Homework 4

Math 228 **Solutions** Due: Wednesday, Feb 11

(6pts) 1. Suppose you own x fezzes and y bow ties. Of course, x and y are both greater than 1.

(a) How many combinations of fez and bow tie can you make? You can wear only one fez and one bow tie at a time. Explain.

> **Solution:** You have x choices for the fez, and for each choice of fez you have y choices for the bow tie. Thus you have $x \cdot y$ choices for fez and bow tie combination.

(b) Explain why the answer is *also* $\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2}$. (If this is what you claimed the answer was in part (a), try it again.)

> **Solution:** Line up all $x + y$ quirky clothing items - the x fezzes and y bow ties. Now pick 2 of them. This can be done in $\binom{x+y}{2}$ ways. However, we might have picked 2 fezzes, which is not allowed. There are $\binom{x}{2}$ ways to pick 2 fezzes. Similarly, the $\binom{x+y}{2}$ ways to pick two items includes $\binom{y}{2}$ ways to select 2 bow ties, also not allowed. Thus the total number of ways to pick a fez and a bow ties is
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> $$\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2}$$

(c) Use your answers to parts (a) and (b) to give a combinatorial proof of the identity

$$\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2} = xy$$

> **Solution:**
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> *Proof.* The question is how many ways can you select one of x fezzes and one of y bow ties. We answer this question in two ways. First, the answer could be $a \cdot b$. This is correct as described in part (a) above. Second, the answer could be $\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2}$. This is correct as described in part (b) above. Therefore
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> $$\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2} = xy$$
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(6pts) 2. Consider the identity:

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

(a) Is this true? Try it for a few values of n and k.

> **Solution:** Yes. For example, if $n = 7$ and $k = 4$, we have
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> $$4 \cdot \binom{7}{4} = 4 \cdot 35 = 140 = 7 \cdot 20 = 7 \cdot \binom{6}{3}$$

(b) Use the formula for $\binom{n}{k}$ to give an algebraic proof of the identity.

Solution:

$$k\binom{n}{k} = k\frac{n!}{(n-k)!\,k!} = \frac{n!}{(n-k)!(k-1)!} = n\frac{(n-1)!}{(n-1-(k-1))!(k-1)!} = n\binom{n-1}{k-1}$$

(c) Give a combinatorial proof of the identity. Hint: How many ways can you select a team of k people from a group of n people *and* select one of them to be the team captain?

Solution:

Proof. Question: How many ways can you select a chaired committee of k people from a group of n people? That is, you need to select k people to be on the committee and one of them needs to be in charge. How many ways can this happen?

Answer 1: First select k of the n people to be on the committee. This can be done in $\binom{n}{k}$ ways. Now select one of those k people to be in charge - this can be done in k ways. So there are a total of $k\binom{n}{k}$ ways to select the chaired committee.

Answer 2: First select the chair of the committee. You have n people to choose from, so this can be done in n ways. Now fill the rest of the committee. There are $n-1$ people to choose from (you cannot select the person you picked to be the chair) and $k-1$ spots to fill (the chair's spot is already taken). So this can be done in $\binom{n-1}{k-1}$ ways. Therefore there are $n\binom{n-1}{k-1}$ ways to select the chaired committee. $\square$

(6pts) 3. After a late night of math studying, you and your friends decide to go to your favorite tax-free fast food Mexican restaurant, *Burrito Chime*. You decide to order off of the dollar menu, which has 7 items. Your group has \$16 to spend (and will spend all of it).

(a) How many different orders are possible? Explain. (The *order* in which the order is placed does not matter - just which and how many of each item that is ordered.)

Solution: $\binom{22}{6}$ - there are 16 stars and 6 bars.

(b) How many different orders are possible if you want to get at least one of each item? Explain.

Solution: $\binom{15}{6}$ - buy one of each item, using \$7. This leaves you \$11 to distribute to the 7 items, so 11 stars and 6 bars.

(c) How many different orders are possible if you don't get more than 4 of any one item? Explain. Hint: get rid of the bad orders using PIE.

Solution:

$$\binom{22}{6} - \left[\binom{7}{1}\binom{17}{6} - \binom{7}{2}\binom{12}{6} + \binom{7}{3}\binom{7}{6}\right]$$

(6pts) 4. Consider functions $f : \{1, 2, 3, 4, 5\} \to \{0, 1, 2, \ldots, 9\}$.

(a) How many of these functions are strictly increasing? Explain. (A function is strictly increasing provided if $a < b$, then $f(a) < f(b)$.)

Solution: $\binom{10}{5}$. Note that a strictly increasing function is automatically injective. So the five outputs must all be different. So let's first pick which five outputs we will use: there are $\binom{10}{5}$ ways to do this. Now how many ways are there to assign those outputs to the inputs 1 through 5? Only one way, since there is only one way to arrange numbers in increasing order.

(b) How many of the functions are non-decreasing? Explain. (A function is non-decreasing provided if $a < b$, then $f(a) \leq f(b)$.)

Solution: $\binom{14}{5}$. This is in fact a stars and bars problem. The stars are the 5 inputs and the bars are the 9 spots between the 10 possible outputs. Think of it this way - we will specify $f(1)$, then $f(2)$, then $f(3)$, and so on in that order. Start with the possible output 0. We can use it as the output of $f(1)$, or we can switch to 1 as a potential output. Say we put $f(1) = 1$. Now we are at 1 (can't go back to 0). Should $f(2) = 1$? If yes, then we are putting down another star. If no, put down a bar and switch to 2. Maybe you switch to 3, then assign $f(2) = 3$ and $f(3) = 3$ (two more stars) before switching to 4 as a possible output. And so on.

(6pts) 5. The Grinch sneaks into a room with 6 Christmas presents to 6 different people. He proceeds to switch the name-labels on the presents. How many ways could he do this if:

(a) No present is allowed to end up with its original label? Explain what each term in your answer represents.

Solution:

$$6! - \left[\binom{6}{1}5! - \binom{6}{2}4! + \binom{6}{3}3! - \binom{6}{4}2! + \binom{6}{5}1! - \binom{6}{6}0!\right]$$

(b) Exactly 2 presents keep their original labels? Explain.

Solution:

$$\binom{6}{2}\left(4! - \left[\binom{4}{1}3! - \binom{4}{2}2! + \binom{4}{3}1! - \binom{4}{4}0!\right]\right)$$

(c) Exactly 5 presents keep their original labels? Explain.

Solution: 0. Once 5 presents have their original label, there is only one present left and only one label left, so the 6th present must get its own label.