

# **Richards Tests**

Andy Wilkins  
CSIRO

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# 1 Introduction

The Richards' equation describes slow fluid flow through a porous medium. This document describes the test suite associated with the Richards MOOSE code, both brief unit-style tests, and more complicated benchmark verifications. There are two other accompanying documents: (1) The theoretical and numerical foundations of the code; (2) Examples of input syntax that users can utilise when building models.

## 2 UserObject tests

The Richards' UserObjects define the nonlinear functions that form the core of all models. The tests of these UserObjects involve checking whether the functions and their derivatives are correctly coded. This is done by comparing the values of the UserObjects with ParsedFunctions that are coded into a MOOSE input file, and the values of the UserObject derivatives with finite-differences of the same ParsedFunction. These are simple tests and are part of the automatic test suite. The following tests are performed.

- That the 'power' form of the relative permeability:

$$\kappa_{\text{rel}}(S) = (n+1)S^n - nS^{n+1}, \quad (2.1)$$

is correctly coded, and also that its first and second derivatives with respect to  $S$  are correctly coded.

- That the 'van Genuchten' form of the relative permeability:

$$\kappa_{\text{rel}}(S) = \sqrt{S} \left( 1 - \left( 1 - S^{1/m} \right)^m \right)^2, \quad (2.2)$$

is correctly coded, and also that its first and second derivatives with respect to  $S$  are correctly coded.

- That the 'modified van Genuchten' form of the relative permeability:

$$\kappa_{\text{rel}}(S) = \begin{cases} \sqrt{S} \left( 1 - \left( 1 - S^{1/m} \right)^m \right)^2 & \text{for } S < S_{\text{cut}} \\ \text{cubic} & \text{for } S \geq S_{\text{cut}} \end{cases} \quad (2.3)$$

is correctly coded, and also that its first and second derivatives with respect to  $S$  are correctly coded.

- That the 'constant bulk modulus' form of the density

$$\rho(P) = \rho_0 e^{P/B}, \quad (2.4)$$

is correctly coded, and also that its first and second derivatives with respect to  $P$  are correctly coded.

- That the 'ideal gas' form of the density

$$\rho(P) = s(P - P_0), \quad (2.5)$$

is correctly coded, and also that its first and second derivatives with respect to  $P$  are correctly coded.

- That the ‘van Genuchten’ effective saturation

$$S_{\text{eff}} = \left( 1 + (\alpha P_c)^{\frac{1}{1-m}} \right)^{-m}, \quad (2.6)$$

is correctly coded, and also that its first and second derivatives with respect to  $P_c$  are correctly coded.

### 3 Jacobian tests

The UserObjects and their derivatives need to be combined to form a residual and a Jacobian during the solution process. Correctly coding the Jacobian leads to rapid convergence to the correct solution, so tests of the Jacobian are important. These are simple tests and are part of the automatic test suite.

In MOOSE parlance, the Jacobian consists of ‘Diagonal’ and ‘OffDiagonal’ terms. The former consist of the derivative of a variable’s residual with respect to the same variable (at the same quadrature point or a different one); while the latter consist of the derivatives with respect to another variable (at the same quadrature point or a different one).

The ‘Diagonal’ terms are tested using a single-phase single-element model with random initial conditions. Sixteen different tests are performed which are all possible combinations of:

- Fully-saturated or unsaturated initial conditions
- With or without gravity
- With or without SUPG
- With or without time derivatives

Of course, the unsaturated case with gravity, SUPG and time derivatives is the most complicated, but the other cases are useful for debugging.

The ‘OffDiagonal’ terms only appear if there is more than one pressure variable, that is for multi-phase problems. No tests have been performed yet.

## 4 Long-time behaviour

These tests concern the steadystate pressure distribution obtained either by running a transient model for a long time, or by running a steady-state analysis, both of which should lead to the same result. Without fluxes, the steadystate pressure distribution is just

$$P(x) = P_0 + \rho_0 g x , \quad (4.1)$$

if the fluid bulk modulus is large enough compared with  $P$ . Here  $\rho_0$  is the constant reference fluid density,  $g$  is the acceleration due to gravity (a vector), and  $x$  is position. These are simple tests and are part of the automatic test suite.

This is verified by running sixteen single-phase single-element models with random initial conditions. The sixteen cases are all possible combinations of:

- Fully-saturated or unsaturated initial conditions
- With or without gravity
- With or without SUPG
- Transient or Steadystate

In addition to these cases, a number of more complicated scenarios are also part of the automatic test suite:

1. A single-phase situation with nonzero immobile saturation. There are 5 elements in the  $x$  direction along which gravity acts. The  $x$  direction has length 20 m, and the initial condition is sufficiently unsaturated so that after some time the saturation at the ‘top’ of the model ( $x \sim 20$ ) would reduce below immobile saturation if the relative permeability weren’t preventing it. Two cases are studied: with SUPG and without SUPG. The results are shown in the top two pictures of Figure 4.1 and show that SUPG reduces oscillations and prevents  $S < S_{\text{imm}}$  in this example at least.
2. The same situation is in item 1, but with 50 elements in the  $x$  direction. Figure 4.1 shows once again the stabilising nature of SUPG is seen, as well as the hydrostatic pressure head for  $S > S_{\text{imm}}$ .

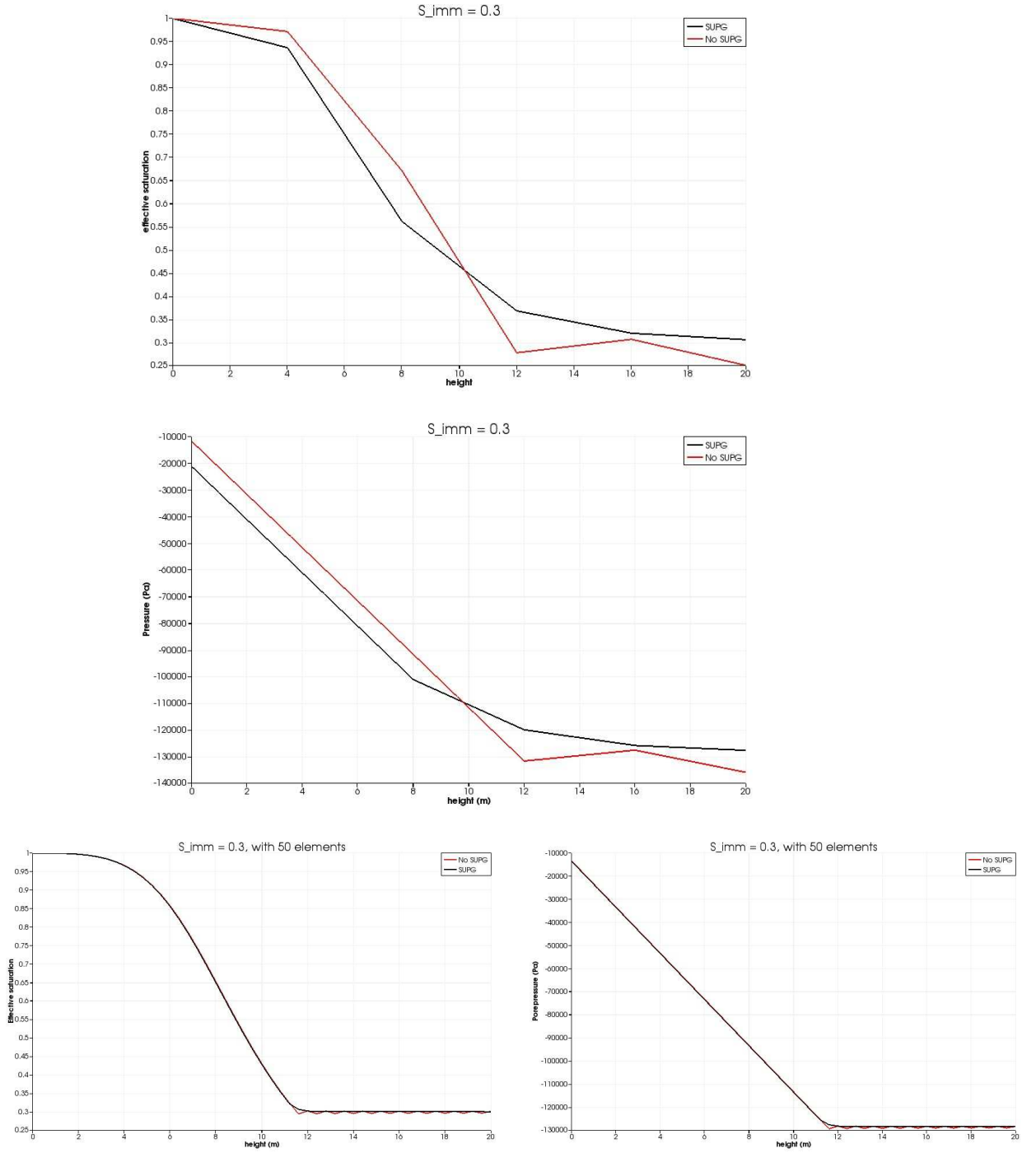


Figure 4.1: Results for  $S_{imm} = 0.3$ . Gravity points to the left. Top picture: Effective saturation. Middle picture: Pore pressure. Bottom pictures: The situation with 50 elements in the  $x$  direction instead of just 5. In each picture the red line is without SUPG, and oscillatory results can be observed in addition to  $S_{eff} < S_{imm}$ .



## 5 A pressure pulse in the fully saturated situation - NEEDS UPDATE

Darcy's equation for flow through a fully saturated medium without gravity and without sources is

$$\frac{\partial}{\partial t} \phi \rho = \nabla_i \left( \frac{\rho \kappa_{ij}}{\mu} \nabla_j P \right) , \quad (5.1)$$

with notation described in the Theory Manual. Using  $\rho \propto \exp(P/K)$ , where  $K$  is the fluid bulk modulus, Darcy's equation becomes

$$\frac{\partial}{\partial t} \rho = \nabla_i \alpha_{ij} \nabla_j \rho , \quad (5.2)$$

with

$$\alpha_{ij} = \frac{\kappa_{ij} B}{\mu \phi} . \quad (5.3)$$

Here I've assumed the porosity and bulk modulus are constant in space and time.

Consider the one-dimensional case where the spatial dimension is the semi-infinite line  $x \geq 0$ . Suppose that initially the pressure is constant, so that

$$\rho(x, t = 0) = \rho_0 \quad \text{for } x \geq 0 . \quad (5.4)$$

Then apply a fixed-pressure Dirichlet boundary condition at  $x = 0$  so that

$$\rho(x = 0, t > 0) = \rho_\infty \quad (5.5)$$

The solution of the above differential equation is well known to be

$$\rho(x, t) = \rho_\infty + (\rho_0 - \rho_\infty) \operatorname{Erf} \left( \frac{x}{\sqrt{4\alpha t}} \right) , \quad (5.6)$$

where Erf is the error function.

This is verified in Wallaby using the following tests on a line of 100 elements.

1. Steady state analysis with SUPG, in 3D, to demonstrate that the steady-state of  $\rho = \rho_\infty$  is achieved.
2. Transient analysis with SUPG in 3D.

An example verification is shown in Figure 5.1.

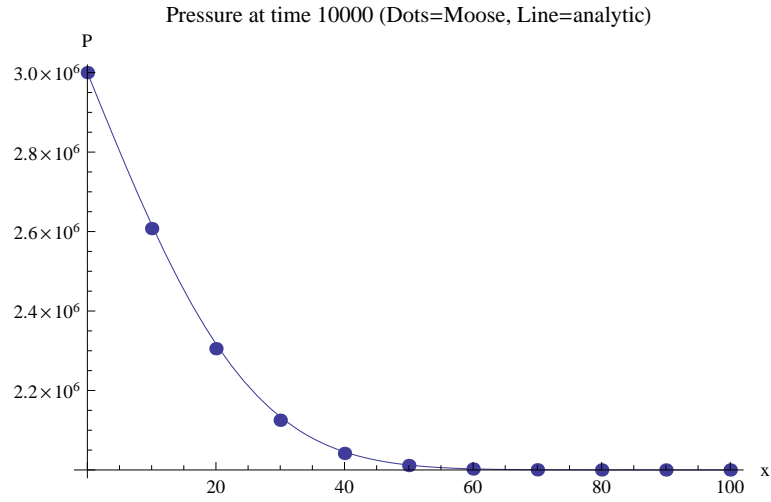


Figure 5.1: Comparison between the Wallaby result (in dots), and the exact analytic expression given by Eqn (5.6). This test had 10 elements in the  $x$  direction, with  $0 \leq x \leq 100$  m, and ran for a total of  $10^4$  seconds with 10 timesteps. The parameters were  $B = 2$  GPa,  $\kappa_{xx} = 10^{-15}$  m<sup>2</sup>,  $\mu = 10^{-3}$  Pa.s,  $\phi = 0.1$ , with initial pressure  $P = 2$  MPa, and applied pressure  $P = 3$  MPa at  $x = 0$ . For greater spatial resolution and smaller timesteps the agreement increases.

## 6 Buckley-Leverett - NEEDS UPDATE

Wallaby is compared with a simple one dimensional Buckley-Leverett problem<sup>1</sup>. The fluid moves in a region  $0 \leq x \leq 15$  m. A fully-saturated front initially sits at position  $x = 5$ , while the region  $x > 5$  is initially unsaturated. With zero suction function  $P_c = 0$ , there is no diffusion of the sharp front, and it progresses towards the right. This is a difficult problem to simulate numerically as maintaining the sharp front is hard. The speed is independent of the relative permeability, since the fluid is flowing from a fully-saturated region. This problem is therefore a good test of the upwinding.

In the simulation below, the pressure at the left boundary is kept fixed at  $P_0 = 0.98$  MPa, while the right boundary is kept fixed at  $P_{15} = -20000$  Pa. The medium's permeability is set to  $\kappa = 10^{-10}$  m<sup>2</sup> and its porosity is  $\phi = 0.15$ . It is not possible to use a zero suction function in Wallaby, but using  $\alpha = 10^{-4}$  Pa<sup>-1</sup> and  $m = 0.8$  approximates it. The fluid viscosity is  $\mu = 10^{-3}$  Pa.s.

The initial condition is

$$P(t=0) = \begin{cases} P_0 - (P_0 - P_{15})x/5 & \text{for } x < 5 \\ P_{15} & \text{for } x \geq 5 \end{cases}, \quad (6.1)$$

which is shown in Figure 6.1. With the suction function defined above this gives

$$S(t=0) = \begin{cases} 1 & \text{for } x \leq 4.9 \\ 0.061 & \text{for } x \geq 5 \end{cases} \quad (6.2)$$

Good approximations for the pressure  $P(x, t)$  and the front position  $f(t)$  may be determined from

$$\begin{aligned} \frac{df}{dt} &= -\frac{\kappa}{\phi\mu} \left. \frac{\partial P}{\partial x} \right|_{x=f}, \\ P(x, t) &= \begin{cases} P_0 - (P_0 - P_{15})x/f & \text{for } x \leq f \\ P_{15} & \text{for } x > f \end{cases}, \end{aligned} \quad (6.3)$$

which has solution

$$f(t) = \sqrt{f(0)^2 + \frac{2\kappa}{\phi\mu}(P_0 - P_{15})t}. \quad (6.4)$$

For the parameters listed above, the front will be at position  $f = 9.6$  m at  $t = 50$  s. This solution is only valid for zero capillary suction. A nonzero suction function will tend to diffuse the sharp front away.

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<sup>1</sup>SE Buckley and MC Leverett (1942) "Mechanism of fluid displacements in sands". Transactions of the AIME **146** 107–116

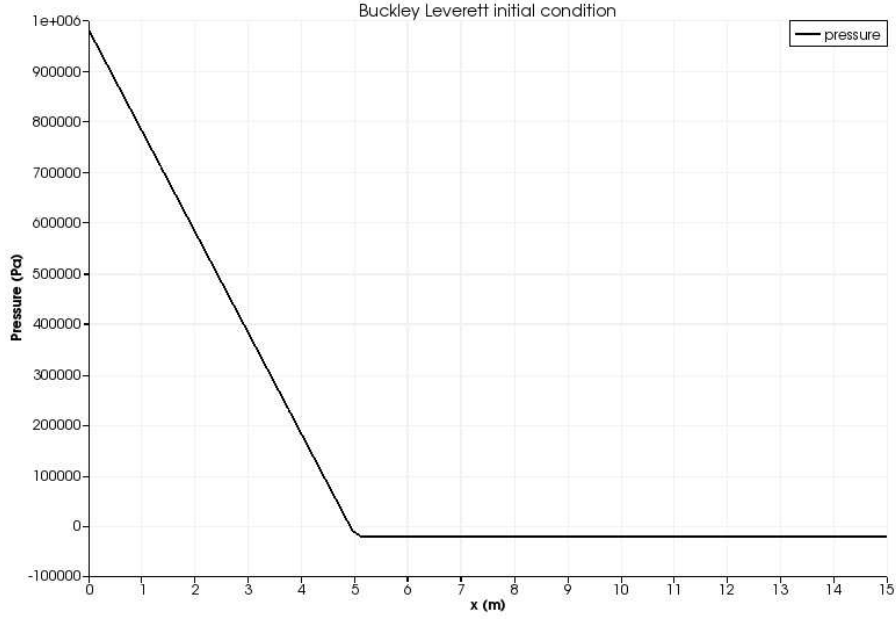


Figure 6.1: Initial setup of the Buckley-Leverett problem where porepressure is a piecewise linear function. The region  $x \leq 4.9$  is fully saturated, while the region  $x > 5$  has saturation 0.061. During simulation the value  $P(x = 0) = 0.98 \times 10^6$  MPa is held fixed.

With coarse meshes it is impossible to simulate a sharp front, of course, since the front is spread over at least one element. It is therefore quite advantageous to use mesh adaptivity in this test, since the mesh can be fine around the front where all the interesting dynamics occurs, and coarse elsewhere.

Figure 6.2 shows the results from a Wallaby simulation with an initial mesh of element size 1 m, and a minimum size of 0.125 m, with a maximum timestep of 0.3 s. (Reducing the minimum element size or the maximum timestep size keeps the front sharper.) The front in this simulation is between  $x = 9.9$  m and  $x = 10.35$  m, in fair agreement with the predicted value of 9.6 m.

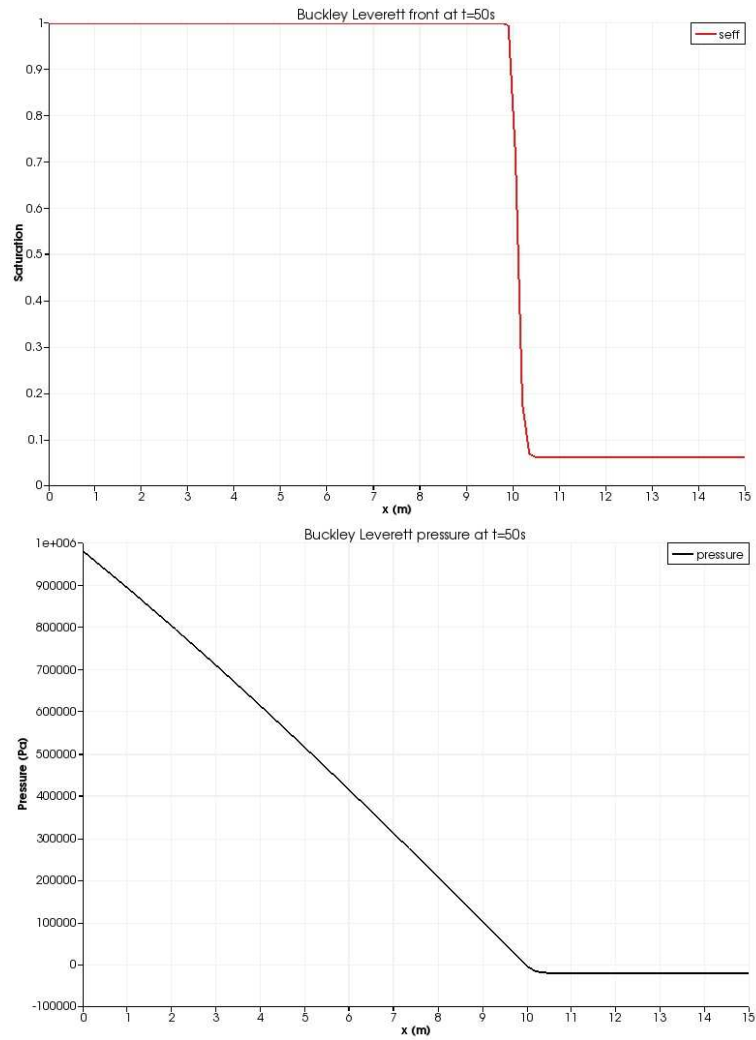


Figure 6.2: The Wallaby solution of the Buckley-Leverett problem at  $t = 50$  s. Top: saturation. Bottom: porepressure. The front sits between  $x = 9.9$  m and  $x = 10.35$  m.

## 7 Unsaturated flow in a bar - NEEDS UPDATE

This problem is one of cosflow's benchmark problems. Water inside a porous “bar” of material length 10m, and width and depth 1m is initialised to porepressure  $P_0 < 0$ , corresponding to water saturation  $S_0 < 1$ . The porepressure left-hand end (at  $x = 0$ ) is raised and fixed at  $P_1 < 0$ , corresponding to water saturation  $S_1$  with  $S_0 < S_1 < 1$ , and the evolution of porepressure at the right-hand end ( $x = 10$  m) is recorded. Apart from the left-hand end, the other boundaries of the bar are impermeable. The setup is shown in Figure 7.1.

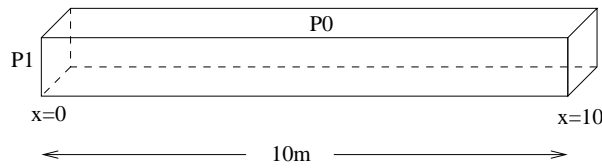


Figure 7.1: The unsaturated problem involves a porous “bar” of material of length 10m with initial porepressure  $P_0$ . The left-hand end is raised to porepressure  $P_1$  and held fixed. The other parts of the bar’s exterior surface are impermeable.

This problem exhibits quite severe mesh dependency, and since the upwinding in cosflow and Wallaby are different, the results are not expected to be the same, except in the limit of zero element size.

The following parameters are used

Bar porosity	0.1
Bar permeability	$10^{-12} \text{ m}^2$
Gravity	0
Water density	$1000 \text{ kg.m}^{-3}$
Water viscosity	$0.001 \text{ Pa.s}$
Water bulk modulus	2 GPa
Water immobile saturation	0.0
Water residual saturation	0.0
Air residual saturation	0.0
van Genuchten $\alpha$	$10^{-4} \text{ Pa}^{-1}$
van Genuchten $a$	0.35
Initial porepressure $P_0$	-197347.0503 Pa
Initial saturation $S_0$	0.2
Applied pressure $P_1$	-9283.000501 Pa
Applied saturation $S_1$	0.8

Figure 7.2 shows agreement between Wallaby and cosflow for a variety of different mesh densities, including an adaptive mesh example. The Wallaby results appear to be closer to the zero-element-size result than cosflow, but for low mesh density exhibit spurious oscillations as the high saturation region moves into the low saturation region. (This oscillation is almost definitely due to my current inability to lump the mass term.)

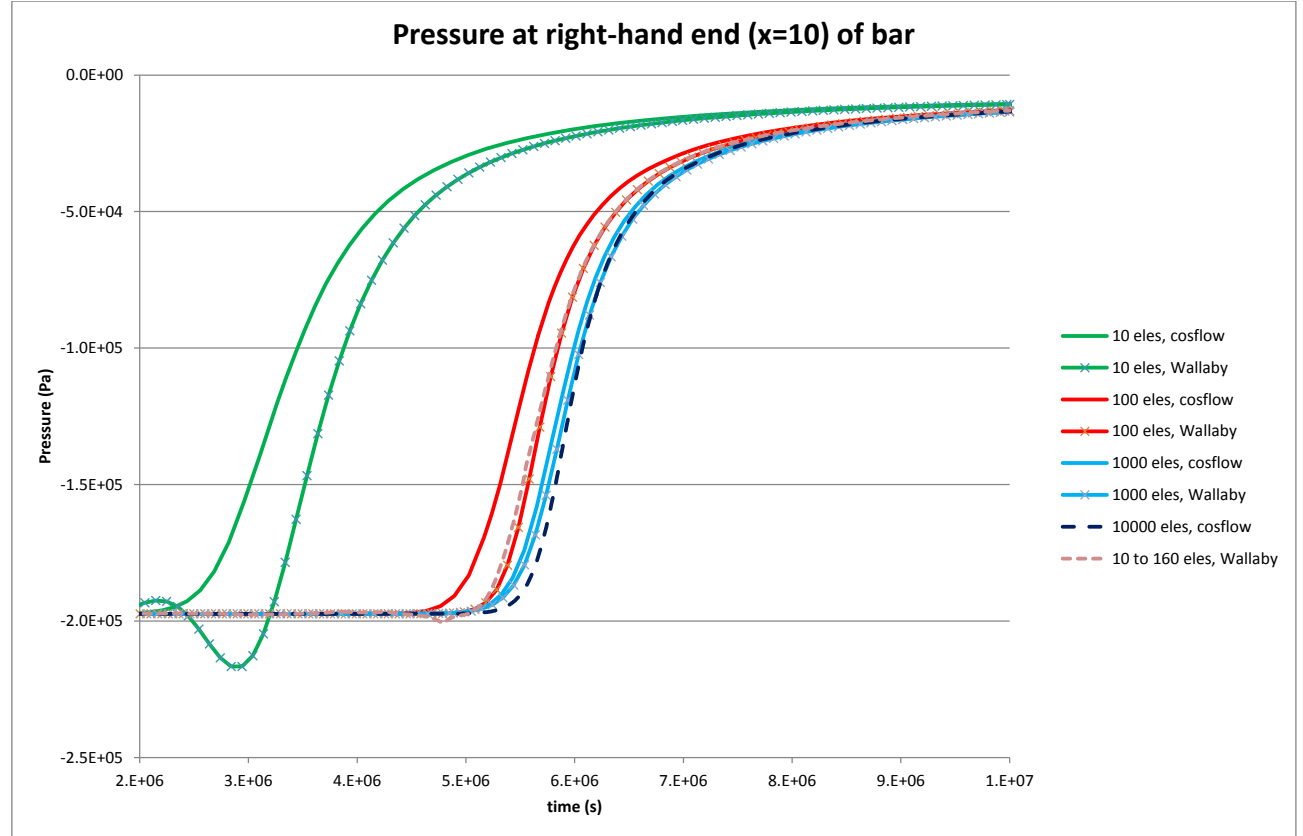


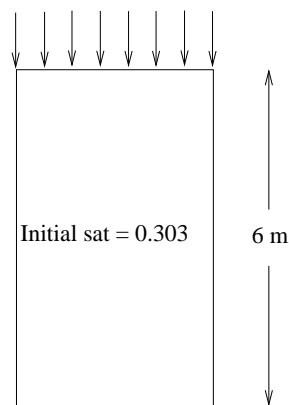
Figure 7.2: The porepressure at the right-hand end ( $x = 10$ ) of the bar as a function of time for various different meshes.

## 8 Infiltration and drainage - NEEDS UPDATE

Forsyth, Wu and Pruess<sup>1</sup> describe a HYDRUS simulation of an experiment involving infiltration (experiment 1) and subsequent drainage (experiment 2) in a large caisson. The simulation is effectively one dimensional, and is shown in Figure 8.1.

Experiment 1

Infiltration  $0.002315 \text{ kg/m}^2/\text{s}$



Experiment 2

gravity

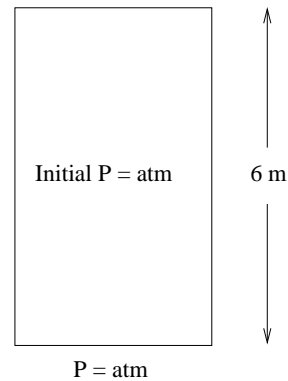


Figure 8.1: Two experimental setups from Forsyth, Wu and Pruess. Experiment 1 involves infiltration of water into an initially unsaturated caisson. Experiment 2 involves drainage of water from an initially saturated caisson.

The properties common to each experiment are:

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<sup>1</sup>PA Forsyth, YS Wu and K Pruess, "Robust numerical methods for saturated-unsaturated flow with dry initial conditions in heterogeneous media", *Advances in Water Resources* 18 (1995) 25–38



Caisson	0.33
Caisson permeability	$2.95 \times 10^{-13} \text{ m}^2$
Gravity	$10 \text{ m.s}^{-2}$
Water density	$1000 \text{ kg.m}^{-3}$
Water viscosity	$0.00101 \text{ Pa.s}$
Water bulk modulus	20 MPa
Water immobile saturation	0.0
Water residual saturation	0.0
Air residual saturation	0.0
Air pressure	0.0
van Genuchten $\alpha$	$1.43 \times 10^{-4} \text{ Pa}^{-1}$
van Genuchten $a$	0.336
van Genuchten_1 cutoff	0.99

In each experiment 120 finite elements were used along the length of the Caisson. The modified van-Genuchten relative permeability curve was employed in order to improve convergence significantly. Hydrus also uses a modified van-Genuchten curve, although I couldn't find any details on the modification.

In experiment 1, the caisson is initially at saturation 0.303 ( $P = -72620.4 \text{ Pa}$ ), and water is pumped into the top with a rate  $0.002315 \text{ kg.m}^{-2}.\text{s}^{-1}$ . This causes a front of water to advance down the caisson. Figure 8.2 shows the agreement between Wallaby and the published result (this result was obtained by extracting data by hand from online graphics).

In experiment 2, the caisson is initially fully saturated at  $P = 0$ , and the bottom is held at  $P = 0$  to cause water to drain via the action of gravity. Figure 8.2 shows the agreement between Wallaby and the published result.

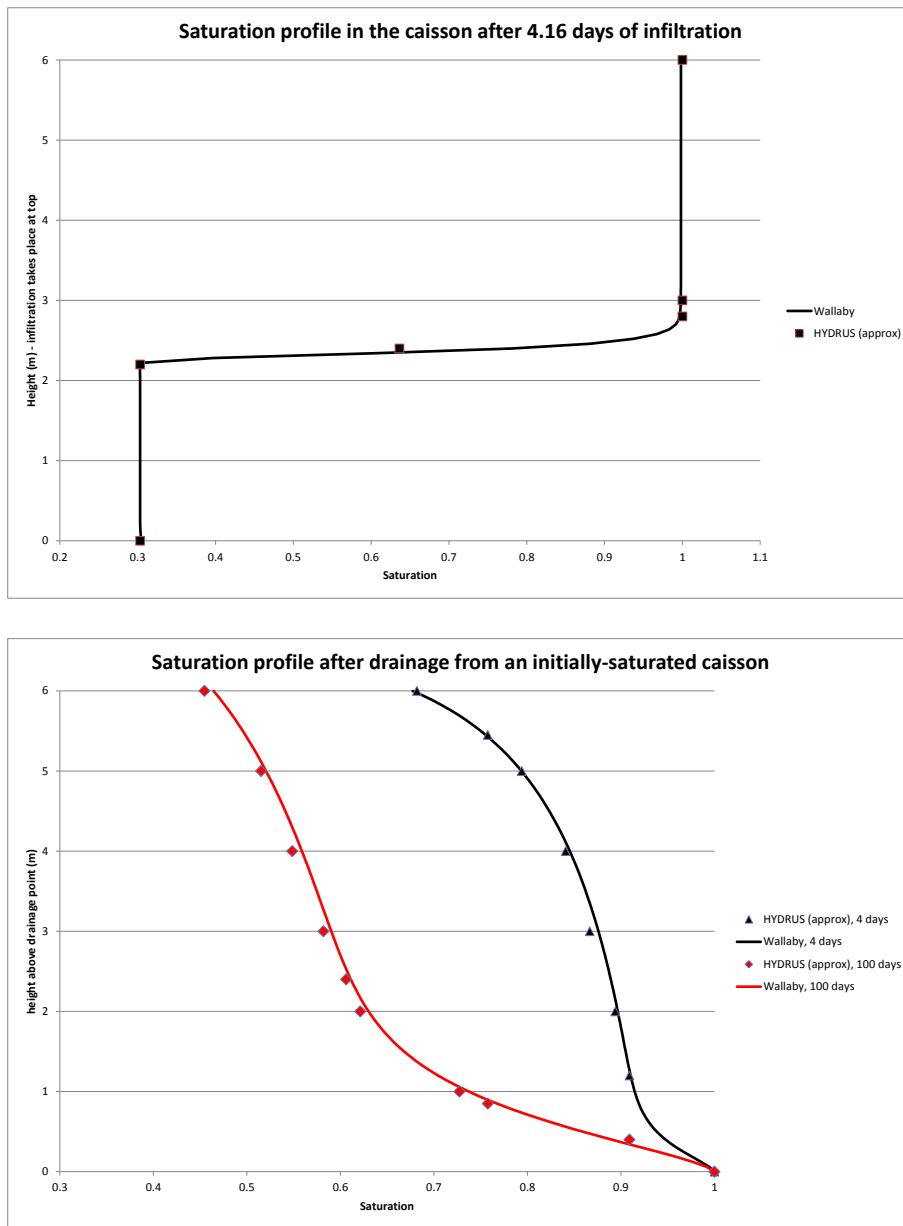


Figure 8.2: Saturation profiles in the caisson. Top: After 4.16 days of infiltration. Bottom: After drainage from an initially-saturated simulation (4 days and 100 days profiles). Note that the HYDRUS results are only approximate as I extrated the data by hand from online graphics.

## 9 Newton cooling from a bar - NEEDS UPDATE

Darcy's equation for flow through a fully saturated medium without gravity and without sources is

$$\frac{\partial}{\partial t} \phi \rho = \nabla_i \left( \frac{\rho \kappa_{ij}}{\mu} \nabla_j P \right), \quad (9.1)$$

with notation described in the Theory Manual. Using  $\rho \propto \exp(P/K)$ , where  $K$  is the fluid bulk modulus, Darcy's equation becomes

$$\frac{\partial}{\partial t} \rho = \nabla_i \alpha_{ij} \nabla_j \rho, \quad (9.2)$$

with

$$\alpha_{ij} = \frac{\kappa_{ij} B}{\mu \phi}. \quad (9.3)$$

Here I've assumed the porosity and bulk modulus are constant in space and time.

Consider the one-dimensional case where a bar sits between  $x = 0$  and  $x = L$  with initial pressure distribution so  $\rho(x, t = 0) = \rho_0(x)$ . Maintain the end  $x = 0$  at constant pressure, so that  $\rho(x = 0, t) = \rho_0(0)$ . At the end  $x = L$ , prescribe a sink flux

$$\left. \frac{\partial \rho}{\partial x} \right|_{x=L} = -C(\rho - \rho_e)_{x=L}, \quad (9.4)$$

where  $\rho_e$  is a fixed quantity ("e" stands for "external"), and  $C$  is a constant conductance. This corresponds to the flux

$$\left. \frac{\partial P}{\partial x} \right|_{x=L} = -CB \left( 1 - e^{(P_e - P)/B} \right)_{x=L}, \quad (9.5)$$

which can easily be coded into a Wallaby input file: the flux is  $\rho \kappa \nabla P / \mu = -CB \kappa (e^{P/B} - e^{P_e/B}) / \mu$ .

The solution of this problem is well known and is

$$\rho(x, t) = \rho_0(0) - \frac{\rho_0(0) - \rho_e}{1 + LC} Cx + \sum_{n=1}^{\infty} a_n \sin \frac{k_n x}{L} e^{-k_n^2 \alpha t / L^2}, \quad (9.6)$$

where  $k_n$  is the  $n^{\text{th}}$  positive root of the equation  $LC \tan k + k = 0$  ( $k_n$  is a little bigger than  $(2n - 1)\pi/2$ ), and  $a_n$  is determined from

$$a_n \int_0^L \sin^2 \frac{k_n x}{L} dx = \int_0^L \left( \rho_0(x) - \rho_0(0) + \frac{\rho_0(0) - \rho_e}{1 + LC} Cx \right) \sin \frac{k_n x}{L} dx, \quad (9.7)$$

which may be solved numerically.

The problem is solved in Wallaby using the following parameters:

Bar length	100 m
Bar porosity	0.1
Bar permeability	$10^{-15} \text{ m}^2$
Gravity	0
Water density	$1000 \text{ kg.m}^{-3}$
Water viscosity	0.001 Pa.s
Water bulk modulus	1 MPa
Initial porepressure $P_0$	2 MPa
Environmental pressure $P_e$	0
Conductance $C$	$0.05389 \text{ m}^{-1}$

This conductance is chosen so at steadystate  $\rho(x = L) = 2000 \text{ kg.m}^{-3}$ .

The problem is solved using 1000 elements along the  $x$  direction ( $L = 100 \text{ m}$ ), and using 100 time-steps of size  $10^6 \text{ s}$ . Using fewer elements or fewer timesteps means the agreement with the theory is marginally poorer. The problem is also solved using the steadystate solver. In this case the initial condition is  $P = 2 - x/L \text{ MPa}$ , since the uniform  $P = 2 \text{ MPa}$  does not converge. The results are shown in Figure 9.1.

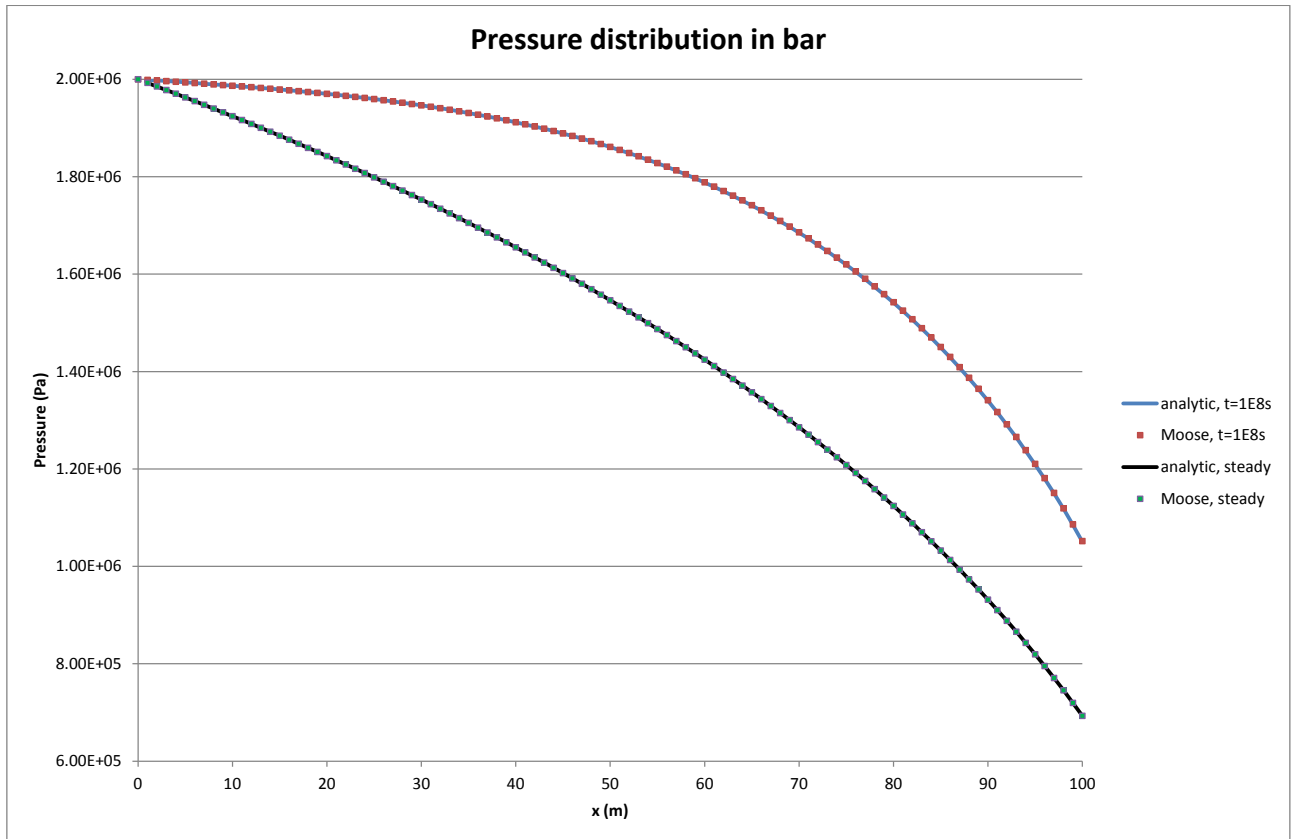


Figure 9.1: The porepressure in the bar at  $t = 10^8$  s, and at steadystate. Wallaby agrees well with theory.

## 10 Future tests

See "Benchmarking of Richards Model" Yanlian Du, Wenqing Wang and Olaf Kolditz for a summary of the usual suspects.

See Forsyth, Wu and Pruess for more tests.