Mathematical Logic

Logic is the study of consequence. Given a few mathematical statements or facts, we would like to be able to draw some conclusions. For example, if I told you that a particular real valued function was continuous on the interval $[0, 1]$, and $f(0) = -1$ and $f(1) = 5$, can we conclude that there is some point between $[0, 1]$ where the graph of the function crosses the x-axis? Yes, we can, thanks to the Intermediate Value Theorem from Calculus. Can we conclude that there is exactly one point? No. Whenever we find an “answer” in math, we really have a (perhaps hidden) argument - given the situation we are in, we can conclude the answer is the case. Of course real mathematics is about proving general statements (like the Intermediate Value Theorem), and this too is done via an argument, usually called a proof. We start with some given conditions - these are the premises of our argument. From these we find a consequence of interest - our conclusion.

The problem is, as I’m sure you are aware from arguing with friends, not all arguments are good arguments. A “bad” argument is one in which the conclusion does not follow from the premises - the conclusion is not a consequence of the premises. Logic is the study of what makes an argument good or bad. In other words, logic aims to determine in which cases a conclusion is, or is not, a consequence of a set of premises.

By the way, “argument” is actually a technical term in math (and philosophy, another discipline which studies logic):

Definition 1. An *argument* is a set of statements, on of which is called the *conclusion* and the rest of which are called *premises*. An argument is said to be *valid* if the conclusion must be true whenever the premises are all true. An argument is *invalid* if it is not valid - if it is possible for all the premises to be true and the conclusion to be false.

For example, consider the following two arguments:

	If Edith eats her vegetables, then she can have a cookie.
	Edith eats her vegetables.
$\therefore$	Edith gets a cookie.

	Florence must eat her vegetables in order to get a cookie.
	Florence eats her vegetables.
$\therefore$	Florence gets a cookie.

(The symbol “$\therefore$” means “therefore”)

Are these arguments valid? Hopefully you agree that the first one is but the second one is not. Logic tells us why. How? By analyzing the structure of the statements in the argument. Notice that the two arguments above look almost identical. Edith and Florence both eat their vegetables. In both cases there is a connection between eating of vegetables and cookies. But we claim that it is valid to conclude that Edith gets a cookie, but not that Florence does. The difference must be in the connection between eating vegetables and getting cookies. We need to be good at reading and comprehending these English sentences. Do the two sentences mean the same thing? Unfortunately, when talking in everyday language, we are often sloppy, and you might be tempted to say they are equivalent. But notice that just because Florence *must* eat her vegetables, we have not said that doing so would be enough - she might also need to clean her room, for example. In everyday (non-mathematical) practice, you might say that the "other direction" was implied. We don't ever get to say that.

Our goal in studying logic is to gain intuition on which arguments are valid and which are invalid. This will require us to become better at reading and writing mathematics - a worthy goal in its own right. So let's get started.

1 Propositional Logic

A proposition is simply a statement. Propositional logic studies the ways statements can interact with each other. It is important to remember that propositional logic does not really care about the content of the statements. For example, from a propositional logic statement point, the claims, "if the moon is made of cheese then basketballs are round," and, "if spiders have eight legs then Sam walks with a limp" are exactly the same. They are both statements of the form, "if ⟨something⟩, then ⟨something else⟩."

Here's a question: is it true that when playing Monopoly, if you get more doubles than any other player you will lose, or that if you lose you must have bought the most properties? We will answer this question, and won't need to know anything about Monopoly. Instead we will look at the logical form of the statement. First though, let's back up and make sure we are very clear on some basics.

Definition 2. A *statement* is any declarative sentence which is either true or false.

Example: These are statements:

- Telephone numbers in the USA have 10 digits.
- The moon is made of cheese.
- 42 is a perfect square.
- Every even number greater than 2 can be expressed as the sum of two primes.

And these are not:

- Would you like some cake?
- The sum of two squares.
- $1 + 3 + 5 + 7 + \cdots + 2n + 1$.
- Go to your room!
- This sentence is false.[1]
- That's what she said.

The reason the last sentence is not a statement is because it contains variables ("that" and "she"). Unless those are specified, the sentence cannot be true or false, and as such not

[1] This is a tricky one. Remember, a sentence is only a statement if it is either true or false. Here, the sentence is not false, for if it were, it would be true. It is not true, for that would make it false.

a statement. Other examples of this: $x + 3 = 7$. Depending on x, this is either true or false, but as it stands it it neither.

You can build more complicated statements out of simpler ones using *logical connectives*. For example, this is a statement: Telephone numbers in the USA have 10 digits and 42 is a perfect square. Note that we can break this down into two smaller statements. The two shorter statements are *connected* by an "and". We will consider 5 connectives: "and" (Sam is a man and Chris is a woman), "or" (Sam is a man or Chris is a woman), "if. . . then. . . " (if Sam is a man, then Chris is a woman), "if and only if" (Sam is a man if and only if Chris is a woman), and "not" (Sam is not a man).

Since we rarely care about the content of the individual statements, we can replace them with variables. We use capital letters in the middle of the alphabet for these *propositional* (or *sentential*) variables: $P, Q, R, S \ldots$. We also have symbols for the logical connectives: $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\neg$.

Logical Connectives

- $P \wedge Q$ means P and Q, called a *conjunction*.
- $P \vee Q$ means P or Q, called a *disjunction*.
- $P \rightarrow Q$ means if P then Q, called an *implication* or *conditional*.
- $P \leftrightarrow Q$ means P if and only if Q, called a *biconditional*.
- $\neg P$ means not P, called a *negation*.

The logical connectives allow us to construct longer statements out of simpler statements. But the result is still a statement - it is either true or false. The *truth value* can be determined by the truth or falsity of the parts, depending on the connectives.

Truth Conditions for Connectives

- $P \wedge Q$ is true when both P and Q are true
- $P \vee Q$ is true when P or Q or both are true.
- $P \rightarrow Q$ is true when P is false or Q is true or both.
- $P \leftrightarrow Q$ is true when P and Q are both true, or both false.
- $\neg P$ is true when P is false.

I think all of these are obvious, except for $P \rightarrow Q$. Consider the statement, "If Bob gets a 90 on the final, then Bob will pass the class." This is definitely an implication: P is the statement "Bob gets a 90 on the final" and Q is the statement "Bob will pass the class." Suppose I said that to you - in what circumstances would it be fair to call me a liar? What if Bob really did get a 90 on the final, and he did pass the class? Then I have not lied to you, so my statement is true. But if Bob did get a 90 on the final and did not pass the class, then I lied, making the statement false. The tricky case is this: what if Bob did not get a 90 on the final? Maybe he passes the class, maybe he doesn't - did I lie in either case? I think not. In these last two cases, P was false, and so was the statement $P \to Q$. In the first case, Q was true, and so was $P \to Q$. So $P \to Q$ is true when either P is false or Q is true. Perhaps an easier way to look at it is this: $P \to Q$ is *false* in only one case: if P is true and Q is false. Otherwise, $P \to Q$ is true. Admittedly, there are times in English when this is not how "if…, then…" works. However, in mathematics, we *define* the implication to work this way.

1.1 Truth Tables

In order to answer our question about monopoly, we need to decide when the statement $(P \to Q) \vee (Q \to R)$ is true. Using the rules above, we see that either $P \to Q$ is true or $Q \to R$ is true (or both). Those are true if either P is false or Q is true (in the first case) and Q is false or R is true (in the second case). So... yeah, it gets kind of messy. Luckily, we can make a chart to keep track of all the possibilities. Enter truth tables. The idea is this: on each row, we list a possible combination of T's and F's (standing of course, for true and false) for each of the sentential variables, and then mark down whether the statement in question is true or false in that case. We do this for every possible combination of T's and F's. Then we can clearly see in which cases the statement is true or false. For complicated statements, we will first fill in values for each part of the statement, as a way of breaking up our task into smaller, more manageable pieces.

All you really need to know is the truth tables for each of the logical connectives. Here they are:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

The truth table for negation looks like this:

P	$\neg P$
T	F
F	T

None of these truth tables should come as a surprise - they are all just restating the definitions of the connectives. Let's try another one:

Example: Make a truth table for the statement $\neg P \vee Q$.

Solution: Note that this statement is not $\neg(P \vee Q)$, the negation belongs to P alone. Here is the truth table:

P	Q	$\neg P$	$\neg P \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

We added a column for $\neg P$ to make filling out the last column easier. The entries in the $\neg P$ column were determined by the entries in the P column. Then to fill in the final column, I look only at the column for Q and the column for $\neg P$ and use the rule of $\vee$.

You might notice that the final column is identical to the final column in the truth table for $P \to Q$. Since we listed the possible values for P and Q in the same (in fact, standard) order, this says that $\neg P \vee Q$ and $P \to Q$ are *logically equivalent*

Now let's answer our question about monopoly:

Example: Analyze the statement, "if you get more doubles than any other player you will lose, or that if you lose you must have bought the most properties," using truth tables.

Solution: Represent the statement in symbols as $(P \to Q) \vee (Q \to R)$, where P is the statement "you get more doubles than any other player," Q is the statement "you will lose," and R is the statement "you must have bought the most properties." Now make a truth table.

The truth table needs to contain 8 rows in order to account for every possible combination of truth and falsity among the three statements. Here is the full truth table:

P	Q	R	$P \to Q$	$Q \to R$	$(P \to Q) \vee (Q \to R)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

The first three columns are simply a systematic listing of all possible combinations of T and F for the three statements (do you see how you would list the 16 possible combinations for four statements?). The next two columns are determined by the values of P, Q, and R and the definition of implication.

> Then, the last column is determined by the values in the previous two columns and the definition of $\vee$. It is this final column we care about.
>
> Notice that in each of the eight possible cases, the statement in question is true. So our statement about monopoly is true (regardless of how many properties you own, how many doubles you roll, or whether you win or lose).

The statement about monopoly is an example of a *tautology* - a statement which is true on the basis of its logical form alone. Tautologies are always true but they don't tell us much about the world. No knowledge about monopoly was required to determine that the statement was true. In fact, it is equally true that "If the moon is made of cheese, then Elvis is still alive, or if Elvis is still alive, then unicorns have 5 legs."

1.2 Deductions

In the introduction, we claimed that the following was a valid argument:

> If Edith eats her vegetables, then she can have a cookie. Edith ate her vegetables. Therefore Edith gets a cookie.

How do we know this is valid? Let's look at the form of the statements. Let P denote "Edith eats her vegetables" and Q denote "Edith can have a cookie." The logical form of the argument is then:

$$\begin{array}{rc} & P \to Q \\ & P \\ \hline \therefore & Q \end{array}$$

This is an example of a *deduction rule* - a logical form of an argument which is always valid. This one is a particularly famous rule called *modus ponens*. Are you convinced that it is a valid deduction rule? If not, consider the following truth table:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

In this case, the truth table is identical to the truth table of $P \to Q$, but what matters here is that all the lines in the deduction rule have their own column in the truth table. Now remember that an argument is valid provided the conclusion must be true given that the premises are true. The premises in this case are $P \to Q$ and P. Which *rows* of the truth table correspond to both of these being true? P is true in the first two rows, and of those, only the first row has $P \to Q$ true as well. And low-and-behold, in this one case, Q is true as well. So if $P \to Q$ and P are both true, we see that Q must be true as well.

Here are a few more examples.

Example: Show that

$$\begin{array}{rc} & P \to Q \\ & \neg P \to Q \\ \hline \therefore & Q \end{array}$$

is a valid deduction rule.

Solution: We make a truth table which contains all the lines of the argument form:

P	Q	$P \to Q$	$\neg P$	$\neg P \to Q$
T	T	T	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	F

(we include a column for $\neg P$ just as a step to help getting the column for $\neg P \to Q$).

Now look at all the rows for which both $P \to Q$ and $\neg P \to Q$ are true. This happens only in rows 1 and 3. Hey! In those rows Q is true as well, so the argument form is valid (it is a valid deduction rule).

Example: Decide whether

$$\begin{array}{rc} & (P \to R) \vee (Q \to R) \\ \hline \therefore & (P \vee Q) \to R \end{array}$$

is a valid deduction rule.

Solution: Let's make a truth table containing both statements.

P	Q	R	$P \vee Q$	$P \to R$	$Q \to R$	$(P \vee Q) \to R$	$(P \to R) \vee (Q \to R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Look at the fourth row. In this case, $(P \to R) \vee (Q \to R)$ is true, but $(P \vee Q) \to R$ is false. Therefore the argument form is not valid (it is not a valid deduction rule). The truth table tells us more: our premise is true when P is true and Q and R are both false, but the conclusion is false in this case. The same problem occurs when Q is true and P and R are false (row 6).

Notice that if we switch the premise and conclusion, then we do have a valid argument - whenever $(P \vee Q) \to R$ is true, so is $(P \to R) \vee (Q \to R)$. Another way to say this is to state that the statement

$$[(P \vee Q) \to R] \to [(P \to R) \vee (Q \to R)]$$

is a tautology.

2 Rephrasing - Logical Equivalence

When reading or writing a proof, or even just trying to understand a mathematical statement, it can be very helpful to rephrase the statement. But how do you know you are doing so correctly? How do you know the two statements are equivalent?

One way is to make a truth table for each and ensure that the final columns of both are identical. We saw earlier that $P \to Q$ is logically equivalent to $\neg P \vee Q$ because their truth tables agreed. Now though we can just remember this fact - if we see the statement, "if Sam is a man then Chris is a woman," we can instead think of it as "Sam is a woman or Chris is a woman." You might also be tempted to rephrase further: "Sam or Chris is a woman." This is okay of course, but that this second rephrasing is allowed is due to the meaning of "is," not any of our logical connectives.

Here are some common logical equivalences which can help rephrase mathematical statements:

Double Negation

$$\neg\neg P \text{ is logically equivalent to } P$$

Example: "It is not the case that c is not odd" means "c is odd."

No surprise there. Now let's see how negation plays with conjunctions and disjunctions.

De Morgan's Laws

$$\neg(P \wedge Q) \text{ is logically equivalent to } \neg P \vee \neg Q$$

$$\neg(P \vee Q) \text{ is logically equivalent to } \neg P \wedge \neg Q$$

Example: "c is not even or c is not prime" means "c is not both odd and prime"

Do you believe De Morgan's laws? If not, make a truth table for each of them. I think most of us get these right most of the time without thinking about them too hard. If I told you that I had popcorn and goobers at the movies, but then you found out it was opposite day (so my statement was false) then you would agree, I hope, that I either did not have popcorn *or* did not have goobers (or didn't have either). You would not insist that I could not have had either.

I should warn you that often in English, we are sloppy about our and's, or's and not's. When you write about mathematics, you should be careful and write what you mean. If you are not sure what to write, rephrasing carefully using De Morgan's can help you make sure that statement matches your intended meaning.

Now some rules for implications:

Negation of Implication

$$\neg(P \to Q) \text{ is logically equivalent to } P \wedge \neg Q$$

In words: the only way for an implication to be false is for the "if" part to be true and the "then" part to be false.

This is very important, and not obvious - implications are tricky. But look at the truth table for $P \to Q$:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

There is only one way for the implication to be false - P is true and Q is false. Another way to see that this is true is by using De Morgan's Laws. We saw earlier that $P \to Q$ can be rephrased as $\neg P \vee Q$ so we have

$$\neg(P \to Q) \text{ is logically equivalent to } \neg(\neg P \vee Q)$$

But by De Morgan's laws, $\neg(\neg P \vee Q)$ is equivalent to $\neg\neg P \wedge \neg Q$. By double negation $\neg\neg P$ is the same as P.

While we are thinking about implications, we should talk about the converse and contrapositive:

Converse and Contrapositive

- The *converse* of an implication $P \to Q$ is the implication $Q \to P$. The converse is **NOT** logically equivalent to the original implication.
- The *contrapositive* of an implication $P \to Q$ is the statement $\neg Q \to \neg P$. An implication and its contrapositive are logically equivalent.

A related, but lesser used, term is the *inverse* of an implication. The inverse of $P \to Q$ is $\neg P \to \neg Q$. Notice that the inverse of an implication is the contrapositive of the converse. Read that one more time. Good. So since implications and their contrapositives are logically equivalent, the inverse and converse of an implication are logically equivalent to each other, but not to the original implication.

Example: Suppose I tell Sue that if she gets a 93% on her final, she will get an A in the class. Assuming that what I said is true, what can you conclude in the following cases:

(a) Sue gets a 93% on her final.

(b) Sue gets an A in the class.

(c) Sue does not get a 93% on her final.

(d) Sue does not get an A in the class.

Solution: Note first that whenever $P \to Q$ and P are both true statements, Q must be true as well. For this problem, take P to mean "Sue gets a 93% on her final" and Q to mean "Sue will get an A in the class."

(a) We have $P \to Q$ and P so Q follows - Sue gets an A.

(b) You cannot conclude anything - Sue could have gotten the A because she did extra credit for example. Notice that we do not know that if Sue gets an A, then she gets a 93% on her final - that is the converse of the original implication, so it might or might not be true.

(c) The inverse of $P \to Q$ is $\neg P \to \neg Q$, which states that if Sue does not get a 93% on the final then she will not get an A in the class. But this does not follow from the original implication. Again, we can conclude nothing - Sue could have done extra credit.

(d) What would happen if Sue does not get an A but *did* get a 93% on the final. Then P would be true and Q would be false. But this makes the implication $P \to Q$ false! So it must be that Sue did not get a 93% on the final. Notice now we have the implication $\neg Q \to \neg P$ - the contrapositive of $P \to Q$. Since $P \to Q$ is assumed to be true, we know $\neg Q \to \neg P$ is true as well - they are equivalent.

As we said above, an implication is not logically equivalent to its converse. Given particular statements P and Q, the statements $P \to Q$ and $Q \to P$ could both be true, both be false, or one could be true and the other false (in either order). Now if both $P \to Q$ and $Q \to P$ are true, then we say that P and Q are equivalent. In fact, we have:

If and only if

$$P \leftrightarrow Q \text{ is logically equivalent to } (P \to Q) \wedge (Q \to P).$$

Example: given an integer n, it is true that n is even if and only if n^2 is even. That is, if n is even, then n^2 is even, as well as the converse: if n^2 is even then n is even.

You can think of "if and only if" statements as having two parts: an implication as its converse. We might say one is the "if" part, and the other is the "only if" part. We also sometimes say that "if and only if" statements have two directions: a forward direction

($P \to Q$) and a backwards direction ($P \leftarrow Q$, which is really just sloppy notation for $Q \to P$).

Let's think a little about which part is which. Is $P \to Q$ the "if" part or the "only if" part? Perhaps we should look at an example.

Example: Suppose it is true that I sing if and only if I'm in the shower. We know this means that both if I sing, then I'm in the shower, and also the converse - that if I'm in the shower, then I sing. Let P be the statement, "I sing," and Q be, "I'm in the shower." So $P \to Q$ is the statement "if I sing, then I'm in the shower." Which part of the if and only if statement is this?

What we are really asking is what is the meaning of "I sing if I'm in the shower" and "I sing only if I'm in the shower." When is the first one (the "if" part) *false*? Well if I am in the shower but not singing. That is the same condition on being false as the statement "if I'm in the shower, then I sing." So the "if" part is $Q \to P$. On the other hand, to say, "I sing only if I'm in the shower" is equivalent to saying "if I sing, then I'm in the shower," so the only if part is $P \to Q$.

It is not terribly important to know which part is the if or only if part, but this does get at something very very important: THERE ARE MANY WAYS TO STATE AN IMPLICATION! The problem is, since these are all different ways of saying the same implication, we cannot use truth tables to analyze the situation. Instead, we just need good English skills.

Example: Rephrase the implication, "if I dream, then I am asleep" in as many different ways as possible. Then do the same for the converse.

Solution: The following are all equivalent to the original implication:

1. I am asleep if I dream.
2. I dream only if I am asleep.
3. In order to dream, I must be asleep.
4. To dream, it is necessary that I am asleep.
5. To be asleep, it is sufficient to dream.
6. I am not dreaming unless I am asleep.

The following are equivalent to the converse - if I am asleep, then I dream:

1. I dream if I am asleep.
2. I am asleep only if I dream.
3. It is necessary that I dream in order to be asleep.
4. It is sufficient that I be asleep in order to dream.
5. If I don't dream, then I'm not asleep.

Hopefully you agree with the above example. We include the "necessary and sufficient" versions because those are common when discussing mathematics. In fact, let's agree once and for all what they mean:

Necessary and Sufficient

- "P is necessary for Q" means $Q \to P$.
- "P is sufficient for Q" means $P \to Q$.
- If P is necessary and sufficient for Q, then $P \leftrightarrow Q$.

To be honest, I have trouble with these if I'm not very careful. I find it helps to have an example in mind:

Example: Recall from calculus, if a function is differentiable at a point c, then it is continuous at c, but that the converse of this statement is not true (for example, $f(x) = |x|$ at the point 0). Restate this fact using necessary and sufficient language.

Solution: It is true that in order for a function to be differentiable at a point c, it is necessary for the function to be continuous at c. However, it is not necessary that a function be differentiable at c for it to be continuous at c.

It is true that to be continuous at a point c, is is sufficient that the function be differentiable at c. However, it is not the case that being continuous at c is sufficient for a function to be differentiable at c.