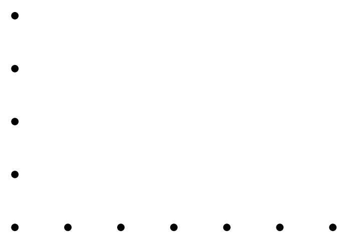
Math 228 **Homework 8 Solutions** April 4, 2012

(5pts) 1. How many triangles are there with vertices from the points shown below? Note, we are not allowing degenerate triangles - ones with all three vertices on the same line. Explain why your answer is correct. (HINT: you need at exactly two points on either the x- or y-axis, but don't over-count the right triangles.)



Solution: There are 120 triangles. Here are two ways (there are others as well) to get this:

1. First count the triangles with the base on the x-axis. There are $\binom{7}{2}$ ways to pick the base. The third vertex of the triangle must be one of the 4 dots on the y-axis (not the origin) so there are a total of $\binom{7}{2}4$ of these triangles. The triangles with base on the y axis can be counted similarly: $\binom{5}{2}6$. However, we have counted all the right triangles twice - they have a base on the x-axis and also on the y-axis. There are $4 \cdot 6$ right triangles. Thus the total number of triangles is:
$$\binom{7}{2}4 + \binom{5}{2}6 - 6 \cdot 4 = 120$$
2. We must select 3 of the 11 dots. This can be done in $\binom{11}{3}$ ways. However, this will also give us degenerate triangles when all three vertices are on the x-axis or on the y-axis. There are $\binom{7}{3}$ ways we could have picked all three vertices on the x-axis. There are $\binom{5}{3}$ ways we could have picked all three vertices on the y-axis. Therefore the total number of triangles is
$$\binom{11}{3} - \binom{7}{3} - \binom{5}{3} = 120$$

(6pts) 2. Suppose you own a fezzes and b bow ties. Of course, a and b are both greater than 1.

(a) How many combinations of fez and bow tie can you make? You can wear only one fez and one bow tie at a time. Explain.

Solution: You have a choices for the fez, and for each choice of fez you have b choices for the bow tie. Thus you have $a \cdot b$ choices for fez and bow tie combination.

(b) Explain why the answer is *also* $\binom{a+b}{2} - \binom{a}{2} - \binom{b}{2}$. (If this is what you claimed the answer was in part (a), try it again.)

Solution: Line up all $a + b$ quirky clothing items - the a fezzes and b bow ties. Now pick 2 of them. This can be done in $\binom{a+b}{2}$ ways. However, we might have picked 2 fezzes, which is not allowed. There are $\binom{a}{2}$ ways to pick 2 fezzes. Similarly, the $\binom{a+b}{2}$ ways to pick two items includes $\binom{b}{2}$ ways to select 2 bow ties, also not allowed. Thus the total number of ways to pick a fez and a bow ties is
$$\binom{a+b}{2} - \binom{a}{2} - \binom{b}{2}$$

(c) Use your answers to parts (a) and (b) to give a combinatorial proof of the identity

$$\binom{a+b}{2} - \binom{a}{2} - \binom{b}{2} = ab$$

Solution:

Proof. The question is how many ways can you select one of a fezzes and one of b bow ties. We answer this question in two ways. First, the answer could be $a \cdot b$. This is correct as described in part (a) above. Second, the answer could be $\binom{a+b}{2} - \binom{a}{2} - \binom{b}{2}$. This is correct as described in part (b) above. Therefore

$$\binom{a+b}{2} - \binom{a}{2} - \binom{b}{2} = ab$$

□

3. Consider the identity:

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

(2pts) (a) Is this true? Try it for a few values of n and k.

Solution: Yes. For example, if $n = 7$ and $k = 4$, we have

$$4 \cdot \binom{7}{4} = 4 \cdot 35 = 140 = 7 \cdot 20 = 7 \cdot \binom{6}{3}$$

(3pts) (b) Use the formula for $\binom{n}{k}$ to give an algebraic proof of the identity.

Solution:

$$k\binom{n}{k} = k\frac{n!}{(n-k)!\,k!} = \frac{n!}{(n-k)!(k-1)!} = n\frac{(n-1)!}{(n-1-(k-1))!(k-1)!} = n\binom{n-1}{k-1}$$

(4pts) (c) Give a combinatorial proof of the identity. Hint: How many ways can you select a chaired committee of k people from a group of n people?

Solution:

Proof. Question: How many ways can you select a chaired committee of k people from a group of n people? That is, you need to select k people to be on the committee and one of them needs to be in charge. How many ways can this happen?
Answer 1: First select k of the n people to be on the committee. This can be done in $\binom{n}{k}$ ways. Now select one of those k people to be in charge - this can be done in k ways. So there are a total of $k\binom{n}{k}$ ways to select the chaired committee.
Answer 2: First select the chair of the committee. You have n people to choose from, so this can be done in n ways. Now fill the rest of the committee. There are $n-1$ people to choose from (you cannot select the person you picked to be the chair) and $k-1$ spots to fill (the chair's spot is already taken). So this can be done in $\binom{n-1}{k-1}$ ways. Therefore there are $n\binom{n-1}{k-1}$ ways to select the chaired committee. □