(4pts) 1. Solve the recurrence relation $a_n = a_{n-1} + 3$ using telescoping or iteration. Show your work.

Solution: Both telescoping and iteration work. For telescoping:

$$\begin{aligned} a_1 - a_0 &= 3 \\ a_2 - a_1 &= 3 \\ a_3 - a_2 &= 3 \\ &\vdots \\ \underline{+a_n - a_{n-1}} &\underline{= 3} \\ a_n - a_0 &= 3n \end{aligned}$$

Thus the solution is $a_n = 3n + a_0$.

(6pts) 2. Let a_n be the number of $1 \times n$ tile designs can you make using 1×1 tiles available in 4 colors and 1×2 tiles available in 5 colors.

(a) First, find a recurrence relation to describe the problem.

Solution: $a_n = 4a_{n-1} + 5a_{n-2}$

(b) Write out the first 6 terms of the sequence $a_1, a_2, \ldots$.

Solution: 4, 21, 104, 521, 2604, 13021

(c) Solve the recurrence relation. That is, find a closed formula for a_n.

Solution: The characteristic equation is $x^2 - 4x - 5 = 0$ so the characteristic roots are $x = 5$ and $x = -1$. Therefore the general solution is

$$a_n = a5^n + b(-1)^n$$

We solve for a and b using the fact that $a_1 = 4$ and $a_2 = 21$. We get $a = \frac{5}{6}$ and $b = \frac{1}{6}$. Therefore the solution is

$$a_n = \frac{5}{6}5^n + \frac{1}{6}(-1)^n$$

(5pts) 3. Consider the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$.

(a) Find the general solution to the recurrence relation (beware the repeated root).

Solution: The characteristic polynomial is $x^2 - 4x + 4$ which factors as $(x-2)^2$, so the only characteristic root is $x = 2$. Thus the general solution is

$$a_n = a2^n + bn2^n$$

(b) Find the solution when $a_0 = 1$ and $a_1 = 2$.

Solution: Since $1 = a2^0 + b \cdot 0 \cdot 2^0$ have have $a = 1$. Then $2 = 2^1 + b2^1$ so $b = 0$. We have the solution

$$a_n = 2^n$$

(c) Find the solution when $a_0 = 1$ and $a_1 = 8$.

Solution: Again, we have $a = 1$. Now when we plug in $n = 1$ we bet $8 = 2 + 2b$ so $b = 3$. The solution:

$$a_n = 2^n + 3n2^n$$

(5pts) 4. Prove, by mathematical induction, that $F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$, where F_n is the nth Fibonacci number ($F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$).

Solution:

Proof. Let $P(n)$ be the statement $F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$. We will prove that $P(n)$ is true for all $n \geq 0$.

Base case: $P(0)$ states that $F_0 = F_2 - 1$, which is true because $F_0 = 0$ and $F_2 = 1$.

Inductive case: Assume $P(k)$ is true for an arbitrary fixed $k \geq 0$. That is,

$$F_0 + F_1 + F_2 + \cdots + F_k = F_{k+2} - 1$$

We must prove that $P(k+1)$ is true as well (i.e. that $F_0 + F_1 + \cdots + F_{k+1} = F_{k+3} - 1$). Start with the left hand side:

$$\begin{aligned} F_0 + F_1 + F_2 + \cdots + F_k + F_{k+1} &= F_{k+2} - 1 + F_{k+1} && \text{by the inductive hypothesis} \\ &= F_{k+3} - 1 && \text{by the definition of the Fibonacci numbers} \end{aligned}$$

Thus $P(k+1)$ is true.

Therefore by the principle of mathematical induction, $P(n)$ is true for all $n \geq 0$. □