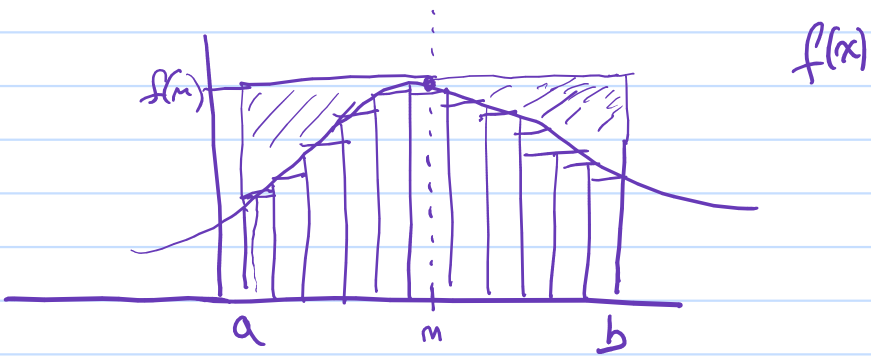
Numerical Integration

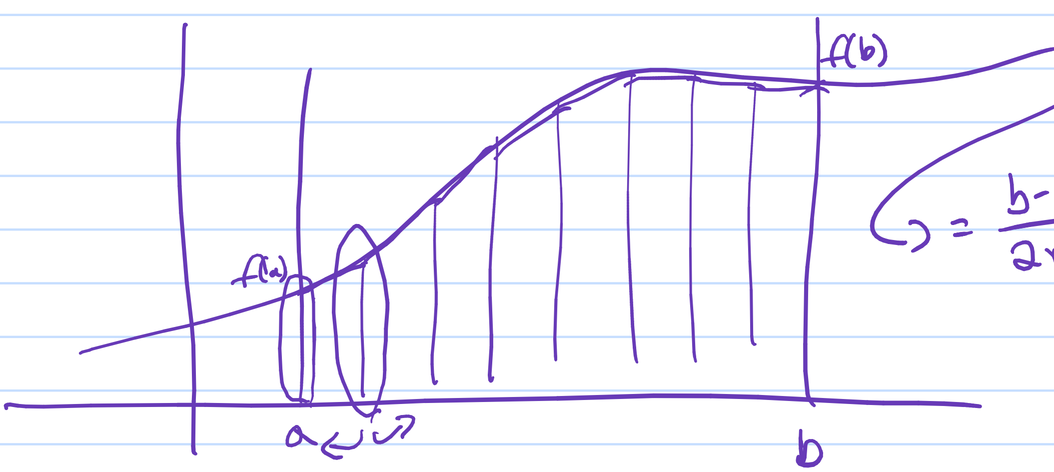
$$\int_I f(x)\,\omega(x)\,dx \approx \sum_{i=0}^{n} \underbrace{\omega_i}_{\text{weights}} f(\overbrace{x_i}^{\text{nodes}})$$

Midpoint Rule



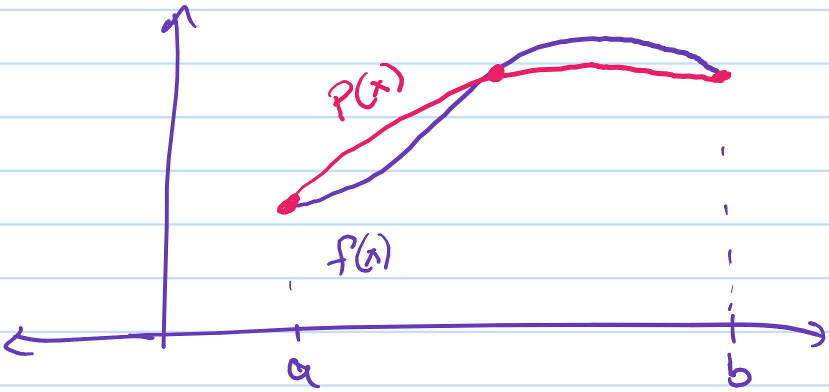
Trapezoid Rule

$$\int_a^b f(x)\,dx \approx \boxed{\sum_{i=1}^{n} \omega_i f(x_i)} = \frac{b-a}{2n}\Big(f(x_1) + 2f(x_2) + 2f(x_3) \dots + f(x_{n+1})\Big)$$



Simpson's Rule

- evenly spaced nodes



$$\int_a^b P(x)\,dx = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$\downarrow$

$$\int_a^b f(x)\,dx \simeq \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \dots f(x_n)\right]$$

$h = \frac{b-a}{n}$ and $x_i = a + ih$

$$\int_a^b f(x)\,dx \simeq \sum_{i=0}^{n} w_i f(x_i)$$

where $w_0 = w_n = \frac{h}{3}$

$w_i = \frac{4h}{3}$ if i = even

$w_i = \frac{2h}{3}$ if i = odd

Exactly compute integral of any <u>cubic polynomial.</u>

Monte Carlo Methods

SLLN $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(x_i) = \mathbb{E} f(\tilde{X})$

$\Rightarrow \quad \mathbb{E} f(\tilde{X}) \approx \boxed{\frac{1}{n}\sum_{i=1}^{n} f(x_i)}$

random draws from underlying pdf from $\mathbb{E}$

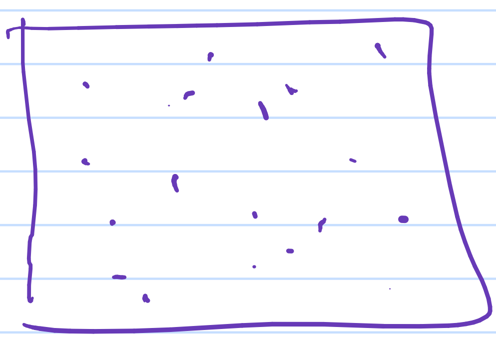
Suppose $\tilde{X}$ has cdf $F(x) = \Pr(\tilde{X} < x)$ and F^{-1} well defined.

If $\tilde{U} \sim U[0,1]$ then $F^{-1}(\tilde{U})$ has the same dist as $\tilde{X}$

$\rightarrow F^{-1}(\tilde{U}) \sim F(x)$

Quasi Monte Carlo Methods

MC



Q-MC

1. Halton sequences
2. anti-thetic acceleration $\rightarrow$ if draw x_1 then also use $-x_1$

Gaussian Quadrature

In the case of $\boxed{\text{Simpson's / Trapezoid}}$ the nodes (x_i) were equally spaced. $w_i =$ ____

In the case of MC, x_i was "randomly" spaced. $w_i = 1/n$

Quasi Mont Carlo → equally spaced nodes with $w_i =$ ~

Can we choose nodes and weights "efficiently"?

↪ approximate the integral with fewest nodes possible?

→ Can we choose $\boxed{x_i}$ and $\boxed{w_i}$ so that the approximation is exact with n nodes if f is P_{2n-1}.

Matlab → gnwnorm(   )

Gaussian Quad Example

Choose $x_1 \dots x_n$ and $w_1 \dots w_n$ to exactly match the $2n$ moments of $g(x)$

$$2n \text{ eqns } \left\{ \int_a^b x^k g(x)\,dx = \sum_{i=1}^{n} w_i x_i^k \quad k = 0 \dots 2n-1 \right.$$

if $g(x)$ is a density then the k equations are the k uncentered moments of a cont. random var.

$$k=0 \qquad E(x^0) = \sum_{i=1}^{n} w_i$$

$$k=1 \qquad E(x^1) = \sum_{i=1}^{n} w_i x_i$$

$$\vdots$$

$$k=2n-1 \qquad E(x^{2n-1}) = \sum_{i=1}^{n} w_i x_i^{2n-1}$$

Example: $X \sim N(0,1)$ and $\phi(x)$ is the pdf (weights)

$\boxed{\text{first 6 moments}}$

$$E(x^0) = 1 = \sum_{i=1}^{3} w_i$$

$$E(x^1) = 0 = \sum_{i=1}^{3} w_i x_i$$

$$E(x^2) = 1 = \sum_{i=1}^{3} w_i x_i^2$$

$$\vdots$$

$$E(x^5) = 0 = \sum_{i=1}^{3} w_i x_i^5$$

solve 6 non-linear eqns for 6 unknowns

x	w
$x_1 = -1.732$	0.166
$x_2 = 0$	0.667
$x_3 = 1.732$	0.166

so if you want to find $E(x^2) = var(x)$ $\int_{-\infty}^{\infty} x^2 \phi(x) \Rightarrow$

$$\boxed{E(x^2) = var(x) = \int_{-\infty}^{\infty} x^5 \phi(x)\, dx}$$

$$= \sum_{i=1}^{3} x_i^5 \omega_i$$

$$= \underline{(-1.732)^5(0.166) + (0)^5(0.667) + (1.732)^5(0.166)}$$