

# MOOSE's PorousFlow module

CSIRO, INL

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## Abstract

This document describes the physics captured by MOOSE's PorousFlow module, as well as describing various numerical aspects and tips to ensure good convergence.

Each nontrivial test in the test suite *has its own separate documentation* and these should help users understand more clearly how to build a PorousFlow input file.

Within this document, black text indicates functionality that has been implemented and tested, blue text indicates functionality that should be implemented by March 2017, and red text indicates functionality that should be implemented by December 2017.

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# Chapter 1

## Physical equations

The equations for heat flow, fluid flow, solid mechanics, and chemical reactions are defined here. Notation is introduced throughout, but summarised in Section 1.5. The Lagrangian coordinate system is used since it co-moves with the mesh.

### 1.1 Fluid flow

Mass conservation for fluid species  $\kappa$  is described by the continuity equation

$$0 = \frac{\partial M^\kappa}{\partial t} + M^\kappa \nabla \cdot \mathbf{v}_s + \nabla \cdot \mathbf{F}^\kappa - q^\kappa . \quad (1.1)$$

Here  $M$  is the mass of fluid per volume of rock (measured in  $\text{kg.m}^{-3}$ ),  $\mathbf{v}_s$  is the velocity of the porous solid skeleton (measured in  $\text{m.s}^{-1}$ ),  $\mathbf{F}$  is the flux (a vector, measured  $\text{kg.s}^{-1}.\text{m}^{-2}$ ), and  $q$  is a source (measured in  $\text{kg.m}^{-3}.\text{s}^{-1}$ ).

The coupling to the solid mechanics is via the  $M \nabla \cdot \mathbf{v}_s$  term, as well as via changes in porosity and permeability described in Section 3.13 and Section 3.12. Coupling to heat flow and chemical reactions is via the equations of state used within the terms of Eqn (1.1), as well as the source term  $q^\kappa$ .

The species are parameterised by  $\kappa = 1, \dots$ . For example,  $\kappa$  might parameterise water, air,  $\text{H}_2$ , a solute, and so on.  $\kappa$  parameterises things which cannot be decomposed into other species, but can change phase. For instance, sometimes it might be appropriate to consider air as a single species (say  $\kappa = 1$ ), while other times it might be appropriate to consider it to be a mixture of nitrogen and oxygen ( $\kappa = 1$  and  $\kappa = 2$ ). To model chemical precipitation, a solid phase, which is distinct from the porous skeleton, may also be used (its relative permeability will be zero: see below).

### 1.1.1 Mass density: $M$

The mass of species  $\kappa$  per volume of rock is written as a sum over all phases present in the system:

$$M^\kappa = \phi \sum_{\beta} S_{\beta} \rho_{\beta} \chi_{\beta}^{\kappa} + (1 - \phi) \rho^R A^{\kappa} \quad (1.2)$$

The solid's porosity is  $\phi$ .  $S_{\beta}$  is the saturation of phase  $\beta$  (solid, liquid, gas, NAPL).  $\rho_{\beta}$  is the density of phase  $\beta$ .  $\chi_{\beta}^{\kappa}$  is the mass fraction of component  $\kappa$  present in phase  $\beta$ . The final term represents fluid absorption into the porous-rock skeleton:  $A^{\kappa}$  is the mass of absorbed species per mass of rock grain material.

The density  $\rho_{\beta}$  is a function of pressure, mass fraction, and so on, as described by the equation of state used (Chapter 3).

The saturation and mass fractions must obey

$$\sum_{\beta} S_{\beta} = 1, \quad (1.3)$$

$$\sum_{\kappa} \chi_{\beta}^{\kappa} = 1 \quad \forall \beta. \quad (1.4)$$

### 1.1.2 Flux: $\mathbf{F}$

The flux is a sum of advective flux and diffusive-and-dispersive flux:

$$\mathbf{F}^{\kappa} = \sum_{\beta} \chi_{\beta}^{\kappa} \mathbf{F}_{\beta}^{\text{advective}} + \mathbf{F}_{\text{diffusion+dispersion}}^{\kappa}. \quad (1.5)$$

#### Advection

Advective flux is governed by Darcy's law. Each phase is assumed to obey Darcy's law. Each phase has its own density,  $\rho_{\beta}$ , relative permeability  $k_{r,\beta}$ , viscosity  $\mu_{\beta}$ , and pressure  $P_{\beta}$ . These may all be nonlinear functions of the independent variables. With them, we can form the advective Darcy flux:

$$\mathbf{F}_{\beta}^{\text{advective}} = \rho_{\beta} \mathbf{v}_{\beta} = -\rho_{\beta} \frac{k k_{r,\beta}}{\mu_{\beta}} (\nabla P_{\beta} - \rho_{\beta} \mathbf{g}) \quad (1.6)$$

In this equation  $\mathbf{v}_{\beta}$  is the Darcy velocity (volume flux, measured in  $\text{m.s}^{-1}$ ) in phase  $\beta$ . It is used below in the diffusive-and-dispersion flux too.

The absolute permeability is denoted by  $k$  and it is a tensor. The relative permeability<sup>1</sup> of phase  $\beta$  is denoted by  $k_{r,\beta}$ . It is always a function of the saturation(s), **but with Klinkenberg effects**,

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<sup>1</sup>In reality relative permeability is actually a tensor (for example it's usually different in lateral and vertical directions) but is most often treated as a scalar, since it's hard to get parameters for the tensorial case. In the PorousFlow module it is treated as a scalar.

it may also be a function of the gas pressure. Relative permeability can also be hysteretic, so that it depends on the history of saturation.

In some circumstances  $K_{ij}\nabla_j T$  is added to the above Darcy flux to model thermo-osmosis (with some  $K_{ij}$  tensor parameterising its strength), i.e. a gradient of temperature induces fluid flow.

Also note:

- The pressure in phase  $\beta$  is

$$P_\beta = P + P_{c,\beta} \quad (1.7)$$

where  $P$  is the reference pressure (usually taken to be the gas phase), and  $P_{c,\beta}$  is the capillary pressure (nonpositive if the reference phase is the gas phase). The capillary pressure is often a function of saturation, and various forms have been coded into the PorousFlow module: see Section 3.7. **The capillary pressure relationship used in a model can have a great bearing on both the speed of convergence and the results obtained.**

The capillary pressure relationship can also be hysteretic, in that it can depend on the history of the saturation.

The pressure in a gas phase is the sum of the gas partial pressures:  $P_{\text{gas}} = \sum_\kappa P_{\text{gas}}^\kappa$ . (The partial pressure of a gaseous species is  $P_{\text{gas}}^\kappa = P_{\text{gas}} N^\kappa / N$  where  $N$  is the number of molecules. This is Dalton's law.) The partial pressure concept is reasonable for dilute gases, but is less useful for dense gases.

- The mass-fraction of a species in the aqueous phase is often computed using Henry's law:

$$\chi_{\text{aqueous}}^\kappa = P_{\text{gas}}^\kappa / H_\kappa, \quad (1.8)$$

where  $H_\kappa$  is the Henry coefficient<sup>2</sup>. Occasionally this law is not accurate enough, and there are more complicated alternatives. Chapter 3 discusses various equations of state that include computations of Henry's coefficients.

- When a liquid and a gaseous phase exist, the simplest equation for vapour pressure is

$$P_{\text{vapour}} = P_{\text{sat}}(T), \quad (1.9)$$

which is just the saturated vapour pressure of the liquid phase. This can set the temperature  $T$  from the gaseous pressure, or vice-versa, depending on the choice of independent variables. A more complicated alternative is the Kelvin equation for vapour pressure that takes into account vapour pressure lowering due to capillarity and phase adsorption affects

$$P_{\text{vapour}} = \exp\left(\frac{M_w P_{c,l}(S_l)}{\rho_l R(T + 273.15)}\right) P_{\text{sat}}(T). \quad (1.10)$$

Here  $M_w$  is the molecular weight of water;  $P_{c,l} = P_{c,l}(S_l) \leq 0$  is the capillary pressure — the difference between aqueous (liquid water) and gas phase pressures — a function of  $S_l$ ;  $\rho_l$  is the aqueous (liquid water) density;  $R$  is the universal gas constant;  $T$  is the temperature;  $P_{\text{sat}}$  is the saturated vapour pressure of the bulk aqueous (liquid water) phase.

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<sup>2</sup>Unfortunately, the Henry coefficient has multiple definitions which makes looking up numerical values complicated.



## Diffusion and hydrodynamic dispersion

Diffusion and dispersion are proportional to the gradient of  $\chi_\beta^\kappa$ . The TOUGH2 manual<sup>3</sup>, in Appendix D, contains a nice discussion of diffusion-dispersion. Here we use the common expression

$$\mathbf{F}_{\text{diffusion+dispersion}}^\kappa = - \sum_\beta \rho_\beta \mathcal{D}_\beta^\kappa \nabla \chi_\beta^\kappa \quad (1.11)$$

Note  $\mathbf{F}$  is a vector,  $\mathcal{D}$  is a 2-tensor and  $\nabla$  a vector. The hydrodynamic dispersion tensor is

$$\mathcal{D}_\beta^\kappa = D_{\beta,T}^\kappa \mathcal{I} + \frac{D_{\beta,L}^\kappa - D_{\beta,T}^\kappa}{\mathbf{v}_\beta^2} \mathbf{v}_\beta \mathbf{v}_\beta, \quad (1.12)$$

where

$$D_{\beta,L}^\kappa = \phi \tau_0 \tau_\beta d_\beta^\kappa + \alpha_{\beta,L} |\mathbf{v}|_\beta, \quad (1.13)$$

$$D_{\beta,T}^\kappa = \phi \tau_0 \tau_\beta d_\beta^\kappa + \alpha_{\beta,T} |\mathbf{v}|_\beta. \quad (1.14)$$

These are called the longitudinal and transverse dispersion coefficients.  $d_\beta^\kappa$  is the molecular diffusion coefficient for component  $\kappa$  in phase  $\beta$ .  $\tau_0 \tau_\beta$  is the tortuosity which includes a porous medium dependent factor  $\tau_0$  and a coefficient  $\tau_\beta = \tau_\beta(S_\beta)$ , and  $\alpha_L$  and  $\alpha_T$  are the longitudinal and transverse dispersivities. It is common to set the hydrodynamic dispersion to zero by setting  $\alpha_{\beta,T} = 0 = \alpha_{\beta,L}$ .

## 1.2 Heat flow

Energy conservation for heat is described by the continuity equation

$$0 = \frac{\partial \mathcal{E}}{\partial t} + \mathcal{E} \nabla \cdot \mathbf{v}_s + \nabla \cdot \mathbf{F}^T - \nu(1 - \phi) \sigma_{ij}^{\text{eff}} \frac{\partial}{\partial t} \epsilon_{ij}^{\text{plastic}} - q^T \quad (1.15)$$

Here  $\mathcal{E}$  is the energy per unit volume in the rock-fluid system,  $\mathbf{v}_s$  is velocity of the porous solid skeleton,  $\mathbf{F}^T$  is the heat flux,  $\nu$  describes the ratio of plastic-deformation energy that gets transferred to heat energy,  $\sigma_{ij}^{\text{eff}}$  is the effective stress (see Eqn (1.25)),  $\epsilon_{ij}^{\text{plastic}}$  is the plastic strain, and  $q^T$  is a heat source.

The coupling to the solid mechanics is via the  $\mathcal{E} \nabla \cdot \mathbf{v}_s$  term, the  $\nu(1 - \phi) \sigma_{ij}^{\text{eff}} \frac{\partial}{\partial t} \epsilon_{ij}^{\text{plastic}}$  term, as well as via changes in porosity and permeability described in Section 3.13 and Section 3.12. Coupling to the fluid flow and chemical reactions is via the equations of state used within the terms of Eqn (1.15), as well as the source term  $q^T$ . Joule-Thompson effects<sup>4</sup> may be included via the fluid properties (Chapter 3).

<sup>3</sup>TOUGH2 Users's Guide, Version 2. Karsten Pruess, Curt Oldenburg, George Moridis. Earth Sciences Division, LBNL, Univ California, Berkeley. November 1999, revised August 2011. LBNL-43134 (revised).

<sup>4</sup>See for instance Eqn (1) of SA Mathias et al. "Analytical solution for Joule-Thomson cooling during CO<sub>2</sub> geo-sequestration in depleted oil and gas reservoirs" <http://www.osti.gov/scitech/servlets/purl/985333>

Here it is assumed the liquids and solid are in local thermal equilibrium i.e. there is a single local temperature in all phases. If this doesn't hold, one is also normally in the high-flow regime where the flow is non-Darcy as well.

Sometimes a term is added that captures the thermal power due to volumetric expansion of the fluid,  $K\beta T\nabla \cdot \mathbf{v}$ , where  $K$  is the bulk modulus of the fluid or solid,  $\beta$  is the thermal expansion coefficient, and  $\mathbf{v}$  is the Darcy velocity. This is not included in the Porous Flow module currently.

### 1.2.1 Energy density: $\mathcal{E}$

The energy per unit volume is

$$\mathcal{E} = (1 - \phi)\rho_R C_R T + \phi \sum_{\beta} S_{\beta} \rho_{\beta} \mathcal{E}_{\beta} + \sum_{\kappa} (1 - \phi) \rho^R \mathcal{E}_{\text{abs } \kappa} A^{\kappa} \quad (1.16)$$

The notation is: porosity  $\phi$ ; grain density  $\rho_R$ ; specific heat capacity of rock  $C_R$ ; temperature  $T$ ; saturation of phase  $S_{\beta}$  (solid, liquid, gas, NAPL); density of phase  $\rho_{\beta}$ ; internal energy in phase  $\mathcal{E}_{\beta}$ ; internal energy of absorbed species  $\mathcal{E}_{\text{abs } \kappa}$ .

### 1.2.2 Heat flux: $\mathbf{F}^T$

The heat flux is a sum of heat conduction and convection with the fluid:

$$\mathbf{F}^T = -\lambda \nabla T + \sum_{\beta} h_{\beta} \mathbf{F}_{\beta} . \quad (1.17)$$

Here  $\lambda$  is the tensorial thermal conductivity of the rock-fluid system, which is a function of the thermal conductivities of rock and fluid phases. Usually  $\lambda$  will be diagonal but in anisotropic porous materials it may not be. The specificy enthalpy of phase  $\beta$  is denoted by  $h_{\beta}$ , and  $\mathbf{F}_{\beta}$  is the advective Darcy flux.

### 1.2.3 A simple example

Consider just the fluid terms of the heat-flow Eqn (1.15). For notational simplicity, consider just one fully-saturated phase so that  $\mathcal{E} = \phi \rho_{\beta} \mathcal{E}_{\beta}$ . Here  $\mathcal{E}_{\beta}$  has SI units  $\text{J.kg}^{-1}$ . Also consider the fluid flux with no conduction so that  $\mathbf{F}^T = h_{\beta} \rho_{\beta} \mathbf{v}_{\beta}$ . Integrating Eqn (1.15) over an arbitrary volume  $\Omega$ , and employing integration by parts yields

$$0 = \frac{\partial}{\partial t} \int_{\Omega} \phi \rho_{\beta} \mathcal{E}_{\beta} + \int_{\partial\Omega} h_{\beta} \rho_{\beta} \mathbf{n} \cdot \mathbf{v}_{\beta} . \quad (1.18)$$

The first term on the RHS is the heat energy of the fluid in the fixed volume  $\Omega$ . Since the volume is fixed, it is appropriate to use the specific heat at constant volume,  $C_v^\beta$ , in the expression for the fluid energy-per-mass:

$$\mathcal{E}_\beta = C_v^\beta T . \quad (1.19)$$

The second term on the RHS describes the change in heat energy due to fluid entering or leaving the volume with velocity  $\mathbf{v}_\beta$ . This change in heat energy can be decomposed into two parts in the following way. Imagine that fluid is entering the volume  $\Omega$ . Firstly, imagine a new, slightly bigger volume  $\Omega'$  that contains all the fluid that will enter  $\Omega$  during the small time  $dt$ . The heat energy contained in the fluid within the small volume  $\Omega' - \Omega$  is the extra heat energy added during fluid flow. This is simply

$$\int_{\partial\Omega} C_v T \rho \mathbf{n} \cdot \mathbf{v} dt . \quad (1.20)$$

We now know the total heat energy within the new volume,  $\Omega'$ . The second step must be to squash this new volume back to the original  $\Omega$ . This costs energy, which is just calculated using force $\times$ distance. The force must be equal to the fluid pressure (multiplied by the infinitesimal area element), and the distance is exactly  $\mathbf{n} \cdot \mathbf{v} dt$ , meaning the second contribution is

$$\int_{\partial\Omega} P \mathbf{n} \cdot \mathbf{v} dt . \quad (1.21)$$

Collecting these ideas together Eqn (1.15) reads, in this simple example

$$0 = \frac{\partial}{\partial t} \phi \rho C_v T + \nabla \cdot \left( C_v T + \frac{P}{\rho} \right) \rho \mathbf{v} . \quad (1.22)$$

The superscript  $\beta$  denoting the phase number has been omitted.

Finally, two small observations might be useful.

1. Given the above discussion, it is clear that the enthalpy can be written as

$$h_\beta = \mathcal{E}_\beta + P_\beta / \rho_\beta . \quad (1.23)$$

2. For an ideal gas,  $P/\rho = RT$ , so the enthalpy,

$$h_\beta = C_v T + \frac{P}{\rho} = C_{\beta,v} T + RT = C_{\beta,p} T , \quad (1.24)$$

where  $C_{\beta,p}$  is the specific heat capacity at constant pressure of the ideal gas phase.

## 1.3 Solid mechanics

Most of the solid mechanics used by the Porous Flow module is handled by the Tensor Mechanics module. This section provides a brief overview, concentrating on the aspects that differ from pure solid mechanics.

Denote the total stress tensor by  $\sigma^{\text{tot}}$ . An externally applied mechanical force will create a nonzero  $\sigma^{\text{tot}}$ , and conversely, resolving  $\sigma^{\text{tot}}$  into forces yields the forces on nodes in the finite-element mesh.

Denote the effective stress tensor by  $\sigma^{\text{eff}}$ . It is defined by

$$\sigma_{ij}^{\text{eff}} = \sigma_{ij}^{\text{tot}} + \alpha_B \delta_{ij} P_f . \quad (1.25)$$

The notation is as follows.

- $P_f$  is a measure of porepressure. In single-phase, fully-saturated situations it is traditional to use  $P_f = P_\beta$ . However, for multi-phase situations  $P_f = \sum_\beta S_\beta P_\beta$  is also used. **Yet other expressions involve Bishop's parameter.**
- $\alpha_B$  is the “Biot coefficient”. This obeys  $0 \leq \alpha_B \leq 1$ . For a multi-phase system, the Biot coefficient is often chosen to be  $\alpha_B = 1$ . The Biot coefficient is interpreted physically by the following. If, by pumping fluid into a porous material, the  $P_f$  porepressure is increased by  $\Delta P_f$ , and at the same time a mechanical external force applies an incremental pressure equaling  $\alpha_B \Delta P_f$ , then the volume of the porous solid remains static. (During this process, the porevolume and porosity will change, however, as quantified in Section 3.13.)

It is assumed that the elastic constitutive law reads

$$\sigma_{ij}^{\text{eff}} = E_{ijkl}(\epsilon_{kl}^{\text{elastic}} - \delta_{kl} \alpha_T T) , \quad (1.26)$$

with  $\alpha_T$  being the thermal expansion coefficient of the drained porous skeleton, and  $\epsilon_{kl} = (\nabla_k u_l + \nabla_l u_k)/2$  being the usual total strain tensor ( $u$  is the deformation of the porous solid), which can be split into the elastic and plastic parts,  $\epsilon = \epsilon^{\text{elastic}} + \epsilon^{\text{plastic}}$ , and  $E_{ijkl}$  being the elasticity tensor (the so-called “drained” version). The generalisation to large strain is obvious. The inverse of the constitutive law is

$$\epsilon_{ij}^{\text{elastic}} - \delta_{ij} \alpha_T T = C_{ijkl} \sigma_{kl}^{\text{eff}} , \quad (1.27)$$

with  $C$  being the compliance tensor.

It is assumed that the conservation of momentum is

$$\rho_{\text{mat}} \frac{\partial v_s^j}{\partial t} = \nabla_i \sigma_{ij}^{\text{tot}} + \rho_{\text{mat}} b_j = \nabla_i \sigma_{ij}^{\text{eff}} - \alpha_B \nabla_j P_f + \rho_{\text{mat}} b_j , \quad (1.28)$$

where  $\mathbf{v}_s = \partial \mathbf{u} / \partial t$  is the velocity of the solid skeleton,  $\rho_{\text{mat}}$  is the mass-density of the material (this is the “undrained” density:  $\rho_{\text{mat}} = (1 - \phi) \rho^R + \phi \sum_\beta S_\beta \rho_\beta$ ), and  $b_j$  are the components of the external force density (for example, the gravitational acceleration). Here any terms of  $O(v^2)$  have been explicitly ignored, and it's been assumed that to this order the velocity of each phase is identical to the velocity of the solid skeleton (otherwise there are terms involving  $\partial \mathbf{F} / \partial t$  on the left-hand side).

It is assumed that the *effective stress not the total stress* enters into the constitutive law (as above), and the plasticity, and the insitu stresses, and almost everywhere else. One exception

is specifying Neumann boundary conditions for the displacements where the total stresses are being specified, as can be seen from Eqn (1.28). Therefore, MOOSE will use effective stress, and not total stress, internally. If one needs to input or output total stress, one must subtract  $\alpha_B \nabla_j P_f$  from MOOSE's stress.

## 1.4 Chemical reactions

The first step in defining the chemistry is to choose a set of basis species in the liquid phase, such that every other chemical species, including minerals (now all referred to as secondary species) can be expressed via combinations of these basis species.

Let  $j$  index the basis species in the liquid phase, denoted by  $\mathcal{A}_l^j$ , and  $i$  index the secondary species,  $\mathcal{A}_\beta^i$ . Then the reactions that connect these are given by

$$\mathcal{A}_\beta^i \rightleftharpoons \sum_j \nu_\beta^{ji} \mathcal{A}_l^j \quad (1.29)$$

where the  $\nu_\beta^{ji}$  are the stoichiometric coefficients.

Now define the total concentration of basis species  $j$  in fluid phase  $\beta$ , denoted  $\Psi_\beta^j$ , by

$$\Psi_\beta^j = \delta_{l\beta} C_l^j + \sum_i \nu_\beta^{ji} C_\beta^i \quad (1.30)$$

where the  $C_\beta^i$  is the concentration of secondary species  $i$  in phase  $\beta$ ,  $C_l^j$  is the concentration of basis species  $j$  in the liquid phase, and  $\delta_{l\beta}$  is the Kronecker delta ( $= 1$  if  $\beta = l$ , else  $= 0$ ).

The concentrations of the secondary species  $C_\beta^i$  are obtained from the basis species concentrations by mass action equations of the form

$$C_\beta^i = \frac{K_\beta^i}{\gamma_\beta^i} \prod_j (\gamma_l^j C_l^j)^{\nu_\beta^{ji}} \quad (1.31)$$

where  $K_\beta^i$  is the equilibrium constant and  $\gamma_\beta^i$  is the activity coefficient of species  $i$  in phase  $\beta$ . There are various possible relationships for computing activity coefficients that depend both on properties of that species (e.g. valence and ionic radius) and the total ionic strength (which depends on all the ion concentrations).

In all these, the concentration units are moles per litre.

Now define the total concentration accumulation of basis species  $j$  by  $M^j$ , which is given by

$$M^j = \phi \sum_\beta S_\beta \Psi_\beta^j \quad (1.32)$$

where the sum in Eqn (1.32) is over the fluid phases only, and as before  $\phi$  is the porosity and  $S_\beta$  is the phase saturation.

The continuity equation is then

$$0 = \frac{\partial M^j}{\partial t} + M^j \nabla \cdot \mathbf{v}_s + \nabla \cdot \mathbf{F}^j - q^j . \quad (1.33)$$

Here  $\mathbf{F}$  is the flux (a vector, measured  $\text{mol.s}^{-1}.\text{L}^{-1}$ ), and  $q^j$  is a source (measured in  $\text{mol.L}^{-1}.\text{s}^{-1}$ ).

For minerals indexed by  $m$  one has

$$\frac{\partial \phi_m}{\partial t} = \bar{V}_m I_m \quad (1.34)$$

where  $\phi_m$  is the volume fraction of that mineral,  $\bar{V}_m$  is the molar volume, and  $I_m$  is the mineral reaction rate. The porosity is related to the mineral volume fractions by

$$\phi = 1 - \sum_m \phi_m \quad (1.35)$$

The total flux  $\mathbf{F}^j$  in Eqn (1.33) is given by

$$\mathbf{F}^j = \sum_\beta (\mathbf{v}_\beta - \phi S_\beta \mathcal{D}_\beta^j \nabla) \Psi_\beta^j \quad (1.36)$$

which incorporates the advective and diffusive transport of chemical species in the sum over the fluid phases.  $\mathcal{D}_\beta^j$  is the hydrodynamic dispersion tensor in phase  $\beta$  of species  $j$ , including effects of tortuosity, dispersion etc, as in Eqn (1.12). It's common to assume the this tensor is the same for all species in that phase.

First define the mineral saturation ratio  $\Omega_m$  by

$$\Omega_m = K_m^{-1} \prod_j (C^j)^{\nu_{mj}} (\gamma^j)^{\nu_{mj}} \quad (1.37)$$

where  $K_m$  is the equilibrium constant for the reaction. There are a variety of possible rate expressions. One of the simplest is to write

$$I_m = \pm k_m A_m |1 - \Omega_m^\theta|^\eta \quad (1.38)$$

where the rate  $I_m$  is positive for dissolution and negative for precipitation,  $A_m$  is the specific reactive surface area per litre for mineral  $m$ , and the exponents  $\theta$  and  $\eta$  are specific to each mineral reaction, but are often both taken to be unity. The rate constant  $k_m$  ( $\text{mole}/(\text{m}^2 \text{ s})$ ) is a function of temperature.

## 1.5 Notation

Symbol	Units	Physical description
$A^\kappa$	$\text{kg.kg}^{-1}$	Mass of absorbed species $\kappa$ per mass of rock grain material.
$\mathcal{A}_\beta^i$	symbolic	Chemical species $i$ in phase $\beta$
$A_m$	$\text{m}^2.\text{L}^{-1}$	specific reactive surface area for mineral $m$
$\alpha_{\beta,L}$	m	Longitudinal dispersivity of phase $\beta$ .
$\alpha_{\beta,T}$	m	Transverse dispersivity of phase $\beta$ .
$\alpha_B$	dimensionless	Biot coefficient. I believe we can treat this as a constant Material parameter.
$\alpha_T$	$\text{K}^{-1}$	Volumetric coefficient of thermal expansion of the drained porous skeleton (ie, the porous rock without fluid, or which a fluid that is free to move in and out of the rock).
$\mathbf{b}$	$\text{m.s}^{-2}$	External force density acting on the porous solid. This could be gravitational acceleration, or a load-density from a platten.
$\beta$	dimensionless	Index representing phase. For example, $\beta$ might parameterise liquid ( $\beta = 0$ ), gas ( $\beta = 1$ ) and NAPL ( $\beta = 2$ ).
$C_R$	$\text{J.kg}^{-1}.\text{K}^{-1}$	Specific heat capacity of rock grains.
$C_{ijkl}$	Pa	Drained compliance tensor of the porous solid (ie, inverse of $E_{ijkl}$ ).
$C_\beta^i$	moles per litre	concentration of species $i$
$\chi_\beta^\kappa$	$\text{kg.kg}^{-1}$	Mass fraction of component $\kappa$ present in phase $\beta$ .
$\mathcal{D}_\beta^\kappa$	$\text{m}^2.\text{s}^{-1}$	Fluid dispersion tensor for species $\kappa$ in phase $\beta$ .
$D_{\beta,L}^\kappa$	$\text{m}^2.\text{s}^{-1}$	Longitudinal dispersion coefficient for species $\kappa$ in phase $\beta$ .
$D_{\beta,T}^\kappa$	$\text{m}^2.\text{s}^{-1}$	Transverse dispersion coefficient for species $\kappa$ in phase $\beta$ .
$d_\beta^\kappa$	$\text{m}^2.\text{s}^{-1}$	Molecular diffusion coefficient for component $\kappa$ in phase $\beta$ .
$\delta_{l\beta}$	dimensionless	Kronecker delta, unity if $\beta = l$ , else zero.
$\mathcal{E}$	$\text{J.m}^{-3}$	Energy density of the rock-fluid system.
$\mathcal{E}_\beta$	$\text{J.kg}^{-1}$	Internal energy of fluid phase $\beta$ . In the simplest case this is just $c_\beta T$ , where $c_\beta$ is the fluid's specific heat capacity.
$\epsilon_{ij}$	dimensionless	Strain tensor of the porous solid $((\nabla_k u_l + \nabla_l u_k)/2)$ .
$\epsilon_{ij}^{\text{elastic}}$	dimensionless	Elastic strain tensor of the porous solid. The total strain $\epsilon = \epsilon^{\text{elastic}} + \epsilon^{\text{plastic}}$ .
$\epsilon_{ij}^{\text{plastic}}$	dimensionless	Plastic strain tensor of the porous solid. The total strain $\epsilon = \epsilon^{\text{elastic}} + \epsilon^{\text{plastic}}$ .
$\eta$	dimensionless	exponent in rate expression
$E_{ijkl}$	Pa	“Drained” elasticity tensor of the porous skeleton (ie, this enters the stress-strain relation when fluid is allowed to freely drain from the skeleton, or when the skeleton is dry).
$\phi$	dimensionless	Porosity of the solid.
$\phi_m$	dimensionless	volume fraction of mineral in solid

Symbol	Units	Physical description
$\mathbf{F}$	$\text{kg.s}^{-1}.\text{m}^{-2}$	Fluid flux. This is a sum of the advective (Darcy) flux, and a diffusive-and-dispersive flux.
$\mathbf{F}^T$	$\text{J.s}^{-1}.\text{m}^{-2}$	Heat flux. This is a sum of heat conduction through the rock-fluid system, and convection with the fluid.
$\mathbf{g}$	$\text{m.s}^{-2}$	Acceleration due to gravity. It is a vector pointing downwards (eg $(0, 0, -10)$ ).
$\gamma_\beta^i$	dimensionless	activity coefficient of species $i$ in phase $\beta$
$h_\beta$	$\text{J.kg}^{-1}$	Specific enthalpy of fluid phase $\beta$ . In the simplest case this is just $c_\beta T$ where $c_\beta$ is the fluid's specific heat capacity.
$H_\kappa$	Pa	Henry coefficient for species $\kappa$ which describes the solubility of the species in the aqueous phase.
$I_m$	$\text{mol.L}^{-1}.\text{s}^{-1}$	mineral reaction rate
$\kappa$	dimensionless	Index representing species. For example, $\kappa$ might parameterise water ( $\kappa = 0$ ), air ( $\kappa = 1$ ), and $\text{H}_2$ ( $\kappa = 2$ ). It parameterises things that cannot be decomposed into other species, but can change phase. For instance, sometimes it might be appropriate to consider air as a single species, while at other times it might be appropriate to consider it to be a mixture of nitrogen and oxygen ( $\kappa = 0$ and $\kappa = 1$ , say).
$k$	$\text{m}^2$	Permeability tensor of rock.
$k_{r,\beta}$	dimensionless	Relative permeability of phase $\beta$ . This is a nonlinear function of the independent variables. Often it is just a function of the phase's saturation, but with Klinkenberg effects it will be a function of the gas pressure too. In the single-phase, fully-saturated case it is unity.
$K$	Pa	Bulk modulus of the drained porous skeleton. I think in the anisotropic situation $1/K == \delta_{ij}\delta_{kl}C_{ijkl}$ .
$K_\beta^i$	depends on reaction	equilibrium constant for secondary species $i$ in phase $\beta$
$k_m$	$\text{mol.m}^{-2}.\text{s}^{-1}$	mineral rate constant
$\lambda$	$\text{J.s}^{-1}.\text{m}^{-1}.\text{K}^{-1}$	Thermal conductivity of the rock-fluid system ( $\text{J.s}^{-1}.\text{m}^{-1}.\text{K}^{-1} = \text{kg.m.s}^{-3}.\text{K}^{-1}$ ). It is a tensorial quantity to allow modelling of anisotropic situations, and is a function of the rock and fluid-phase's thermal conductivities
$\lambda_{dry}$	$\text{J.s}^{-1}.\text{m}^{-1}.\text{K}^{-1}$	Thermal conductivity of the rock-fluid system when aqueous phase saturation is zero.
$\lambda_{wet}$	$\text{J.s}^{-1}.\text{m}^{-1}.\text{K}^{-1}$	Thermal conductivity of the rock-fluid system when aqueous phase saturation is unity.
$M$	$\text{kg.m}^{-3}$	Mass density
$\mu$	Pa.s	Dynamic viscosity measured in Pa.s or $\text{kg.m}^{-1}.\text{s}^{-1}$ . This is a nonlinear function of the independent variables.
$\nu$	dimensionless	Fraction of plastic-deformation energy that becomes heat energy. Probably $\nu = 1$ is correct.



Symbol	Units	Physical description
$\nu_{\beta}^{ij}$	dimensionless	stoichiometric coefficient of basis species $j$ in equilibrium reaction for secondary species $j$
$\nabla$	$\text{m}^{-1}$	Spatial differential operator
$\Omega_m$	dimensionless	mineral saturation ratio
$P$	Pa	Fluid porepressure
$P_f$	Pa	Measure of porepressure used in the effective stress. Often this is chosen to be $\sum_{\beta} S_{\beta} P_{\beta}$ .
$\Phi_{\beta}^j$	moles per litre	Total concentration of basis species $j$ in phase $\beta$
$q$	$\text{kg.m}^{-3}.\text{s}^{-1}$	Fluid source
$q^j$	$\text{mol.L}^{-1}.\text{s}^{-1}$	source of chemical species
$q^T$	$\text{J.m}^{-3}.\text{s}^{-1}$	Heat source.
$\rho$	$\text{kg.m}^{-3}$	Fluid density
$\rho_R$	$\text{kg.m}^{-3}$	Grain density of the rock (so that $(1-\phi)\rho_R$ is the density of the dry porous rock).
$\rho_{\text{mat}}$	$\text{kg.m}^{-3}$	The mass-density of the fluid-filled porous solid $\rho_{\text{mat}} = (1-\phi)\rho^R + \phi \sum_{\beta} S_{\beta} \rho_{\beta}$ .
$S$	dimensionless	Saturation
$S_l$	dimensionless	saturation of aqueous phase
$\sigma_{ij}^{\text{tot}}$	Pa	Total stress. An externally applied mechanical force will create a nonzero $\sigma^{\text{tot}}$ , and conversely, resolving $\sigma^{\text{tot}}$ into forces yields the forces on nodes in the finite-element mesh.
$\sigma_{ij}^{\text{eff}}$	Pa	Effective stress.
$t$	s	Time
$T$	K	Temperature.
$\tau_0 \tau_{\beta}$	dimensionless	The phase tortuosity, which includes a porous-medium dependent factor $\tau_0$ and a coefficient $\tau_{\beta} = \tau_{\beta}(S_{\beta})$ .
$\theta$	dimensionless	exponent in rate expression
$\mathbf{u}$	$\text{m.s}^{-1}$	Deformation vector of the porous solid.
$\mathbf{v}$	$\text{m.s}^{-1}$	Darcy velocity (volume flux)
$\mathbf{v}_s$	$\text{m.s}^{-1}$	Velocity of the solid = $\partial \mathbf{u} / \partial t$ , where $\mathbf{u}$ is the solid mechanical displacement vector of the porous solid.
$\bar{V}_m$	$\text{L.mol}^{-1}$	molar volume

Table 1.1: Notation introduced throughout Section 1.

## 1.6 Example use case: Single-phase single-component fully-saturated fluid

Since the number of phases and components is unity, the subscripts  $\beta$  and  $\kappa$  may be dropped. Since  $\chi = 1$  there is no diffusion nor dispersion. Assume that the relative permeability is unity. Mass conservation of the fluid (Eqns (1.1), (1.2), (1.5), (1.6), (1.11), (1.12), (1.13) and (1.14))

reads

$$0 = \frac{\partial}{\partial t} \phi \rho + \nabla \cdot (\phi \rho \mathbf{v}_s) - \nabla_i \cdot \left( \frac{\rho k_{ij}}{\mu} (\nabla_j P - \rho g_j) \right) - q^{\text{fluid}} , \quad (1.39)$$

where  $P$  is the porepressure of the fluid and will be one of MOOSE's variables in this case.

Make the following assumptions:

- the fluid's internal energy is  $\mathcal{E} = cT$ , where  $c$  is the fluid's specific heat capacity;
- the fluid's specific enthalpy is  $h = cT$ ;
- the rock-fluid thermal conductivity is a weighted sum of the rock-grain and fluid thermal conductivities:  $\lambda = (1 - \phi)\lambda^R + \phi\lambda^f$

The heat-energy conservation equation (Eqn (1.15), (1.16) and (1.17)) is

$$\begin{aligned} 0 = & \frac{\partial}{\partial t} ((1 - \phi)\rho_R C_R T + \phi \rho c T) + \nabla \cdot (((1 - \phi)\rho_R C_R T + \phi \rho c T) \mathbf{v}_s) \\ & - \nabla \cdot (((1 - \phi)\lambda^R + \phi\lambda^f) \nabla T) - \nabla \cdot \left( cT \frac{\rho k_{ij}}{\mu} (\nabla_j P - \rho g_j) \right) - q^T , \end{aligned} \quad (1.40)$$

where  $T$  is the temperature of the fluid-rock system and will be one of MOOSE's variables in this case. Under the additional assumptions that the following parameters are constant (or very nearly so):  $\phi$ ,  $\rho_R$ ,  $C_R$ ,  $c$ ,  $\lambda^R$ ,  $\lambda^f$ , this equations reads

$$0 = C \frac{\partial T}{\partial t} + C \nabla \cdot (T \mathbf{v}_s) - \lambda \nabla^2 T - \nabla \cdot \left( \rho c T \frac{k_{ij}}{\mu} (\nabla_j P - \rho g_j) \right) - q^T , \quad (1.41)$$

with  $C = (1 - \phi)\rho_R C_R + \phi \rho c$  being the macroscopic heat capacity.

Eqns (1.39) and (1.41) with  $\mathbf{v}_s = 0$  are the equations of FALCON. Using the assumption  $\rho = \rho_0 \exp(P/K_f - \alpha_T^f T)$ , where  $K_f$  is the constant bulk modulus of the fluid, and  $\alpha_T^f$  is its constant thermal expansion coefficient, these two equations are those of FLAC's THM.<sup>5</sup>

The effective stress is given in Eqn (1.25), with  $P_f = P$ . The elastic constitutive equation is Eqn (1.26). Conservation of momentum is given in Eqn (1.28), with  $\rho_{\text{mat}} = (1 - \phi)\rho^R + \phi\rho$ . The variables are the 3 displacements (in 3 dimensional problems),  $u_i$ .

The evolution of porosity is given in Eqn (3.17). In many situations of interest, the change in porosity is very small.

In summary, the variables for this case can be the porepressure  $P$ , the temperature  $T$ , and the three displacements  $u_i$ .

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<sup>5</sup>There is one possible inconsistency: FLAC's Eqn(12) of FLAC7.0 "Thermal Analysis (2011) writes  $\beta = n\beta_f + (1 - n)\beta_g$ , whereas the formulation here would give  $\beta = n\beta_f + (\alpha_B - n)\beta_g$ . FLAC has a habit of sneakily putting  $\alpha_B = 1$  without saying, because some of their formulation is only correct to  $\alpha_B = 1$ , so I'm not sure whether this is FLAC's error/assumption, or an error in the porosity evolution equation Eqn (3.17) here.

The following parameters depend, in principle, on the porepressure, temperature and (derivatives of) the displacements: porous-rock insitu permeability tensor  $k_{ij}$ ; rock-grain density  $\rho_R$ ; rock-grain specific heat capacity  $C_R$ ; rock-grain thermal conductivity  $\lambda^R$ ; fluid density  $\rho$ ; fluid viscosity  $\mu$ ; fluid thermal conductivity  $\lambda^f$ ; fluid specific heat capacity  $c$ . All these can be coded into a suite of Materials, as described in Chapter 3.

# Chapter 2

## Boundaries and sinks

### 2.1 Boundaries

MOOSE's Dirichlet and Neumann boundary conditions enable simulation of simple scenarios. The Porous Flow module includes a very flexible boundary condition that allows many different scenarios to be modelled. The boundary condition is built by adding various options to a basic sink in the following way.

The basic sink is

$$s = f(t, x) , \quad (2.1)$$

where  $f$  is a MOOSE Function of time and position on the boundary. If  $f > 0$  then the boundary condition will act as a sink, while if  $f < 0$  the boundary condition acts as a source. If applied to a fluid-component equation, the function  $f$  has units  $\text{kg.m}^{-2}.\text{s}^{-1}$ . If applied to the heat equation, the function  $f$  has units  $\text{J.m}^{-2}.\text{s}^{-1}$ . These units are potentially modified if the extra building blocks enumerated below are used.

The basic sink may be multiplied by a MOOSE Functions of the pressure of a fluid phase *or* the temperature:

$$s = f(t, x) \times g(P^\beta) \quad \text{or} \quad s = f(t, x) \times g(T) . \quad (2.2)$$

Here the units of  $f \times g$  are  $\text{kg.m}^{-2}.\text{s}^{-1}$  (for fluids) or  $\text{J.m}^{-1}.\text{s}^{-1}$  (for heat). Some commonly use forms have been hard-coded into the Porous Flow module for ease of use in simulations:

- a piecewise linear function (for simulating fluid or heat exchange with an external environment via a conductivity term, for instance);
- a half-gaussian (for simulating evapotranspiration, for instance)
- a half-cubic (for simulating evapotranspiration).

In addition, the sink may be multiplied by any or all of the following quantities

- Fluid relative permeability
- Fluid mobility ( $k_{ij}n_in_j\rho/\nu$ , where  $n$  is the normal vector to the boundary)
- Fluid mass fraction
- Fluid enthalpy
- Fluid internal energy
- Thermal conductivity

It is likely that multiplication by generalised concentrations will also be needed.

For example, a boundary condition corresponding to injection of fluid at a fixed temperature could involve: (1) using a Dirichlet condition for temperature; (2) using  $f = -1$  without any multiplicative factors. More complicated examples with heat and fluid injection and production are detailed in the test suite documentation `newton_cooling` and `sinks`.

Commonly, if fluid or heat is exiting the porous material, multiplication by relative permeability, mobility, mass fraction, enthalpy or thermal conductivity is necessary, while if fluid or heat is entering the porous material this multiplication is not necessary. Two sinks can be constructed in a MOOSE input file: one involving “production” which is only active for  $P > P_e$  using a Piecewise-linear sink (here  $P_e$  is the environmental pressure); and one involving “injection”, which is only active for  $P < P_e$  using a piecewise-linear sink multiplied by the appropriate factors.

## 2.2 Boreholes

Peaceman formulation

# Chapter 3

## Equations of state and other functions

This chapter briefly outlines the Porous Flow Materials.

### 3.1 Brine

A high-precision equation of state for brine can be used to obtain the following quantities:

1. density
2. viscosity
3. internal energy
4. enthalpy

### 3.2 CO<sub>2</sub>

A high-precision equation of state for CO<sub>2</sub> can be used to obtain the following quantities:

1. density
2. viscosity
3. internal energy
4. enthalpy

### 3.3 Water and steam

A high-precision equation of state for water and steam can be used to obtain the following quantities:

1. density
2. viscosity
3. internal energy
4. enthalpy

### 3.4 An ideal gas

The following quantities can be obtained for an ideal gas

1. density
2. viscosity
3. internal energy
4. enthalpy

### 3.5 Methane

A high-precision equation of state for methane can be used to obtain the following quantities:

1. density
2. viscosity
3. internal energy
4. enthalpy

## 3.6 CO<sub>2</sub> and brine

## 3.7 Capillary pressure relationships

### 3.7.1 Van Genuchten

van Genuchten's capillary-pressure relationship<sup>1</sup> is

$$S_{\text{eff}} = \begin{cases} 1 & \text{if } P \geq 0, \\ (1 + (-\alpha P)^{1/(1-m)})^{-m} & \text{if } P < 0. \end{cases} \quad (3.1)$$

or

$$P_c = \begin{cases} 0 & \text{if } S_{\text{eff}} \geq 1.0, \\ \frac{1}{\alpha} (S_{\text{eff}}^{-1/m} - 1)^{1-m} & \text{if } S_{\text{eff}} < 1 \end{cases} \quad (3.2)$$

The effective saturation has been denoted by  $S_{\text{eff}}$  and  $P$  is the porepressure, which is the *negative* of the capillary pressure:  $P = -P_c$ . Here  $\alpha$  and  $m$  are user-defined parameters. The parameter  $m$  must satisfy

$$0 < m < 1. \quad (3.3)$$

Sometimes the van Genuchten function is defined in terms of the parameter

$$n = \frac{1}{1-m} > 1. \quad (3.4)$$

In van Genuchten's paper, he finds good fits with experimental data for various soils and rock when the parameter  $m$  ranges between about 0.5 and 0.9 (meaning  $2 < n < 10$ , roughly), and  $\alpha$  is between  $4 \times 10^{-5} \text{ Pa}^{-1}$  and  $2 \times 10^{-4} \text{ Pa}^{-1}$ . Figure 3.1 shows the shape of the van Genuchten suction,  $P_c$ , as a function of  $S_{\text{eff}}$ .

Numerically there are three important features of Eqn (3.1).

1.  $S_{\text{eff}}$  is a monotonically decreasing function of  $P_c$ , which is necessary for a unique solution.
2.  $P_c \rightarrow \infty$  as  $S_{\text{eff}} \rightarrow 0$ . As mentioned above, this is not justifiable physically, but numerically it is extremely advantageous over  $P_c \rightarrow P_c^0 < \infty$ , as this latter version often causes algorithms to “get stuck” around  $S_{\text{eff}} = 0$ . As also mentioned above, because of the low relative permeability around  $S_{\text{eff}}$ , physically realistic problems rarely explore the  $S_{\text{eff}} \sim 0$  region.
3.  $S_{\text{eff}} \rightarrow 1$  and  $dS_{\text{eff}}/dP_c \rightarrow 0^+$  as  $P_c \rightarrow 0$ , for all  $m$ . This ensures that there is continuity in the porepressure,  $P$ , and the derivative  $dS/dP$  around full saturation (remember that by definition  $S_{\text{eff}} = 1$  for  $P_c < 0$ ). Also  $d^2S_{\text{eff}}/dP_c^2 \rightarrow 0$  as  $P_c \rightarrow 0^+$  if  $m > 0.5$ .

I encourage users to set  $m > 0.5$ .

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<sup>1</sup>van Genuchten, M. Th., “A closed for equation for predicting the hydraulic conductivity of unsaturated soils” Soil Sci. Soc., 44, 892-898 (1980)



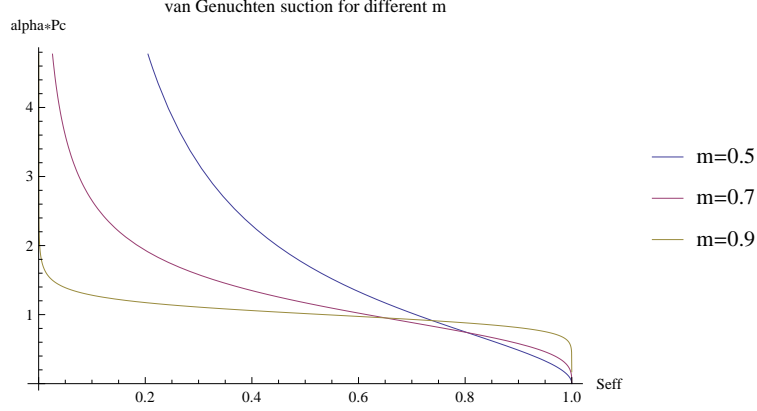


Figure 3.1:  $\alpha P_c$  as a function of  $S_{\text{eff}}$  as given by van Genuchten's expression Eqn (3.1). Three values of  $m$  are shown: 0.5, 0.7 and 0.9 (the  $m = 0.5$  case has largest  $P_c$  around  $S_{\text{eff}} \sim 0$ ).

### 3.7.2 Broadbridge-White

The Broadbridge-White<sup>2</sup> capillarity relationship valid for small  $K_n$  is

$$S_{\text{eff}} = S_n + (S_s - S_n) \frac{c}{1 + L(x)} . \quad (3.5)$$

where

$$x = (c - 1)e^{c-1-cP/\lambda} , \quad (3.6)$$

and  $L(x)$  is the Lambert W-function that satisfies  $L(z)e^{L(z)} = z$ . This is of limited use in real simulations, and is only used in the Porous Flow module for comparison with the analytical solutions of Broadbridge and White, and Warrick, Lomen and Islas<sup>3</sup> for multi-phase infiltration and drainage problems.

### 3.7.3 Rogers-Stallybrass-Clements

The Rogers-Stallybrass-Clements capillary relationship<sup>4</sup> is

$$S_{\text{eff}} = \frac{1}{\sqrt{1 + \exp((P_c - A)/B)}} , \quad (3.7)$$

when the oil viscosity is exactly twice the water viscosity. This is of limited use in real simulations, and is only used in the Porous Flow module for comparison with the analytical solutions offered by the authors for multi-phase infiltration and drainage problems.

<sup>2</sup>P Broadbridge, I White "Constant rate rainfall infiltration: A versatile nonlinear model, 1 Analytical solution". Water Resources Research 24 (1988) 145–154.

<sup>3</sup>AW Warrick, DO Lomen and A Islas, "An analytical solution to Richards' Equation for a Draining Soil Profile", Water Resources Research 26 (1990) 253–258.

<sup>4</sup>C Rogers, MP Stallybrass and DL Clements "On two phase filtration under gravity and with boundary infiltration: application of a Backlund transformation" Nonlinear Analysis, Theory, Methods and Applications 7 (1983) 785–799

### 3.8 Density

For simple calculations with water, the fluid bulk modulus,  $B$ , may be taken to be constant, so that the fluid density is

$$\rho = \rho_0 e^{P/B} , \quad (3.8)$$

where  $P$  is the fluid pressure. It is common to use  $\rho_0 = 1000 \text{ kg.m}^{-3}$  and  $B = 2 \text{ GPa}$ .

### 3.9 Tortuosity and diffusion coefficients

The tortuosity may be constant, or it may follow the Millington-Quirk<sup>5</sup> form

$$\tau_\beta = \phi^{1/3} S_\beta^{10/3} \quad (3.9)$$

The molecular diffusion coefficients,  $d_\beta^\kappa$ , are assumed to be constant.

### 3.10 Effective fluid pressure

The effective fluid pressure used in the definition of the fluid effective stress Eqn (1.25) is

$$P = \sum_\beta S_\beta P_\beta . \quad (3.10)$$

### 3.11 Enthalpy and energy-density

The specific enthalpy  $h_\beta$ , for a phase  $\beta$ , may be defined using one of the high-precision equation of states (eg see Section 3.3), or the relationship

$$h_\beta = \mathcal{E}_\beta + a P_\beta / \rho_\beta . \quad (3.11)$$

It is usually appropriate to use  $a = 1$ , however for comparison with other codes it may be useful to use  $a = 0$ .

The internal energy density  $\mathcal{E}_\beta$  for a phase may be defined using one of the high-precision equation of states, or the relationship

$$\mathcal{E}_\beta = C_v T . \quad (3.12)$$

may be used.

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<sup>5</sup>Millington and Quirk, Permeability of Porous Solids, Trans. Faraday Soc., 57, 1200- 1207, 1961

## 3.12 Permeability

The porous-material's insitu permeability tensor can be constant, or it can take one of the following forms

### 3.12.1 Exponential

$$k_{ij} = k_{ij}^0 e^{a\phi} , \quad (3.13)$$

where  $\phi$  is the porosity, and  $a$  is a user-defined constant.

### 3.12.2 Kozeny-Carman

Using the Kozeny-Carman relationship<sup>6</sup>

$$k_{ij} = k_{ij}^0 \frac{\phi^n}{(1 - \phi)^m} . \quad (3.14)$$

Here  $n$  and  $m$  are user-defined constants.

### 3.12.3 Permeability with a solid phase

A solid phase (from chemical precipitation, for instance) can be included in the framework described herein simply by setting its relative permeability to zero. However, in this case, the absolute permeability of the porous material should be

$$k = k^{\text{without solid phase}} (1 - S_s)^2 , \quad (3.15)$$

where  $S_s$  is the solid-phase saturation.

## 3.13 Porosity

Porosity may be fixed at a constant value, or it may be a function of the effective porepressure, the volumetric strain and/or the temperature, as discussed below.

The evolution of the porosity<sup>7</sup> is governed by

$$\frac{\partial \phi}{\partial t} = (\alpha_B - \phi) \frac{\partial}{\partial t} (\epsilon_{ii}^{\text{total}} - \alpha_T T) + \frac{(1 - \alpha_B)(\alpha_B - \phi)}{K} \frac{\partial P_f}{\partial t} . \quad (3.16)$$

---

<sup>6</sup>Oelkers 1996: Reviews in Mineralogy v. 34, p. 131-192

<sup>7</sup>This is a generalisation of Eqn(18) in Y Chen, C Zhou and L Jing “Modeling coupled THM processes of geological porous media with multiphase flow: Theory and vlication against laboratory and field scale experiments” Computers and Geotechnics 36 (2009) 1308-1329

Here  $K$  is the bulk modulus of the drained porous skeleton:  $1/K = \delta_{ij}\delta_{kl}C_{ijkl}$ . This has solution

$$\phi = \alpha_B + (\phi_0 - \alpha_B) \times \exp\left(\frac{\alpha_B - 1}{K}P_f - \epsilon_{ii}^{\text{total}} + \alpha_T T\right) , \quad (3.17)$$

where  $\phi_0$  is the porosity at zero porepressure, zero elastic strain and zero temperature. Note this porosity can become negative.

The evolution of porosity is motivated further in the appendix Chapter C.

Without porepressure effects, the correct expression for porosity as a function of volumetric strain and temperature is

$$\phi = 1 + (\phi_0 - 1) \times \exp\left(-\epsilon_{ii}^{\text{total}} + \alpha_T T\right) . \quad (3.18)$$

These expressions may be modified to include the effects of plasticity.

## 3.14 Relative permeability relationships

The relative permeability of a phase is a function of its effective saturation:

$$S_{\text{eff}}(S) = \frac{S - S_{\text{res}}^{\beta}}{1 - \sum_{\beta'} S_{\text{res}}^{\beta'}} \quad (3.19)$$

In this equation  $S_{\text{res}}^{\beta}$  is the residual saturation for phase  $\beta$ . If  $S_{\text{eff}} < 0$  then the relative permeability is zero, while if  $S_{\text{eff}} > 1$  then the relative permeability is unity. Otherwise, the relative permeability is given by the expressions below.

### 3.14.1 Constant

$$k_r = C . \quad (3.20)$$

This is not recommended because there is nothing to discourage phase disappearance, which manifests itself in poor convergence. Usually  $k_r(S) \rightarrow 0$  as  $S \rightarrow 0$  is a much better choice.

### 3.14.2 Corey

$$k_r = S_{\text{eff}}^n . \quad (3.21)$$

where  $n$  is a user-defined quantity.

### 3.14.3 Broadbridge-White

$$k_r = K_n + \frac{K_s - K_n}{(c - 1)(c - S_{\text{eff}})} S_{\text{eff}}^2 . \quad (3.22)$$

### 3.14.4 van Genuchten and cut van Genuchten

$$k_r = \sqrt{S_{\text{eff}}} \left( 1 - (1 - S_{\text{eff}}^{1/m})^m \right)^2 . \quad (3.23)$$

This has the numerical disadvantage that its derivative as  $S_{\text{eff}} \rightarrow 1$  is crappy. This means that simulations where the saturation oscillates around  $S_{\text{eff}} = 1$  do not converge well. Therefore, a “cut” version of the van-Genuchten expression is also offered, which is almost definitely indistinguishable experimentally from the original expression:

$$k_r = \begin{cases} \text{van Genuchten} & \text{for } S_{\text{eff}} < S_c \\ \text{cubic} & \text{for } S_{\text{eff}} \geq S_c \end{cases} . \quad (3.24)$$

Here the cubic is chosen so that its value and derivative match the van Genuchten expression at  $S = S_c$ , and so that it is unity at  $S_{\text{eff}} = 1$ .

### 3.14.5 FLAC

$$k_r = (n + 1) S_{\text{eff}}^n - n S_{\text{eff}}^{n+1} . \quad (3.25)$$

This has the distinct advantage over the Corey formulation that its derivative is continuous at  $S_{\text{eff}} = 1$ .

## 3.15 Thermal conductivity

Currently the Porous Flow module implements a simple approach to the functional dependence of thermal conductivity on the rock and fluid-phase conductivities. A “wet” thermal conductivity,  $\lambda_{\text{wet}}$ , and a “dry” thermal conductivity  $\lambda_{\text{dry}}$ , are defined (both of which are tensors), and

$$\lambda = \lambda_{\text{dry}} + S_{\text{aqueous}}^n (\lambda_{\text{wet}} - \lambda_{\text{dry}}) \quad (3.26)$$

where  $S_{\text{aqueous}}$  is the aqueous phase saturation, and  $n$  is a user-defined exponent.

The dependence of fluid thermal conductivity on temperature may be handled within the fluid property modules (to accommodate the kind of large changes that can occur when water flashes to steam). The solid thermal conductivities also have a slight dependence on pressure, but it is sometimes difficult to obtain reliable data for these dependencies.

## 3.16 Viscosity

Viscosity may be taken as constant, or the high-precision equations of state may be used to define it.

# Chapter 4

## Implementation details and numerical issues

### 4.1 Independent (nonlinear) variables

The Porous Flow module was designed from the ground up to be able to employ variable switching, or persistent variables, instead of the more common pressure-temperature-displacement set of variables. Most simulations in the test-suite use these common set of variables, but it is up to the user to specify which variables are independent, and which variables are dependent (usually the dependent variables will be MOOSE AuxVariables). A good choice of variables can lead to much improved convergence.

### 4.2 Lumping and upwinding

The Porous Flow module employs fluid-mass and heat-energy lumping to the nodes, as well as full upwinding of the advective and diffusive flow terms. This ensures far superior numerical convergence, especially in situations where mass-fractions of phases are close to disappearing.

### 4.3 Preconditioners and linear solvers

# Appendix A

## The continuity equation

The derivation of the continuity equation is fundamental to the fluid and heat flow DEs, and is included in this section. The notation in this appendix is different from that in the main report.

### A.1 Eulerian coordinates

Introduce the notion of the “spatial coordinate frame”. It is the coordinate frame of a stationary observer who is looking at the deforming porous solid from the outside. Denote its coordinates by  $\mathbf{x}$ , and the derivatives with respect those coordinates by  $\nabla$ .

Let  $\Omega$  be a volume that is attached to particles of the porous-solid skeleton. As the porous solid deforms, so too will  $\Omega$ . Denote the velocity of the porous solid is  $\mathbf{v}_s$ , measured in the spatial coordinate frame:  $\mathbf{v}_s = \mathbf{v}_s(x, t)$ . Then the change of a small volume element  $d\Omega$  is computed by calculating the Jacobian, and it is

$$\frac{d}{dt}(d\Omega) = \nabla \cdot \mathbf{v}_s d\Omega . \quad (\text{A.1})$$

Remember the derivative  $\nabla$  is differentiating with respect to the coordinates of the spatial coordinate frame. This formula is easy to motivate because  $\nabla \cdot \mathbf{v}_s = \dot{\epsilon}_{ii}$  which is the time derivative of the volumetric strain.

Let  $M$  represent a quantity that is attached to the porous-solid skeleton, for instance the mass density of the solid. Express  $M$  in the spatial coordinate frame:  $M = M(x, t)$ . As the porous solid deforms

$$\frac{d}{dt}M = \frac{\partial}{\partial t}M + \mathbf{v}_s \cdot \nabla M . \quad (\text{A.2})$$

The second term is easy to motivate if you consider a constant velocity with the spatially-dependent but temporally-constant  $M$ .



The continuity equation is

$$\frac{d}{dt} \int_{\Omega} M d\Omega + \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} dA = 0 , \quad (\text{A.3})$$

where  $\mathbf{F}$  is the flux of  $M$  out of  $\Omega$ ,  $n$  is the outward unit normal, and  $dA$  is the area element on  $\partial\Omega$  (which is the surface of  $\Omega$ ). Using the above expressions, and the divergence theorem, the continuity equation reads

$$\int_{\Omega} \left( \frac{\partial}{\partial t} M + \nabla \cdot (M \mathbf{v}_s) + \nabla \cdot \mathbf{F} \right) d\Omega = 0 . \quad (\text{A.4})$$

Specialising  $M$  and  $\mathbf{F}$  to fluids and heat gives the equations mentioned in the main text.

## A.2 Lagrangian coordinates

Introduce the notion of the “material coordinate frame”. It is the coordinate frame of an observer who is fixed to a certain point in the porous solid (eg, a particular finite-element node). Denote the coordinates in this frame by  $X$ . This is the frame used throughout this report. Fluid properties (pressures, mass fractions), the temperature, etc, are all stored at the finite-element nodes or the quadpoints, and move with the mesh. At the very least, an Eulerian description would be inconvenient when visualising with paraview.

Introduce the material derivative  $D/Dt$ . If  $M$  is any property that is expressed in terms of the “spatial coordinate frame” (Eulerian coordinates):  $M = \tilde{M}(x, t)$ , then the material derivative is

$$\frac{D}{Dt} \tilde{M}(x, t) = \frac{\partial}{\partial t} \tilde{M}(x, t) + \mathbf{v}_s \cdot \nabla \tilde{M}(x, t) . \quad (\text{A.5})$$

The continuity Eqn (A.4) can be re-written as

$$0 = \frac{D}{Dt} M + M \nabla \cdot \mathbf{v}_s . \quad (\text{A.6})$$

However, if  $M$  is expressed in terms of the Lagrangian coordinates:

$$M = M(X, t) , \quad (\text{A.7})$$

(generally  $M(X, t)$  will have a different functional form than  $\tilde{M}(x, t)$ , thus the tilde to emphasise the difference) then the material derivative is expressed by

$$\frac{D}{Dt} M(X, t) = \frac{\partial}{\partial t} M(X, t) , \quad (\text{A.8})$$

but the continuity equation has the identical form:

$$0 = \frac{D}{Dt} M + M \nabla \cdot \mathbf{v}_s . \quad (\text{A.9})$$

# Appendix B

## Benchmark studies to perform

### B.1 Use case: Two-component, two-phase, nonisothermal

This solution will benchmark the two-phase immiscible flow and thermal equations in of the code against the semi-analytical solutions in LaForce et al (AWR, 73 (2014a) 227–241). (Note: These will have to be written because they are copy/pasted from manuscript)

The analytical solutions assume that pressure changes in the reservoir are small, so that fluid volume, density, and viscosity are all constant. Fluids and the solid are treated as incompressible. It is also assumed that changes in temperature do not impact any of the solid or fluid properties. With these assumptions the thermal and saturation equations become

$$[S_1 + \beta] \frac{\partial T_{D1}}{\partial t_D} + [f_1 + \alpha] \frac{\partial D1}{\partial r_D} = k_{Tt} \gamma_j \frac{\partial^2 D1}{\partial z_D^2} \quad (\text{B.1})$$

$$\frac{\partial S_1}{\partial t_D} + \frac{\partial f_1}{\partial r_D} = 0 \quad (\text{B.2})$$

$$\alpha = \frac{\rho_2 C_2}{(\rho_1 C_1 - \rho_2 C_2)} \quad (\text{B.3})$$

$$\beta = \frac{\rho_2 C_2 + \frac{1-\phi_r}{\phi_r} \rho_s C_s}{(\rho_1 C_1 - \rho_2 C_2)} \quad (\text{B.4})$$

where  $S_1$  is the saturation of the water phase,  $f_1$  is the fractional flow of the water (resident) phase,  $T_{D1}$  is temperature and  $r_D$  is radial distance. The  $C_j$  are the mass-based heat capacity of fluid 1, 2, or the formation rock. The parameters  $\alpha$  and  $\beta$  in Eqns B.3—(B.4) are constant throughout the displacement. Dimensionless variables are defined in LaForce et al (2014a).

The relative permeability of the phases are from one of the TOUGH2 models

$$\begin{aligned} k_{r1} &= \hat{S}^4 \\ k_{r2} &= \left(1 - \hat{S}\right)^2 \left(1 - \hat{S}^2\right) \\ \hat{S} &= \frac{S_1 - S_{1r}}{1 - S_{1r} - S_{2r}} \end{aligned} \tag{B.5}$$

In the analytical model, heat loss from the reservoir to the over/under burden is handled via a heat sink term that approximates the diffusive process.

$$k_{Tt} \frac{\partial^2 T_D}{\partial z^2} \rightarrow -UT_D. \tag{B.6}$$

Note: It may be useful to solve these equations with  $U = 0$  initially.

In this formulation the thermal properties of the reservoir and the adjacent formations may be different, but the over- and under-burden must have the same thermal properties for vertical symmetry. It will be interesting to test the simulator on the full Eqn (B.1) with vertical heat diffusion vs a simulated solution with the heat sink.

## B.2 Two-component, two-phase, nonisothermal, forward coupled stress

This solution will benchmark the code against the semi-analytical solutions in LaForce et al (AWR, 73 (2014) 242 – 253), which takes the saturation and temperature solution from above and also calculates pressure and stress in the reservoir. The semi-analytical solutions are strictly forward-coupled, so that pressure does not impact the saturation or temperature profiles, and stress has no impact on pressure, saturation or temperature. In that work, pressure and stress are calculated at a given snapshot in time,  $t_D^*$ . This gives pressure and stress equations with no time dependence:

$$\frac{\partial P}{\partial r_D} = \frac{q}{4\pi h k r_D} \frac{1}{M} \tag{B.7}$$

where  $M$  is the mobility of the overall fluid composition at  $r_D$ .

The displacement  $u$ , of the porous medium at a fixed time is given by **Note: Notation may not be consistent with the above!!**:

$$\frac{d}{d\xi} \left( \frac{1}{\xi} \frac{d}{d\xi} (\xi u_D) \right) = \frac{\partial T_{D2}}{\partial \xi} + \frac{(1-2\nu)}{(1-\nu)} \frac{\partial P_D}{\partial \xi} \quad (\text{B.8})$$

where  $\xi = \frac{r}{r_w}$  for well radius  $r_w$ , and  $\nu$  is the drained Poisson ratio. All other dimensionless groups given in LaForce et al (2014b).

In these equations stresses are defined as positive if they are tensile and negative if they are compressive. The stress at the wall of the well will have equal magnitude as the fluid pressure, but opposite sign so that at the wellbore the effective radial stress is always zero. These equations can be solved analytically on a finite reservoir domain for either zero stress or zero displacement outer boundary conditions.

# Appendix C

## Evolution of porosity for an isothermal situation

The evolution of porosity is fundamental to the coupling between flow and solid mechanics. Consider the isothermal situation with no plasticity.

Denote the change of a quantity,  $q$ , by  $\Delta q$ . Recall that the porosity is defined by  $\phi = V_f/V$ , where  $V$  is an arbitrary volume of the porous material, and  $V_f$  is the porevolume within that volume. Also, by definition of the effective stress,

$$\Delta\epsilon_{ij} = C_{ijkl}(\Delta\sigma_{ij}^{\text{tot}} + \alpha_B\delta_{ij}\Delta P_f) . \quad (\text{C.1})$$

Taking the trace of this equation, and using  $V^{-1}\Delta V = \Delta\epsilon_{ii}$  yields

$$\frac{\Delta V}{V} = \delta_{ij}C_{ijkl}(\Delta\sigma_{ij}^{\text{tot}} + \alpha_B\delta_{ij}\Delta P_f) . \quad (\text{C.2})$$

In most instances it is appropriate to write this equation as

$$\frac{\Delta V}{V} = -\frac{1}{K}(P_{\text{mech}} - \alpha_B\delta_{ij}\Delta P_f) . \quad (\text{C.3})$$

where the total mechanical pressure is

$$P_{\text{mech}} = -\text{Tr}\sigma/3 . \quad (\text{C.4})$$

and  $K$  is the so-called “drained” bulk modulus  $K = \delta_{ij}C_{ijkl}\delta_{kl}$ . To find the evolution equation for porosity, a similar equation for  $\Delta V_f/V_f$  must be derived.

Assuming linearity

$$\frac{\Delta V_f}{V_f} = A_{ij}(\Delta\sigma_{ij}^{\text{tot}} + B\delta_{ij}\Delta P_f) . \quad (\text{C.5})$$

The Betti-Maxwell reciprocal theorem yields  $A_{ij}$  and  $B$ , as is now shown.

The work increment is

$$dW = -P_{\text{mech}}dV + P_f dV_f , \quad (\text{C.6})$$

So during some deformation that takes  $P_{\text{mech}}$  from  $P_{\text{mech}}^i$  to  $P_{\text{mech}}^f$ , and  $P_f$  from  $P_f^i$  to  $P_f^f$ , the total work is

$$\begin{aligned} W &= - \int P_{\text{mech}} dV + \int P_f dV_f, \\ &= \frac{V}{K} \int_{P_{\text{mech}}^i}^{P_{\text{mech}}^f} P_{\text{mech}} dP_{\text{mech}} - \frac{V\alpha_B}{K} \int_{P_f^i}^{P_f^f} P_{\text{mech}} dP_f + V_f A_{ii} \int_{P_{\text{mech}}^i}^{P_{\text{mech}}^f} P_f dP_{\text{mech}} + V_f A_{ii} B \int_{P_f^i}^{P_f^f} P_{\text{mech}} dP_f. \end{aligned} \quad (\text{C.7})$$

Now consider two experiments:

1. First take  $P_{\text{mech}}$  from 0 to  $P_{\text{mech}}$  with  $P_f$  fixed at 0. Then, leaving  $P_{\text{mech}}$  fixed, take  $P_f$  from 0 to  $P_f$ . The first takes work  $V P_{\text{mech}}^2 / (2K)$ , while the second takes work  $-\alpha_B V P_{\text{mech}} P_f / K + V_f A_{ii} B P_f^2 / 2$ .
2. First take  $P_f$  from 0 to  $P_f$  with  $P_{\text{mech}}$  fixed at 0. Then, leaving  $P_f$  fixed, take  $P_{\text{mech}}$  from 0 to  $P_{\text{mech}}$ . The first takes work  $V_f A_{ii} B P_f^2 / 2$ , and the second takes work  $V P_{\text{mech}}^2 / (2K) + V_f P_{\text{mech}} A_{ii} P_f$ .

The two experiments must give the same work done (this is called the Betti-Maxwell reciprocal theorem), which yields

$$A_{ij} = \alpha_B C_{ijkl} \delta_{kl} / \phi. \quad (\text{C.8})$$

Now to identify  $B$ . Consider a so-called “ideal porous material”, which is characterised by a fully-connected pore space and a homogeneous and isotropic matrix material. In this case, applying a uniform porepressure,  $P_f$ , and an equal mechanical pressure,  $P_{\text{mech}} = P_f$ , the solid material will experience a uniform pressure throughout its skeleton. This means it will deform uniformly without any shape change, and

$$\frac{\Delta V_f}{V_f} = \frac{\Delta V}{V}. \quad (\text{C.9})$$

Substituting this equation, this specific pressure condition, and Eqn (C.8) into Eqns (C.3) and (C.5), yields

$$B = 1 + \phi - \phi / \alpha_B. \quad (\text{C.10})$$

Now that  $A_{ij}$  and  $B$  have been identified, they may be substituted into Eqn (C.5). Rearranging yields

$$\frac{\Delta V_f}{V_f} = \frac{\alpha_B}{\phi} \delta_{ij} C_{ijkl} [\Delta \sigma_{kl}^{\text{tot}} + \alpha_B \delta_{kl} \Delta P_f] + \frac{\delta_{ij} \delta_{kl} C_{ijkl}}{\phi} (1 - \alpha_B) (\alpha_B - \phi) \Delta P_f \quad (\text{C.11})$$

Using the expression for  $\Delta V/V$  yields

$$\Delta V_f = V \alpha_B \Delta \epsilon_{ii} + V \delta_{ij} \delta_{kl} C_{ijkl} (1 - \alpha_B) (\alpha_B - \phi) \Delta P_f \quad (\text{C.12})$$

Now  $\Delta \phi = V^{-1} \Delta V_f - V_f V^{-2} \Delta V$ , so using the definition of  $K$  yields

$$\frac{\partial \phi}{\partial t} = (\alpha_B - \phi) \frac{\partial \epsilon_{ii}}{\partial t} + \frac{(1 - \alpha_B) (\alpha_B - \phi)}{K} \frac{\partial P_f}{\partial t}, \quad (\text{C.13})$$

as written in Eqn (3.16).