Practice Problems 8: Binomial Identities

Math 228 **Solutions** Spring 2012

1. *Proof.* Question: How many subsets of $A = 1, 2, 3, \ldots, n+1$ contain exactly two elements?

   Answer 1: We must choose 2 elements from $n+1$ choices, so there are $\binom{n+1}{2}$ subsets.

   Answer 2: We break this question down into cases, based on what the larger of the two elements in the subset is. The larger element can't be 1, since we need at least one element smaller than it.

   Larger element is 2: there is 1 choice for the smaller element.

   Larger element is 3: there are 2 choices for the smaller element.

   Larger element is 4: there are 3 choices for the smaller element.

   And so on. When the larger element is $n+1$, there are n choices for the smaller element. Since each two element subset must be in exactly one of these cases, the total number of two element subsets is $1 + 2 + 3 + \cdots + n$.

   Answer 1 and answer 2 are both correct, so they must be equal. Therefore

   $$1 + 2 + 3 + \cdots + n = \binom{n+1}{2}$$

   $\square$

2. (a) She has $\binom{15}{6}$ ways to select the 6 bride's maids, and then for each way, has 6 choices for the maid of honor. Thus she has $\binom{15}{6}6$ choices.

   (b) She has 15 choices for who will be her maid of honor. Then she needs to select 5 of the remaining 14 friends to be bride's maids, which she can do in $\binom{14}{5}$ ways. Thus she has $15\binom{14}{5}$ choices.

   (c) We have answered the question (how many wedding parties can the bride choose from) in two ways. The first way gives the left hand side of the identity and the second way gives the right hand side of the identity. Therefore the identity holds.

3. (a) After the 1, we need to find a 5-bit string with one 1. There are $\binom{5}{1}$ ways to do this.

   (b) $\binom{4}{1}$ (we need to pick 1 of the remaining 4 slots to be the second 1).

   (c) $\binom{3}{1}$

   (d) Yes. We still need strings starting with 0001 (there are $\binom{2}{1}$ of these) and strings starting 00001 (there is only $\binom{1}{1} = 1$ of these).

   (e) $\binom{6}{2}$

   (f) An example of the Hockey Stick Theorem:

   $$\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \binom{4}{1} + \binom{5}{1} = \binom{6}{2}$$

4. (a) 3^n, since there are 3 choices for each of the n digits.

   (b) 1, since all the digits need to be 2's. However, we might write this as $\binom{n}{0}$.

   (c) There are $\binom{n}{1}$ places to put the non-2 digit. That digit can be either a 0 or a 1, so there are $2\binom{n}{1}$ such strings.

   (d) We must choose two slots to fill with 0's or 1's. There are $\binom{n}{2}$ ways to do that. Once the slots are picked, we have two choices for the first slot (0 or 1) and two choices for the second slot (0 or 1). So there are a total of $2^2\binom{n}{2}$ such strings.

   (e) There are $\binom{n}{k}$ ways to pick which slots don't have the 2's. Then those slots can be filled in 2^k ways (0 or 1 for each slot). So there are $2^k\binom{n}{k}$ such strings.

   (f) These strings contain just 0's and 1's - so they are bit strings. There are 2^n bit strings. But keeping with the pattern above, we might write this as $2^n\binom{n}{n}$.

(g) We answer the question of how many length n ternary digit strings there are in two ways. First, each digit can be one of three choices, so the total number of strings is 3^n. On the other hand, we could break the question down into cases by how many of the digits are 2's. If they are all 2's, then there are $\binom{n}{0}$ strings. If all but one is a 2, then there are $2\binom{n}{1}$ strings. If all but 2 of the digits are 2's, then there are $2^2\binom{n}{2}$ strings - we choose 2 of the n digits to be non-2, and then there are 2 choices for each of those digits. And so on for every possible number of 2's in the string.

5. *Proof.* Question: How many k-letter words can you make using n different letters without repeating any letter?

Answer 1: There are n choices for the first letter, $n-1$ choices for the second letter, $n-2$ choices for the third letter, and so on until $n-(k-1)$ choices for the kth letter (since $k-1$ letters have already been assigned at that point). The product of these numbers can be written $\frac{n!}{(n-k)!}$ which is $P(n,k)$.

Answer 2: First pick k letters to be in the word from the n choices. This can be done in $\binom{n}{k}$ ways. Now arrange those letters into a word - there are k choices for the first letter, $k-1$ choices for the second, and so on, for a total of $k!$ arrangements of the k letters. Thus the total number of words is $\binom{n}{k}k!$. $\square$