Instructions: Same rules as usual - turn in your work on separate sheets of paper. You must justify all your answers for full credit.

(3pts) 1. Prove that if $a \mid b$ and $b \mid c$, then $a \mid c$.

Solution:

Proof. Suppose that $a \mid b$ and $b \mid c$. This means that b is a multiple of a, and that c is a multiple of b. In other words, $a = kb$ and $b = jc$ for some integers k and j. But then $a = kjc$, so a is a multiple of c. In other words, $a \mid c$. □

(3pts) 2. Prove that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$. Hint: rewrite each congruence as an equation, then multiply the equations.

Solution:

Proof. Suppose $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. That means $a = b + kn$ and $c = d + jn$ for some integers k and j. Then

$$ac = (b + kn)(d + jn) = bd + bjn + dkn + kjn^2 = bd + (bj + dk + kjn)n$$

In other words, $ac = bd + hn$ for some integer h, so $ac \equiv bd \pmod{n}$. □

(6pts) 3. Find the remainder when $42^{2013} + 2013^{42}$ is divided by 5. Use modular arithmetic, and explain how you got your answer.

Solution: We want to reduce $42^{2013} + 2013^{42}$ modulo 5. First, note that $42 \equiv 2 \pmod{5}$ and $2013 \equiv 3 \pmod{5}$, so we know:

$$42^{2013} + 2013^{42} \equiv 2^{2013} + 3^{42} \pmod{5}$$

Let's simplify the 2^{2013} first. Look for a power of 2 which is congruent to 1 modulo 5. We could use $2^4 = 16$. 2013 is not a multiple of 4, so first pull out a 2:

$$2^{2013} = 2(2^4)^5 3 \equiv 2 \cdot 1^5 2 \pmod{5}$$

so $2^{2013} \equiv 2 \pmod{5}$. Now for the $3^4 2$. We could again look at $3^4 = 81$, or we could simply do $3^2 = 9 \equiv -1 \pmod{5}$. We get

$$3^{42} = 9^{21} \equiv (-1)^{21} \pmod{5}$$

Of course $(-1)^2 1 = -1$.

Putting this all together we get:

$$42^{2013} + 2013^{42} \equiv 2 - 1 \pmod{5}$$

Thus the remainder is simply 1.

(6pts) 4. Solve the following congruences for x. Give the general solution, as well as all solutions with $0 \le x \le 20$.

(a) $75x \equiv 41 \pmod 4$

Solution: First reduce everything modulo 4. We get,

$$3x \equiv 1 \pmod 4$$

Then continue to add 4 to the right side (which is like adding 0 to both sides, since $0 \equiv 4 \pmod 4$) until we get a multiple of 3. So $1 + 4 = 5$, $5 + 4 = 9$ - bingo.

$$3x \equiv 9 \pmod 4$$

Divide both sides by 3:

$$x \equiv 3 \pmod 4$$

The general solution is $x = 3 + 4k$. The solutions between 0 and 20 are:

$$x \in \{3, 7, 11, 15, 19\}$$

(b) $10x + 7 \equiv 3 \pmod{12}$

Solution: Subtract 7 from both sides:

$$10x \equiv -4 \pmod{12}$$

Of course $-4 \equiv 8 \equiv 20 \pmod{12}$. Now when we divide both sides by 10, we must also reduce the modulus, since 10 and 12 have a common divisor. We get:

$$x \equiv 2 \pmod 6$$

So the general solution is $x = 2 + 6k$, and the set of solutions between 0 and 20 is:

$$x \in \{2, 8, 14, 20\}$$

(6pts) 5. Solve the following Diophantine equations. Describe all solutions.

(a) $7x + 46y = 100$.

Solution: First convert to a congruence modulo 7:

$$46y \equiv 100 \pmod 7$$

This reduces to

$$4y \equiv 2 \pmod 7$$

Since $2 \equiv 16 \pmod 7$, if we divide by 4, we get

$$y \equiv 4 \pmod 7$$

so $y = 4 + 7k$. Then $7x + 46(4 + 7k) = 100$, so $x = -12 + 46k$

(b) $55x + 42y = 47$

Solution: We can rewrite this as

$$55x \equiv 47 \pmod{42}$$

This reduces to

$$13x \equiv 5 \pmod{42}$$

Now we could keep adding 42 to 5 until we get a multiple of 13. Or we can switch back to a Diophantine equation:

$$13x = 5 + 42z$$

Now reduce modulo 13:

$$0 \equiv 5 + 42z \pmod{13}$$

This is the same as:

$$-3z \equiv 5 \pmod{13}$$

which becomes $-3z \equiv 18 \pmod{13}$ reducing to $z \equiv -6 \pmod{13}$. So $z = -6 + 13k$. Now go back and find x and y:

$$13x = 5 + 42(-6 + 13k)$$

so $x = -19 + 42k$. Then plug in to find y:

$$55(-19 + 42k) + 42y = 47$$

so $y = 26 - 55k$.

(6pts) 6. The hugely popular math-rock band *Fibonacci's Rabbits* recently performed on campus. Tickets were \$21, unless you knew the password (the closed formula for the nth Fibonacci number), in which case tickets were only \$13. All together, the revenues from ticket sales came to \$12855. How many total tickets were sold, assuming the number of discounted tickets was as large as possible.

Solution: The total ticket sales revenue is some multiple of 34 plus some multiple of 21. That is, we are trying to solve the Diophantine equation:

$$21x + 13y = 12855$$

Let's reduce this modulo 13. We get

$$21x + 13y \equiv 12855 \pmod{13}$$

Simplifying gives:

$$8x \equiv 11 \pmod{13}$$

But $11 \equiv 24 \pmod{13}$, so

$$8x \equiv 24 \pmod{13}$$

$$x \equiv 3 \pmod{13}$$

So $x = 3 + 13k$. We can then solve for y:

$$21(3 + 13k) + 13y = 12855$$

$$y = 984 - 21k$$

We want to make x, the number of full price tickets, as small as possible, which in this case would be to take $k = 0$. So $x = 3$ and $y = 984$. Therefore the total number of tickets sold was 987 (which, of course, is the 16th Fibonacci number).