Practice Problems 10: Graph Theory

Math 228 **Solutions** Spring 2012

1. This is asking for the number of edges in K_{10}. Each vertex (person) has degree (shook hands with) 9 (people). So the sum of the degrees is 90. However, the degrees count each edge (handshake) twice, so there are 45 edges in the graph. That is how many handshakes took place.

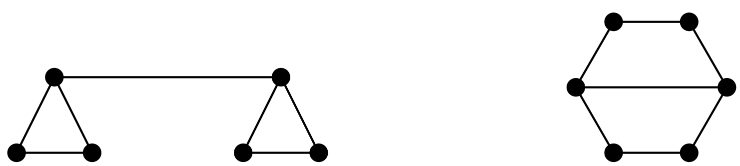
2. It is possible for everyone to be friends with exactly 2 people - you could arrange the 5 people in a circle and say that everyone is friends with the two people on either side of them (so you get the graph C_5). However, it is not possible for everyone to be friends with 3 people - that would lead to a graph with an odd number of odd degree vertices which is impossible - the sum of the degrees must be even.

3. This is a question about finding Euler paths. Draw a graph with a vertex in each state, and connect vertices if their states share a border. Exactly two vertices will have odd degree - the vertices for Nevada and Utah. Thus you must start your road trip at in one of those states and end it in the other.

4. The first and the third graphs are the same, but the middle graph is different.

5. The first (and third) graphs contain an Euler path. All the graphs are planar.

6. Yes. For example, both graphs below contain 6 vertices, 7 edges, and have degrees (2,2,2,2,3,3).



7. (a) K_4 does not have an Euler path or circuit.
   (b) K_5 has an Euler circuit (so also an Euler path).
   (c) $K_{5,7}$ does not have an Euler path or circuit.
   (d) $K_{2,7}$ has an Euler path but not an Euler circuit.
   (e) C_7 has an Euler circuit (it is a circuit graph!)
   (f) P_7 has an Euler path but no Euler circuit.

8. When n is odd, K_n contains an Euler circuit. This is because every vertex has degree $n-1$, so an odd n results in all degrees being even.

9. If both m and n are even, then $K_{m,n}$ has an Euler circuit. When both are odd, there is no Euler path or circuit. If one is 2 and the other is odd, then there is an Euler path but not an Euler circuit.

10. Three of the graphs are bipartite. The one which is not is C_7 (second from the right).

11. C_n is bipartite if and only if $n = 1$ or is even.

12. For example, K_5.

13. For example, $K_{3,3}$.

14. No. A (connected) planar graph must satisfy Euler's formula: $V-E+F=2$. Here $V-E+F=6-10+5=1$.

15. Yes. According to Euler's formula it would have 2 faces. It does. The only such graph is C_{10}.

16. G has 10 edges. It could be planar, and then it would have 6 faces.

17. 2, since the graph is bipartite. One color for the top set of vertices, another color for the bottom set of vertices.

18. For example, K_6. If the chromatic number is 6, then the graph is not planar - the 4-color theorem states that all planar graphs can be colored with 4 or fewer colors.

19. The chromatic numbers are 2, 3, 4, 5, and 3 respectively from left to right.

20. (a) Only if $n \geq 6$ and is even.

    (b) None.

    (c) 12. Such a graph would have $\frac{5n}{2}$ edges. If the graph is planar, then $n-\frac{5n}{2}+F=2$ so there would be $\frac{4+3n}{2}$ faces. Also, we must have $3F \leq 2E$, since the graph is simple. So we must have $3\frac{4+3n}{2} \leq 5n$. Solving for n gives $n \geq 12$.