Practice Problems 9: Advanced Counting

Math 228 **Solutions** Spring 2012

1. (a) $6^4 = 1296$, since there are six choices of where to send each of the 4 elements of the domain.
   (b) $P(6,4) = 6 \cdot 5 \cdot 4 \cdot 3 = 360$, since outputs cannot be repeated.
   (c) None.
   (d) There are $5 \cdot 6^3$ functions for which $f(1) \neq a$ and another $5 \cdot 6^3$ functions for which $f(2) \neq b$. There are $5^2 \cdot 6^2$ functions for which both $f(1) \neq a$ and $f(2) \neq b$. So the total number of functions for which $f(1) \neq a$ or $f(2) \neq b$ or both is

$$5 \cdot 6^3 + 5 \cdot 6^3 - 5^2 \cdot 6^2 = 1260$$

2. (a) 17^{10}
   (b) $P(17, 10)$
3. $5^{10} - \left[\binom{5}{1}4^{10} - \binom{5}{2}3^{10} + \binom{5}{3}2^{10} - \binom{5}{4}1^{10}\right]$
4. $5! - \left[\binom{5}{1}4! - \binom{5}{2}3! + \binom{5}{3}2! - \binom{5}{4}1! + \binom{5}{5}0!\right]$. This is a sneaky way to as for the number of derangements on 5 elements.
5. $\binom{10}{6}\left(4! - \left[\binom{4}{1}3! - \binom{4}{2}2! + \binom{4}{3}1! - \binom{4}{4}0!\right]\right)$. We choose 6 of the 10 ladies to get their own hat, and the multiply by the number of ways the remaining hats can be deranged.
6. (a) $\binom{18}{4}$. Each outcome can be represented by a sequence of 14 stars and 4 bars.
   (b) $\binom{13}{4}$. First put one ball in each bin. This leaves 9 stars and 4 bars.
   (c) $\binom{18}{4} - \left[\binom{5}{1}\binom{11}{4} - \binom{5}{2}\binom{4}{4}\right]$. Subtract all the distributions for which one or more bins contain 7 or more balls.
7. (a) $\binom{7}{2}$. After each variable gets 1 star for free, we are left with 5 stars and 2 bars.
   (b) $\binom{10}{2}$. We have 8 stars and 2 bars.
   (c) $\binom{19}{2}$. This problem is equivalent to finding the number of solutions to $x'+y'+z' = 17$ where x', y' and z' are non-negative. (In fact, we really just do a substitution. Let $x = x' - 3$, $y = y' - 3$ and $z = z' - 3$).
8. $\binom{10}{5}$. We have 5 stars (the five dice) and 5 bars (the five switches between the number 1-6).
9. $\binom{18}{3}$. Distribute 10 units to the variables before finding all solutions to $x_1' + x_2' + x_3' + x_4' = 15$ in non-negative integers.
10. (a) $\binom{8}{3}$, after giving one present to each kid, you are left with 5 presents (stars) which need to be divide among the 4 kids (giving 3 bars).
    (b) $\binom{12}{3}$. You have 9 stars and 3 bars.
    (c) 4^9. You have 4 choices for whom to give each present. This is like making a function from the set of present to the set of kids.
    (d) $4^9 - \left[\binom{4}{1}3^9 - \binom{4}{2}2^9 + \binom{4}{3}1^9\right]$. Now the function from the set of present to the set of kids must be onto.

11. (a) Neither. $\binom{14}{4}$.
    (b) $\binom{10}{4}$
    (c) $P(10, 4)$, since order is important.
    (d) Neither. Assuming you will wear each of the 4 ties on just 4 of the 7 days, without repeats: $\binom{10}{4}P(7, 4)$.
    (e) $P(10, 4)$
    (f) $\binom{10}{4}$
    (g) Neither. Since you could repeat letters: 10^4. If no repeats are allowed, it would be $P(10, 4)$.
    (h) Neither. Actually, "k" is the 11th letter of the alphabet, so the answer is 0. If "k" was among the first 10 letters, there would only be 1 way - write it down.
    (i) Neither. Either $\binom{9}{3}$ (if every kid gets an apple) or $\binom{13}{3}$ (if appleless kids are allowed).
    (j) Neither. Note that this could not be $\binom{10}{4}$ since the 10 things and 4 things are from different groups. 4^{10}
    (k) $\binom{10}{4}$ - don't be fooled by the "arrange" in there - you are picking 4 out of 10 *spots* to put the 1's.
    (l) $\binom{10}{4}$ (assuming order is irrelevant).
    (m) Neither. 16^{10} (each kid chooses yes or no to 4 varieties).
    (n) Neither. 0.
    (o) Neither. $4^{10} - [\binom{4}{1}3^{10} - \binom{4}{2}2^{10} + \binom{4}{3}1^{10}]$
    (p) Neither. $10 \cdot 4$.
    (q) Neither. 4^{10}.
    (r) $\binom{10}{4}$ (which is the same as $\binom{10}{6}$).
    (s) Neither. If all the kids were identical, and you wanted no empty teams, it would be $\binom{10}{4}$. Instead, this will be the same as the number of onto functions from a set of size 11 to a set of size 5.
    (t) $\binom{10}{4}$
    (u) $\binom{10}{4}$
    (v) Neither. 4!.
    (w) Neither. It's $\binom{10}{4}$ if you won't repeat any choices. If repetition is allowed, then this becomes $x_1 + x_2 + \cdots + x_{10} = 4$, which has $\binom{13}{9}$ solutions in non-negative integers.
    (x) Neither. Since repetition of cookie type is allowed, the answer is 10^4. Without repetition, you would have $P(10, 4)$.
    (y) $\binom{10}{4}$ since that is equal to $\binom{9}{4} + \binom{9}{3}$.
    (z) Neither. It will be a complicated (possibly PIE) counting problem.