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# On approximating the modified Bessel function of the second kind

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## Abstract

In the article, we prove that the double inequalities

$$\frac{\sqrt{\pi}e^{-x}}{\sqrt{2(x+a)}} < K_0(x) < \frac{\sqrt{\pi}e^{-x}}{\sqrt{2(x+b)}}, \quad 1 + \frac{1}{2(x+a)} < \frac{K_1(x)}{K_0(x)} < 1 + \frac{1}{2(x+b)}$$

hold for all  $x > 0$  if and only if  $a \geq 1/4$  and  $b = 0$  if  $a, b \in [0, \infty)$ , where  $K_\nu(x)$  is the modified Bessel function of the second kind. As applications, we provide bounds for  $K_{n+1}(x)/K_n(x)$  with  $n \in \mathbb{N}$  and present the necessary and sufficient condition such that the function  $x \mapsto \sqrt{x + \rho}e^x K_0(x)$  is strictly increasing (decreasing) on  $(0, \infty)$ .

**MSC:** 33B10; 26A48

**Keywords:** modified Bessel function; gamma function; monotonicity

## 1 Introduction

The modified Bessel function of the first kind  $I_\nu(x)$  is a particular solution of the second-order differential equation

$$x^2 y''(x) + xy'(x) - (x^2 + \nu^2)y(x) = 0,$$

and it can be expressed by the infinite series

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(\nu + n + 1)} \left(\frac{x}{2}\right)^{2n+\nu}.$$

While the modified Bessel function of the second kind  $K_\nu(x)$  is defined by

$$K_\nu(x) = \frac{\pi(I_{-\nu}(x) - I_\nu(x))}{2 \sin(\pi \nu)}, \quad (1.1)$$

where the right-hand side of the identity of (1.1) is the limiting value in case  $\nu$  is an integer.

The following integral representation formula and asymptotic formulas for the modified Bessel function of the second kind  $K_\nu(x)$  can be found in the literature [1], 9.6.24, 9.6.8, 9.6.9, 9.7.2:

$$K_\nu(x) = \int_0^\infty e^{-x \cosh(t)} \cosh(\nu t) dt \quad (x > 0), \quad (1.2)$$

$$K_0(x) \sim -\log x \quad (x \rightarrow 0), \quad (1.3)$$

$$K_\nu(x) \sim \frac{1}{2} \Gamma(\nu) \left(\frac{x}{2}\right)^{-\nu} \quad (\nu > 0, x \rightarrow 0), \quad (1.4)$$

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \left[ 1 + \frac{4\nu^2 - 1}{8x} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{2!(8x)^2} + \cdots \right] \quad (x \rightarrow \infty). \quad (1.5)$$

From (1.2) we clearly see that

$$K_0(x) = \int_0^\infty e^{-x \cosh(t)} dt = \int_1^\infty \frac{e^{-xt}}{\sqrt{t^2 - 1}} dt, \quad (1.6)$$

$$K_0'(x) = - \int_1^\infty \frac{te^{-xt}}{\sqrt{t^2 - 1}} dt = -K_1(x). \quad (1.7)$$

Recently, the bounds for the modified Bessel function of the second kind  $K_\nu(x)$  have attracted the attention of many researchers. Luke [2] proved that the double inequality

$$\frac{8\sqrt{x}}{8x+1} < \sqrt{\frac{2}{\pi}} e^x K_0(x) < \frac{16x+7}{(16x+9)\sqrt{x}} \quad (1.8)$$

holds for all  $x > 0$ .

Gaunt [3] proved that the double inequality

$$\frac{1}{\sqrt{x + \frac{1}{2}}} < \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x+1)} < \sqrt{\frac{2}{\pi}} e^x K_0(x) < \frac{1}{\sqrt{x}} \quad (1.9)$$

takes place for all  $x > 0$ , where  $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$  is the classical gamma function.

In [4], Segura proved that the double inequality

$$\frac{\nu + \sqrt{x^2 + \nu^2}}{x} < \frac{K_{\nu+1}(x)}{K_\nu(x)} < \frac{\nu + \frac{1}{2} + \sqrt{x^2 + (\nu + \frac{1}{2})^2}}{x} \quad (1.10)$$

holds for all  $x > 0$  and  $\nu \geq 0$ .

Bordelon and Ross [5] and Paris [6] provided the inequality

$$\frac{K_\nu(x)}{K_\nu(y)} > e^{y-x} \left(\frac{x}{y}\right)^\nu \quad (1.11)$$

for all  $\nu > -1/2$  and  $y > x > 0$ .

Laforgia [7] established the inequality

$$\frac{K_\nu(x)}{K_\nu(y)} > e^{y-x} \left(\frac{x}{y}\right)^{-\nu} \quad (1.12)$$

for all  $y > x > 0$  if  $\nu \in (0, 1/2)$ , and inequality (1.12) is reversed if  $\nu \in (1/2, \infty)$ .

Baricz [8] presented the inequality

$$\frac{K_\nu(x)}{K_\nu(y)} > e^{y-x} \left(\frac{x}{y}\right)^{-1/2}$$

for all  $y > x > 0$  and  $\nu \in (-\infty, -1/2) \cup (1/2, \infty)$ .

Motivated by inequality (1.9), in the article, we prove that the double inequality

$$\frac{\sqrt{\pi}e^{-x}}{\sqrt{2(x+a)}} < K_0(x) < \frac{\sqrt{\pi}e^{-x}}{\sqrt{2(x+b)}}$$

holds for all  $x > 0$  if and only if  $a \geq 1/4$  and  $b = 0$  if  $a, b \in [0, \infty)$ . As applications, we provide bounds for  $K_{n+1}(x)/K_n(x)$  with  $n \in \mathbb{N}$  and present the necessary and sufficient condition such that the function  $x \mapsto \sqrt{x+pe^x}K_0(x)$  is strictly increasing (decreasing) on  $(0, \infty)$ .

## 2 Lemmas

In order to prove our main results, we need two lemmas which we present in this section.

**Lemma 2.1** (See [9]) *Let  $-\infty \leq a < b \leq \infty$ ,  $f, g : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ , and  $g'(x) \neq 0$  on  $(a, b)$ . If  $f'(x)/g'(x)$  is increasing (decreasing) on  $(a, b)$ , then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

*If  $f'(x)/g'(x)$  is strictly monotone, then the monotonicity in the conclusion is also strict.*

**Lemma 2.2** *The function*

$$x \mapsto f(x) = \frac{K_0(x)}{2[K_1(x) - K_0(x)]} - x \quad (2.1)$$

*is strictly increasing from  $(0, \infty)$  onto  $(0, 1/4)$ .*

*Proof* Let  $\omega(t) = \sqrt{(t-1)/(t+1)}$ . Then it follows from (1.6), (1.7) and (2.1) that

$$\begin{aligned} K_1(x) - K_0(x) &= \int_1^\infty \omega(t)e^{-xt} dt, \\ x[K_1(x) - K_0(x)] &= - \int_1^\infty \omega(t)d(e^{-xt}) \\ &= \omega(t)e^{-xt} \Big|_{t=\infty}^{t=1} + \int_1^\infty \omega'(t)e^{-xt} dt \\ &= \int_1^\infty \frac{t-1}{(t^2-1)^{3/2}} e^{-xt} dt, \\ K_0(x) - 2x[K_1(x) - K_0(x)] &= \int_1^\infty \frac{\omega(t)}{t+1} e^{-xt} dt, \\ f(x) &= \frac{K_0(x) - 2x[K_1(x) - K_0(x)]}{2[K_1(x) - K_0(x)]} = \frac{\int_1^\infty \frac{\omega(t)}{t+1} e^{-xt} dt}{2 \int_1^\infty \omega(t)e^{-xt} dt}, \\ f'(x) &= \frac{- \int_1^\infty \frac{t\omega(t)}{t+1} e^{-xt} dt \int_1^\infty \omega(t)e^{-xt} dt + \int_1^\infty \frac{\omega(t)}{t+1} e^{-xt} dt \int_1^\infty t\omega(t)e^{-xt} dt}{2(\int_1^\infty \omega(t)e^{-xt} dt)^2} \\ &= \frac{\int_1^\infty (\int_1^\infty \frac{s-t}{t+1} \omega(t)\omega(s)e^{-x(s+t)} dt) ds}{2(\int_1^\infty \omega(t)e^{-xt} dt)^2} = \frac{\int_1^\infty (\int_1^\infty \frac{t-s}{s+1} \omega(s)\omega(t)e^{-x(t+s)} ds) dt}{2(\int_1^\infty \omega(t)e^{-xt} dt)^2} \end{aligned} \quad (2.2)$$

$$\begin{aligned}
 &= \frac{\int_1^\infty \left( \int_1^\infty \frac{s-t}{t+1} \omega(t) \omega(s) e^{-x(s+t)} dt \right) ds + \int_1^\infty \left( \int_1^\infty \frac{t-s}{s+1} \omega(s) \omega(t) e^{-x(t+s)} ds \right) dt}{4 \left( \int_1^\infty \omega(t) e^{-xt} dt \right)^2} \\
 &= \frac{\int_1^\infty \int_1^\infty \frac{(s-t)^2}{(t+1)(s+1)} \omega(t) \omega(s) e^{-x(t+s)} dt ds}{4 \left( \int_1^\infty \omega(t) e^{-xt} dt \right)^2} > 0
 \end{aligned}$$

for all  $x > 0$ .

Note that (1.3)-(1.5) and (2.1) lead to

$$\begin{aligned}
 \lim_{x \rightarrow 0} xK_0(x) &= 0, & \lim_{x \rightarrow 0} xK_1(x) &= 1, \\
 \lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \left[ \frac{xK_0(x)}{2(xK_1(x) - xK_0(x))} - x \right] = 0,
 \end{aligned} \tag{2.3}$$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left[ \frac{K_0(x) - 2x(K_1(x) - K_0(x))}{2(K_1(x) - K_0(x))} \right] = \lim_{x \rightarrow \infty} \frac{\frac{1}{4x} + o(\frac{1}{x})}{\frac{1}{x} + o(\frac{1}{x})} = \frac{1}{4}. \tag{2.4}$$

Therefore, Lemma 2.2 follows easily from (2.2)-(2.4).  $\square$

### 3 Main results

**Theorem 3.1** *Let  $a, b \geq 0$ . Then the double inequality*

$$\frac{1}{\sqrt{x+a}} < \sqrt{\frac{2}{\pi}} e^x K_0(x) < \frac{1}{\sqrt{x+b}}$$

*holds for all  $x > 0$  if and only if  $a \geq 1/4$  and  $b = 0$ .*

*Proof* Let  $x > 0$ ,  $f(x)$  be defined by Lemma 2.2, and  $f_1(x)$ ,  $f_2(x)$  and  $F(x)$  be respectively defined by

$$f_1(x) = \frac{\pi}{2} - xe^{2x} K_0^2(x), \quad f_2(x) = e^{2x} K_0^2(x) \tag{3.1}$$

and

$$F(x) = \frac{\frac{\pi}{2} - xe^{2x} K_0^2(x)}{e^{2x} K_0^2(x)} = \frac{f_1(x)}{f_2(x)}. \tag{3.2}$$

Then from (1.5), (1.7) and (3.1) we clearly see that

$$\lim_{x \rightarrow \infty} f_1(x) = \lim_{x \rightarrow \infty} f_2(x) = 0, \tag{3.3}$$

$$\frac{f_1'(x)}{f_2'(x)} = \frac{-e^{2x} K_0^2(x) + 2xe^{2x} K_0(x)[K_1(x) - K_0(x)]}{-2e^{2x} K_0(x)[K_1(x) - K_0(x)]} = f(x). \tag{3.4}$$

It follows from (3.2)-(3.4), Lemmas 2.1 and 2.2 together with L'Hôpital's rule that the function  $F(x)$  is strictly increasing on  $(0, \infty)$  and

$$\lim_{x \rightarrow \infty} F(x) = \frac{1}{4}. \tag{3.5}$$

Note that (1.3) and (3.2) lead to

$$\lim_{x \rightarrow 0} F(x) = \lim_{x \rightarrow 0} \left[ \frac{\pi}{2e^{2x}K_0^2(x)} - x \right] = 0. \quad (3.6)$$

Therefore, Theorem 3.1 follows easily from (3.2), (3.5), (3.6) and the monotonicity of  $F(x)$ .  $\square$

**Remark 3.2** From Lemma 2.2 we clearly see that the double inequality

$$p < \frac{K_0(x)}{2[K_1(x) - K_0(x)]} - x < q$$

holds for all  $x > 0$  if and only if  $p \leq 0$  and  $q \geq 1/4$ .

From (1.7) and Remark 3.2 we get Corollary 3.3 immediately.

**Corollary 3.3** *Let  $p, q \geq 0$ . Then the double inequalities*

$$1 + \frac{1}{2(x+p)} < \frac{K_1(x)}{K_0(x)} < 1 + \frac{1}{2(x+q)}$$

and

$$[\log(e^x \sqrt{x+p})]' < -[\log K_0(x)]' < [\log(e^x \sqrt{x+q})]' \quad (3.7)$$

hold for all  $x > 0$  if and only if  $p \geq 1/4$  and  $q = 0$ .

**Remark 3.4** Let  $p \geq 0$ . Then from inequality (3.7) we know that the function  $x \mapsto \sqrt{x+p}e^x K_0(x)$  is strictly increasing (decreasing) on  $(0, \infty)$  if and only if  $p = 0$  ( $p \geq 1/4$ ). We clearly see that the bounds for  $K_1(x)/K_0(x)$  given in Corollary 3.3 are better than the bounds given in (1.10) for  $\nu = 0$ .

From (1.3), (1.5) and Remark 3.4 we get Corollary 3.5 immediately.

**Corollary 3.5** *The double inequality*

$$\sqrt{\frac{\pi}{2}} = \lim_{x \rightarrow \infty} [\sqrt{x+p}e^x K_0(x)] < [\sqrt{x+p}e^x K_0(x)] < \lim_{x \rightarrow 0} [\sqrt{x+p}e^x K_0(x)] = \infty \quad (3.8)$$

holds for all  $x > 0$  if  $p \geq 1/4$ , and inequality (3.8) is reversed if  $p = 0$ .

Remark 3.4 also leads to Corollary 3.6.

**Corollary 3.6** *Let  $p, q \geq 0$ . Then the double inequality*

$$\sqrt{\frac{y+p}{x+p}} e^{y-x} < \frac{K_0(x)}{K_0(y)} < \sqrt{\frac{y+q}{x+q}} e^{y-x}$$

holds for all  $0 < x < y$  if and only if  $p \geq 1/4$  and  $q = 0$ .

**Remark 3.7** We clearly see that the lower bound for  $K_0(x)/K_0(y)$  in Corollary 3.6 is better than the bounds given in (1.11) and (1.12) for  $\nu = 0$ .

**Remark 3.8** From the inequality

$$\frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} < \frac{1}{\sqrt{x + \frac{1}{4}}}$$

given in [10], (2.20), and the fact that

$$\frac{1}{\sqrt{x + \frac{1}{4}}} > \frac{8\sqrt{x}}{8x + 1}$$

for all  $x > 0$  we clearly see that the lower bound given in Theorem 3.1 for  $\sqrt{2/\pi}e^x K_0(x)$  is better than that given in (1.8) and (1.9). But the upper bound given in Theorem 3.1 is weaker than that given in (1.8).

**Remark 3.9** From Theorem 3.1 and Corollary 3.3 we clearly see that there exist  $\theta_1 = \theta_1(x) \in (0, 1/4)$  and  $\theta_2 = \theta_2(x) \in (0, 1/4)$  such that

$$K_0(x) = \sqrt{\frac{\pi}{2(x + \theta_1)}} e^{-x}, \quad K_1(x) = \left[1 + \frac{1}{2(x + \theta_2)}\right] \sqrt{\frac{\pi}{2(x + \theta_1)}} e^{-x}$$

for all  $x > 0$ .

**Theorem 3.10** Let  $x > 0$ ,  $n \in \mathbb{N}$ ,  $R_n(x) = K_{n+1}(x)/K_n(x)$ ,  $u_0(x) = 1 + 1/(2x)$ ,  $v_0(x) = 1 + 1/(2x + 1/2)$ , and  $u_n(x)$  and  $v_n(x)$  be defined by

$$u_n(x) = \frac{1}{v_{n-1}(x)} + \frac{2n}{x}, \quad v_n(x) = \frac{1}{u_{n-1}(x)} + \frac{2n}{x} \quad (n \geq 1). \quad (3.9)$$

Then the double inequality

$$v_n(x) < R_n(x) = \frac{K_{n+1}(x)}{K_n(x)} < u_n(x) \quad (3.10)$$

holds for all  $x > 0$  and  $n \in \mathbb{N}$ .

*Proof* We use mathematical induction to prove inequality (3.10). From Corollary 3.3 we clearly see that inequality (3.10) holds for all  $x > 0$  and  $n = 0$ .

Suppose that inequality (3.10) holds for  $n = k - 1$  ( $k \geq 1$ ), that is,

$$v_{k-1}(x) < R_{k-1}(x) < u_{k-1}(x). \quad (3.11)$$

Then it follows from (3.9) and (3.11) together with the formula

$$\frac{K'_v(x)}{K_v(x)} = -\frac{K_{v-1}(x)}{K_v(x)} - \frac{v}{x} = -\frac{K_{v+1}(x)}{K_v(x)} + \frac{v}{x}$$

given in [11] that

$$\begin{aligned} R_k(x) &= \frac{1}{R_{k-1}(x)} + \frac{2k}{x}, \\ v_k(x) &= \frac{1}{u_{k-1}(x)} + \frac{2k}{x} < R_k(x) < \frac{1}{v_{k-1}(x)} + \frac{2k}{x} = u_k(x). \end{aligned} \quad (3.12)$$

Inequality (3.12) implies that inequality (3.10) holds for  $n = k$ , and the proof of Theorem 3.10 is completed.  $\square$

**Remark 3.11** Let  $n = 1, 2, 3$ . Then Theorem 3.10 leads to

$$\begin{aligned} \frac{2(x+1)^2}{x(2x+1)} &< \frac{K_2(x)}{K_1(x)} < \frac{4x^2+9x+6}{x(4x+3)}, \\ \frac{4x^3+19x^2+36x+24}{x(4x^2+9x+6)} &< \frac{K_3(x)}{K_2(x)} < \frac{2x^3+9x^2+16x+8}{2x(x+1)^2}, \\ \frac{2(x^4+8x^3+28x^2+48x+24)}{x(2x^3+9x^2+16x+8)} &< \frac{K_4(x)}{K_3(x)} < \frac{4x^4+33x^3+120x^2+216x+144}{x(4x^3+19x^2+36x+24)} \end{aligned}$$

for all  $x > 0$ .

**Remark 3.12** It is not difficult to verify that

$$\begin{aligned} \frac{2(x+1)^2}{x(2x+1)} &> \frac{1+\sqrt{x^2+1}}{x}, & \frac{4x^2+9x+6}{x(4x+3)} &< \frac{\frac{3}{2}+\sqrt{x^2+\frac{9}{4}}}{x}, \\ \frac{4x^3+19x^2+36x+24}{x(4x^2+9x+6)} &> \frac{2+\sqrt{x^2+4}}{x}, & \frac{2x^3+9x^2+16x+8}{2x(x+1)^2} &< \frac{\frac{5}{2}+\sqrt{x^2+\frac{25}{4}}}{x}, \\ \frac{2(x^4+8x^3+28x^2+48x+24)}{x(2x^3+9x^2+16x+8)} &> \frac{3+\sqrt{x^2+9}}{x}, \\ \frac{4x^4+33x^3+120x^2+216x+144}{x(4x^3+19x^2+36x+24)} &< \frac{\frac{7}{2}+\sqrt{x^2+\frac{49}{4}}}{x} \end{aligned}$$

for  $x > 0$ . Therefore, the bounds given in Remark 3.11 are better than the bounds given in (1.10) for  $v = 1, 2, 3$ .

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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