Homework 5

Math 228 **Solutions** Due: Wednesday, Feb 25

(8pts) 1. For each sequence given below, find a closed formula for a_n, the nth term of the sequence (assume the first terms are a_0) by relating it to another sequence for which you already know the formula. In each case, briefly say how you got your answers.

(a) 4, 5, 7, 11, 19, 35, ...

Solution: If we subtract 3 from each term, we get $1, 2, 4, 8, 16, 32, \ldots$, which are the powers of 2. So we must shift this sequence up by 3 to get our sequence. Thus

$$a_n = 2^n + 3$$

(b) 0, 3, 8, 15, 24, 35, ...

Solution: Add 1 to each term - we get $1, 4, 9, 16, 25, 36, \ldots$, the square numbers. So maybe our sequence as formula $a_n = n^2 - 1$. Does this work? $a_3 = 8$. That is a term of our sequence, but it should be a_2. So we want

$$a_n = (n+1)^2 - 1$$

(c) 6, 12, 20, 30, 42, ...

Solution: Each term is even, so let's see what happens when we divide by 2: we get $3, 6, 10, 15, 21, \ldots$. These are the triangle numbers, but starting at T_2. We know $T_n = \frac{n(n+1)}{2}$. We want $a_0 = 2T_2$, $a_1 = 2T_3$, and so on. In general, $a_n = 2T_{n+2}$, so

$$a_n = (n+2)(n+3)$$

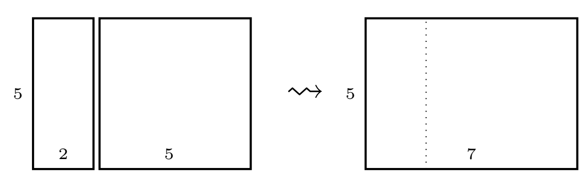
(This is like a shift by 2 units to the left and a stretch by a factor of 2.)

(d) 0, 2, 7, 15, 26, 40, 57, ... (Cryptic Hint: these might be called "house numbers")

Solution: How far off from triangular number are these? The triangular numbers (starting with T_0) are $0, 1, 3, 6, 10, 15, 21, \ldots$. The given sequence differs from this by $0, 1, 4, 9, 16, 25, 36, \ldots$ the square numbers! Thus

$$a_n = \frac{n(n+1)}{2} + n^2$$

(8pts) 2. Starting with any rectangle, we can create a new, larger rectangle by attaching a square to the longer side. For example, if we start with a 2×5 rectangle, we would glue on a 5×5 square, forming a 5×7 rectangle:



(a) Create a sequence of rectangles using this rule starting with a 1×2 rectangle. Then write out the sequence of *perimeters* for the rectangles (the first term of the sequence would be 6, since the perimeter of a 1×2 rectangle is 6 - the next term would be 10).

> **Solution:** The rectangles are 1×2, 2×3, 3×5, 5×8, 8×13, and so on. The sequence of perimeters is
> $$6, 10, 16, 26, 42, \ldots$$

(b) Repeat the above part this time starting with a 1×3 rectangle.

> **Solution:** The sequence of rectangles have dimensions 1×3, 3×4, 4×7, 7×11, 11×18, and so on. The sequence of perimeters is
> $$8, 14, 22, 36, 58, \ldots$$

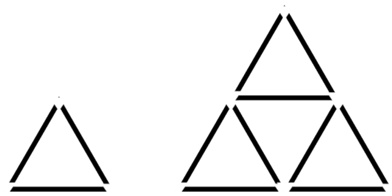
(c) Find recursive formulas for each of the sequences of perimeters you found in parts (a) and (b). Don't forget to give the initial conditions as well.

> **Solution:** For the sequence from (a), the recursive formula is $a_1 = 6$, $a_2 = 10$, and $a_n = a_{n-1} + a_{n-2}$.
> For the sequence from (b), the recursive formula is $a_1 = 8$, $a_2 = 14$, and $a_n = a_{n-1} + a_{n-2}$.
> Notice that both sequence have the same rule for getting terms from the previous ones, it is just the initial conditions that are different.

(d) Are the sequences arithmetic? Geometric? If not, are they *close* to being either of these (i.e., are the differences or ratios *almost* constant)? Explain.

> **Solution:** The sequences are not arithmetic because the differences between terms is not constant. Similarly the ratio between terms is not constant, so the sequences are not geometric either.
> However, look at the ratio between terms: $10/6 \approx 1.66$, $16/10 = 1.6$, $26/16 \approx 1.625$, $42/26 \approx 1.61$, ... In fact, the ratio between terms of the second sequence also floats around this same number.
> That number (and in fact, the limit of the ratios of either sequence as the terms increase) is $\frac{1+\sqrt{5}}{2} \approx 1.618$, also known as the golden ratio.

(8pts) 3. If you have enough toothpicks, you can make a large triangular grid. Below, are the triangular grids of size 1 and of size 2. The size 1 grid requires 3 toothpicks, the size 2 grid requires 9 toothpicks.



(a) Let t_n be the number of toothpicks required to make a size n triangular grid. Write out the first 5 terms of the sequence $t_1, t_2, \ldots$.

> **Solution:** $3, 9, 18, 30, 45, \ldots$

(b) Find a recursive definition for the sequence. Explain why you are correct.

> **Solution:** $t_n = t_{n-1} + 3n$ with $t_1 = 3$. This works because to get the next larger triangular grid, we must add a row of n triangles, each requiring 3 toothpicks.

(c) Is the sequence arithmetic or geometric? If not, is it the sequence of partial sums of an arithmetic or geometric sequence? Explain why your answer is correct.

> **Solution:** The sequence is not arithmetic (since $9 - 3 = 6$ but $18 - 9 = 9$). It is also not geometric (since $9/3 = 3$ but $18/9 = 2$). To decide whether the sequence is a sequence of partial sums, we look at the differences. The sequence of differences is $6, 9, 12, 15, \ldots$. This *is* an arithmetic sequence. So we see that
>
> $$t_1 = 3$$
>
> $$t_2 = 3 + 6$$
>
> $$t_3 = 3 + 6 + 9$$
>
> $$t_4 = 3 + 6 + 9 + 12$$
>
> $$t_n = \sum_{k=1}^{n} 3k$$

(d) Use your results from part (c) to find a closed formula for the sequence. Show your work.

> **Solution:** We have $t_n = 3+6+9+\cdots+3n$. If we reverse these and add corresponding terms we get
>
> $$2t_n = (3n+3) + (3n+3) + (3n+3) + \cdots + (3n+3)$$
>
> On the right hand side of the equation we have the sum of n copies of $(3n+3)$ so we get
>
> $$2t_n = n(3n+3)$$
>
> or
>
> $$t_n = \frac{n(3n+3)}{2}$$

This is not a huge surprise. Notice that each term in the sequence is a multiple of 3. Dividing each term by 3 gives the sequence $1, 3, 6, 10, 15, \ldots$ - the triangular numbers. This makes sense because we are forming triangles out of 3-toothpick collections. So a closed formula is therefore $t_n = 3\frac{n(n+1)}{2}$