3 + \cdots$ which is the generating series for $1, 2, 3, 4, \ldots$. This says

The generating function for $1, 2, 3, 4, 5, \ldots$ is $\dfrac{1}{(1-x)^2}$

What happens if we take a second derivative: $\frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + 20x^3 + \cdots$. So $\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + \cdots$ is a generating function for the triangular numbers.

1.2 Differencing

We have seen how to find generating functions from $\frac{1}{1-x}$ using multiplication (by a constant or by x), substitution, addition, and differentiation. To use each of these, you must notice a way to transform the sequence $1, 1, 1, 1, 1 \ldots$ into your desired sequence. This is not always easy. It is also not really the way we have been looking at analyzing sequences. One thing we have considered often is the sequence of differences between terms of a sequence. This will turn out to be helpful in finding generating functions as well. The idea is that the sequence of differences is often simpler than the original sequence. So if we know a generating function for the differences, we would like to use this to find a generating function for the original sequence. First, we must figure out how to relate a generating series to the generating series for the sequence of differences.

For example, consider the sequence $2, 4, 10, 28, 82, \ldots$. How could we move to the sequence of first differences: $2, 6, 18, 54, \ldots$? We want to subtract 2 from the 4, 4 from the 10, 10 from the 28, and so on. So if we subtract (term by term) the sequence $0, 2, 4, 10, 28, \ldots$ from $2, 4, 10, 28 \ldots$, we will be set. Of course it is easy to find the generating function for $0, 2, 4, 10, 28 \ldots$ (multiply the generating function for $2, 4, 10, 28, \ldots$ by x) - then just subtract. Use A to represent the generating function for $2, 4, 10, 28, 82, \ldots$ Then:

$$\begin{aligned} A &= 2 + 4x + 10x^2 + 28x^3 + 82x^4 + \cdots \\ \underline{-xA} &\underline{= 0 + 2x + 4x^2 + 10x^3 + 28x^4 + 82x^5 + \cdots} \\ (1-x)A &= 2 + 2x + 6x^2 + 18x^3 + 54x^4 + \cdots \end{aligned}$$

Now we don't get exactly the sequence of differences - but we get something close. In this particular case, we already know the generating function A (we found it in the previous section) but most of the time we will use this differencing technique to *find* A: if we have the generating function for the sequence of differences, we can then solve for A. Here is an example.

Example: Find a generating function for $1, 3, 5, 7, 9, \ldots$.

Solution: We notice that the sequence of differences is constant, and we know how to find the generating function for any constant sequence. So call the generating function for $1, 3, 5, 7, 9, \ldots$ simply A. We have

$$\begin{aligned} A &= 1 + 3x + 5x^2 + 7x^3 + 9x^4 + \cdots \\ \underline{-xA} &\underline{= 0 + x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \cdots} \\ (1-x)A &= 1 + 2x + 2x^2 + 2x^3 + 2x^4 + \cdots \end{aligned}$$

Now we know that $2x + 2x^2 + 2x^3 + 2x^4 + \cdots = \dfrac{2x}{1-x}$. Thus

$$(1-x)A = 1 + \frac{2x}{1-x}$$

Now solve for A:

$$A = \frac{1}{1-x} + \frac{2x}{(1-x)^2} = \frac{1+x}{(1-x)^2}$$

Does this makes sense? Before we simplified the two fractions into one, we were adding the generating function for the sequence $1, 1, 1, 1, \ldots$ to the generating function for the sequence $0, 2, 4, 6, 8, 10, \ldots$ (remember $\frac{1}{(1-x)^2}$ generates $1, 2, 3, 4, 5, \ldots$ - multiplying by $2x$ shifts it over, putting the zero out front, and doubles each term). If we add these term by term, we get the correct sequence $1, 3, 5, 7, 9, \ldots$.

Now that we have a generating function for the odd numbers, we can use that to find the generating function for the squares.

Example: Find the generating function for $1, 4, 9, 16, \ldots$.

Solution: Again we call the generating function for the sequence A. Use differencing:

$$\begin{aligned} A &= 1 + 4x + 9x^2 + 16x^3 + \cdots \\ \underline{-xA} &\underline{= 0 + x + 4x^2 + 9x^3 + 16x^4 + \cdots} \\ (1-x)A &= 1 + 3x + 5x^2 + 7x^3 + \cdots \end{aligned}$$

Since $1 + 3x + 5x^2 + 7x^3 + \cdots = \dfrac{1+x}{(1-x)^2}$ we have $A = \dfrac{1+x}{(1-x)^3}$

In each of the examples above, found the difference between consecutive terms which gave use a sequence of differences we knew a generating function for. We can generalize this to more complicated relationships between terms of the sequence. For example, what if we know that the sequence satisfies the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$? In other words, if we take a term of the sequence and subtracted 3 times the previous term and then added 2 times the term before that, we would get 0 (since $a_n - 3a_{n-1} + 2a_{n-2} = 0$). That will will for all but the first two terms of the sequence. So after the first two terms, the sequence of results of these calculations would be a sequence of 0's, which we definitely know a generating function for.

Example: The sequence $1, 3, 7, 15, 31, 63, \ldots$ satisfies the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$. Find the generating function for the sequence.

Solution: Call the generating function for the sequence A. We have

$$\begin{array}{rl} A &= 1 + 3x + 7x^2 + 15x^3 + 31x^4 + \cdots + a_n x^n + \cdots \\ -3xA &= 0 - 3x - 9x^2 - 21x^3 - 45x^4 - \cdots - 3a_{n-1}x^n - \cdots \\ + \quad 2x^2A &= 0 + 0x + 2x^2 + 6x^3 + 14x^4 + \cdots + 2a_{n-2}x^n + \cdots \\ \hline (1 - 3x + 2x^2)A &= 1 \end{array}$$

Let us see what happened there - we multiplied A by $-3x$ which shifts every term over one spot and multiplies them by -3. On the third line, we multiplied x by $2x^2$, which shifted every term over two spots and multiplied them by 2. When we add the corresponding terms up, we are taking each term, subtracting 3 times the previous term, and adding 2 times the term before that. You can see that for the initial terms this does indeed give $0x^n$. This will happen for each term because $a_n - 3a_{n-1} + 2a_{n-2} = 0$. In general, we might have two terms from the beginning of the generating series, although in this case the second term happens to be 0 as well.

Now we just need to solve for A:

$$A = \frac{1}{1 - 3x + 2x^2}$$

1.3 Multiplication - Partial Sums

What happens to the sequences when you multiply two generating functions? Let's see: $A = a_0 + a_1x + a_2x^2 + \cdots$ and $B = b_0 + b_1x + b_2x^2 + \cdots$. To multiply A and B, we need to do a lot of distributing (infinite FOIL?) but keep in mind we will regroup and only need to write down the first few terms to see the pattern. What is the constant term? a_0b_0. What is the coefficient of x? $a_0b_1 + a_1b_0$. And so on. We get:

$$AB = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + \cdots$$

Example: "Multiply" the sequence $1, 2, 3, 4, \ldots$ by the sequence $1, 2, 4, 8, 16, \ldots$.

Solution: The new constant term is just $1 \cdot 1$. The next term will be $1 \cdot 2 + 2 \cdot 1 = 4$. The next term: $1 \cdot 4 + 2 \cdot 2 + 3 \cdot 1 = 11$. One more: $1 \cdot 8 + 2 \cdot 4 + 3 \cdot 2 + 4 \cdot 1 = 28$. The resulting sequence is

$$1, 4, 11, 28, 57, \ldots$$

Since the generating function for $1, 2, 3, 4, \ldots$ is $\frac{1}{(1-x)^2}$ and the generating function for $1, 2, 4, 8, 16, \ldots$ is $\frac{1}{1-2x}$, we have that the generating function for $1, 4, 11, 28, 57, \ldots$ is $\frac{1}{(1-x)^2(1-2x)}$

Now what happens when you multiply a sequence by $1, 1, 1, \ldots$? Try it with $1, 2, 3, 4, 5 \ldots$. The first term is $1 \cdot 1 = 1$. Then $1 \cdot 2 + 1 \cdot 1 = 3$. Then $1 \cdot 3 + 1 \cdot 2 + 1 \cdot 1 = 6$. The next term will be 10. We are getting the triangular numbers. More precisely we get the sequence of partial sums of $1, 2, 3, 4, 5, \ldots$. In terms of generating functions, we take $\frac{1}{1-x}$ (generating $1, 1, 1, 1, 1 \ldots$) and multiply it by $\frac{1}{(1-x)^2}$ (generating $1, 2, 3, 4, 5, \ldots$) and this give $\frac{1}{(1-x)^3}$. This should not be a surprise - we found the same generating function for the triangular numbers earlier.

The point is, if you need to find a generating function for the sum of the first n terms of a particular sequence, and you know the generating function for *that* sequence, you can multiply it by $\frac{1}{1-x}$. This makes sense - to go back from the sequence of partial sums to the original sequence, you look at the sequence of differences. When you get the sequence of differences you end up multiplying by $1 - x$ - that is, dividing by $\frac{1}{1-x}$. Multiplying by $\frac{1}{1-x}$ gives partial sums, dividing by $11 - x$ gives differences.

1.4 Solving Recurrence Relations with Generating Functions

We end with an example of one of the many reasons studying generating functions is helpful - we can use generating functions to solve recurrence relations.

Example: Solve the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 3$.

Solution: We saw in an example above that this recurrence relation gives the sequence $1, 3, 7, 15, 31, 63, \ldots$ which has generating function $\dfrac{1}{1 - 3x + 2x^2}$. We did this by calling the generating function A and then computing $A - 3xA + 2x^2A$ which was just 1, since every other term canceled out.

But how does knowing the generating function help us? Well, we must first break up the generating function into two simpler ones. For this, we need partial fraction decomposition. Start by factoring the denominator:

$$\frac{1}{1 - 3x + 2x^2} = \frac{1}{(1-x)(1-2x)}$$

Now partial fraction decomposition tells us that we can write this faction as the sum of two fractions (we decompose the given fraction):

$$\frac{1}{(1-x)(1-2x)} = \frac{a}{1-x} + \frac{b}{1-2x} \quad \text{for some constants } a \text{ and } b$$

To find a and b we add the two decomposed fractions using a common denominator. This gives

$$\frac{1}{(1-x)(1-2x)} = \frac{a(1-2x) + b(1-x)}{(1-x)(1-2x)}$$

so

$$1 = a(1-2x) + b(1-x)$$

This must be true for all values of x. If $x = 1$, then the equation becomes $1 = -a$ so $a = -1$. When $x = \frac{1}{2}$ we get $1 = b/2$ so $b = 2$. This tells us that we can decompose the fraction like this:

$$\frac{1}{(1-x)(1-2x)} = \frac{-1}{1-x} + \frac{2}{1-2x}$$

This completes the partial fraction decomposition. But now notice that these two fractions look like generating functions we know. In fact, we should be able to expand each of them.

$$\frac{-1}{1-x} = -1 - x - x^2 - x^3 - x^4 - \cdots \text{ which generates } -1, -1, -1, -1, -1, \ldots$$

$$\frac{2}{1-2x} = 2 + 4x + 8x^2 + 16x^3 + 32x^4 + \cdots \text{ which generates } 2, 4, 8, 16, 32, \ldots$$

We can in fact give a closed formula for the nth term of each of these sequences. The first is just $a_n = -1$. The second is $a_n = 2^{n+1}$. The sequence we are interested in is just the sum of these, so the solution to the recurrence relation is

$$a_n = 2^{n+1} - 1$$

So now we can add generating functions to our list of methods for solving recurrence relations - although we do need to know how to do partial fraction decomposition.