Practice Problems 3: Functions

Math 228 **Hints and Answers** Spring 2012

1. There are 8 different functions. For example, $f(1) = a$, $f(2) = a$, $f(3) = a$; or $f(1) = a$, $f(2) = b$, $f(3) = a$, and so on. None of the functions are one-to-one. Exactly 6 of the functions are onto. No functions are both (since no functions here are one-to-one).

2. There are nine functions - you have a choice of three outputs for $f(1)$, and for each, you have three choices for the output $f(2)$. Of these functions, 6 are one-to-one, 0 are onto, and 0 are both.

3. (a) f is not one-to-one, since $f(2) = f(5)$ - two different inputs have the same output.
   (b) f is onto, since every element of the codomain is an element of the range.

4. (a) f is not one-to-one, since $f(1) = 3$ and $f(4) = 3$.
   (b) f is not onto, since there is no input which gives 2 as an output.

5. (a) f is one-to-one, but not onto.
   (b) f is one-to-one and onto.
   (c) f is one-to-one, but not onto.
   (d) f is not one-to-one, but is onto.

6. (a) f is not one-to-one. To prove this, we must simply find two different elements of the domain which map to the same element of the codomain. Since $f(\{1\}) = 1$ and $f(\{2\}) = 1$, we see that f is not one-to-one.
   (b) f is not onto. The largest subset of A is A itself, and $|A| = 10$. So no natural number greater than 10 will ever be an output.
   (c) $f^{-1}(1) = \{\{1\}, \{2\}, \{3\}, \ldots \{10\}\}$ (the set of all the singleton subsets of A).
   (d) $f^{-1}(0) = \{\emptyset\}$. Note, it would be wrong to write $f^{-1}(0) = \emptyset$ - that would claim that there is no input which has 0 as an output.
   (e) $f^{-1}(12) = \emptyset$, since there are no subsets of A with cardinality 12.

7. (a) $f^{-1}(3) = \{003, 030, 300, 012, 021, 102, 201, 120, 210, 111\}$
   (b) $f^{-1}(28) = \emptyset$ (since the largest sum of three digits is $9 + 9 + 9 = 27$)
   (c) Part (a) proves that f is not one-to-one - the output 3 is assigned to 10 different inputs.
   (d) Part (b) proves that f is not onto - there is an element of the codomain (28) which is assigned to no inputs.

8. X can really be any set, as long as $f(x) = 0$ or $f(x) = 1$ for every $x \in X$. For example, $X = \mathbb{N}$ and $f(n) = 0$ works.

9. (a) $|X| \leq |Y|$ - otherwise two or more of the elements of X would need to map to the same element of Y.
   (b) $|X| \geq |Y|$ - otherwise there would be one or more elements of Y which were never an output.
   (c) $|X| = |Y|$. This is the only way for both of the above to occur.

10. (a) Yes. (Can you give an example?)
    (b) Yes.

(c) Yes.
(d) Yes.
(e) No.
(f) No.

11. (a) f is one-to-one.

Proof. Let x and y be elements of the domain $\mathbb{Z}$. Assume $f(x) = f(y)$. If x and y are both even, then $f(x) = x + 1$ and $f(y) = y + 1$. Since $f(x) = f(y)$, we have $x + 1 = y + 1$ which implies that $x = y$. Similarly, if x and y are both odd, then $x - 3 = y - 3$ so again $x = y$. The only other possibility is that x is even an y is odd (or visa-versa). But then $x + 1$ would be odd and $y - 3$ would be even, so it cannot be that $f(x) = f(y)$. Therefore if $f(x) = f(y)$ we then have $x = y$, which proves that f is one-to-one. $\square$

(b) f is onto.

Proof. Let y be an element of the codomain $\mathbb{Z}$. We will show there is an element n of the domain ($\mathbb{Z}$) such that $f(n) = y$. There are two cases. First, if y is even, then let $n = y+3$. Since y is even, n is odd, so $f(n) = n - 3 = y + 3 - 3 = y$ as desired. Second, if y is odd, then let $n = y - 1$. Since y is odd, n is even, so $f(n) = n + 1 = y - 1 + 1 = y$ as needed. Therefore f is onto. $\square$