Instructions: Same rules as usual - turn in your work on separate sheets of paper. You must justify all your answers for full credit.

(6pts) 1. Consider the following two graphs:
G_1: $V_1 = \{a, b, c, d, e, f, g\}$, $E_1 = \{\{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{b, e\}, \{b, f\}, \{c, g\}, \{d, e\}, \{e, f\}, \{f, g\}\}$.
G_2: $V_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$,
$E_2 = \{\{v_1, v_4\}, \{v_1, v_5\}, \{v_1, v_7\}, \{v_2, v_3\}, \{v_2, v_6\}, \{v_3, v_5\}, \{v_3, v_7\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_5, v_7\}\}$

(a) Let $f : G_1 \to G_2$ be a function that takes the vertices of Graph 1 to vertices of Graph 2. The function is given by the following table:

x	a	b	c	d	e	f	g
$f(x)$	v_4	v_5	v_1	v_6	v_2	v_3	v_7

Does f define an isomorphism between Graph 1 and Graph 2? Explain.

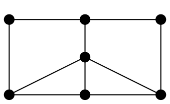
Solution: Recall that in order for f to define an isomorphism between $G1$ and $G2$, it must preserve relationships between vertices. To put this into context, this means that since a and b are joined via an edge in $G1$ that their corresponding vertices in $G2$ must also be joined by an edge. This must be true for all of the vertices and edges. When examining the function, we can see that the vertex g goes to v_7, that is $f(g) = v_7$. BUT, g has exactly 2 edges (so g is degree 2) and v_7 is degree 3. This means that f cannot possibly be an isomorphism. Similarly, we can see that f does not take c to the correct vertex either c is degree 2 and v_1 has degree 3.

(b) Define a new function g (with $g \neq f$) that defines an isomorphism between Graph 1 and Graph 2.

Solution:

x	a	b	c	d	e	f	g
$f(x)$	v_4	v_5	v_6	v_1	v_7	v_3	v_2

(c) Is the graph pictured below isomorphic to Graph 1 and Graph 2? Explain.

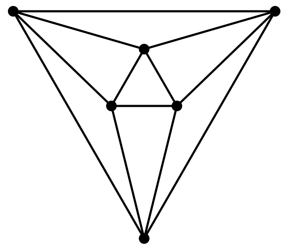


Solution: No, it could not possibly be isomorphic. If you count up the degrees of each vertex in this picture, you can see that the highest degree is 4 (the center vertex). In order to be isomorphic to either $G1$ or $G2$ we would definitely need a vertex of degree 5, which we don't have.

(9pts) 2. Recall that in class we proved that every (connected) planar graph with V vertices, E edges and F faces satisfied $V - E + F = 2$. We also saw that every convex polyhedron could be represented as a planar graph.

(a) An *octahedron* is a regular polyhedron made up of 8 equilateral triangles (it sort of looks like two pyramids with their bases glued together). Draw a planar graph representation of an octahedron. How many vertices, edges and faces does an octahedron (and your graph) have?

Solution: Since there are 8 triangles, there must be 8 faces. We can count the number of edges by taking $8 \cdot 3 = 24$, but this is double counting since each edge corresponds to two faces. Thus there are 12 edges. We can use Euler's formula to find that there are 6 vertices (and this shows that each vertex is the joining of 4 triangles).
The planar representation of the graph is:



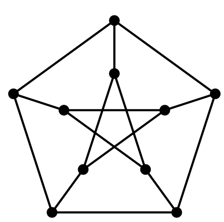
(b) The traditional design of a soccer ball is in fact a (spherical projection of a) truncated icosahedron. This consists of 12 regular pentagons and 20 regular hexagons. No two pentagons are adjacent (so the edges of each pentagon are shared only by hexagons). How many vertices, edges, and faces does a truncated icosahedron have? Explain how you arrived at your answers. Bonus: draw the planar graph representation of the truncated icosahedron.

Solution: Well, right off we know that the truncated icosahedron has $12 + 20 = 32$ faces by counting the number of pentagons and hexagons. Now, because we know that every connected planar graph with V vertices, E edges and F faces satisfied $V - E + F = 2$, we only really need to find out the number of edges or the number of vertices since $V - E = -30$. So, let's maybe try to figure out the number of edges we have. If we think about the number of total edges when the pentagons and hexagons are not attached, we know that we have $5 \times 12 + 6 \times 20 = 180$. But each of these edges is shared with another edge, which means that we have cut the number of edges in half. So, we have 90 edges, which then gives us 60 vertices.

(c) Your "friend" claims that he has constructed a convex polyhedron out of 2 triangles, 2 squares, 6 pentagons and 5 octagons. Prove that your friend is lying. Hint: each vertex of a convex polyhedron must border at least three faces.

Solution: So, let's assume for a contradiction that your friend really has constructed a convex polyhedron. Then, we would know that the polyhedron has 15 faces, $(2 \times 3 + 2 \times 4 + 6 \times 5 + 5 \times 8)/2 = 42$ edges, and $V = 2 + 42 - 15 = 29$ vertices. Now, using the hint, we can calculate the total number of vertices that this figure would have $2 \times 3 + 2 \times 4 + 6 \times 5 + 8 \times 5 = 84$. Now, when you put the polygon 'together' you need to assign at least three faces to one vertex which means that we should get that $29 < \frac{84}{3}$ which is not in fact true. Therefore, we have a contradiction.

(6pts) 3. Prove that the *Petersen graph* (below) is not planar. Hint: what is the length of the shortest cycle?



Solution:

Proof. Suppose, for contradiction, that the Petersen graph were planar. Then it would satisfy Euler's formula: $V - E + F = 2$. Since the graph has 10 vertices and 15 edges, this says that there must be 7 faces.

Now let B be the total number of boundaries around all faces when the graph is drawn in a planar way. Since each edge is used in two boundaries we have $B = 2E$. On the other hand, each face is surrounded by *at least* 5 boundaries, since the shortest cycle (circuit) in the graph contains 5 edges. Thus $F \leq \frac{B}{5}$. Putting these two facts together we get

$$F \leq \frac{2E}{5}$$

This is a contradiction, since $7 \not\leq \frac{30}{5}$. Alternatively, the above relationship says that $F \leq 6$, but we said $F = 7$ above.

Therefore the Petersen graph is not planar. □

(9pts) 4. Remember that a *tree* is a connected graph with no cycles.

(a) Conjecture a relationship between a tree graph's vertices and edges. (For instance, can you have a tree with 5 vertices and 7 edges?)

Solution: After drawing a few trees, you should notice that there is always exactly one more vertices than edges. That is $V = E + 1$

(b) Explain why every tree with at least 3 vertices has a leaf (i.e., a vertex of degree 1).

Solution: Assume for a contradiction that every tree with at least 3 vertices does not have a leaf. Namely, that there are no vertices of degree 1. Then, every vertex must have a degree of 2 or more, but this would imply that we have a cycle! Why? Think about taking a path from one vertex to another. Choose the first vertex and a route to leave it (remember it should have at least two ways to leave). Once I leave that vertex I reach another vertex which also has at least one more way to leave. Keep doing this. Eventually you will get back to one of the vertices that you have already visited and therefore you will have completed a cycle.

(c) Prove your conjecture from part (a) by induction on the number of vertices. Hint: For the inductive step, you will assume that your conjecture is true for all trees with k vertices, and show it is also true for an arbitrary tree with $k + 1$ vertices. Consider what happens when you cut off a leaf and then let it regrow.

Solution: Recall that we need two parts for an induction proof: a base case and an inductive case.

Proof. Let $P(n)$ be the statement "a tree graph T_n with n vertices has $n - 1$ edges. Base Case: Draw a tree with three vertices. Clearly you have only 2 edges (otherwise you would have a cycle).
Inductive Case: Assume $P(k)$ is true for some arbitrary $3 < k < n$.
NTS: $P(k + 1)$ is true. That is T_{k+1} has k edges.
Let T_{k+1} be a tree graph with $k+1$ vertices. By part (b) we know that every tree with at least 3 vertices has a leaf, so cut that one leaf off of T_{k+1}. Then our tree graph has only k vertices, and by our inductive case has $k - 1$ edges. Well, if we let that leaf regrow we will add both one edge and one vertex, which means that we will have a tree graph with $k + 1$ vertices and $k - 1 + 1 = k$ edges. TBTPOMI we have shown that $P(n)$ is true.