Holmes owns two suits: one blue and one brown. He always wears either a blue suit or white socks. Whenever he wears his blue suit and a blue shirt, he also wears a blue tie. He never wears the blue suit unless he is also wearing either a blue shirt or white socks. Whenever he wears white socks, he also wears a blue shirt. Today, Holmes is wearing a gold tie. What else is he wearing?

Consider the following statement:

If a times b is an even number, then a is even or b is even.

Based on the ideas that we discussed in class today, decide whether the following proofs of the above statement are valid or invalid.

1. Suppose $a = 2k + 1$ (a is odd) and $b = 2m + 1$ (b is odd). Then

$$\begin{aligned} ab &= (2k + 1)(2m + 1) \\ &= 4km + 2k + 2m + 1 \\ &= 2(2km + k + m) + 1 \end{aligned}$$

   Which proves that ab is odd if a and b are odd. Therefore, if ab is even, then a or b is be even.

2. Assume that a or b is even - say it is a. That is, $a = 2k$ for some integer k. Then

$$\begin{aligned} ab &= (2k)b \\ &= 2(kb) \end{aligned}$$

   Which means that ab is even. The case where b is even is identical. Therefore, if ab is even then a is even or b is even.

3. Suppose that ab is even but a and b are both odd. Namely, $ab = 2n$, $a = 2k+1$ and $b = 2j+1$ for some integers n, k, and j. Then

$$\begin{aligned} 2n &= (2k + 1)(2j + 1) \\ 2n &= 4kj + 2k + 2j + 1 \\ n &= 2kj + k + j + \frac{1}{2} \end{aligned}$$

   But since $2kj + k + j$ is an integer, this says that the integer n is equal to a non-integer, which is impossible. Therefore, if ab is even then a or b must be even.

4. Let ab be an even number, $ab = 2n$, and a be an odd number, $a = 2k + 1$. Then

$$\begin{aligned} ab &= (2k + 1)b \\ 2n &= 2kb + b \\ 2n - 2kb &= b \\ 2(n - kb) &= b \end{aligned}$$

   Therefore, b must be even. So, if ab is even then either a or b must also be even.