**Instructions**: Same rules as usual - turn in your work on separate sheets of paper. You must justify all your answers for full credit.

(6pts) 1. Solve the recurrence relation $a_n = a_{n-1} + 3$ using:

(a) Telescoping. Show your work.

**Solution:**

$$\begin{aligned}
a_1 - a_0 &= 3 \\
a_2 - a_1 &= 3 \\
a_3 - a_2 &= 3 \\
\vdots \quad &\quad \vdots \\
+ \quad a_n - a_{n-1} &= 3 \\
\hline
a_n - a_0 &= 3n
\end{aligned}$$

Thus the solution is $a_n = 3n + a_0$.

(b) Iteration. Show you work.

**Solution:**

$$\begin{aligned}
a_1 &= a_0 + 3 \\
a_2 &= a_1 + 3 = (a_0 + 3) + 3 \\
a_3 &= a_2 + 3 = (a_0 + 3 + 3) + 3 \\
a_4 &= a_3 + 3 = (a_0 + 3 + 3 + 3) + 3 \\
\vdots &= \quad \vdots \\
a_n &= a_{n-1} + 3 = (a_0 + 3 + 3 + \cdots + 3) + 3
\end{aligned}$$

Every iteration just adds another 3. So the $n$th iteration takes $a_0$ and adds $n$ 3's. Thus $a_n = a_0 + 3n$.

2. Let $a_n$ be the number of $1 \times n$ tile designs can you make using $1 \times 1$ tiles available in 4 colors and $1 \times 2$ tiles available in 5 colors.

(3pts) (a) First, find a recurrence relation to describe the problem. Explain why the recurrence relation is correct (in the context of the problem).

> **Solution:** $a_n = 4a_{n-1} + 5a_{n-2}$. Each path of length $n$ must either start with one of the 4 $1 \times 1$ tiles, in each case there are then $a_{n-1}$ ways to finish the path, or start with one of the 5 $1 \times 2$ tiles, in each case there are then $a_{n-2}$ ways to finish the path.

(2pts) (b) Write out the first 6 terms of the sequence $a_1, a_2, \ldots$.

> **Solution:** 4, 21, 104, 521, 2604, 13021

(3pts) (c) Solve the recurrence relation. That is, find a closed formula for $a_n$.

> **Solution:** The characteristic equation is $x^2 - 4x - 5 = 0$ so the characteristic roots are $x = 5$ and $x = -1$. Therefore the general solution is
>
> $$a_n = a5^n + b(-1)^n$$
>
> We solve for $a$ and $b$ using the fact that $a_1 = 4$ and $a_2 = 21$. We get $a = \frac{5}{6}$ and $b = \frac{1}{6}$. Therefore the solution is
>
> $$a_n = \frac{5}{6}5^n + \frac{1}{6}(-1)^n$$

(6pts) 3. Consider the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$.

(a) Find the general solution to the recurrence relation (beware the repeated root).

> **Solution:** The characteristic polynomial is $x^2 - 4x + 4$ which factors as $(x-2)^2$, so the only characteristic root is $x = 2$. Thus the general solution is
>
> $$a_n = a2^n + bn2^n$$

(b) Find the solution when $a_0 = 1$ and $a_1 = 2$.

> **Solution:** Since $1 = a2^0 + b \cdot 0 \cdot 2^0$ have have $a = 1$. Then $2 = 2^1 + b2^1$ so $b = 0$. We have the solution
>
> $$a_n = 2^n$$

(c) Find the solution when $a_0 = 1$ and $a_1 = 8$.

> **Solution:** Again, we have $a = 1$. Now when we plug in $n = 1$ we bet $8 = 2 + 2b$ so $b = 3$. The solution:
>
> $$a_n = 2^n + 3n2^n$$

(10pts) 4. Write down first 6 or so terms of the sequences generated by each of the following generating functions, using the fact that $\frac{1}{1-x}$ generates $1, 1, 1, 1, \ldots$. In each case, briefly explain how you arrived at your answer.

(a) $\dfrac{5}{1-x}$

> **Solution:** $5, 5, 5, 5, 5, 5, \ldots$. We multiplied the power series by 5 - each term got multiplied by 5.

(b) $\dfrac{1}{1+2x}$

> **Solution:** $1, -2, 4, -8, 16, -32, \ldots$. We substituted $-2x$ in for $x$. This gives the power series $1+(-2x)+(-2x)^2+(-2x)^3+\cdots$. Thus we get the powers of $-2$ as coefficients.

(c) $\dfrac{1}{(1-x^2)^2}$

> **Solution:** $1, 0, 2, 0, 3, 0, 4, 0, 5, \ldots$. We know $\frac{1}{(1-x)^2}$ generates $1, 2, 3, 4, \ldots$ (by taking the derivative of $\frac{1}{1-x}$ for example). Then substituting $x^2$ for $x$ spaces out the sequence - we have the same coefficients, but now only on even powers of $x$ - the coefficients of odd powers of $x$ are odd.

(d) $\dfrac{1}{1+2x} + \dfrac{5}{1-x}$

> **Solution:** $6, 3, 9, -3, 21, -27, \ldots$. Here we just added the sequences from above, term by term.

(e) $\dfrac{1}{1+2x} \cdot \dfrac{5}{1-x}$

> **Solution:** $5, -5, 15, -25, 55, -105, \ldots$. When multiplying two generating functions, the $n$th term is the sum of the first $n$ terms of one, each multiplied by the first $n$ terms in reverse order of the other. So here we get this sequence like this: $(1 \cdot 5), (1 \cdot 5 + (-2) \cdot 5), (1 \cdot 5 + (-2) \cdot 5 + 4 \cdot 5)$ and so on.
> Alternatively, multiplying a generating function by $\frac{1}{1-x}$ gives the sequence of partial sums. Then multiply each term by 5.