Advanced Counting

1 Advanced Counting using PIE

Counting surjections used the Principle of Inclusion/Exclusion (PIE), which gives a method for finding the cardinality of the union of not necessarily disjoint sets. The idea is that to find how many things are in *one or more* of the sets A, B, and C we should just add up the number of things in each of these sets. However, if there is any overlap among the sets, those elements are counted multiple times. So we subtract the things in each intersection of a pair of sets. But doing this removes elements which are in all three sets once too often, so we need to add it back in.

For four or more sets, we do not write down a formula for PIE. Instead, we just think of the principle: add up all the elements in single sets, then subtract out things you counted twice (elements in the intersection of a *pair* of sets), then add back in elements you removed too often (elements in the intersection of groups of three sets), then take back out elements you added back in too often (elements in the intersection of groups of four sets), then add back in, take back out, add back in, etc.

Here are some additional examples of how it can be used.

1.1 Counting Derangements

A *derangement* of n elements $\{1, 2, 3, \ldots, n\}$ is a permutation in which no element is fixed. For example, there are 6 permutations of the three elements $\{1, 2, 3\}$:

$$123 \quad 132 \quad 213 \quad 231 \quad 312 \quad 321$$

but most of these have one or more elements fixed: 123 has all three elements fixed, 132 has the first element fixed (1 is in its original first position), and so on. In fact, the only derangements of three elements are

$$231 \text{ and } 312$$

If we go up to 4 elements, there are 24 permutations (because we have 4 choices for the first element, 3 choices for the second, 2 choices for the third leaving only 1 choice for the last - $4! = 24$). How many of these are derangements? If you list out all 24 permutations and eliminate those which are not derangements, you will be left with just 9 derangements. Let's see how we can get that number using PIE.

> **Example**: How many derangements are there of 4 elements?
>
> *Solution*: We count all permutations, and subtract those which are not derangements. There are $4! = 24$ permutations of 4 elements. Now for a permutation to *not* be a derangement, at least one of the 4 elements must be fixed. There are $\binom{4}{1}$ choices for which single element we fix. Once fixed, we need to find a permutation of the other three elements - there are $3!$ permutations on 3 elements. But now we have counted too many non-derangements, so we must subtract those permutations which fix two elements. There are $\binom{4}{2}$ choices for which two elements we fix, and then for each pair, $2!$ permutations of the

remaining elements. But this subtracts too many, so add back in permutations which fix 3 elements, all $\binom{4}{3}1!$ of them. Finally subtract the $\binom{4}{4}0!$ permutations (recall $0! = 1$) which fix all four elements. All together we get that the number of derangements of 4 elements is:

$$4! - \left[\binom{4}{1}3! - \binom{4}{2}2! + \binom{4}{3}1! - \binom{4}{4}0!\right] = 24 - 15 = 9$$

Of course we can use a similar formula to count the derangements on any number of elements. However, the more elements we have, the longer the formula gets. It turns out that there is no closed formula for counting derangements or surjective functions for that matter. We need to use PIE.

Before moving on to another advanced counting technique, here is another example.

Example: Five gentlemen attend a party, leaving their hats at the door. At the end of the party, they hastily grab hats on their way out. How many different ways could this happen so that none of the gentlemen leave with their own hat?

Solution: We are counting derangements on 5 elements. There are 5! ways for the gentlemen to grab hats in any order - but many of these permutations will result in someone getting their own hat. So we subtract all the ways in which one or more of the men get their own hat. In other words, we subtract the non-derangements. Doing so requires PIE. Thus the answer is:

$$5! - \left[\binom{5}{1}4! - \binom{5}{2}3! + \binom{5}{3}2! - \binom{5}{4}1! + \binom{5}{5}0!\right]$$

1.2 Stars, Bars, and Pie

Another time PIE crops up is in stars and bars problems. Let's go back to our 7 cookies and 4 kids problem. Recall we want to distribute the 7 identical cookies to 4 (non-identical) kids. Now we place a restriction on it

Example: How many ways can you distribute 7 cookies to 4 kids so that no kid gets more than 2 cookies?

Solution: To answer this, we will subtract all the outcomes in which a kid gets 3 or more cookies. How many outcomes are there like that? We can force kid A to eat 3 or more cookies by giving him 3 cookies before we start. Doing so reduces the problem to one in which we have 4 cookies to give to 4 kids without any restrictions. In that case, we have 4 stars (the 4 remaining cookies) and 3 bars (one less than the number of kids) so we can distribute the cookies in

$\binom{7}{3}$ ways. Of course we could choose any one of the 4 kids to give too many cookies, so it would appear that there are $\binom{4}{1}\binom{7}{3}$ ways to distribute the cookies giving too many to one kid. But in fact, we have over counted. We must get rid of the outcomes in which two kids have too many cookies. There are $\binom{4}{2}$ ways to select to 2 kids to give extra cookies. It takes 6 cookies to do this, leaving only 1 cookie. So we have 1 star and still 3 bars. The remaining 1 cookie can thus be distributed in $\binom{4}{3}$ ways (for each of the $\binom{4}{2}$ choices of which 2 kids to over-feed). We could continue in this fashion (using PIE) but there are no ways to give too many cookies to 3 kids - you don't have enough cookies.

All together we get that the number of ways to distribute 7 cookies to 4 kids without giving any kid more than 2 cookies is:

$$\binom{10}{3} - \left[\binom{4}{1}\binom{7}{3} - \binom{4}{2}\binom{4}{3}\right] = 120 - [140 - 24] = 4$$

This makes sense: the only way to distribute cookies with this restriction is for 3 kids to get 2 cookies and one kid to get 1. There are 4 choices for which kid gets the 1 cookie.

Example: Earlier we counted the number of solutions to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13.$$

How many of those solutions have $0 \leq x_i \leq 3$ for each x_i?

Solution: We must subtract off the number of solutions in which one or more of the variables has a value greater than 3. We will need to use PIE because counting the number of solutions for which each of the five variables separately are greater than 3 counts solutions multiple times. Here is what we get:

- Total solutions: $\binom{17}{4}$
- Solutions where $x_1 > 3$: $\binom{13}{4}$ - give x_1 4 units first, then distribute the remaining 9 units to the 5 variables.
- Solutions where $x_1 > 3$ and $x_2 > 3$: $\binom{9}{4}$ - after you give 4 units to x_1 and another 4 to x_2, you only have 5 units left to distribute.
- Solutions where $x_1 > 3$, $x_2 > 3$ and $x_3 > 3$: $\binom{5}{4}$.
- Solutions where $x_1 > 3$, $x_2 > 3$, $x_3 > 3$, and $x_4 > 3$: 0.

We also need to account for the fact that we could choose any of the five variables in the place of x_1 above, any pair of variables in the place of x_1 and x_2 and so on. It is because of this that the double counting occurs, so we need to use PIE. All together we have that the number of solutions with $0 \leq x_i \leq 3$ is

$$\binom{17}{4} - \left[\binom{5}{1}\binom{13}{4} - \binom{5}{2}\binom{9}{4} + \binom{5}{3}\binom{5}{4}\right]$$