1 Probability Distributions

Probability Density Function (PDF):

$$\mathbb{P}\left[a \leq X \leq b\right] = \int_a^b f_X(x) dx \tag{1}$$

Important properties include $\int_{-\infty}^{\infty} f_X(x) dx = 1$ and $f_X(x) \geq 0$.

Example: *Uniform Distribution*. $f_X(x) = \frac{1}{b-a}$ for $x \in [a, b]$, 0 otherwise.

Cumulative distribution function (CDF):

$$\mathbb{P}\left[X \leq a\right] = \int_{-\infty}^{a} f_X(x) dx = F_X(a) \tag{2}$$

2 Population Statistics

Expected Value and Mean (1st moment):

$$\mu_X = \mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx \tag{3}$$

Example. What is $\mathbb{E}[X]$ if X is *uniformly distributed* over $[a, b]$?

Variance (2nd moment):

$$\sigma_X^2 = \mathbb{V}\left[X\right] = \mathbb{E}[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx \tag{4}$$

Example. What is $\mathbb{V}[X]$ if X is *uniformly distributed* over $[a, b]$?

Exam Practice: Using the above definitions, show that $\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.

3 Sample Statistics

We cannot directly measure population distributions; instead we gather samples and *infer* distribution properties.

Sample Mean:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \quad (5)$$

Sample Variance:

$$s_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2 \quad (6)$$

Sample Standard Deviation:

$$s_x = \sqrt{s_x^2} \quad (7)$$

Sample Covariance:

$$s_{x,y} = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y}) \quad (8)$$

4 Error Propagation

Note: The following formulas are written for random variables and population statistics. The sample formulas also apply for sample statistics.

Addition. Consider $Z = c_1 X + c_2 Y$,

$$\sigma_Z^2 = c_1^2\sigma_X^2 + c_2^2\sigma_Y^2 + 2c_1c_2\sigma_{X,Y} \quad (9)$$

Subtraction. Consider $Z = c_1 X - c_2 Y$,

$$\sigma_Z^2 = c_1^2\sigma_X^2 + c_2^2\sigma_Y^2 - 2c_1c_2\sigma_{X,Y} \quad (10)$$

Multiplication. Consider $Z = X \cdot Y$,

$$\sigma_Z^2 \approx (\mathbb{E}[Z])^2 \left[\left(\frac{\sigma_X}{\mathbb{E}[X]}\right)^2 + \left(\frac{\sigma_Y}{\mathbb{E}[Y]}\right)^2 + \frac{2\sigma_{X,Y}}{\mathbb{E}[X]\,\mathbb{E}[Y]}\right] \quad (11)$$

Division. Consider $Z = X/Y$,

$$\sigma_Z^2 \approx (\mathbb{E}[Z])^2 \left[\left(\frac{\sigma_X}{\mathbb{E}[X]}\right)^2 + \left(\frac{\sigma_Y}{\mathbb{E}[Y]}\right)^2 - \frac{2\sigma_{X,Y}}{\mathbb{E}[X]\,\mathbb{E}[Y]}\right] \quad (12)$$

Differentiable Function. Consider $Z = g(X,Y)$,

$$\sigma_Z^2 \approx \left|\frac{\partial g}{\partial X}\right|^2 \sigma_X^2 + \left|\frac{\partial g}{\partial Y}\right|^2 \sigma_Y^2 + 2\frac{\partial g}{\partial X}\frac{\partial g}{\partial Y}\sigma_{X,Y} \quad (13)$$

Exam Practice. Use the definition of mean and variance to prove the addition and subtraction rules, e.g., $\mathbb{V}[c_1X + c_2Y] = c_1^2\,\mathbb{V}[X] + c_2^2\,\mathbb{V}[Y] + 2c_1c_2\mathrm{cov}(X,Y)$. For simplicity, assume X and Y are linearly independent, i.e., $f_Z(x,y) = f_X(x)f_Y(y)$ and $\mathrm{cov}(X,Y) = 0$.