Practice Problems 6: Generating Functions

Math 228 **Solutions** Spring 2012

1. (a) $\dfrac{4}{1-x}$

   (b) $\dfrac{2}{(1-x)^2}$

   (c) $\dfrac{2x^3}{(1-x)^2}$

   (d) $\dfrac{1}{1-5x}$

   (e) $\dfrac{1}{1+3x}$

   (f) $\dfrac{1}{1-5x^2}$

   (g) $\dfrac{x}{(1-x^3)^2}$

2. (a) $0, 4, 4, 4, 4, 4, \ldots$

   (b) $1, 4, 16, 64, 256, \ldots$

   (c) $0, 1, -1, 1, -1, 1, -1, \ldots$

   (d) $0, 3, -6, 9, -12, 15, -18, \ldots$

   (e) $1, 3, 6, 9, 12, 15, \ldots$

3. (a) The second derivative of $\dfrac{1}{1-x}$ is $\dfrac{2}{(1-x)^3}$ which expands to $2 + 6x + 12x^2 + 20x^3 + 30x^4 + \cdots$. Dividing by 2 gives the generating function for the triangular numbers.

   (b) Compute $A - xA$ and you get $1 + 2x + 3x^2 + 4x^3 + \cdots$ which can be written as $\dfrac{1}{(1-x)^2}$. Solving for A gives the correct generating function.

   (c) The triangular numbers are the sum of the first n numbers $1, 2, 3, 4, \ldots$. To get the sequence of partial sums, we multiply by $\frac{1}{1-x}$. so this gives the correct generating function again.

4. Call the generating function A. Compute $A - xA = 4 + x + 2x^2 + 3x^3 + 4x^4 + \cdots$. Thus $A - xA = 4 + \dfrac{x}{(1-x)^2}$. Solving for A gives $\dfrac{4}{1-x} + \dfrac{x}{(1-x)^3}$.

5. $\dfrac{1+2x}{1-3x+x^2}$

6. Compute $A - xA - x^2A$ and the solve for A. The generating function will be $\dfrac{x}{1-x-x^2}$.

7. $\dfrac{x}{(1-x)(1-x-x^2)}$

8. $\dfrac{2}{1-5x} + \dfrac{7}{1+3x}$.

9. $a_n = 3 \cdot 4^{n-1} + 1$

10. Hint: you should “multiply” the two sequences. Answer: 158.