Wallaby Tests

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Contents

1 The test suite

Wallaby is a finite-element based groundwater flow code. This document describes the test suite associated with Wallaby.

2 Gravitational head

Without fluxes, the steadystate pressure distribution is just

$$P(x) = P_0 + \rho_0 g x \,, \tag{2.1}$$

where ρ_0 is the constant reference fluid density, g is the acceleration due to gravity (a vector), and x is position.

This is verified in Wallaby using the following tests on a single square element of unit size.

1. Steady state analysis with SUPG, fully-saturated situation.
2. Steady state analysis without SUPG, fully-saturated situation.
3. Transient analysis with SUPG, fully-saturated situation.
4. Transient analysis without SUPG, fully-saturated situation.
5. Steady state analysis with SUPG, partially saturated situation.
6. Steady state analysis without SUPG, partially saturated situation.
7. Transient analysis with SUPG, partially saturated situation.
8. Transient analysis without SUPG, partially saturated situation.
9. Transient analysis with SUPG, partially saturated situation.
10. Transient analysis without SUPG, partially saturated situation, with nonzero immobile saturation, and with 5 elements in the x direction which has length 20 m.
11. Transient analysis with SUPG, partially saturated situation, with nonzero immobile saturation, and with 5 elements in the x direction which has length 20 m.

The final two tests are interesting because they also show that a nonzero S_{imm} means that effective saturation can no longer reduce to zero, and they also illustrate the stabilisating nature of SUPG in reducing oscillatory behaviour in the solution, as shown in Figure 2.1

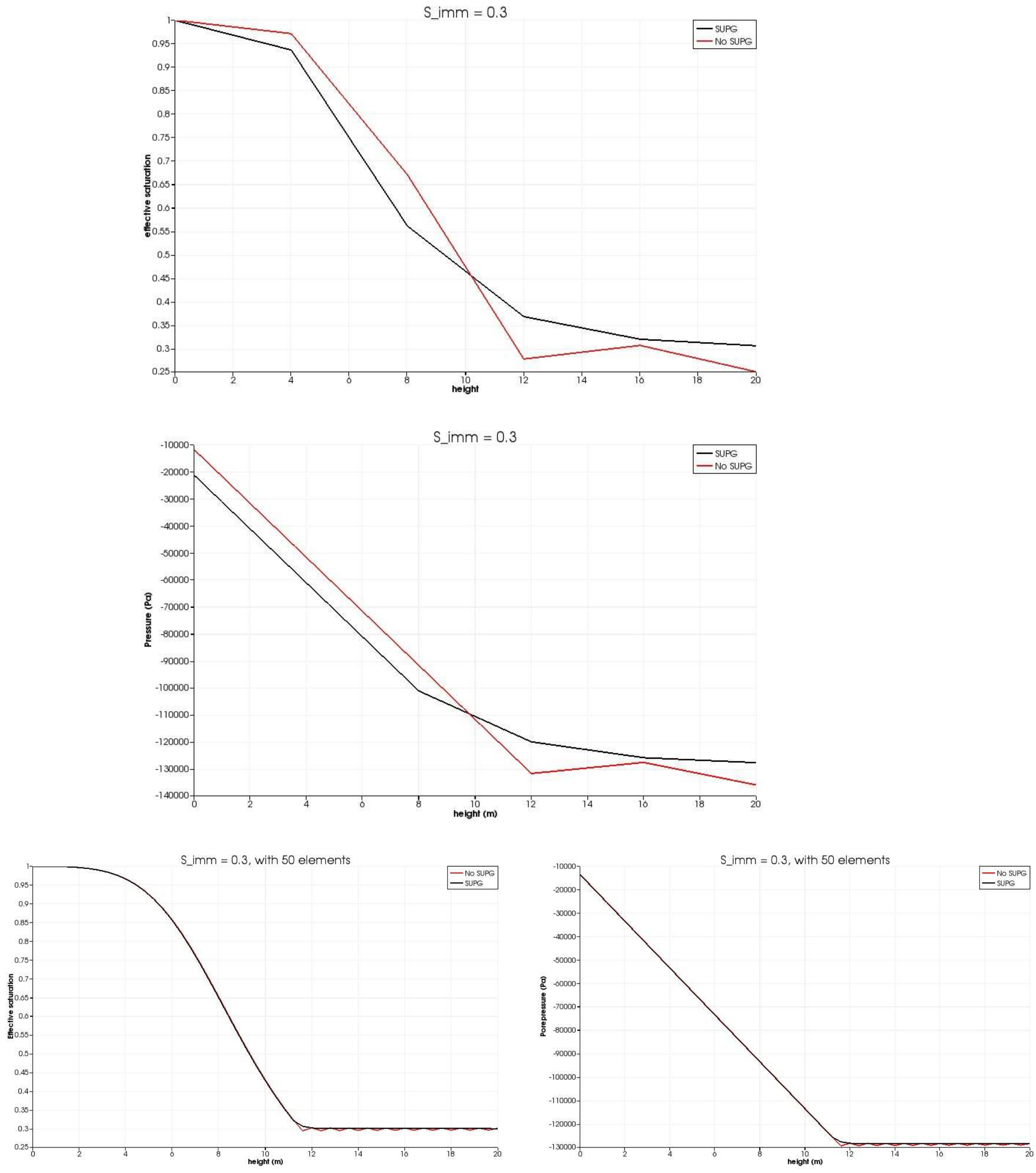

Figure 2.1: Results for $S_{\mathrm{imm}} = 0.3$. Gravity points to the left. Top picture: Effective saturation. Middle picture: Pore pressure. Bottom pictures: The situation with 50 elements in the x direction instead of just 5. In each picture the red line is without SUPG, and oscillatory results can be observed in addition to $S_{\mathrm{eff}} < S_{\mathrm{imm}}$.

3 A pressure pulse in the fully saturated situation

Darcy's equation for flow through a fully saturated medium without gravity and without sources is

$$\frac{\partial}{\partial t}\phi\rho = \nabla_i\left(\frac{\rho\kappa_{ij}}{\mu}\nabla_j P\right) \ , \tag{3.1}$$

with notation described in the Theory Manual. Using $\rho \propto \exp(P/K)$, where K is the fluid bulk modulus, Darcy's equation becomes

$$\frac{\partial}{\partial t}\rho = \nabla_i\alpha_{ij}\nabla\rho \ , \tag{3.2}$$

with

$$\alpha_{ij} = \frac{\kappa_{ij}B}{\mu\phi} \ . \tag{3.3}$$

Here I've assumed the porosity and bulk modulus are constant in space and time.

Consider the one-dimensional case were the spatial dimension is the semi-infinite line $x \geq 0$. Suppose that initially the pressure is constant, so that

$$\rho(x, t=0) = \rho_0 \quad \text{for} \quad x \geq 0 \ . \tag{3.4}$$

Then apply a fixed-pressure Dirichlet boundary condition at $x = 0$ so that

$$\rho(x=0, t>0) = \rho_\infty \tag{3.5}$$

The solution of the above differential equation is well known to be

$$\rho(x,t) = \rho_\infty + (\rho_0 - \rho_\infty)\,\mathrm{Erf}\left(\frac{x}{\sqrt{4\alpha t}}\right) \ , \tag{3.6}$$

where Erf is the error function.

This is verified in Wallaby using the following tests on a line of 100 elements.

1. Steady state analysis with SUPG, in 3D, to demonstrate that the steady-state of $\rho = \rho_\infty$ is achieved.

2. Transient analysis with SUPG in 3D.

An example verification is shown in Figure 3.1.

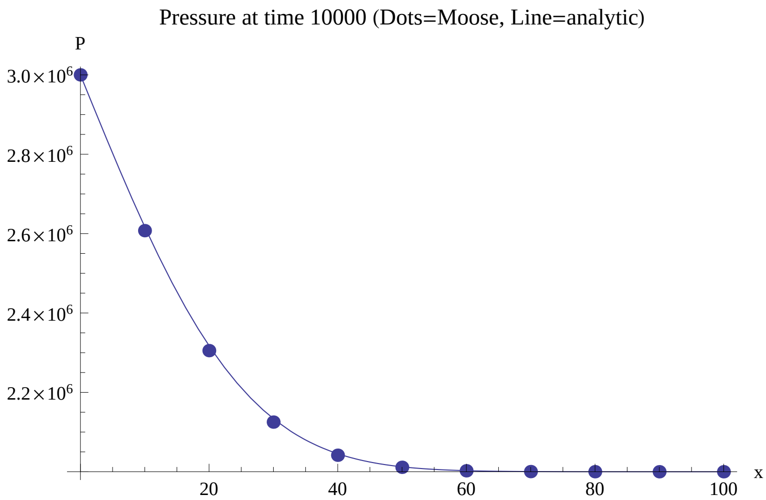

Figure 3.1: Comparison between the Wallaby result (in dots), and the exact analytic expression given by Eqn (3.6). This test had 10 elements in the x direction, with $0 \leq x \leq 100$ m, and ran for a total of 10^4 seconds with 10 timesteps. The parameters were $B = 2$ GPa, $\kappa_{xx} = 10^{-15}$ m^2, $\mu = 10^{-3}$ Pa.s, $\phi = 0.1$, with initial pressure $P = 2$ MPa, and applied pressure $P = 3$ MPa at $x = 0$. For greater spatial resolution and smaller timesteps the agreement increases.

4 Buckley-Leverett

Wallaby is compared with a simple one dimensional Buckley-Leverett problem[1]. The fluid moves in a region $0 \leq x \leq 15\,$m. A fully-saturated front initially sits at position $x = 5$, while the region $x > 5$ is initially unsaturated. With zero suction function $P_c = 0$, there is no diffusion of the sharp front, and it progresses towards the right. This is a difficult problem to simulate numerically as maintaining the sharp front is hard. The speed is independent of the relative permeability, since the fluid is flowing from a fully-saturated region. This problem is therefore a good test of the upwinding.

In the simulation below, the pressure at the left boundary is kept fixed at $P_0 = 0.98\,$MPa, while the right boundary is kept fixed at $P_{15} = -20000\,$Pa. The medium's permeability is set to $\kappa = 10^{-10}\,\mathrm{m}^2$ and its porosity is $\phi = 0.15$. It is not possible to use a zero suction function in Wallaby, but using $\alpha = 10^{-4}\,\mathrm{Pa}^{-1}$ and $m = 0.8$ approximates it. The fluid viscosity is $\mu = 10^{-3}\,$Pa.s.

The initial condition is

$$P(t=0) = \begin{cases} P_0 - (P_0 - P_{15})x/5 & \text{for } x < 5 \\ P_{15} & \text{for } x \geq 5 \end{cases} , \tag{4.1}$$

which is shown in Figure 4.1. With the suction function defined above this gives

$$S(t=0) = \begin{cases} 1 & \text{for } x \leq 4.9 \\ 0.061 & \text{for } x \geq 5 \end{cases} \tag{4.2}$$

Good approximations for the pressure $P(x,t)$ and the front position $f(t)$ may be determined from

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= -\frac{\kappa}{\phi\mu} \left.\frac{\partial P}{\partial x}\right|_{x=f} , \\ P(x,t) &= \begin{cases} P_0 - (P_0 - P_{15})x/f & \text{for } x \leq f \\ P_{15} & \text{for } x > f \end{cases} , \end{aligned} \tag{4.3}$$

which has solution

$$f(t) = \sqrt{f(0)^2 + \frac{2\kappa}{\phi\mu}(P_0 - P_{15})t} \, . \tag{4.4}$$

For the parameters listed above, the front will be at position $f = 9.6\,$m at $t = 50\,$s. This solution is only valid for zero capillary suction. A nonzero suction function will tend to diffuse the sharp front away.

[1] SE Buckley and MC Leverett (1942) "Mechanism of fluid displacements in sands". Transactions of the AIME **146** 107–116

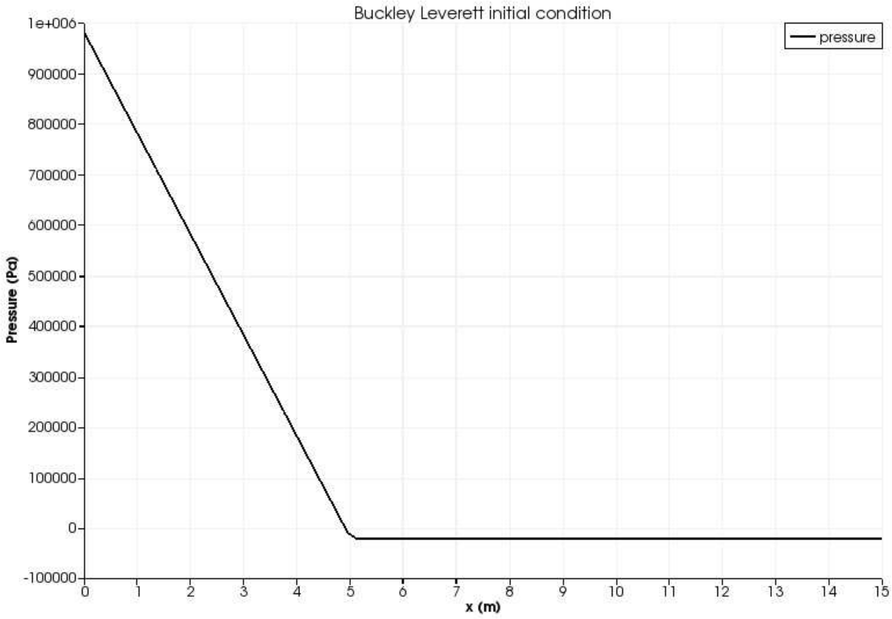

Figure 4.1: Initial setup of the Buckley-Leverett problem where porepressure is a piecewise linear function. The region $x \leq 4.9$ is fully saturated, while the region $x > 5$ has saturation 0.061. During simulation the value $P(x=0) = 0.98 \times 10^6$ MPa is held fixed.

With coarse meshes it is impossible to simulate a sharp front, of course, since the front is spread over at least one element. It is therefore quite advantageous to use mesh adaptivity in this test, since the mesh can be fine around the front where all the interesting dynamics occurs, and coarse elsewhere.

Figure 4.2 shows the results from a Wallaby simulation with an initial mesh of element size 1 m, and a minimum size of 0.125 m, with a maximum timestep of 0.3 s. (Reducing the minimum element size or the maximum timestep size keeps the front sharper.) The front in this simulation is between $x = 9.9$ m and $x = 10.35$ m, in fair agreement with the predicted value of 9.6 m.

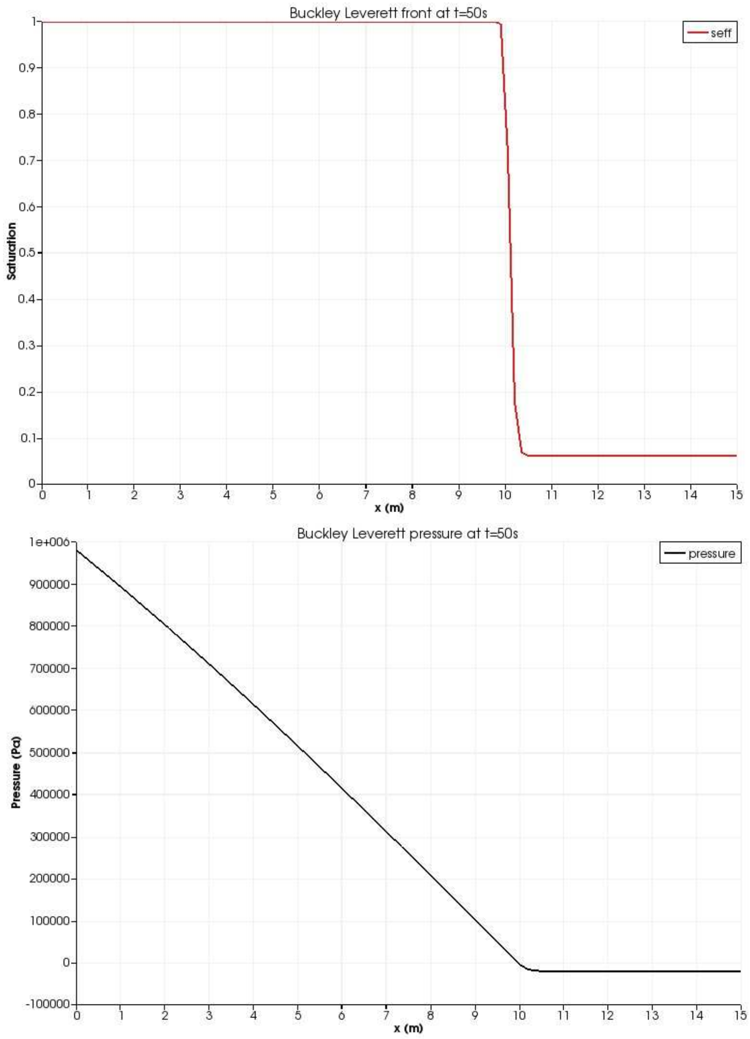

Figure 4.2: The Wallaby solution of the Buckley-Leverett problem at $t = 50\,\mathrm{s}$. Top: saturation. Bottom: porepressure. The front sits between $x = 9.9\,\mathrm{m}$ and $x = 10.35\,\mathrm{m}$.

5 Unsaturated flow in a bar

This problem is one of cosflow's benchmark problems. Water inside a porous "bar" of material length 10m, and width and depth 1m is initialised to porepressure $P_0 < 0$, corresponding to water saturation $S_0 < 1$. The porepressure left-hand end (at $x = 0$) is raised and fixed at $P_1 < 0$, corresonding to water saturation S_1 with $S_0 < S_1 < 1$, and the evolution of porepressure at the right-hand end ($x = 10\,\mathrm{m}$) is recorded. Apart from the left-hand end, the other boundaries of the bar are impermeable. The setup is shown in Figure 5.1.

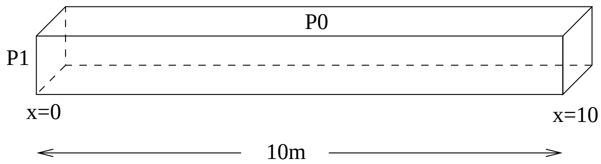

Figure 5.1: The unsaturated problem involves a porous "bar" of material of length 10m with initial porepressure P_0. The left-hand end is raised to porepressure P_1 and held fixed. The other parts of the bar's exterior surface are impermeable.

This problem exhibits quite severe mesh dependency, and since the upwinding in cosflow and Wallaby are different, the results are not expected to be the same, except in the limit of zero element size.
The following parameters are used

Bar porosity	0.1
Bar permeability	10^{-12} m^2
Gravity	0
Water density	1000 kg.m^{-3}
Water viscosity	0.001 Pa.s
Water bulk modulus	2 GPa
Water immobile saturation	0.0
Water residual saturation	0.0
Air residual saturation	0.0
van Genuchten α	10^{-4} Pa^{-1}
van Genuchten a	0.35
Initial porepressure P_0	-197347.0503 Pa
Initial saturation S_0	0.2
Applied pressure P_1	-9283.000501 Pa
Applied saturation S_1	0.8

Figure 5.2 shows agreement between Wallaby and cosflow for a variety of different mesh densities, including an adaptive mesh example. The Wallaby results appear to be closer to the zero-element-size result than cosflow, but for low mesh density exhibit spurious oscillations as the high saturation region moves into the low saturation region. (This oscillation is almost definitely due to my current inability to lump the mass term.)

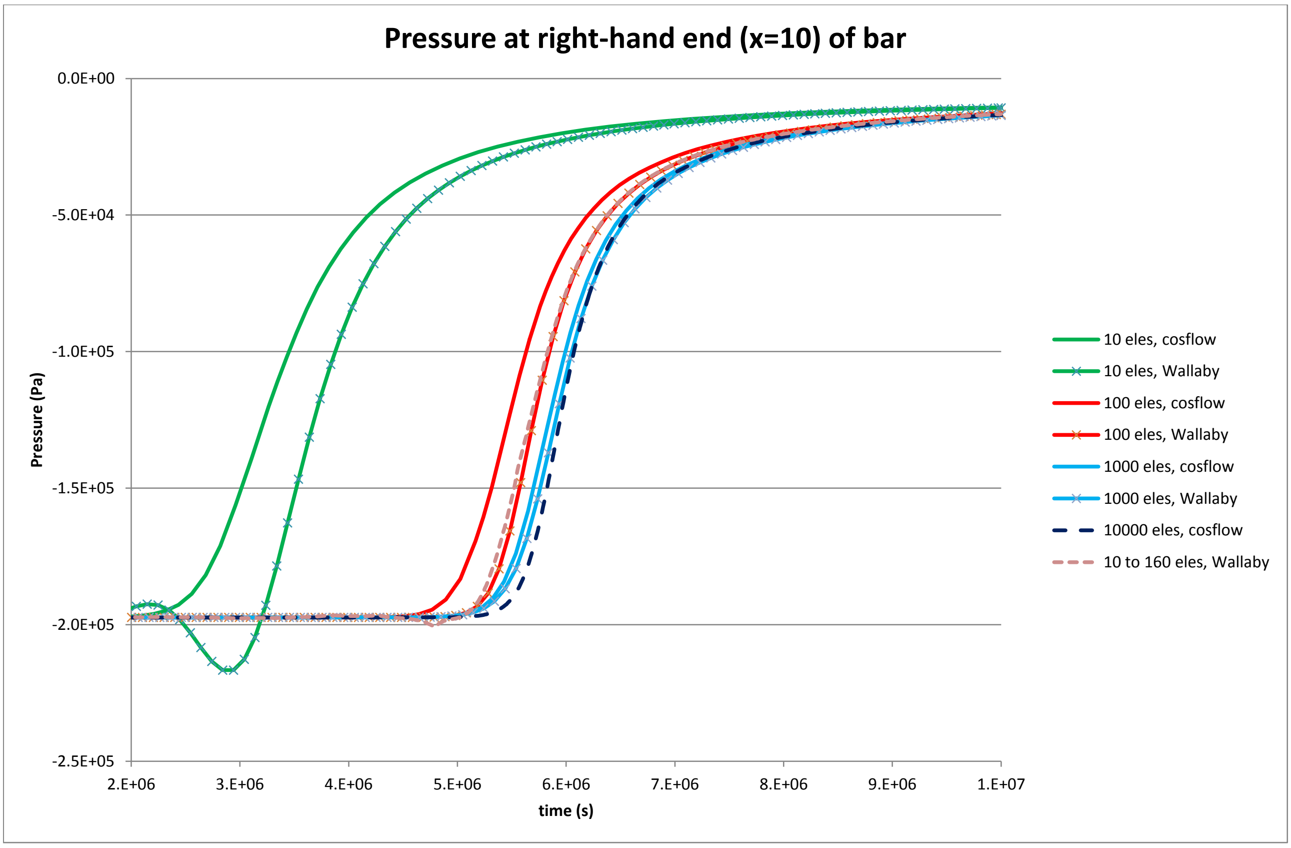

Figure 5.2: The porepressure at the right-hand end ($x = 10$) of the bar as a function of time for various different meshes.

6 Infiltration and drainage

Forsyth, Wu and Pruess[1] describe a HYDRUS simulation of an experiment involving infiltration (experiment 1) and subsequent drainage (experiment 2) in a large caisson. The simulation is effectively one dimensional, and is shown in Figure 6.1.

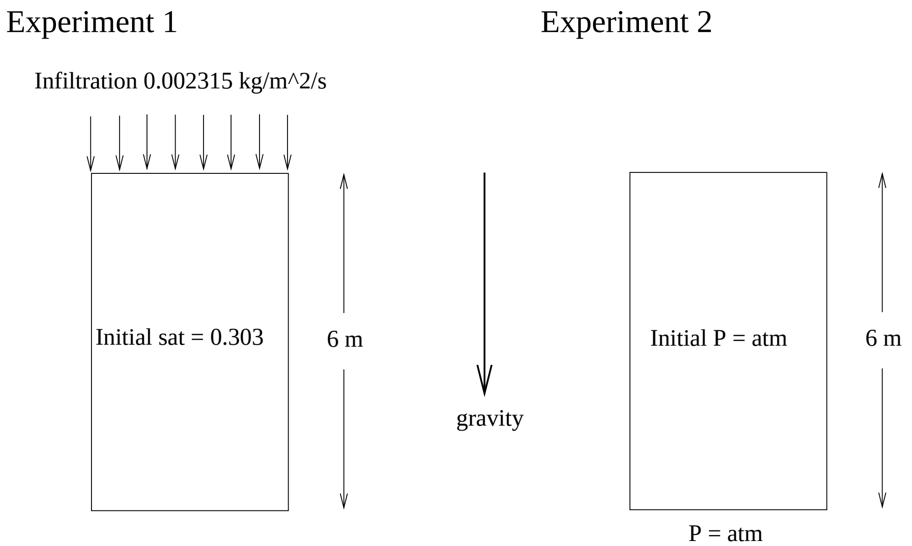

Figure 6.1: Two experimental setups from Forsyth, Wu and Pruess. Experiment 1 involves infiltration of water into an initially unsaturated caisson. Experiment 2 involves drainage of water from an initially saturated caisson.

The properties common to each experiment are:

[1] PA Forsyth, YS Wu and K Pruess, "Robust numerical methods for saturated-unsaturated flow with dry initial conditions in heterogeneous media", Advances in Water Resources 18 (1995) 25–38

Caisson	0.33
Caisson permeability	$2.95 \times 10^{-13}\,\mathrm{m}^2$
Gravity	$10\,\mathrm{m.s}^{-2}$
Water density	$1000\,\mathrm{kg.m}^{-3}$
Water viscosity	0.00101 Pa.s
Water bulk modulus	20 MPa
Water immobile saturation	0.0
Water residual saturation	0.0
Air residual saturation	0.0
Air pressure	0.0
van Genuchten α	$1.43 \times 10^{-4}\,\mathrm{Pa}^{-1}$
van Genuchten a	0.336
van Genuchten_1 cutoff	0.99

In each experiment 120 finite elements were used along the length of the Caisson. The modified van-Genuchten relative permeability curve was employed in order to improve convergence significantly. Hydrus also uses a modified van-Genuchten curve, although I couldn't find any details on the modification.

In experiment 1, the caisson is initially at saturation 0.303 ($P = -72620.4\,\mathrm{Pa}$), and water is pumped into the top with a rate $0.002315\,\mathrm{kg.m}^{-2}.\mathrm{s}^{-1}$. This causes a front of water to advance down the caisson. Figure 6.2 shows the agreement between Wallaby and the published result (this result was obtained by extracting data by hand from online graphics).

In experiment 2, the caisson is initially fully saturated at $P = 0$, and the bottom is held at $P = 0$ to cause water to drain via the action of gravity. Figure 6.2 shows the agreement between Wallaby and the published result.

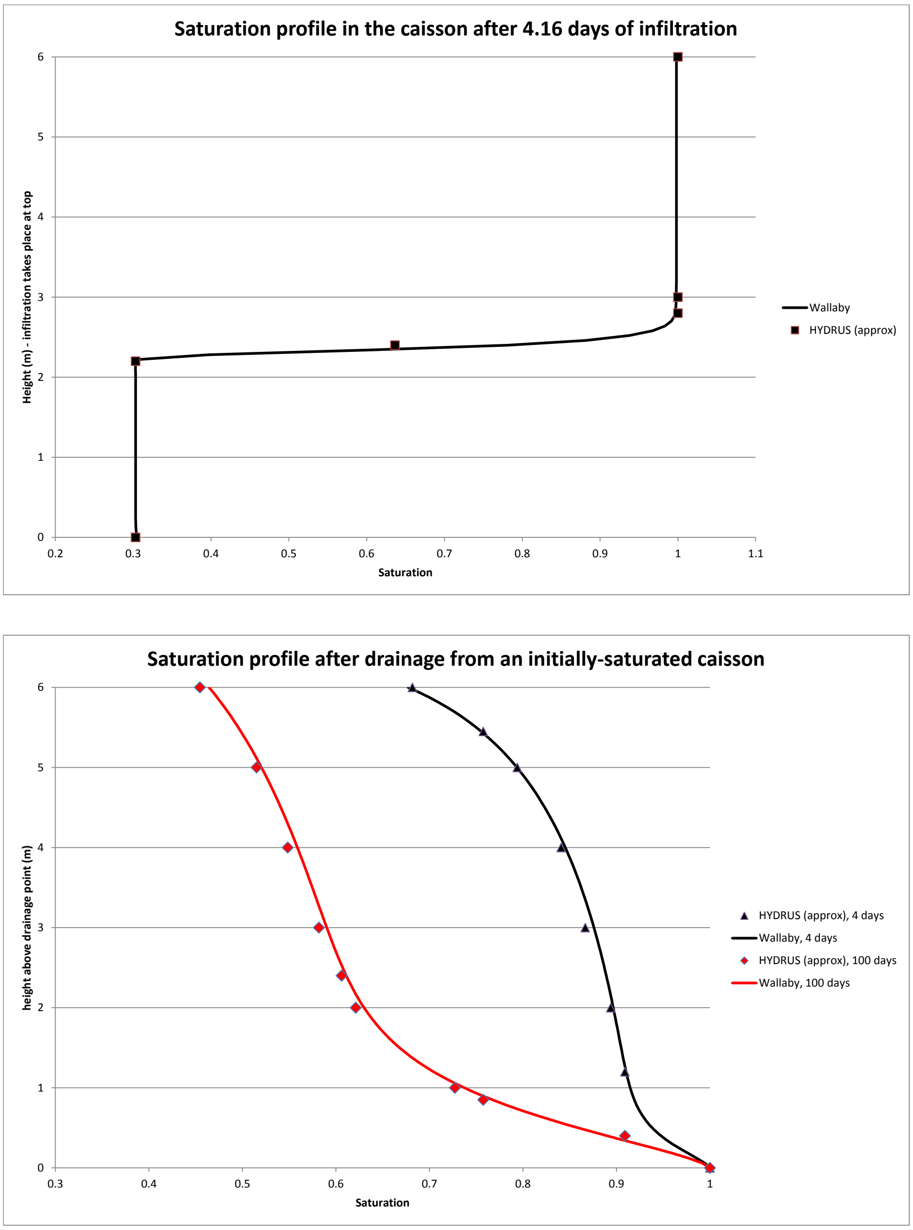

Figure 6.2: Saturation profiles in the caisson. Top: After 4.16 days of infiltration. Bottom: After drainage from an initially-saturated simulation (4 days and 100 days profiles). Note that the HYDRUS results are only approximate as I extrated the data by hand from online graphics.

7 Newton cooling from a bar

Darcy's equation for flow through a fully saturated medium without gravity and without sources is

$$\frac{\partial}{\partial t}\phi\rho = \nabla_i \left(\frac{\rho\kappa_{ij}}{\mu} \nabla_j P \right) , \tag{7.1}$$

with notation described in the Theory Manual. Using $\rho \propto \exp(P/K)$, where K is the fluid bulk modulus, Darcy's equation becomes

$$\frac{\partial}{\partial t}\rho = \nabla_i \alpha_{ij} \nabla\rho , \tag{7.2}$$

with

$$\alpha_{ij} = \frac{\kappa_{ij} B}{\mu\phi} . \tag{7.3}$$

Here I've assumed the porosity and bulk modulus are constant in space and time.

Consider the one-dimensional case where a bar sits between $x = 0$ and $x = L$ with initial pressure distribution so $\rho(x, t = 0) = \rho_0(x)$. Maintain the end $x = 0$ at constant pressure, so that $\rho(x = 0, t) = \rho_0(0)$. At the end $x = L$, prescribe a sink flux

$$\left.\frac{\partial\rho}{\partial x}\right|_{x=L} = -C\left(\rho - \rho_e\right)_{x=L} , \tag{7.4}$$

where ρ_e is a fixed quantity ("e" stands for "external"), and C is a constant conductance. This corresponds to the flux

$$\left.\frac{\partial P}{\partial x}\right|_{x=L} = -CB\left(1 - e^{(P_e - P)/B}\right)_{x=L} , \tag{7.5}$$

which can easily be coded into a Wallaby input file: the flux is $\rho\kappa\nabla P/\mu = -CB\kappa(e^{P/B} - e^{P_e/B})/\mu$.

The solution of this problem is well known and is

$$\rho(x,t) = \rho_0(0) - \frac{\rho_0(0) - \rho_e}{1 + LC} Cx + \sum_{n=1}^{\infty} a_n \sin\frac{k_n x}{L} e^{-k_n^2 \alpha t/L^2} , \tag{7.6}$$

where k_n is the n^{th} positive root of the equation $LC\tan k + k = 0$ (k_n is a little bigger than $(2n - 1)\pi/2$), and a_n is determined from

$$a_n \int_0^L \sin^2\frac{k_n x}{L}\,\mathrm{d}x = \int_0^L \left(\rho_0(x) - \rho_0(0) + \frac{\rho_0(0) - \rho_e}{1 + LC} Cx \right) \sin\frac{k_n x}{L}\,\mathrm{d}x , \tag{7.7}$$

which may be solved numerically.

The problem is solved in Wallaby using the following parameters:

Bar length	100 m
Bar porosity	0.1
Bar permeability	10^{-15} m^2
Gravity	0
Water density	1000 kg.m^{-3}
Water viscosity	0.001 Pa.s
Water bulk modulus	1 MPa
Initial porepressure P_0	2 MPa
Environmental pressure P_e	0
Conductance C	0.05389 m^{-1}

This conductance is chosen so at steadystate $\rho(x = L) = 2000\,\mathrm{kg.m}^{-3}$.

The problem is solved using 1000 elements along the x direction ($L = 100\,$m), and using 100 time-steps of size 10^6 s. Using fewer elements or fewer timesteps means the agreement with the theory is marginally poorer. The problem is also solved using the steadystate solver. In this case the initial condition is $P = 2 - x/L\,$MPa, since the uniform $P = 2\,$MPa does not converge. The results are shown in Figure 7.1.

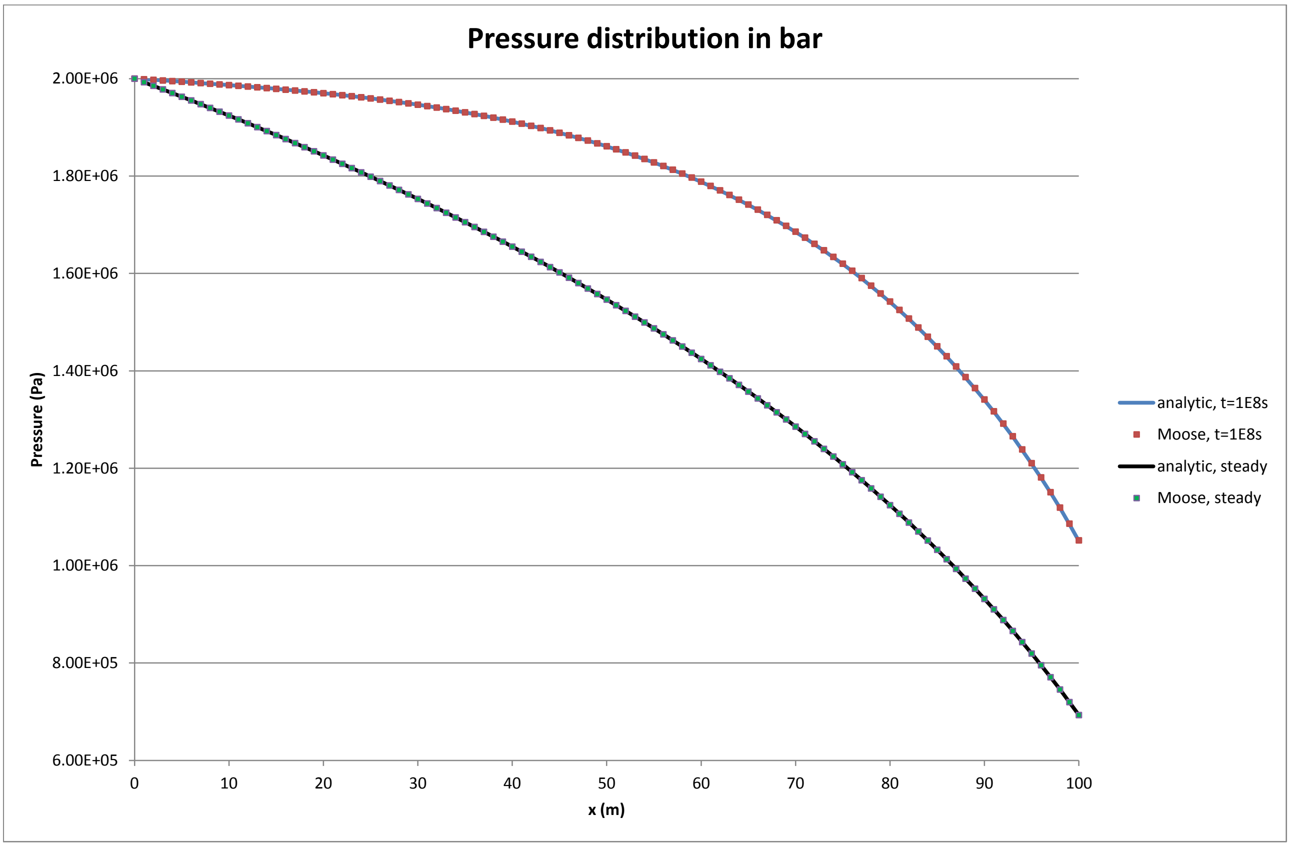

Figure 7.1: The porepressure in the bar at $t = 10^8$ s, and at steadystate. Wallaby agrees well with theory.

8 Future tests

See ”Benchmarking of Richards Model” Yanlian Du, Wenqing Wang and Olaf Kolditz for a summary of the usual suspects.

See Forsyth, Wu and Pruess for more tests.