Conjugate Gradient method

Abstract

Conjugate gradient method (CG) is an algorithm for the numerical solution of particular systems of linear equations, namely those whose matrix is symmetric and positive-definite. The conjugate gradient method is often implemented as an iterative algorithm, applicable to sparse systems that are too large to be handled by a direct implementation or other direct methods such as the Cholesky decomposition.

I. Description of the problem

Suppose we want to solve the system of linear equations

$$(P1)\ A * x = b \qquad : \text{matrix ver.}$$

or,

$$(P2)\ A(x) = b \qquad : \text{function ver.}$$

for the vector x, where the known n x n matrix A is symmetric (i.e., $A^T = A$), positive-definite (i.e., $x^T A x > 0$ for all non-zero vectors x in R^n), and real, and b is known as well. We denote the unique solution of this system by x^*.

II. Implementation

A. The basic iteration CG for solving problem (matrix ver.)

```
function [x] = conjgrad(A, b, x)
    r = b - A * x;
    p = r;
    rsold = r' * r;

    for i = 1:length(b)
        Ap = A * p;
        alpha = rsold / (p' * Ap);
        x = x + alpha * p;
        r = r - alpha * Ap;
        rsnew = r' * r;
        if sqrt(rsnew) < 1e-10
            break;
        end
        p = r + (rsnew / rsold) * p;
        rsold = rsnew;
    end
end
```

B. The basic iteration CG for solving problem (function ver.)

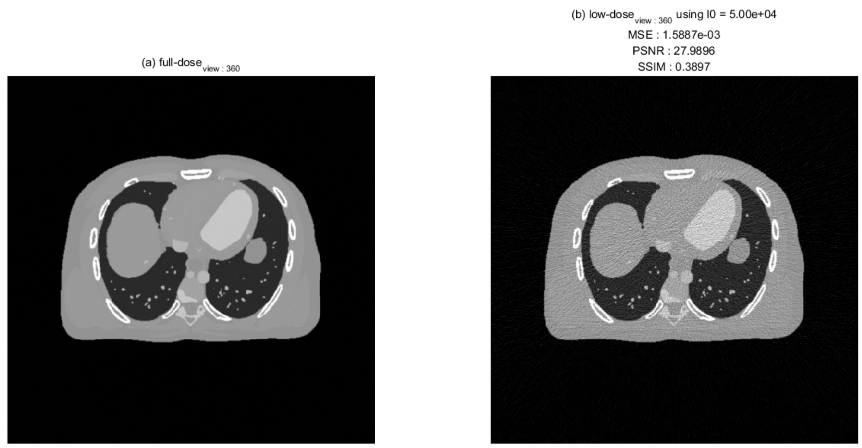
```
function [x] = conjgrad(A, b, x, N)
    r = b - A ( x );
    p = r;
    rsold = r(:)' * r(:);

    for i = 1:N
        Ap = A ( p );
        alpha = rsold / (p(:)' * Ap(:));
        x = x + alpha * p;
        r = r - alpha * Ap;
        rsnew = r(:)' * r(:);
        if sqrt(rsnew) < 1e-10
            break;
        end
        p = r + (rsnew / rsold) * p;
        rsold = rsnew;
    end
end
```

III. EXAMPLE

A. Problem definition

In X-ray computed tomography (CT) system, radiation exposure is critical limitation. To reduce the radiation exposure, X-ray CT system uses a low-dose X-ray source. The low-dose X-ray source generate severe poison noise when X-ray photon are measured at X-ray detector, then a reconstruction image using low-dose measurement is too noisy to diagnosis diseases by doctor. Below image shows (a) full-dose image and (b) low-dose image, respectively.



To reduce the noise, objective function with total variation (TV) regularization can be formulated as follows:

$$L(\mathrm{x}) = \frac{1}{2}\mathrm{y} - A\mathrm{x}_2^2 + \frac{\lambda}{2}(D_x(\mathrm{x})_2^2 + D_y(\mathrm{x})_2^2),$$

where y is measurement and x defines reconstruction image (denoised image). A is system operator (in this case, it is defined as CT system operator, called by **radon transform**). D_x and D_y are differential operators along x-axis and y-axis, respectively. In above equation, since each terms is quadratic function, an optimal point of x is $\frac{d}{d\mathrm{x}}L(x) = 0$:

$$-A^T(\mathrm{y} - A\mathrm{x}) + \lambda(D_x^T D_x(\mathrm{x}) + D_y^T D_y(\mathrm{x})) = 0,$$

$$A^T A(\mathrm{x}) + \lambda(D_x^T D_x(\mathrm{x}) + D_y^T D_y(\mathrm{x})) = A^T\mathrm{y},$$

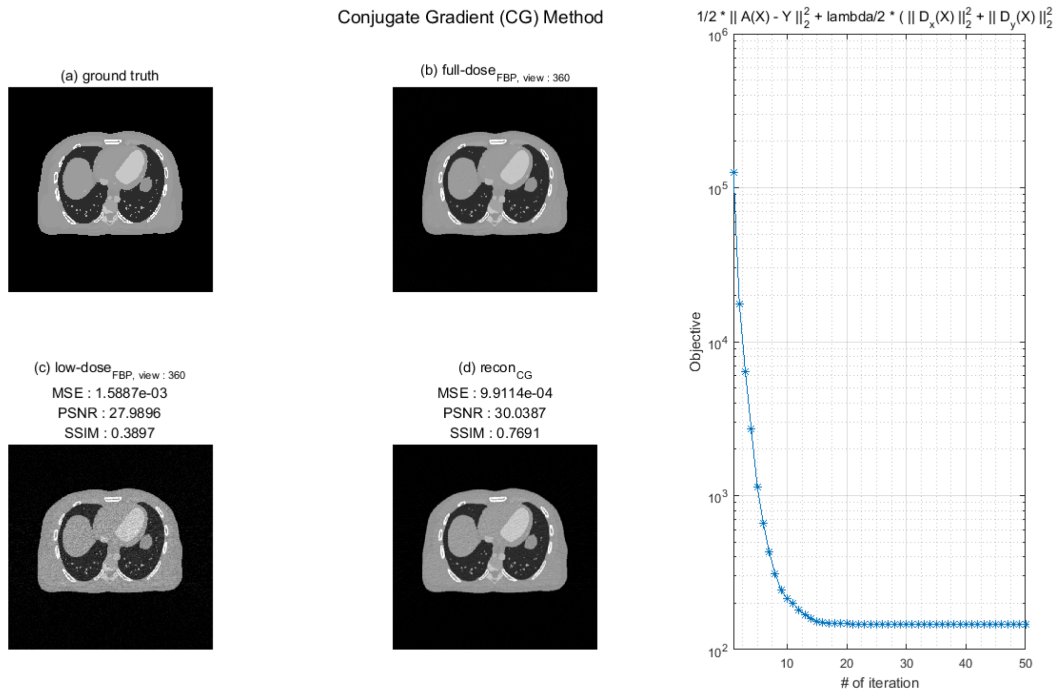
$$[A^T A + \lambda(D_x^T D_x + D_y^T D_y)](\mathrm{x}) = A^T\mathrm{y}.$$

To match the CG formula, $[A^T A + \lambda(D_x^T D_x + D_y^T D_y)]$ is defined as A_{cg} and $A^T\mathrm{y} = b$, then

$$A_{cg}(\mathrm{x}) = b.$$

Now, above equation form $A_{cg}(\mathrm{x}) = b$ is exactly matched with CG equation form $A(x) = b$. Using above reordered formula, optimal point x^* of $L(\mathrm{x})$ can be found.

B. Results


Conjugate Gradient (CG) Method
(a) ground truth
(b) full-dose FBP, view : 360
(c) low-dose FBP, view : 360
MSE : 1.5887e-03
PSNR : 27.9896
SSIM : 0.3897
(d) recon CG
MSE : 9.9114e-04
PSNR : 30.0387
SSIM : 0.7691
1/2 * || A(X) - Y ||_2^2 + lambda/2 * (|| D_x(X) ||_2^2 + || D_y(X) ||_2^2)
Objective
# of iteration