Basic Principles of Polarimetry

Carlos López-Martínez, Eric Pottier

1 Theory of Radar Polarimetry

1.1 Wave Polarimetry

Polarimetry refers specifically to the vector nature of the electromagnetic waves, whereas radar polarimetry is the science of acquiring, processing and analyzing the polarization state of an electromagnetic wave in radar applications. This section summarizes the main theoretical aspects necessary for a correct processing and interpretation of the polarimetric information. As a result, the first part presents the so called wave polarimetry that deals with the representation and the understanding of the polarization state of an electromagnetic wave. The second part introduces the concept of scattering polarimetry. This concept collects the topic of inferring the properties of a given target, from a polarimetric point of view, given the incident and the scattered polarizes electromagnetic waves.

1.1.1 Electromagnetic Waves and Polarization

The generation, the propagation, as well as the interaction with matter of the electric $\vec{\mathbf{E}}(\vec{\mathbf{r}},t)$ and the magnetic fields $\vec{\mathbf{H}}(\vec{\mathbf{r}},t)$ are governed by the Maxwell's equations [R1]. In the most general case, these fields may present any spatial, i.e., $\vec{\mathbf{r}}$, and any time, i.e., t, dependence.

Nevertheless, the interest is on the special case of constant amplitude monochromatic plane fileds which is adapted to the analysis of a wave polarization. In this particular case, the electromagnetic fields that shall be considered to be time-harmonic, i.e., the fields present a time dependence of the type $e^{j\omega t}$, where $\omega = 2\pi f$ is the angular frequency and f is the time frequency. In order to simplify the following analysis, this time dependence can be removed by considering the electric and the magnetic fields, for a specific time and a particular point in the space, in the following way

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},t) = \Re\left\{\underline{\vec{\mathbf{E}}}(\vec{\mathbf{r}})e^{j\omega t}\right\} \tag{1.1}$$

$$\vec{\mathbf{H}}(\vec{\mathbf{r}},t) = \Re\left\{\underline{\vec{\mathbf{H}}}(\vec{\mathbf{r}})e^{j\omega t}\right\} \tag{1.2}$$

where $\Re\{\cdot\}$ denotes the real part and $\underline{\vec{\mathbf{E}}}(\vec{\mathbf{r}})$ and $\underline{\vec{\mathbf{H}}}(\vec{\mathbf{r}})$ represent the time independent complex electric and magnetic field amplitudes, respectively, or simply complex amplitudes. Considering a source free, lossless, isotropic media, the expressions for the electric and magnetic complex fields can be of different form, leading for instance to: travelling waves, standing waves, evanescent waves and attenuating travelling and standing waves. Nevertheless, the interest is on travelling waves of the form

$$\underline{\vec{\mathbf{E}}}(\vec{\mathbf{r}}) = \underline{\mathbf{E}}_0^{+}e^{-j\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}} + \underline{\mathbf{E}}_0^{-}e^{j\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}} \tag{1.3}$$

$$\underline{\vec{\mathbf{H}}}(\vec{\mathbf{r}}) = \underline{\mathbf{H}}_0^{+}e^{-j\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}} + \underline{\mathbf{H}}_0^{-}e^{j\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}} \tag{1.4}$$

In the previous two equations, the terms in $e^{-j\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}$ must be considered as travelling waves propagating in the positive sense of the direction of the vector $\vec{\mathbf{k}}$, whereas the terms in $e^{j\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}$ represent travelling waves propagating in the negative sense of $\vec{\mathbf{k}}$. The vector $\vec{\mathbf{k}}$ receives the name of propagating vector, which amplitude is the wave number k. Without loss of generality, considering the propagation of a time-harmonic field in an infinite (unbounded) media, it can be assumed that $\underline{\mathbf{E}}_0^{-} = 0$ and $\underline{\mathbf{H}}_0^{-} = 0$. In the particular case of an infinite, source free, lossless, isotropic media, and considering the solutions of the wave equations in (1.3) and (1.4), *Transverse Electromagnetic* (TEM) waves, or modes, propagating along a particular direction shall be considered. The most important characteristic of TEM waves is that, both the electric $\vec{\underline{\mathbf{E}}}(\vec{\mathbf{r}})$ and the magnetic $\vec{\underline{\mathbf{H}}}(\vec{\mathbf{r}})$ fields, at every point in space are contained in a local plane, which is independent of time. Without any other restriction, this plane may change from one point to another. If the space orientation of the planes where the fields are contained for a TEM mode is the same, that is, the planes are parallel, then the fields form *plane waves*. If in addition to having planar equiphases, the field has equiamplitude planar surfaces, that is, the amplitude is the same over each plane, then it is called uniform plane wave. The planar nature of a wave can be also observed with the fact that the complex term $e^{-j\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}$ of the electric and magnetic fields is constant in a plane. TEM waves are not restricted to be planar. For instance, when the phase term $e^{-j\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}$ is constant in a sphere, the wave is referred to as *spherical wave*. In this situation, the local plane where the electric and magnetic fields are concentrated is tangent to the sphere where $e^{-j\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}$ is constant.

Finally, it must be mentioned that for the TEM fields in source free, lossless, isotropic media, the magnetic field $\vec{\underline{\mathbf{H}}}(\vec{\mathbf{r}})$ can be directly obtained from the electric field $\vec{\underline{\mathbf{E}}}(\vec{\mathbf{r}})$, provided the propagation vector $\hat{\mathbf{k}}$, as follows

$$\vec{\underline{\mathbf{H}}}(\vec{\mathbf{r}}) = \sqrt{\frac{\mu}{\varepsilon}}\hat{\mathbf{k}} \times \vec{\underline{\mathbf{E}}}(\vec{\mathbf{r}}) \tag{1.5}$$

That is, at any point of the space $\vec{\mathbf{r}}$, the electric and magnetic fields are perpendicular vectors that lie in a plane normal to the propagation direction.

The electric and magnetic waves considered until now, $\vec{\underline{\mathbf{E}}}(\vec{\mathbf{r}})$ and $\vec{\underline{\mathbf{H}}}(\vec{\mathbf{r}})$, are vector quantities. Consequently, the description of these waves must be performed considering a particular coordinate system. The most common coordinate systems employed to describe electromagnetic waves are the rectangular or cartesian, the cylindrical and the spherical coordinate systems. The selection of one of these is normally driven by the geometry of the particular problem under consideration, as the selection of the right coordinate system simplifies the analytical expressions of the electromagnetic waves.

The rectangular coordinate system, represented by the three orthonormal vectors $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$, is employed to simplify the electromagnetic wave description when propagation is considered in an infinite lossless, isotropic media. Without loss of generality, it is possible to assume the propagation vector $\hat{\mathbf{k}}$ parallel to $\hat{\mathbf{z}}$. Therefore, the electric and magnetic fields lie in the $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ plane. Thus, the fields may be decomposed as

$$\vec{\underline{\mathbf{E}}}(\vec{\mathbf{r}}) = E_{0x}e^{-jkz}\hat{\mathbf{x}} + E_{0y}e^{-jkz}\hat{\mathbf{y}} \tag{1.6}$$

$$\vec{\underline{\mathbf{H}}}(\vec{\mathbf{r}}) = H_{0x}e^{-jkz}\hat{\mathbf{x}} + H_{0y}e^{-jkz}\hat{\mathbf{y}} \tag{1.7}$$

where E_{0x}, E_{0y} , H_{0x} and H_{0y} are the complex amplitudes of the fields in each coordinate, which shall be considered constant.

As shown, for a TEM electromagnetic wave propagating in an infinite, lossless, isotropic media, the electric and the magnetic fields lie in a plane orthogonal to the direction of propagation. Consequently, and as observed in (1.6) and (1.7), the electric field vector (and the magnetic field vector) can be decomposed into two orthogonal components, $\hat{\mathbf{x}}$

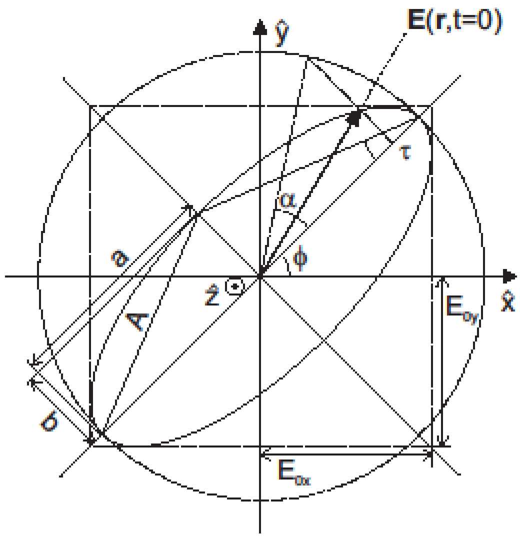


Figure 1.1 – Polarization ellipse.

and $\hat{\mathbf{y}}$ in the previous case. In this framework, polarization refers to the vector nature of the electromagnetic waves. According to the *IEEE Standard definitions for Antennas* [R2], the polarization of a radiated wave is defined as that property of the radiated electromagnetic wave describing a time-varying direction and relative magnitude of the electric field vector; specifically, the figure traced as a function of time by the extremity of the vector at a fixed location in space, and the sense in which it is traced, as observed along the direction of propagation. Hence, polarization is the curve traced out by the end point of the arrow representing the instantaneous electric field.

1.1.2 Wave Polarization Descriptors

For an electromagnetic wave field that is propagating in the $\hat{\mathbf{z}}$ direction, the real electric field can be decomposed into two orthogonal components $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ as indicated in (1.6) and (1.7). The previous description admits also the following vector formulation

$$\vec{\mathbf{E}}(z,t)=\begin{bmatrix}E_x\\E_y\\E_z\end{bmatrix}=\begin{bmatrix}E_{0x}\cos(\omega t-kz+\delta_x)\\E_{0y}\cos(\omega t-kz+\delta_y)\\0\end{bmatrix} \tag{1.8}$$

which may be also considered in a complex form

$$\vec{\underline{\mathbf{E}}}(z,t)=\begin{bmatrix}\underline{E}_x\\\underline{E}_y\\\underline{E}_z\end{bmatrix}=\begin{bmatrix}E_{0x}e^{j\delta_x}e^{-jkz}e^{j\omega t}\\E_{0y}e^{j\delta_y}e^{-jkz}e^{j\omega t}\\0\end{bmatrix} \tag{1.9}$$

where the different parameters of the electromagnetic wave shall be defined in the following.

Let us consider the geometric locus described by the electric field, as a function of time, for a particular point in space, which can be assumed $z=z_0$, without loss of generality. Under these hypotheses, the field components E_x and E_y satisfy the following equation

$$\left(\frac{E_x(z_0,t)}{E_{0x}}\right)^2 - 2\frac{E_x(z_0,t)E_y(z_0,t)}{E_{0x}E_{0y}}\cos\left(\delta_y-\delta_x\right)+\left(\frac{E_y(z_0,t)}{E_{0y}}\right)^2 = \sin\left(\delta_y-\delta_x\right) \tag{1.10}$$

The previous equation describes an ellipse that is called *polarization ellipse*. As one may deduce from the previous equation, the electric field, as a function of time, describes in the most general case an ellipse, which shape does depend neither on time nor on space. This lack of space- or time-dependence must be kept in mind, since it is of critical importance when considering scattered waves from natural targets. The polarization ellipse, for some particular configurations, may reduce to a circle or to a line as it will be detailed in the following.

As it may be deduced from (1.10), the polarization state is completely characterized by three independent parameters: the field amplitudes E_{0x}, E_{0y} and the phase difference $\delta = \delta_y - \delta_x$. Figure 1.1 presents the polarization ellipse for a general polarization state. In addition to the previous three parameters, it is also possible to describe the polarization ellipse by a different set of parameters. The following list details these parameters, which can be observed in Figure 1.1:

- Orientation or tilt angle ϕ. This angle gives the orientation of the ellipse major axis respect to the $\hat{\mathbf{x}}$ axis in such a way that $\phi \in [-\pi/2, \pi/2]$. This angle may be obtained as follows

$$\tan 2\phi = 2\frac{E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2}\cos\delta \tag{1.11}$$

- Ellipticity angle τ . This angle represents the ellipse aperture in such a way that $\tau \in [-\pi/4, \pi/4]$. This angle may be obtained as follows

$$\left|\sin 2\tau\right| = 2\frac{E_{0x}E_{0y}}{E_{0x}^2 + E_{0y}^2}\left|\sin\delta\right| \tag{1.12}$$

- The polarization sense or handedness. Determines the sense in which the polarization ellipse is described. This parameter is given by the sign of the ellipticity angle τ . Following the IEEE convention [R2], the polarization ellipse is right-handed if the electric vector tip rotates clockwise for a wave observed in the direction of propagation, given by $\hat{\mathbf{k}}$. On the contrary, it is said to be left-handed. Therefore, for $\tau < 0$ the polarization sense is right-handed whereas for $\tau > 0$ it is left-handed.
- The polarization ellipse amplitude A. For a major and minor ellipse axes amplitudes a and b respectively, $A=\sqrt{a^2+b^2}$. This amplitude may be also obtained as

$$A = \sqrt{E_{0x}^2 + E_{0y}^2} \tag{1.13}$$

- The absolute phase α . This phase represents the initial phase with respect to the phase origin for $t = 0$ in such a way that $\alpha \in [-\pi, \pi]$. This term corresponds to the common phase in δ_x and δ_y .

Considering the previous sets of parameters describing the polarization state of a TEM wave, one can identify some important particular polarization states that can be considered as canonical polarization states:

- Linear polarization state. Considering the expression for the real electric field in (1.8), two canonical linear polarization states can be identified. Table 1.1 details the orientation and the ellipticity angles for these polarization states. These are the linear polarization states according to the $\hat{\mathbf{x}}$ and to the $\hat{\mathbf{y}}$ axes, respectively. The linear polarization states are characterized by presenting a phase difference of

$$\delta = \delta_y - \delta_x = m\pi \qquad m = 0, \pm 1, \pm 2, \ldots \tag{1.14}$$

According to the expression of the real electric field, these polarization states are represented by

$$\vec{\mathbf{E}}_x(z,t) = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} E_{0x}\cos(\omega t - kz + \delta_x) \\ 0 \\ 0 \end{bmatrix} \tag{1.15}$$

$$\vec{\mathbf{E}}_y(z,t) = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} 0 \\ E_{0y}\cos(\omega t - kz + \delta_y) \\ 0 \end{bmatrix} \tag{1.16}$$

As it may be seen, the linear nature of the polarization state is independent of the phase α .

- Circular polarization state. In this particular case also two canonical circular polarization states can be defined. Table Table 1.1 details the orientation and the ellipticity angles for these polarization states. When the ellipticity angle takes a value of $-\pi/4$ the circular polarization state is right-handed, whereas this value is equal to $\pi/4$ when it is left-handed. The circular polarization states are characterized by presenting a phase difference of

$$\delta = \delta_y - \delta_x = m\frac{\pi}{2} \qquad m = \pm 1, \pm 3, \pm 5, \ldots \tag{1.17}$$

and equal amplitudes for the components of the electric field $E_0 = E_{0x} = E_{0y}$. Consequently, a real electric field with circular polarization is described as

$$\vec{\mathbf{E}}_{circ}(z,t) = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} E_0\cos(\omega t - kz + \delta_x) \\ E_0\cos\left(\omega t - kz + \delta_x + m\dfrac{\pi}{2}\right) \\ 0 \end{bmatrix} \qquad m = \pm 1, \pm 3, \pm 5, \ldots \tag{1.18}$$

Also for circular polarization states, the polarization state is independent of the absolute phase α .

- Elliptical polarization state. When there are not restrictions on the orientation and ellipticity angle values, the electric field is said to present an elliptical polarization state.

	Linear $\hat{\mathbf{x}}$	Linear $\hat{\mathbf{y}}$	Linear π/4	Linear 3π/4	Right hand circular	Left hand circular
ϕ	0	π/2	π/4	3π/4	[-π/2, π/2]	[-π/2, π/2]
τ	0	0	0	0	-π/4	π/4

Table 1.1 – Geometrical parameters of the polarization ellipse for canonical wave polarization states in the rectangular coordinate system.

As observed, it is possible to characterize the polarization state of an electric field by means of the so-called polarization ellipse. This polarization ellipse may be completely described by two equivalent sets of three independent parameters: the set of field parameters $\{E_{0x}, E_{0y}, \delta\}$ or the set of ellipse parameters $\{\phi, \tau, A\}$. In addition to the polarization ellipse to describe the polarization state of a TEM field propagating in a source free, lossless, isotropic media, there exist additional equivalent descriptors that are detailed in the following.

Considering the complex formulation, see (1.9), the real electric field vector given in (1.29) can be directly obtained from the complex electric field vector

$$\vec{\mathbf{E}}(z,t) = \begin{bmatrix} E_{0x}\cos(\omega t - kz + \delta_x) \\ E_{0y}\cos(\omega t - kz + \delta_y) \end{bmatrix} = \Re\left\{ \begin{bmatrix} E_{0x}e^{j\delta_x} \\ E_{0y}e^{j\delta_y} \end{bmatrix} e^{-jkz} e^{j\omega t} \right\} = \Re\left\{ \underline{\vec{\mathbf{E}}}(z) e^{j\omega t} \right\} \tag{1.19}$$

As observed, the time dependence has been removed from the field description. This is possible as the polarization state of the field does no change on time. In order to derive a simple and concise description of the polarization state, it is also possible to remove the space dependence of $\underline{\vec{\mathbf{E}}}(z)$ by considering, for example, the polarization state in a particular point of the space. Without loss of generality, this point can be $z = 0$. Hence, $\underline{\vec{\mathbf{E}}}(0)$ reduces to

$$\underline{\mathbf{E}} = \underline{\vec{\mathbf{E}}}(0) = \begin{bmatrix} E_{0x}e^{j\delta_x} \\ E_{0y}e^{j\delta_y} \end{bmatrix} \tag{1.20}$$

The vector $\underline{\mathbf{E}}$ is referred to as the *Jones vector* and it is a concise representation of a monochromatic, uniform plane wave with a constant polarization [R3][R4][R5][R6][R7]. It is worth to mention that the Jones vector is a $\mathbb{C}^2$ vector, that is, it is a two-dimensional complex vector.

In the rectangular coordinate system, the Jones vector can be written as a function of the parameters that describe the polarization ellipse [R8][R9][R10]

$$\underline{\mathbf{E}} = Ae^{j\alpha} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\tau \\ j\sin\tau \end{bmatrix} \tag{1.21}$$

	Linear $\hat{\mathbf{x}}$	Linear $\hat{\mathbf{y}}$	Linear π/4	Linear 3π/4	Right hand circular	Left hand circular
$\underline{\mathbf{E}}_{\{\hat{\mathbf{x}},\hat{\mathbf{y}}\}}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -j \end{bmatrix}$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ j \end{bmatrix}$

Table 1.2 – Jones vector for some polarization states in the rectangular coordinate system, when *A*=1.

Since the Jones vector, considering the unitary vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, it may also be expressed as

$$\underline{\mathbf{E}}_{\{\hat{\mathbf{x}},\hat{\mathbf{y}}\}} = A \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\tau & j\sin\tau \\ j\sin\tau & \cos\tau \end{bmatrix} \begin{bmatrix} e^{j\alpha} & 0 \\ 0 & e^{-j\alpha} \end{bmatrix} \hat{\mathbf{x}} \tag{1.22}$$

Where the subindex $\{\hat{\mathbf{x}},\hat{\mathbf{y}}\}$ indicates that the Jones vector is expressed in the linear basis $\{\hat{\mathbf{x}},\hat{\mathbf{y}}\}$.

The Jones vector describes completely the polarization ellipse shape, as well as the rotation sense of the electric field vector. On the contrary, handedness information cannot be included within the Jones vector as propagation information has been removed. The use of the Jones vector to describe the polarization state of an electric field considering the expression presented previously is of enormous importance as it allows to define a polarization algebra

[R9][R10] that makes possible to perform a mathematical treatment and analysis of the wave polarization. This treatment allows, for instance, the correct definition of orthogonal polarization states.

Finally, Table 1.2 details the Jones vector, in the rectangular basis, i.e., $\underline{\mathbf{E}}_{\{\hat{\mathbf{x}},\hat{\mathbf{y}}\}}$, for som particular polarization states.

Another description of the wave polarization state, equivalent to the Jones vector, is the so-called complex *polarization ratio*. This ratio is obtained as follows

$$\rho = \frac{E_y}{E_x} = \frac{E_{0y}}{E_{0x}} e^{j(\delta_y - \delta_y)} \tag{1.23}$$

As in the case of the Jones vector, the complex polarization ratio is not able to determine the handedness of the polarization state as propagation information is removed.

The Jones vectors, as well as the complex polarization ratio, are complex quantities that describe the polarization state of that TEM wave. Sir G. Stokes introduced a wave polarization and wave amplitude description based on four real measurable quantities in the field of optics polarization [R11]. The stokes vector, in the rectangular coordinate system, is defined as follows [R7][R12]

$$\underline{\mathbf{g}} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} |E_x|^2 + |E_y|^2 \\ |E_x|^2 - |E_y|^2 \\ 2\Re\{E_x E_y^*\} \\ -2\Im\{E_x E_y^*\} \end{bmatrix} \tag{1.24}$$

where the elements of the vector $\underline{\mathbf{g}}$ are simply called Stokes parameters. Consequently, the Stokes vector is a four dimensional real vector. Since the Stokes vector describes the polarization state of an electromagnetic wave, it can be directly obtained from the geometrical parameters that describe the polarization ellipse, i.e., $\{\phi, \tau, A\}$

$$\underline{\mathbf{g}} = \begin{bmatrix} A \\ A\cos(2\phi)\cos(2\tau) \\ A\sin(2\phi)\cos(2\tau) \\ A\sin(2\tau) \end{bmatrix} \tag{1.25}$$

As indicated, the polarization state of an electromagnetic wave is completely characterized by means of three independent parameters. These statements also hold for the Stokes parameters, since as it may be deduced from (1.25), the following relation applies

$$g_0^2 = g_1^2 + g_2^2 + g_3^2 \tag{1.26}$$

Table 1.3 details the Stokes vector, in the rectangular basis, i.e., $\mathbf{g}_{\{\hat{\mathbf{x}},\hat{\mathbf{y}}\}}$, for some particular polarization states.

	Linear $\hat{\mathbf{x}}$	Linear $\hat{\mathbf{y}}$	Linear π/4	Linear 3π/4	Right hand circular	Left hand circular
$\mathbf{g}_{\{\hat{\mathbf{x}},\hat{\mathbf{y}}\}}$	$\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$

Table 1.3 – Stokes vector for some polarization states in the rectangular coordinate system, when *A*=1.

1.1.3 Totally and Partially Polarized Waves

Single-frequency or monochromatic waves, are completely polarized, that is, the tip of the electric field vector describes an ellipse in the plane orthogonal to the propagation direction. The shape of this ellipse, neglecting attenuation propagation effects which affect only the overall power, does not change in time or space and hence, the wave polarization is constant. Completely polarized waves appear when the different parameters of the wave: ω , E_{0x} , E_{0y} , δ_x and δ_y are constant. Nevertheless, many waves present in the nature are characterized by the fact that the previous parameters depend on time or on space in a random form. As a consequence, the tip of the electric field vector does not longer describe an ellipse in the plane orthogonal to the propagation direction. These waves are referred to as *partially polarized waves*. This loss of polarization is due to the randomness of the illuminated scene, to the presence of noise, etc...

The different parameters that characterize the electric field, i.e., ω , E_{0x}, E_{0y} , δ_x and δ_y may vary in a random form. This type of variation makes the electric field to be modulated and therefore to present a finite bandwidth, so waves cannot be longer be considered as being monochromatic, but *polychromatic*. Under this circumstance, it would be also desirable to have a complex representation of the electromagnetic wave as shown in (1.1) and (1.2). Under the hypothesis of a polychromatic real electric field $\vec{\mathbf{E}}(\vec{\mathbf{r}},t)$ to be square integrable, this field may be rewritten as [20]

$$\begin{aligned}\vec{\mathbf{E}}(\vec{\mathbf{r}},t)=&\left(\int_0^\infty E_x(\omega)\cos\left(\omega t-\Theta_x(\vec{\mathbf{r}},\omega)\right)d\omega+j\int_0^\infty E_x(\omega)\sin\left(\omega t-\Theta_x(\vec{\mathbf{r}},\omega)\right)d\omega\right)\underline{\hat{\mathbf{x}}}\\&+\left(\int_0^\infty E_y(\omega)\cos\left(\omega t-\Theta_y(\vec{\mathbf{r}},\omega)\right)d\omega+j\int_0^\infty E_y(\omega)\sin\left(\omega t-\Theta_y(\vec{\mathbf{r}},\omega)\right)d\omega\right)\underline{\vec{\mathbf{y}}}\end{aligned} \tag{1.27}$$

For the previous polychromatic wave, it is not possible to define a unique polarization state beyond the possibility to define it for every spectral component of the field. Nevertheless, in most of the applications we are interested into, the spectral amplitudes will only have appreciable values in a frequency range $\Delta\omega$ which is small compared to the mean frequency ω . Under this situation, waves are referred to as *quasi-monochromatic waves*, and the expression given by (1.27) admits a simpler representation

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},t)=\Re\left\{E_x(t)e^{j\Theta_x(\vec{\mathbf{r}},t)}e^{j\bar{\omega}t}e^{jkz}\right\}\underline{\hat{\mathbf{x}}}+\Re\left\{E_y(t)e^{j\Theta_y(\vec{\mathbf{r}},t)}e^{j\bar{\omega}t}e^{jkz}\right\}\underline{\vec{\mathbf{y}}} \tag{1.28}$$

For such signals, $E_x(t)$ and $E_y(t)$ vary slowly with respect to $\cos(\bar{\omega}t)$ and $\sin(\bar{\omega}t)$, and the phase terms $\Theta_x(\vec{\mathbf{r}},t)$ and $\Theta_y(\vec{\mathbf{r}},t)$ change slowly when compared to $\bar{\omega}$. Considering a time variation of the type $\bar{\omega}t$, one may represent the electric field in a similar way as done in (1.9)

$$\vec{\underline{\mathbf{E}}}(\vec{\mathbf{r}},t)=\begin{bmatrix}E_x(t)e^{j\Theta_x(\vec{\mathbf{r}},t)}e^{jkz}\\E_y(t)e^{j\Theta_y(\vec{\mathbf{r}},t)}e^{jkz}\end{bmatrix} \tag{1.29}$$

Consequently, one may define the Jones vector of a quasi-monochromatic wave as

$$\underline{\mathbf{E}}=\begin{bmatrix}E_x(t)e^{j\Theta_x(\vec{\mathbf{r}},t)}\\E_y(t)e^{j\Theta_y(\vec{\mathbf{r}},t)}\end{bmatrix} \tag{1.30}$$

As one may see in the previous equation, the Jones Vector of a quasi-monochromatic electric field depends on time and on space. Consequently, this vector is no longer constant. When the time dependence of the Jones vector is deterministic, the polarimetric properties of the wave also change in a deterministic way through time. In this case, the description of the wave polarization is not problematic and may by performed considering the different descriptors detailed in Section 1.1.2. Nevertheless, if the time dependence is random, the analysis of the polarization state of the electromagnetic wave must be carefully addressed, as this description must take into account the stochastic nature of the electric field.

As previously mentioned, the variation of the parameters E_{0x}, E_{0y} , δ_x and δ_y may be random, so the Jones Vector will be also random. In order to characterize the polarization of the quasi-monochromatic electromagnetic wave expressed by the variable Jones vector in (1.30) it is necessary to address this characterization from a stochastic point of view. In the frame of radar remote sensing, the wave transmitted by the radar system may be considered monochromatic and hence totally polarized. Nevertheless, the scattered wave represented by the Jones vector in (1.30), results from the combination of many different ways originated by the different elementary scatters that form the scattering media. The complex addition of these elementary waves resulting from the scattering process, for one component of the electric field can be represented as

$$\mathbf{A}=Ae^{j\theta}=\frac{1}{\sqrt{N}}\sum_{n=1}^{n}a_ne^{j\theta_n} \tag{1.31}$$

where $\mathbf{A}$ represents the total field and $a_n e^{j\theta_n}$ is originated from the scattering from every elementary scatter. Under the assumption of N, i.e., the total number of scattered waves, to be large enough and certain relations that may be established between the amplitude and the phase of the elementary waves [R13][R14][R15], it is possible to demonstrate that the mean value of the electric field, as well as the Jones vector are zero. Consequently, the Jones vector cannot be employed to characterize the polarization state of a quasi-monochromatic wave. This characterization shall be performed considering higher statistical moments.

The second order moments may be arranged in a vector form, giving rise to the so-called *coherency vector* of a quasi-monochromatic vector, which is defined in the following way

$$\mathbf{J}=\left\langle\underline{\mathbf{E}}\otimes\underline{\mathbf{E}}^*\right\rangle=\begin{bmatrix}\left\langle E_xE_x^*\right\rangle\\\left\langle E_xE_y^*\right\rangle\\\left\langle E_yE_x^*\right\rangle\\\left\langle E_yE_y^*\right\rangle\end{bmatrix}=\begin{bmatrix}J_{xx}\\J_{xy}\\J_{yx}\\J_{yy}\end{bmatrix} \tag{1.32}$$

where $\mathbf{J}$ stands for the temporal averaging, assuming the wave is stationary, $\otimes$ is the Kroneker product and * represents complex conjugation. This vector is not zero for quasi-monochromatic waves. The arrangement of the second order moments can be also done in a matrix as proposed in [20], giving rise to the *coherency matrix* of the wave

$$\mathbf{J} = \left\langle \underline{\mathbf{E}} \cdot \underline{\mathbf{E}}^{T*} \right\rangle = \begin{bmatrix} \left\langle E_x E_x^* \right\rangle & \left\langle E_x E_y^* \right\rangle \\ \left\langle E_y E_x^* \right\rangle & \left\langle E_y E_y^* \right\rangle \end{bmatrix} = \begin{bmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{bmatrix} \tag{1.33}$$

where the superscript ^{T} denotes complex conjugation. As observed, the coherency matrix of the wave is Hermitian. Both representations of the quasi-monochromatic field contain the same information.

In the previous section it was mentioned that monochromatic waves are completely polarized, this is not the case for quasi-monochromatic waves. Indeed, completely polarized waves present a polarization state that can be considered as a limit in the sense that it is constant. The opposed extreme is a completely unpolarized wave which polarization state is completely random. The best example of a completely unpolarized wave is the natural light originated in the Sun. Between both extremes, waves are said to present a partial polarization state, that is, the polarization state is neither completely random nor completely deterministic. In order to characterize the degree of polarization, one may consider the *degree of polarization* defined as

$$DoP = \left(1 - 4\frac{|\mathbf{J}|}{Trace(\mathbf{J})}\right)^{\frac{1}{2}} = \frac{\sqrt{\langle g_1 \rangle^2 + \langle g_2 \rangle^2 + \langle g_3 \rangle^2}}{\langle g_0 \rangle^2} \tag{1.34}$$

which is also expressed in terms of the Stokes vector component.

1.1.4 Change of Polarization Basis

As seen in Section 1.1.2, an electromagnetic wave, considering the coordinate system $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$, that propagates in $\hat{\mathbf{z}}$, may be decomposed as the sum of two orthogonal components. In this case, the components $\hat{\mathbf{x}}$and $\hat{\mathbf{y}}$. Separately, the electromagnetic wave of each component can be considered as linearly polarized. Therefore, it is possible to consider that the total electromagnetic wave results from the sum of two orthogonal linear polarized fields. Indeed, this representation must be extended in the sense that any TEM electromagnetic wave propagating in an infinite, lossless, isotropic media can be decomposed as the sum of two orthogonal elliptically polarized waves. The advantage of this representation is that the electric field is decomposed in a pair of orthogonal polarization states, so it is possible, through a deterministic transformation, to obtain the electric field for any other pair of orthogonal polarization states. This process is referred to as *change of polarization basis* or *polarization synthesis*.

Before to analyze the concept of polarization synthesis, it is necessary to introduce the idea of orthogonal polarization states. Given two vectors A and B they are considered orthogonal if they verify

$$\langle \mathbf{A}, \mathbf{B} \rangle = \mathbf{A}^{T*}\mathbf{B} = \mathbf{B}^{T*}\mathbf{A} = 0 \tag{1.35}$$

that is, the scalar product of both vectors is zero. In case of two electromagnetic waves, expressed in terms of the corresponding Jones vectors, they are said to be orthogonal if the scalar product of the Jones vectors is zero, considering that both Jones vectors refer to waves propagating in the same direction and sense. The polarization ellipsis corresponding to two orthogonal Jones vectors present the same ellipticity angle, opposite polarization sense and the polarization axis are mutually orthogonal. That is, for a Jones vector representing a polarization state characterized by a orientation angle ϕ, a ellipticity angle τ and an absolute phase α, its orthogonal Jones vector presents a orientation angle of value $\phi + \pi$ an ellipticity angle of value $-\tau$ and an absolute phase $-\alpha$. In terms of (1.22), the corresponding orthogonal vector is

$$\begin{aligned}\underline{\mathbf{E}}_{\perp\{\hat{\mathbf{x}},\hat{\mathbf{y}}\}} &= A\begin{bmatrix}-\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi\end{bmatrix}\begin{bmatrix}\cos\tau & -j\sin\tau \\ -j\sin\tau & \cos\tau\end{bmatrix}\begin{bmatrix}e^{-j\alpha} & 0 \\ 0 & e^{j\alpha}\end{bmatrix}\hat{\mathbf{x}} \\ &= A\begin{bmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\cos\tau & j\sin\tau \\ j\sin\tau & \cos\tau\end{bmatrix}\begin{bmatrix}e^{j\alpha} & 0 \\ 0 & e^{-j\alpha}\end{bmatrix}\hat{\mathbf{y}}\end{aligned} \tag{1.36}$$

The symbol $\perp$ denotes orthogonal Jones vector.

Considering what it has been indicated, a TEM electromagnetic wave propagating in an infinite, lossless, isotropic media may be described in the following way

$$\underline{\mathbf{E}} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} = E_x\hat{\mathbf{u}}_x + E_y\hat{\mathbf{u}}_y \tag{1.37}$$

where the notation referring to the unitary vectors has been generalized. If (1.36) and (1.37) are considered, it may be seen that the unitary Jones vectors corresponding to the linear orthogonal polarization states $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are transformed to the Jones vector of any polarization state and the corresponding orthogonal Jones vector through the transformation matrix $\mathbf{U}$

$$\begin{aligned}\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\} &= \begin{bmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\cos\tau & j\sin\tau \\ j\sin\tau & \cos\tau\end{bmatrix}\begin{bmatrix}e^{-j\alpha} & 0 \\ 0 & e^{j\alpha}\end{bmatrix}\{\hat{\mathbf{x}},\hat{\mathbf{y}}\} \\ &= \mathbf{U}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}}\{\hat{\mathbf{x}},\hat{\mathbf{y}}\}\end{aligned} \tag{1.38}$$

In the previous case, the matrix $\mathbf{U}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}}$ indicates the transformation matrix from the orthogonal basis $\{\hat{\mathbf{x}},\hat{\mathbf{y}}\}$ to the arbitrary basis $\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}$. Considering (1.37), the electromagnetic wave expressed in the orthogonal basis $\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}$ takes the form

$$\underline{\mathbf{E}} = E_u\hat{\mathbf{u}} + E_{u\perp}\hat{\mathbf{u}}_\perp \tag{1.39}$$

Therefore, the Jones vector in the new basis $\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}$, expressed in terms of the Jones vector in the basis $\{\hat{\mathbf{x}},\hat{\mathbf{y}}\}$, takes the form

$$\begin{bmatrix}E_u \\ E_{u\perp}\end{bmatrix} = \mathbf{U}^{-1}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}}\begin{bmatrix}E_x \\ E_y\end{bmatrix} \tag{1.40}$$

The previous equation indicates that if a electromagnetic wave has been measured in the linear orthogonal basis, it is possible to calculate the same electromagnetic wave, but measured in a different polarization basis, just multiplying by the matrix $\mathbf{U}^{-1}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}}$. That is, it is possible to synthesize the electromagnetic wave for any arbitrary polarization basis just measuring it in a particular polarization basis. The transformation indicated in (1.40) can be made more explicit under the following equivalent notation

$$\begin{bmatrix}E_u \\ E_{u\perp}\end{bmatrix} = \mathbf{U}_{\{\hat{\mathbf{u}}_x,\hat{\mathbf{u}}_y\}\to\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}}\begin{bmatrix}E_x \\ E_y\end{bmatrix} \tag{1.41}$$

At this point, it is possible to consider the transformation of an electromagnetic wave measured in the linear polarization basis $\{\hat{\mathbf{x}},\hat{\mathbf{y}}\}$ to a second arbitrary polarization basis $\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_\perp\}$. In this second basis, (1.39) and **Erreur ! Source du renvoi introuvable.** also apply, considering the transformation matrix $\mathbf{U}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_\perp\}}$

$$\underline{\mathbf{E}} = E_{u'}\hat{\mathbf{u}}' + E_{u'\perp}\hat{\mathbf{u}}'_\perp \tag{1.42}$$

$$\begin{bmatrix} E_{u'} \\ E_{u'\perp} \end{bmatrix} = \mathbf{U}^{-1}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}} \begin{bmatrix} E_x \\ E_y \end{bmatrix} \tag{1.43}$$

According to **Erreur ! Source du renvoi introuvable.**, the previous expression simplifies to

$$\begin{bmatrix} E_{u'} \\ E_{u'\perp} \end{bmatrix} = \mathbf{U}_{\{\hat{\mathbf{u}}_x,\hat{\mathbf{u}}_y\}\to\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_\perp\}} \begin{bmatrix} E_x \\ E_y \end{bmatrix} \tag{1.44}$$

Then, one may see that it is possible to transform the Jones vector between the two arbitrary basis in the following way

$$\begin{bmatrix} E_{u'} \\ E_{u'\perp} \end{bmatrix} = \mathbf{U}^{-1}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_\perp\}} \mathbf{U}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}} \begin{bmatrix} E_x \\ E_y \end{bmatrix} \tag{1.45}$$

or

$$\begin{bmatrix} E_{u'} \\ E_{u'\perp} \end{bmatrix} = \mathbf{U}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}\to\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_\perp\}} \begin{bmatrix} E_u \\ E_{u\perp} \end{bmatrix} \tag{1.46}$$

Erreur ! Source du renvoi introuvable. and **Erreur ! Source du renvoi introuvable.** detail the polarization ellipse parameters, the Jones vector and the Stokes vector for different polarization states for the rotated and the linear polarization bases, respectively.

	Linear $\hat{\mathbf{x}}$	Linear $\hat{\mathbf{y}}$	Linear π/4	Linear 3π/4	Right hand circular	Left hand circular
ϕ	- π/4	π/4	0	π/2	?	?
τ	0	0	0	0	π/4	-π/4
$\underline{\mathbf{E}}_{\{\hat{\mathbf{u}}_{-\pi/4},\hat{\mathbf{u}}_{\pi/4}\}}$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1+j\\-1+j\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1-j\\-1-j\end{bmatrix}$
$\mathbf{g}_{\{\hat{\mathbf{u}}_{-\pi/4},\hat{\mathbf{u}}_{\pi/4}\}}$	$\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$

Table 1.4 – Polarization states expressed in the rotated linear polarization basis $\{\hat{\mathbf{u}}_{-\pi/4},\hat{\mathbf{u}}_{\pi/4}\}$, when *A*=1.

	Linear $\hat{\mathbf{x}}$	Linear $\hat{\mathbf{y}}$	Linear π/4	Linear 3π/4	Right hand circular	Left hand circular
ϕ	?	?	π/4	3π/4	0	π/2
τ	- π/4	π/4	0	0	0	0
$\underline{\mathbf{E}}_{\{\hat{\mathbf{u}}_{lc},\hat{\mathbf{u}}_{rc}\}}$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-j\end{bmatrix}$	$\frac{1}{\sqrt{2}}\begin{bmatrix}-j\\1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1-j\\1-j\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}-1-j\\1+j\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$
$\mathbf{g}_{\{\hat{\mathbf{u}}_{lc},\hat{\mathbf{u}}_{rc}\}}$	$\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$

Table 1.5 – Polarization states expressed in the circular polarization basis $\{\hat{\mathbf{u}}_{lc},\hat{\mathbf{u}}_{rc}\}$, when *A*=1.

1.2 Scattering Polarimetry

The previous section was concerned with the characterization and the representation of the polarization state of an electromagnetic wave. Despite this characterization is important when a radar system is considered, as it transmits and receives electromagnetic waves, nevertheless, the interest is on the scattering process itself. The radar system transmits an electromagnetic wave, with a given polarization state, that reaches the scatterer of interest. The energy of the incident wave interacts with the target, and as a result part of this energy is reradiated to the space. The way this energy is reradiated depends on the properties of the incident wave, as well as on the scatterer itself. Consequently, it is possible to infer some information of the scatterer under consideration considering the properties of the scattered electromagnetic wave with respect to the incident field, which is basically the transmitted field by the radar. One possibility that can be studied to characterize distant targets is to consider the change of the polarization state that a scatterer may induce to an incident wave.

In order to analyze the scattering problem from a general point of view, it is worth to start describing the scattering process that occurs when an incident field reaches a flat transition between two dielectric, infinite, lossless and homogeneous media in oblique incidence. This scattering situation is exemplified in **Erreur ! Source du renvoi introuvable.**. In this case, the incident wave that propagates in the first media reaches the transition between media where part of the incident energy in scattered in the same media, and part of the energy is transmitted to the second media. In order to characterize the scattering process, it is necessary to introduce the concept of *plane of scattering*, which is defined as the plane generated by the propagating vectors of the incident and the scattered waves. In the cases of backscattering and forward scattering this plane is undefined, so any of the infinite planes that are solution may be considered as plane of scattering.

In order to examine specifically reflections at oblique angles of incidence for a general wave polarization, it is convenient to decompose the electric field into their perpendicular and parallel components, relative to the plane of scattering. The total scattered and transmitted waves will be the vector sum from each of these two polarizations. When the wave is perpendicular to the plane of scattering, the polarization of the wave is referred to as perpendicular polarization or horizontal polarization as the electric field is parallel to the interface. When the electromagnetic wave is parallel to the plane of scattering, the polarization is referred to as parallel polarization or vertical polarization as the electromagnetic wave is also perpendicular to the interface. As indicated in **Erreur ! Source du renvoi introuvable.**, the total incident field $\underline{\vec{\mathbf{E}}}^i$ can be decomposed into two orthogonal components in the plane orthogonal to the incident propagation vector $\mathbf{k}^i$. These are the parallel $\underline{\vec{\mathbf{E}}}^i_{\parallel}$ and the perpendicular $\underline{\vec{\mathbf{E}}}^i_{\perp}$ components, which can be written as

$$\mathbf{E}_{\parallel}^{i} = E_{\parallel}^{i} e^{-j\mathbf{k}^{i}\cdot\mathbf{r}} \hat{\mathbf{x}}' \tag{1.47}$$

$$\vec{\underline{\mathbf{E}}}_{\perp}^{i} = E_{\perp}^{i} e^{-j\mathbf{k}^{i}\cdot\mathbf{r}} \hat{\mathbf{y}}' \tag{1.48}$$

As observed, the incident wave has been defined with respect to the coordinate system $\{\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'\}$ in such a way that $\mathbf{k}^{i} = \hat{\mathbf{z}}'$. It may be shown that the scattered wave components can be written similarly

$$\vec{\underline{\mathbf{E}}}_{\parallel}^{s} = E_{\parallel}^{s} e^{-j\mathbf{k}^{s}\cdot\mathbf{r}} \hat{\mathbf{x}}'' \tag{1.49}$$

$$\vec{\underline{\mathbf{E}}}_{\perp}^{s} = E_{\perp}^{s} e^{-j\mathbf{k}^{s}\cdot\mathbf{r}} \hat{\mathbf{y}}'' \tag{1.50}$$

but, in this case according to $\{\hat{\mathbf{x}}'', \hat{\mathbf{y}}'', \hat{\mathbf{z}}''\}$.

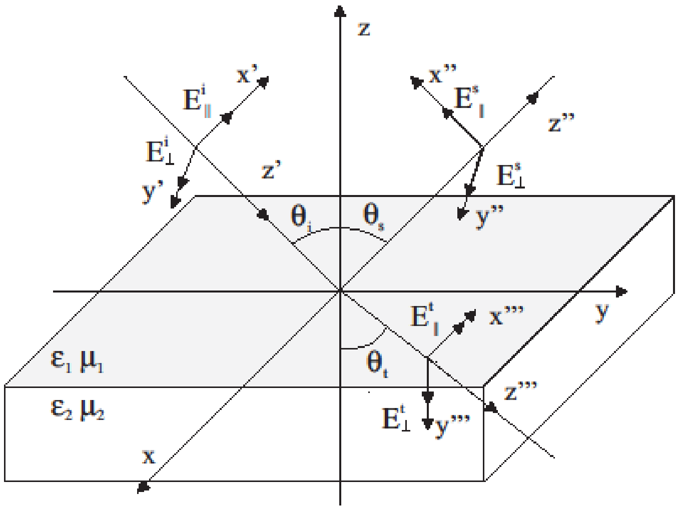


Figure 1.2 – Oblique incidence.

Considering the equations of the incident and the scattered wave, the question rising at this point is to determine whether it is possible or not to express mathematically the scattering process that occurs at the interface between both media. First of all, it is of crucial importance to take into consideration where, in the space, the expressions of the incident and scattered fields are valid. The expressions in **Erreur ! Source du renvoi introuvable.**, **Erreur ! Source du renvoi introuvable.**, (1.49) and (1.50) make reference to uniform plane waves. In the case of the incident wave on the scatterer, such a description for the wave, i.e., the wave originated at the transmitting antenna, is only valid if the scatter as is in the far field zone of the transmitting antenna. In the case of the scattered wave, this wave admits a uniform plane wave formulation if the point where the wave is considered is in the far field of the scatterer. In both cases, the waves in the far field zone may be considered spherical waves, which locally may be considered as a uniform plane waves. Considering a spherical coordinate system centered in the scatterer and under the previous assumptions the incident wave on the scatter can be expressed vectorially as :

$$\mathbf{E}^i = \begin{bmatrix} E_{\parallel}^i \\ E_{\perp}^i \end{bmatrix} \tag{1.51}$$

In the case of the scattered wave, if considered in far field zone of the scatterer, the scattered field admits a uniform plane description so it may be written as

$$\mathbf{E}^s = \begin{bmatrix} E_{\parallel}^s \\ E_{\perp}^s \end{bmatrix} \tag{1.52}$$

As observed, there are different points that need to be considered in the analysis of this problem. The first one is the use of different coordinate systems to characterize, in an unambiguous way, the polarization state of the different waves involved in the scattering process. The second aspect, coupled to the previous one, is to determine the way the scatterer under study changes the different components of the field. This section has studied this entire problem considering the analytical expressions of the fields. Nevertheless, Section 1.1.2 introduced the concept of Jones vector as a way to characterize an electric field, and its polarization state, vectorially. Consequently, it would be interesting to extend this vectorial expression also to the scattering problem.

1.2.1 The Scattering Matrix

Considering the rather simple scattering problem exposed in the previous section, this section will address the generalization of the scattering problem and its study and will introduce those concepts necessary to address it in a vectorial form. The first aspect that needs to be fixed, as observed previously, is to determine the different coordinate systems necessary to characterize the scattering problem and the description of the incident and the reflected waves. The choice of this framework will determine the description of the scattering problem itself as it will be shown in the following.

In the scattering problem, three coordinate systems must be chosen. The first one is the coordinate system located at the center of the scatter under consideration, and refereed in the following as $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$. This coordinate system may be considered as a kind of absolute or global coordinate system. In addition to it, it is necessary to define two additional local coordinates systems in order to determine, in an unambiguous way, the polarization states of the incident and the scattered or reflected waves, respectively. These two coordinate systems, associated with the waves, are defined in terms of the global coordinate system.

Lets us consider an object illuminated by an electromagnetic plane wave which may be described as

$$\vec{\underline{\mathbf{E}}}^i = E_x\hat{\mathbf{x}}' + E_y\hat{\mathbf{y}}' = E_x\hat{\mathbf{h}}_i + E_y\hat{\mathbf{v}}_i \tag{1.53}$$

where the unitary vectors $\hat{\mathbf{x}}'$ and $\hat{\mathbf{y}}'$ are arbitrarily defined. Hence, the propagation direction of the incident field is conveniently selected to be $\mathbf{k}^i = \hat{\mathbf{z}}'$. The incident wave reaches the object of interest and induces currents on it, which in turn reradiates a field. This reradiated field, as shown, is referred to as the scattered wave. In the far-field zone, the scattered wave is an outgoing spherical TEM wave that in the area occupied by the receiving antenna can be

$$\vec{\underline{\mathbf{E}}}^s = E_x\hat{\mathbf{x}}'' + E_y\hat{\mathbf{y}}'' = E_x\hat{\mathbf{h}}_s + E_y\hat{\mathbf{v}}_s \tag{1.54}$$

The propagation direction of the scattered wave is therefore $\mathbf{k}^s = \hat{\mathbf{z}}''$. The scattering process is finally analyzed in terms of the plane of scattering, which is the plane that contains both, the incident and the scattering propagating vectors. The concepts of perpendicular and parallel field components, or horizontal and vertical field components, are defined with respect to the plane of scattering. Consequently, and as indicated in (1.53), the perpendicular components of the field admits to be considered as a horizontal component, i.e., $\hat{\mathbf{x}}' = \hat{\mathbf{h}}_i$, whereas the parallel one admits to be considered a vertical one, i.e., $\hat{\mathbf{y}}' = \hat{\mathbf{v}}_i$. In the case of the scattered wave, the perpendicular component of the wave admits to be considered as a horizontal component, i.e., $\hat{\mathbf{x}}'' = \hat{\mathbf{h}}_s$, whereas the parallel one admits to be considered as a vertical one, i.e., $\hat{\mathbf{y}}'' = \hat{\mathbf{v}}_s$.

The incident and scattered fields in (1.53) and (1.54), respectively, may be also vectorially expressed by means of the Jones vectors

$$\underline{\mathbf{E}}^i = \begin{bmatrix} E_h^i \\ E_v^i \end{bmatrix} \tag{1.55}$$

$$\underline{\mathbf{E}}^s = \begin{bmatrix} E_h^s \\ E_v^s \end{bmatrix} \tag{1.56}$$

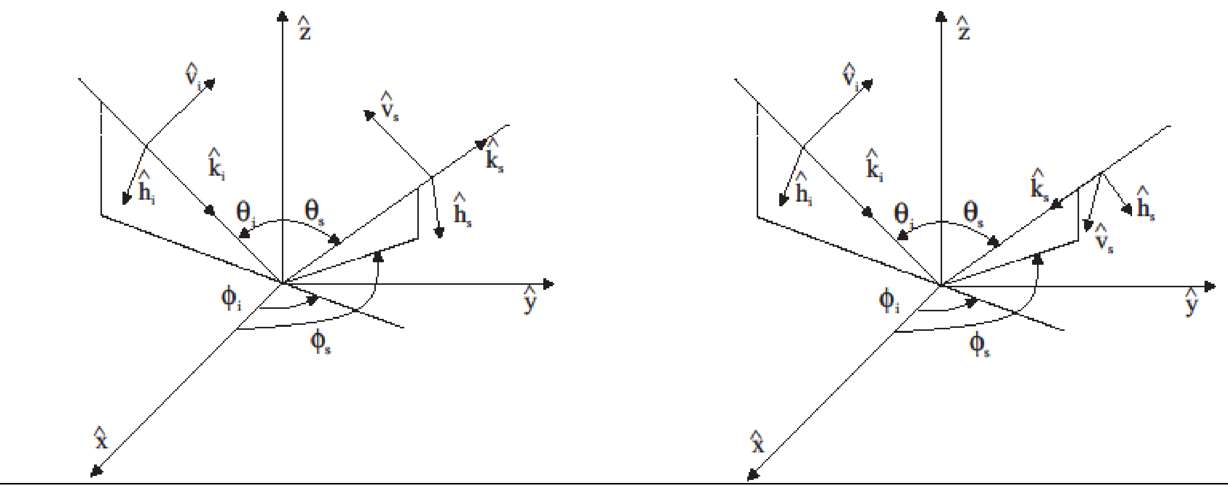
In the definition of the previous two Jones vectors the coordinate systems defined previously are assumed. By using this vector notation for the electromagnetic waves, it is possible to relate the scattered wave with the one of the incident wave by means of a 2 × 2 complex matrix

$$\underline{\mathbf{E}}^s = \frac{e^{-jkr}}{r} \mathbf{S} \underline{\mathbf{E}}^i \tag{1.57}$$

Here, r is the distance between the scatterer and the receiving antenna and k is the wavenumber of the illuminating wave. The coefficient r^{-1} represents the attenuation between the scatterer and the receiving antenna, which is produced by the spherical nature of the scattered wave. On the other hand, the phase factor represents the delay of the travel of the wave from the scatter to the antenna. Eq. (1.57) may be written as

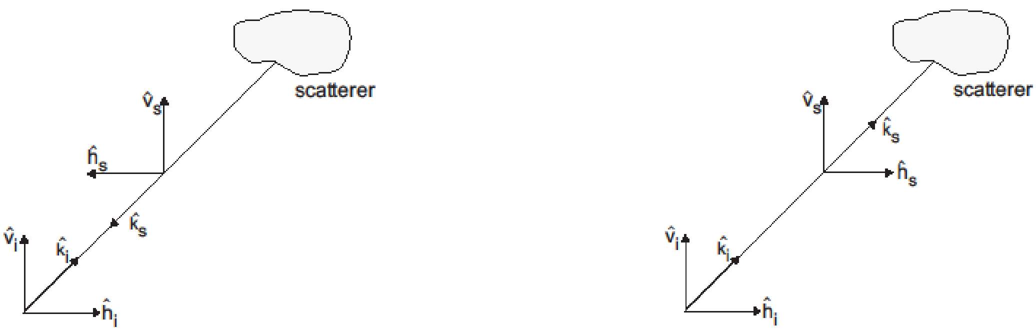
$$\begin{bmatrix} E_h^s \\ E_v^s \end{bmatrix} = \frac{e^{-jkr}}{r} \begin{bmatrix} S_{hh} & S_{hv} \\ S_{vh} & S_{vv} \end{bmatrix} \begin{bmatrix} E_h^i \\ E_v^i \end{bmatrix} \tag{1.58}$$

The matrix $\mathbf{S}$ is referred to as *scattering matrix* whereas its components area known as complex scattering amplitudes. The arrangement of the scattering matrix indicates how these complex scattering amplitudes are measured. The first column of $\mathbf{S}$ is measured by transmitting a horizontally polarized wave and employing two antennas horizontally and vertically polarized to record the scattered waves. The second column is measured in the same form, but transmitting a vertically polarized wave.



(a) (b)

Figure 1.3 – (a) FSA and (b) BSA conventions.



(a) (b)

Figure 1.4 – (a) FSA and (b) BSA conventions in the backscattering case.

It is worth to mention that the scattering matrix characterizes the target under observation for a fixed imaging geometry and frequency. In addition, the four elements must be measured at the same time, especially in those situations where the scatter is not static or fixed. If they are not measured at the same time, the coherency between the elements may be lost as the different elements may refer to a different scatterer. The best example of this situation appears when the ocean surface wants to be measured. At a given frequency, the surface of the ocean is characterized by a coherent time that indicates the amount of time in which the surface can be considered fixed. In this case, the scattering matrix elements should be measured in an interval lower that the coherence time of the ocean surface. Otherwise, the elements refer to a different scatterer.

As indicated, it is important to mention that the scattering matrix represents the scattering process for a particular incident and scattering directions, i.e., $\mathbf{k}^i$ and $\mathbf{k}^s$, respectively. In addition to that, it is also necessary to provide the horizontal and vertical unitary vectors, for the incident and the scattered waves, as they are necessary to define the polarization states of the waves.

In the most general case, which occurs in bistatic configurations where the transmitter and receiver antennas are located in different positions, the scattering matrix contains up to seven independent parameters to characterize the scatterer under observation. These parameters are the four amplitudes and three relative phases. Indeed, any absolute phase in the scattering matrix can be neglected as it does not affect the received power.

$$\begin{bmatrix} E_h^s \\ E_v^s \end{bmatrix} = \underbrace{\frac{e^{-jkr}e^{j\phi_{hh}}}{r}}_{\text{Absolute phase term}} \underbrace{\begin{bmatrix} |S_{hh}| & |S_{hv}|e^{j(\phi_{hv}-\phi_{hh})} \\ |S_{vh}|e^{j(\phi_{vh}-\phi_{hh})} & |S_{vv}|e^{j(\phi_{vv}-\phi_{hh})} \end{bmatrix}}_{\text{Relative scattering matrix}} \begin{bmatrix} E_h^i \\ E_v^i \end{bmatrix} \tag{1.59}$$

As it was already highlighted in the previous two sections of this chapter, the scattering coefficients depend on the direction of the incident and the scattered waves. When considering the matrix $\mathbf{S}$, the analysis of this dependence is of extreme importance since it also involves the definition of the polarization of the incident and the scattered fields. Since (1.58) considers the polarized electromagnetic waves themselves, it is mandatory to assume a frame in which the polarization is defined. There exist two principal conventions concerning the framework where the polarimetric scattering process is considered: *Forward Scatter Alignment* (FSA) and *Backscatter Alignment* (BSA), see Figure 1.3. In both cases, the electric fields of the incident and the scattered waves are expressed in local coordinates systems centred on the transmitting and receiving antennas, respectively. All coordinate systems are defined in terms if a global coordinate system centred inside the target of interest.

The FSA convention, see Figure 1.3, also called *wave-oriented* since it is defined relative to the propagating wave, is normally considered in bistatic problems, that it, in those configurations in which the transmitter and the receiver are not located at the same spatial position.

The bistatic BSA convention framework, see Figure 1.3, is defined, on the contrary, respect to the radar antennas in accordance with the IEEE standard. The advantage of the BSA convention is that for a monostatic configuration, also called backscattering configuration, that is, when the transmitting and receiving antennas are collocated, the coordinated systems of the two antennas coincide, see Figure 1.4. This configuration is preferred in the radar polarimetry community. In the monostatic case, the scattering matrix in the FSA convention, $\mathbf{S}_{FSA}$, can be related with the same matrix referenced to the monostatic BSA convention $\mathbf{S}_{BSA}$ as follows

$$\mathbf{S}_{BSA} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{S}_{FSA} \tag{1.60}$$

As it has been mentioned previously, in the radar polarimetry community, the monostatic BSA convention (backscattering) is considered as the framework to characterize the scattering process. The reason to select this configuration is due to fact that the majority of the existing polarimetric radar systems operate with the same antenna for transmission and reception. One important property of this configuration, for reciprocal targets, is *reciprocity*, which states that

$$S_{hv_{BSA}} = S_{vh_{BSA}} \tag{1.61}$$

$$S_{hv_{FSA}} = -S_{vh_{FSA}} \tag{1.62}$$

Then, the formalization of the scattering process given by (1.58), in the monostatic case under the BSA convention, reduces to

$$\begin{bmatrix} E_h^s \\ E_v^s \end{bmatrix} = \frac{e^{-jkr}}{r} \begin{bmatrix} S_{hh} & S_{hv} \\ S_{hv} & S_{vv} \end{bmatrix} \begin{bmatrix} E_h^i \\ E_v^i \end{bmatrix} \tag{1.63}$$

In the same sense, equation (1.59) takes the form

$$\begin{bmatrix} E_h^s \\ E_v^s \end{bmatrix} = \underbrace{\frac{e^{-jkr} e^{j\phi_{hh}}}{r}}_{\text{Absolute phase term}} \underbrace{\begin{bmatrix} |S_{hh}| & |S_{hv}| e^{j(\phi_{hv}-\phi_{hh})} \\ |S_{hv}| e^{j(\phi_{hv}-\phi_{hh})} & |S_{vv}| e^{j(\phi_{vv}-\phi_{hh})} \end{bmatrix}}_{\text{Relative scattering matrix}} \begin{bmatrix} E_h^i \\ E_v^i \end{bmatrix} \tag{1.64}$$

The main consequence of the previous equation is that in the backscattering direction, a given scatterer is no longer characterized by seven independent parameters, but by five. These are: the three amplitudes and the two relative phases and one additional absolute phase.

A central parameter when considering the scattering process occurring at a given scatterer consists of the scattered power. For single polarization systems, the scattered power is determined by means of the radar cross section or the scattering coefficient. Nevertheless, a polarimetric radar has to be considered as a multi channel system.

Consequently, in order to determine the scattered power, it is necessary to consider all the data channels, that is, all the elements of the scattering matrix. The total scattered power, in the case of a polarimetric radar system is known as *Span*, being defined in the most general case as

$$SPAN(\mathbf{S}) = \text{trace}\left(\mathbf{S}\mathbf{S}^{T*}\right) = |S_{hh}|^2 + |S_{hv}|^2 + |S_{hv}|^2 + |S_{vv}|^2 \tag{1.65}$$

where trace(.) represents the trace of a matrix. In the backscattering case, due to the reciprocity theorem, the Span reduces to

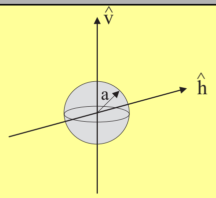
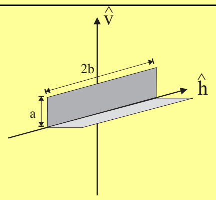
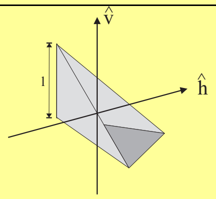
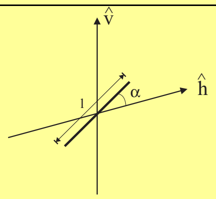
$$SPAN(\mathbf{S}) = |S_{hh}|^2 + 2|S_{hv}|^2 + |S_{vv}|^2 \tag{1.66}$$

The main property of the Span is that it is *polarimetrically invariable*, that is, it does not depend on the polarization basis employed to describe the polarization of the electromagnetic waves.

When the radar wave reaches a scatteter, part of the incident energy is reflected back to the system. If the incident wave is monochromatic, the target is unchanging and the radar-target aspect angle is constant, the scattered wave will be also monochromatic and completely polarized.

Therefore, both, the incident and the scattered waves can be characterized by their corresponding Jones vectors and the scattering process can be characterized by the scattering matrix. These targets are referred to as *point targets*, *single targets* or *deterministic targets*, as when a radar images this type of scatterers, the scattered wave in the far-field zone appears to be originated by a single point. In other words, the target response is not contaminated by additional spurious, so it is possible to infer some information about the target from the single values of the scattering matrix.

Table 1.6 shows the scattering matrix, expressed in the linear polarization basis, for some canonical bodies. These are referred as canonical due to the simplicity of its scattering matrix.

Canonocal body	Diagram	Scattering Matrix
Sphere		$\frac{a}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$
Dihedral Corner Reflector		$\frac{kab}{\pi}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Trihedral Corner Reflector		$\frac{kl^2}{\sqrt{12\pi}}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$
Short Thin Cylinder		$\frac{k^2l^3}{3\left(\ln\left(\frac{4l}{a}\right)-1\right)}\begin{bmatrix}\cos^2\alpha & \sin\alpha\cos\alpha\\ \sin\alpha\cos\alpha & \sin^2\alpha\end{bmatrix}$

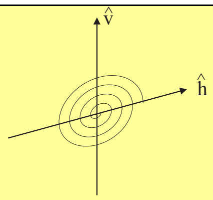
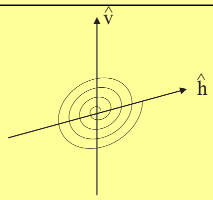
Left Handed Helix		$\frac{1}{2}\begin{bmatrix} 1 & j \\ j & -1 \end{bmatrix}$
Right Handed Helix		$\frac{1}{2}\begin{bmatrix} 1 & -j \\ -j & -1 \end{bmatrix}$

Table 1.6 – Scattering matrix for canonical bodies in the linear polarization basis $\{\hat{\mathbf{h}}, \hat{\mathbf{v}}\}$.

1.2.2 Scattering Polarimetry Descriptors

The scattering matrix introduced in the previous section is indeed a scattering polarimetry descriptor that could be also included in this section. Nevertheless, it merits a separate section as this matrix represents the best vehicle to introduce the description of the scattering process when polarimetry is concerned, as the scattering matrix relates the Jones vectors of the involved electromagnetic fields. Section 1.1.2 introduced additional descriptors for the polarization state of an electromagnetic field. As a consequence, some additional descriptors for the scattering process shall be introduced in the following.

The 2 × 2 complex scattering matrix, as indicated, describes the scattering matrix of a given target. Table 1.6 presented several examples for some simple or canonical scatterers. Nevertheless, a real target presents always a complex scattering response as a consequence of its complex geometrical structure and its reflectivity properties. Consequently, the interpretation of this response is obscure. As it shall be presented later on, a possible solution to interpret this response is to decompose the original scattering matrix into the response of canonical mechanisms. With this idea in mind, but also with the objective to introduce a new formulism to extract physical information, it is possible to transform the scattering matrix into a scattering vector that presents a clearer physical interpretation.

The construction of a target vector $\mathbf{k}$ is performed through the vectorization of the scattering matrix

$$\mathbf{k} = V(\mathbf{S}) = \frac{1}{2} trace(\mathbf{S\Psi}) \tag{1.67}$$

Where *trace*(·) is the matrix trace and $\mathbf{\Psi}$ is a set of 2 × 2 complex basis matrices which are constructed as an orthonormal set under an Hermitian inner product. The interpretation of the target vector $\mathbf{k}$ depends on the selected basis $\mathbf{\Psi}$. The most common matrix bases employed in the context of the radar polarimetry are the so called *lexicographic ordering* basis and the *Pauli basis*. The lexicographic ordering basis consists of the straightforward lexicographic ordering of the elements of the scattering matrix

$$\mathbf{\Psi}_l = \left\{ 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, 2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \tag{1.68}$$

The Pauli basis consists of the set of Pauli spin matrices usually employed in quantum mechanics

$$\mathbf{\Psi}_p = \left\{ \sqrt{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sqrt{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sqrt{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sqrt{2}\begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} \right\} \tag{1.69}$$

Note that the multiplying factor in both bases are necessary in order to keep the total scattered power constant, i.e., $trace\left(\mathbf{SS}^{*T}\right)$.

The selection of the basis to vectorize the scattering matrix depends on the final purpose of the vectorization itself. When the objective is to study the statistical behavior of the SAR data or the radar measurement it is more convenient to consider the lexicographic basis due to its simplicity, as it shall be extended in the next sections. Nevertheless, when the objective is the physical interpretation of the scattering matrix, it is more convenient to consider the Pauli basis. Assuming the Pauli decomposition basis, an arbitrary 2 × 2 scattering matrix may be written in the following terms

$$\mathbf{S}=\begin{bmatrix} a+b & c-jd \\ c+jd & a-b \end{bmatrix}=a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+b\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}+c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}+d\begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} \tag{1.70}$$

and as a consequence, the scattering vector is

$$\mathbf{k}=\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \tag{1.71}$$

It is worth to notice that the elements a, b, c and d are complex. If one considers the decomposition of the scattering matrix as performed in (1.70) , it is possible to identify the four elements of the Pauli basis with some of the scattering matrices of canonical bodies presented in Table 1.6. Therefore, the elements a, b, c and d, i.e., the elements of the target vector $\mathbf{k}$, represent the contribution of every canonical mechanism to the final scattering mechanism. Therefore, the following interpretation is possible:

- a corresponds to the single scattering from a sphere or plane surface.
- b corresponds to dihedral scattering.
- c corresponds to dihedral scattering with a relative orientation of $\pi/4$ rad in the line of sight.
- d corresponds to anti-symmetric helix-type scattering mechanisms that transform the incident wave into its orthogonal circular polarization state (helix related).

All in all, what has been performed in (1.70) is a *target decomposition*. This concept shall be analyzed in-depth in the next. It is also worth to notice that the different components of the Pauli basis, or scattering components, are orthogonal. This means that from a practical point of view, the separation indicated in (1.70) is possible without ambiguities.

Finally, the explicit expression of the target vector in the lexicographic and Pauli decomposition basis, considering the expression of the scattering matrix, in the most general case are

$$\mathbf{k}_l=\begin{bmatrix} S_{hh} \\ S_{hv} \\ S_{vh} \\ S_{vv} \end{bmatrix} \quad \mathbf{k}_p=\frac{1}{\sqrt{2}}\begin{bmatrix} S_{hh}+S_{vv} \\ S_{hh}-S_{vv} \\ S_{hv}+S_{vh} \\ j\left(S_{hv}-S_{vh}\right) \end{bmatrix} \tag{1.72}$$

In the backscattering case, under the BSA convention, the reciprocity property applies. Hence, the previous target vectors admit the following simplification

$$\mathbf{k}_l = \begin{bmatrix} S_{hh} \\ \sqrt{2}S_{hv} \\ S_{vv} \end{bmatrix} \quad \mathbf{k}_p = \frac{1}{\sqrt{2}} \begin{bmatrix} S_{hh}+S_{vv} \\ S_{hh}-S_{vv} \\ 2S_{hv} \end{bmatrix} \tag{1.73}$$

The different 2 and $\sqrt{2}$ factors that appear in the definition of the target vectors are necessary in order to maintain the total scattered power, also referred to as Span. In the most general case, the Span is equal to

$$SPAN(\mathbf{S}) = \operatorname{trace}\left(\mathbf{SS}^{T*}\right) = \left|S_{hh}\right|^2 + \left|S_{hv}\right|^2 + \left|S_{hv}\right|^2 + \left|S_{vv}\right|^2 \tag{1.74}$$

As it is evident, the Span must be constant and independent from the choice of the basis in which the scattering matrix is decomposed. This is known as total *power invariance*.

The concept of target vector, obtained as a vectorization of the scattering matrix makes it possible to obtain a new formulation to describe the information contained in the scattering matrix by mean of the outer product of the target vector with its conjugate transpose, or adjoin vector.

For a vectorization of the scattering matrix through the lexicographic basis, in the most general case, the outer product of the target vector with its transpose conjugate leads to the matrix

$$\mathbf{k}_l \mathbf{k}_l^{T*} \tag{1.75}$$

Introducing (1.72) into **Erreur ! Source du renvoi introuvable.** leads to

$$\mathbf{k}_l \mathbf{k}_l^{T*} = \begin{bmatrix} \left|S_{hh}\right|^2 & S_{hh}S_{hv}^* & S_{hh}S_{vh}^* & S_{hh}S_{vv}^* \\ S_{hv}S_{hh}^* & \left|S_{hv}\right|^2 & S_{hv}S_{vh}^* & S_{hv}S_{vv}^* \\ S_{vh}S_{hh}^* & S_{vh}S_{hv}^* & \left|S_{vh}\right|^2 & S_{vh}S_{vv}^* \\ S_{vv}S_{hh}^* & S_{vv}S_{hv}^* & S_{vv}S_{vh}^* & \left|S_{vv}\right|^2 \end{bmatrix} \tag{1.76}$$

Due to a language abuse, the matrix $\mathbf{k}_l \mathbf{k}_l^{T*}$ is sometimes referred to as covariance matrix and represented by $\mathbf{C}$, but as it will be shown later, the covariance matrix presents a different definition. It is worth to observe that **Erreur ! Source du renvoi introuvable.** is a 4 × 4, complex, Hermitian matrix. The construction of this matrix, through the outer product of the vector $\mathbf{k}_l$ and its transpose conjugate, makes the matrix $\mathbf{k}_l \mathbf{k}_l^{T*}$ to have a rank equal to 1. Consequently, $\mathbf{k}_l \mathbf{k}_l^{T*}$ presents exactly the same information as the scattering matrix, and hence it may have up to seven independent parameters. In case of the backscattering direction under the BSA convention, and due to the fact that the reciprocity relation applies, $\mathbf{k}_l \mathbf{k}_l^{T*}$ matrix can be written, considering (1.73), as

$$\mathbf{k}_l \mathbf{k}_l^{T*} = \begin{bmatrix} \left|S_{hh}\right|^2 & \sqrt{2}S_{hh}S_{hv}^* & S_{hh}S_{vv}^* \\ \sqrt{2}S_{hv}S_{hh}^* & \left|S_{hv}\right|^2 & \sqrt{2}S_{hv}S_{vv}^* \\ S_{vv}S_{hh}^* & \sqrt{2}S_{vv}S_{hv}^* & \left|S_{vv}\right|^2 \end{bmatrix} \tag{1.77}$$

Again, the factor $\sqrt{2}$ appears in order to maintain the Span. As in the previous case, the covariance matrix presents a rank equal to 1 as it is obtained as the outer product of a vector and its transpose conjugate. Nevertheless, in this case, the covariance matrix may present up to five independent parameters, that is, the same number of independent parameters as the scattering matrix from which it derives.

A similar procedure can be applied when the scattering matrix is obtained considering the Pauli basis. In this case, the matrix obtained from the outer product is

$$\mathbf{k}_p \mathbf{k}_p^{T*}. \tag{1.78}$$

Due to a language abuse, the matrix $\mathbf{k}_p \mathbf{k}_p^{T*}$ is sometimes referred to as coherency matrix and represented by $\mathbf{T}$, but as it will be shown later, the covariance matrix presents a different definition. Under the most general imaging configuration, considering (1.72), the coherency matrix can be written as

$$\mathbf{k}_p \mathbf{k}_p^{T*} = \begin{bmatrix} |S_{hh}+S_{vv}|^2 & (S_{hh}+S_{vv})(S_{hh}-S_{vv})^* & (S_{hh}+S_{vv})(S_{hv}+S_{vh})^* & (S_{hh}+S_{vv})(j(S_{hv}-S_{vh}))^* \\ (S_{hh}-S_{vv})(S_{hh}+S_{vv})^* & |S_{hh}-S_{vv}|^2 & (S_{hh}-S_{vv})(S_{hv}+S_{vh})^* & (S_{hh}-S_{vv})(j(S_{hv}-S_{vh}))^* \\ (S_{hv}+S_{vh})(S_{hh}+S_{vv})^* & (S_{hv}+S_{vh})(S_{hh}-S_{vv})^* & |S_{hv}+S_{vh}|^2 & (S_{hv}+S_{vh})(j(S_{hv}-S_{vh}))^* \\ j(S_{hv}-S_{vh})(S_{hh}+S_{vv})^* & j(S_{hv}-S_{vh})(S_{hh}-S_{vv})^* & j(S_{hv}-S_{vh})(S_{hv}+S_{vh})^* & |S_{hv}-S_{vh}|^2 \end{bmatrix} \tag{1.79}$$

As in the case of $\mathbf{k}_l \mathbf{k}_l^{T*}$, $\mathbf{k}_p \mathbf{k}_p^{T*}$ presents a rank equal to 1 and therefore, it may present up to seven independent parameters. Finally, if the backscattering direction is considered under the BSA convention, the coherency matrix reduces to

$$\mathbf{k}_p \mathbf{k}_p^{T*} = \begin{bmatrix} |S_{hh}+S_{vv}|^2 & (S_{hh}+S_{vv})(S_{hh}-S_{vv})^* & 2(S_{hh}+S_{vv})S_{hv}^* \\ (S_{hh}-S_{vv})(S_{hh}+S_{vv})^* & |S_{hh}-S_{vv}|^2 & 2(S_{hh}-S_{vv})S_{hv}^* \\ 2S_{hv}(S_{hh}+S_{vv})^* & 2S_{hv}(S_{hh}-S_{vv})^* & 4|S_{hv}|^2 \end{bmatrix} \tag{1.80}$$

Again, the previous matrix presents a rank equal to one and may have up to five independent parameters.

The lexicographic and the Pauli target vector are just a different transformation of the scattering matrix into a vector. Hence, the covariance and coherency matrices are related by the following unitary transformation in the most general configuration

$$\mathbf{k}_p \mathbf{k}_p^{T*} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & j & -j & 0 \end{bmatrix} \mathbf{k}_l \mathbf{k}_l^{T*} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -j \\ 0 & 0 & 1 & j \\ 1 & -1 & 0 & 0 \end{bmatrix} \tag{1.81}$$

In the case of the backscattering direction under the BSA convention, the previous transformation reduces to

$$\mathbf{k}_p \mathbf{k}_p^{T*} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \mathbf{k}_l \mathbf{k}_l^{T*} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -1 & 0 \end{bmatrix} \tag{1.82}$$

As it may be seen from all this section, the covariance and the coherency matrices contain the same information as the scattering matrix, that is, they are rank-1 matrices. The necessity to introduce these matrices is seen in the next sections. Basically, the covariance and coherency matrices are polarimetric descriptors able to characterize the scattering behaviour of distributed scatterers as these matrices govern the statistics that characterize the polarimetric SAR data.

The complex scattering matrix $\mathbf{S}$ is able to describe a single physical scattering process. As shown previously, the information of this single scattering process can be also represented by the equivalent covariance and coherency matrices. All these descriptors are based on a field representation of the data, which depend on the absolute phase from the target. On the contrary, a power representation of the scattering process eliminates this dependence, as power parameters become incoherently additive parameters. In the most general case, assuming the BSA convention, one may define the 4×4 Kennaugh matrix as follows

$$\mathbf{K} = \mathbf{A}^* \left(\mathbf{S} \otimes \mathbf{S}\right) \mathbf{A}^{-1} \tag{1.83}$$

where $\otimes$ denotes the Kroneker product and

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & j & -j & 0 \end{bmatrix} \tag{1.84}$$

The Kennaugh matrix can be written in the following form

$$\mathbf{K} = \begin{bmatrix} A_0 + B_0 & C_\psi & H_\psi & F_\psi \\ C_\psi & A_0 + B_\psi & E_\psi & G_\psi \\ H_\psi & E_\psi & A_0 - B_\psi & D_\psi \\ F_\psi & G_\psi & D_\psi & -A_0 + B_0 \end{bmatrix} \tag{1.85}$$

where

$$\begin{aligned}
&A_0 = \frac{1}{4}\left|S_{hh} + S_{vv}\right|^2 \\
&B_0 = \frac{1}{4}\left|S_{hh} - S_{vv}\right|^2 + \left|S_{hv}\right|^2 && B_\psi = \frac{1}{4}\left|S_{hh} - S_{vv}\right|^2 - \left|S_{hv}\right|^2 \\
&C_\psi = \frac{1}{2}\left|S_{hh} - S_{vv}\right|^2 && D_\psi = \Im\left\{S_{hh} S_{vv}^*\right\} \\
&E_\psi = \Re\left\{S_{hv}^* \left(S_{hh} - S_{vv}\right)\right\} && F_\psi = \Im\left\{S_{hv}^* \left(S_{hh} - S_{vv}\right)\right\} \\
&G_\psi = \Re\left\{S_{hv}^* \left(S_{hh} + S_{vv}\right)\right\} && H_\psi = \Im\left\{S_{hv}^* \left(S_{hh} + S_{vv}\right)\right\}
\end{aligned} \tag{1.86}$$

In the previous definition, the subindex ψ indicates that the different parameters a roll angle dependent, corresponding to the target rotation along the line-of-sight.

As detailed in Section 1.2.1, the scattering matrix related the scattered with the incident Jones vector. The Kenaugh matrix related the associated Stokes vectors defined in Section 1.1.2. In the forward scattering case, where $\mathbf{S}$ is represented in the FSA coordinate formulation, this matrix is named the 4×4 Mueller matrix and is calculated by

$$\mathbf{M} = \mathbf{A}\left(\mathbf{S}\otimes\mathbf{S}\right)\mathbf{A}^{-1} \tag{1.87}$$

The main difference of $\mathbf{K}$ and $\mathbf{M}$, with respect to $\mathbf{C}$ and $\mathbf{T}$ is that the the Kennaugh and the Mueller matrices are real matrices, whereas the Covariance and Coherency matrices are complex.

1.2.3 Partial Scattering Polarimetry

As indicated previously, radar polarimetry is concerned with two types of waves. The first type is monochromatic, totally polarized electromagnetic waves where the polarization state is perfectly represented by the Jones vectors. Consequently, the scattering process can be completely represented by any of the scattering polarimetry descriptors detailed in the previous section, and especially the scattering matrix. This situation appears when the radar transmits a perfectly monochromatic wave and this wave reaches an unchanging scatterer, resulting into a perfectly polarized scattered wave. As mentioned, these targets are referred to as point targets or coherent targets. The most important point to be considered when coherent scattering is addressed is to determine the number of independent parameters necessary to represent the scattering process. That is, to determine the number of independent parameters necessary to represent the operator able to characterize the change of the polarization state of the scattered wave with respect to the incident wave that occurs in the scattering process. In a monostatic configuration, the scattering operator describing the scattering, i.e., any of the matrix operators indicated in Sections 1.2.1 and 1.2.2, may present up to five independent parameters. In the bistatic case, these descriptors may present up to seven independent parameters.

The situation changes when the scattering properties of the target being imaged by the radar system change in time, as it would be the case for a forest being affected by the wind conditions or, for instance, when the target presents more than one scattering center (a point at which the incident wave can be considered to be reflected). Under this situation, despite the radar system transmits a perfectly polarized wave, the wave scattered by the target is partially polarized. A target of this category is normally referred to as *distributed target*, *depolarizing target* or to a *incoherent scattering target*. The change of the polarization state of the scattered wave makes not possible to use the scattering descriptors presented in Section 1.2.1 and 1.2.2 to describe the scattering process, as these descriptors are not able to describe the variation of the polarization state of the scattered wave.

In case of partially polarized waves, the description of the polarization state must be addressed through polarization descriptors relying on the second order moments of the electromagnetic field. If a field is decomposed into two orthogonal components in the plane perpendicular to the propagation direction, these second order moments refer to the power of each orthogonal component and to the correlation between them. This information is perfectly represented by the vector and the wave coherency matrix or the Stokes vector. In case of the description of the scattering process, this information can be perfectly represented by the covariance and coherency matrices as the mean values of these matrices are not zero.

1.2.4 Change of Polarization Basis

The scattering properties of a given scatterer, as demonstrated, are contained within the scattering matrix $\mathbf{S}$, which, as shown previously, is measured in a particular polarization basis. Since, there exist an infinite number of orthonormal polarization bases, the question rising at this point is that whether it is possible or not to infer the polarimetric properties of the given target in any polarization basis from the response measured at a particular basis. This question presents an affirmative answer. The possibility to synthesize any polarimetric response of a given target from its measurement in a particular orthonormal basis represents the most important property of polarimetric systems in comparison with single-polarization systems. The most important consequence of this process is that the amount of information about a given scatter can be increased, allowing a better characterization and study. This polarization synthesis process is based on the concept of change of polarization basis presented in Section 1.1.4.

Before to describe the polarization synthesis process in the backscattering direction, it is necessary to analyze the scattering process given by (1.58) with respect to the direction of propagation of the incident and the scattered waves.

It must be noticed that the incident wave propagates in the direction given by the unitary vector $\hat{\mathbf{k}}^i$, whereas the scattered one propagates in the opposite direction, given by $\hat{\mathbf{k}}^i$. Consequently, this difference in the propagation direction must be taken into account when defining the polarization state of the wave. Given a Jones vector propagating in the direction $\hat{\mathbf{k}}$, the Jones vector of a wave presenting the same polarization state, but which propagates in the direction $\hat{\mathbf{k}}$ is obtained as

$$\hat{\mathbf{k}} \to -\hat{\mathbf{k}} \quad \underline{\mathbf{E}}\left(\hat{\mathbf{k}}\right) = \underline{\mathbf{E}}^*\left(-\hat{\mathbf{k}}\right) \tag{1.88}$$

where as mentioned previously, the BSA convention is considered. Under this assumption, the scattering matrix is referred to the coordinate system centered in the transmitting/receiving system. Consider a polarimetric radar system, which transmits the electromagnetic waves in the following orthonormal basis $\{\hat{\mathbf{u}}, \hat{\mathbf{u}}_\perp\}$. In this particular basis, the incident and scattered fields are related by the scattering matrix as follows

$$\begin{bmatrix} E_u^s \\ E_{u\perp}^s \end{bmatrix} = \begin{bmatrix} S_{uu} & S_{uu\perp} \\ S_{u\perp u} & S_{u\perp u\perp} \end{bmatrix} \mathbf{U}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}}^{-1} \begin{bmatrix} E_u^i \\ E_{u\perp}^i \end{bmatrix} \tag{1.89}$$

$$\underline{\mathbf{E}}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}}^s = \mathbf{S}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}} \underline{\mathbf{E}}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}}^i \tag{1.90}$$

As shown in Section 1.1.4, given the Jones vector measured in a particular basis, for instance $\{\hat{\mathbf{u}}, \hat{\mathbf{u}}_\perp\}$, it is possible to derive it in any other polarization basis $\{\hat{\mathbf{u}}', \hat{\mathbf{u}}'_\perp\}$ by means of (1.46), which may be rewritten as follows

$$\begin{bmatrix} E_{u'} \\ E_{u'\perp} \end{bmatrix} = \mathbf{U}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}\to\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_\perp\}} \begin{bmatrix} E_u \\ E_{u\perp} \end{bmatrix} \tag{1.91}$$

or simply

$$\underline{\mathbf{E}}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_\perp\}} = \mathbf{U}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}\to\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_\perp\}} \underline{\mathbf{E}}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}} \tag{1.92}$$

Then, considering the incident and the scattered waves transformed in the new basis

$$\underline{\mathbf{E}}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_\perp\}}^i = \mathbf{U}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}\to\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_\perp\}} \underline{\mathbf{E}}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}}^i \tag{1.93}$$

$$\underline{\mathbf{E}}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_\perp\}}^s = \mathbf{U}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}\to\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_\perp\}} \underline{\mathbf{E}}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}}^s \tag{1.94}$$

In order to apply the transformation basis procedure to the scattered fields $\underline{\mathbf{E}}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}}^s$ we need to consider that it propagates in the opposite direction as the incident field $\underline{\mathbf{E}}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}}^i$. The transformation indicated by (1.90) assumes that the incident and the scattered fields propagate in opposite directions, but (1.93) and (1.94) assume that both fields propagate in the same direction. Consequently it is necessary to consider the transformation indicated by (1.88) in (1.94). As a result, the transformation basis procedure applies to the scattered wave as follows

$$\underline{\mathbf{E}}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_\perp\}}^s = \mathbf{U}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}\to\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_\perp\}}^* \underline{\mathbf{E}}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_\perp\}}^s \tag{1.95}$$

or

$$\underline{\mathbf{E}}^{s}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_{\perp}\}} = \mathbf{U}^{*}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_{\perp}\}\to\{\hat{\mathbf{u}},\hat{\mathbf{u}}_{\perp}\}}\underline{\mathbf{E}}^{s}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_{\perp}\}} \tag{1.96}$$

where now, the waves in (1.95) and (1.96) are assumed to propagate in opposite direction with respect to the incident field in (1.93). Now, it is possible to introduce (1.93) and (1.95) (or (1.96)) in (1.90)

$$\mathbf{U}^{*}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_{\perp}\}\to\{\hat{\mathbf{u}},\hat{\mathbf{u}}_{\perp}\}}\underline{\mathbf{E}}^{s}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_{\perp}\}} = \mathbf{S}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_{\perp}\}}\mathbf{U}^{*}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_{\perp}\}\to\{\hat{\mathbf{u}},\hat{\mathbf{u}}_{\perp}\}}\underline{\mathbf{E}}^{i}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_{\perp}\}} \tag{1.97}$$

As the transformation matrix $\mathbf{U}$ is unitary, i.e., $\mathbf{U}^{-1} = \mathbf{U}^{*T}$

$$\underline{\mathbf{E}}^{s}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_{\perp}\}} = \mathbf{U}^{T}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_{\perp}\}\to\{\hat{\mathbf{u}},\hat{\mathbf{u}}_{\perp}\}}\mathbf{S}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_{\perp}\}}\mathbf{U}^{*}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_{\perp}\}\to\{\hat{\mathbf{u}},\hat{\mathbf{u}}_{\perp}\}}\underline{\mathbf{E}}^{i}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_{\perp}\}} \tag{1.98}$$

from where it can be clearly identified the following identity

$$\mathbf{S}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_{\perp}\}} = \mathbf{U}^{T}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_{\perp}\}\to\{\hat{\mathbf{u}},\hat{\mathbf{u}}_{\perp}\}}\mathbf{S}_{\{\hat{\mathbf{u}},\hat{\mathbf{u}}_{\perp}\}}\mathbf{U}^{*}_{\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_{\perp}\}\to\{\hat{\mathbf{u}},\hat{\mathbf{u}}_{\perp}\}} \tag{1.99}$$

The transformation expressed in (1.99) receives the name of con-similarity transformation. This transformation allows to synthesize the scattering matrix in an arbitrary basis $\{\hat{\mathbf{u}}',\hat{\mathbf{u}}'_{\perp}\}$, from its measure in the basis $\{\hat{\mathbf{u}},\hat{\mathbf{u}}_{\perp}\}$.

1.2.5 Scatterers Characterization by Single, Dual, Compact and Full Polarimetry

The main objective behind the use of *polarimetric diversity*, also known as *full polarimetry*, when observing a particular target is that this type of diversity allows a far more complete characterization of the scatterer than the characterization that could be obtained without polarimetric sensitivity, or simply *single polarization* measurements. Although this improved characterization, if compared with single polarization data, the use of polarimetric diversity comes at a price, as the average transmitted power must be doubled and the swath width halved. In addition, a fully polarimetric SAR is technologically more complex that a single polarization SAR system. In order to understand the difference between these two philosophies and the improvement in the characterization of a scatterer provided by polarimetry, it is necessary to introduce two important concepts concerning the idea of scatterer, since they will determine the way in which they shall be characterized.

It may happen the scatterer of interest to be smaller than the coverage of the radar system. In this situation, we consider the scatterer as an isolated scatterer and from a point of view of power exchange this target is characterized by the so-called *radar cross section*. Nevertheless, we can find situations in which the scatterer of interest is significantly larger that the coverage provided by the radar system. In these occasions, it is more convenient to characterize the target independently of his extend. Hence, in these situations, the target is described by the so-called *scattering coefficient*.

The most fundamental form to describe the interaction of an electromagnetic wave with a given target is the so-called *radar equation*. This equation establishes the relation between the power which the target intercepts from the incident electromagnetic wave $\vec{\mathbf{E}}^{i}$ and the power reradiated by the same target in the form of the scattered wave $\vec{\mathbf{E}}^{s}$. The radar equation presents the following form

$$P_r = \frac{P_t G_t}{4\pi R_t^2}\sigma\frac{A_r}{4\pi R_r^2} \tag{1.100}$$

where P_r represents the power detected at the receiving system. The term

$$\frac{P_t G_t}{4\pi R_t^2} \tag{1.101}$$

is determined by the incident field $\vec{\mathbf{E}}^i$ and it consists of its power density expressed in terms of the properties of the transmitting system. The different terms in (1.101) are: the transmitted power P_t, the antenna gain G_t and the distance between the system and the target R_t . On the contrary, the term

$$\frac{A_r}{4\pi R_r^2} \tag{1.102}$$

contains the parameters concerning the receiving system: the effective aperture of the receiving antenna A_r and the distance between the target and the receiving system R_r .

The last term in (1.100), i.e., σ , determines the effects of the scatterer of interest on the balance of powers established by the radar equation. Since (1.101) is a power density, i.e., power par unit area and (1.102) is dimensionless, the parameter σ has units of area. Consequently, σ consists of an effective area which characterizes the scatterer. This parameter determines which amount of power is intercepted from the density (1.101) by the target and reradiated. This reradiated power is finally intercepted by the receiving system (1.102), according to the distance R_t . An important fact which arises at this point is the way the target reradiates the intercepted power in a given direction of the space. In order to be independent of this property, the radar cross section shall be referenced to and idealized isotropic scatterer. Thus, the radar cross section of an object is the cross section of an equivalent isotropic scatterer that generates the same scattered power density as the object in the observed direction

$$\sigma = 4\pi R^2 \frac{\left|\vec{\mathbf{E}}^s\right|^2}{\left|\vec{\mathbf{E}}^i\right|^2} = 4\pi \left|S\right|^2 \tag{1.103}$$

where $\left|\vec{\mathbf{E}}\right|^2$ represents the intensity of the electromagnetic field and S is the complex scattering amplitude of the object. The final value of σ is a function of a large number of parameters which are difficult to consider individually. A first set of these parameters are concerned with the imaging system:

- Wave frequency *f*.
- Wave polarization. This dependence is specially considered later on.
- Imaging configuration, that is, incident and scattering directions.

A second set of parameters are related with the target itself

- Object geometrical structure.
- Object dielectrical properties.

Then, the radar cross section σ is able to characterize the target being imaged for a particular frequency, and imaging system configuration.

The radar equation, as given by (1.100), is valid for those cases in which the target of interest is smaller than the radar coverage, that is, a *point target* or *point scatterer*. For those targets presenting an extend larger than the radar coverage, we need a different model to represent the target. In these situations, a scatterer is represented as an infinite collection of statistically identical point targets. The resulting scattered field $\vec{\mathbf{E}}^s$ results from the coherent addition of the scattered waves from every one of the independent targets which model the extended scatterer. In order to express the scattering properties of the extended target independently of its area extend, we considerer every elementary target as being described by a differential radar cross section $d\sigma$. In order to separate the effects of the target extend, we consider $d\sigma$ as the product of the averaged radar cross section per unit area σ^0 and the differential area occupied by the target ds. Then, the differential power received by the systems due to an elementary scatterer can be written as

$$dP_r = \frac{P_t G_t}{4\pi R_t^2}\sigma^0 ds \frac{A_r}{4\pi R_r^2} \tag{1.104}$$

Hence, to find the total power received from the extended target we need to integrate over the illuminated area A_o

$$P_r = \iint_{A_0} \frac{P_t G_t}{4\pi R_t^2}\sigma^0 \frac{A_r}{4\pi R_r^2} ds \tag{1.105}$$

It must be noted that the radar equation at (1.100) represents a deterministic problem, whereas (1.105) considers a statistical problem. Eq. (1.105) represents the average power returned from the extended target. Hence, the radar cross section per unit area σ^0, or simply *scattering coefficient*, is the ratio of the statistically averaged scattered power density to the average incident power density over the surface of the sphere of radius R_r

$$\sigma^0 = \frac{\langle \sigma \rangle}{A_0} = \frac{4\pi R_r^2}{A_0}\frac{\left\langle \left|\vec{E}^s\right|^2 \right\rangle}{\left|\vec{E}^i\right|^2} \tag{1.106}$$

The scattering coefficient σ^0 is a dimensionless parameter. As in the case of the radar cross section, the scattering coefficient is employed to characterize the scattered being imaged by the radar. This characterization is for a particular frequency *f*, polarization of the incident and scattered waves and incident and scattering directions.

As it has been shown, the characterization of a given scatterer by means of the radar cross section σ or the scattering coefficient σ^0 depends also on the polarization of the incident field $\vec{\mathbf{E}}^i$. As one can observe in (1.103) and (1.106), these two coefficients are expressed as a function of the intensity of the incident and scattered waves. Consequently, σ and σ^0 shall be only sensitive to the polarization of the incident waves through the effects the polarization has over the power of the related electromagnetic waves. Hence, if we denote by *p* the polarization of the incident field and by *q* the polarization of the scattered field, we can define the following polarization dependent radar cross section and scattering coefficient respectively

$$\sigma_{qp} = 4\pi R^2 \frac{\left|\vec{E}_{qp}^s\right|^2}{\left|\vec{E}_{qp}^i\right|^2} = 4\pi \left|S_{qp}\right|^2 \tag{1.107}$$

$$\sigma_{qp}^0 = \frac{\langle \sigma_{qp} \rangle}{A_0} = \frac{4\pi R_r^2}{A_0}\frac{\left\langle \left|\vec{E}_{qp}^s\right|^2 \right\rangle}{\left|\vec{E}_{qp}^i\right|^2} \tag{1.108}$$

As it has been shown, a given target of interest can be characterized by means of the radar cross section or the scattering coefficient depending on the nature of the scatterer itself, see (1.103) and (1.106). Additionally, in (1.107) and (1.108) it has been shown that these two coefficients depend also on the polarization of the incident and the scattered electromagnetic fields. A closer look to these expressions reveals that these two real coefficients depend on the polarization of the electromagnetic fields only through the power associated with them. Thus, they do not exploit, explicitly, the vectorial nature of polarized electromagnetic waves. A SAR system that measures σ or σ^0 is usually referred to as *single polarization* SAR systems as normally, the same polarization is employed for transmission and for reception. In this case, the product delivered by the SAR system are real SAR images containing the information of σ or σ^0.

In order to take advantage of the polarization of the electromagnetic fields, that is, their vectorial nature, the scattering process at the target of interest must be considered as a function of the electromagnetic fields themselves. In Section 1.1.2 , it was shown that the polarization of a plane, monochromatic, electric wave could be represented by the so-called Jones vector. Additionally, a set of two orthogonal Jones vectors form a polarization basis, in which, any polarization state of a given electromagnetic wave can be expressed. Therefore, given the Jones vectors of the incident and the scattered waves, $\underline{\mathbf{E}}^i$ and $\underline{\mathbf{E}}^s$ respectively, the scattering process occurring at the target of interest is represented by the scattering matrix $\mathbf{S}$. In contraposition to a *single polarization* SAR systems, a *fully polarimetric* SAR system measure the complete scattering matrix $\mathbf{S}$. Therefore, the product delivered by this type of SAR systems corresponds to the 2x2 complex scattering matrix and not individual real SAR images.

As it can be observed, the polarimetric sensitivity of a measurement ranges from a complete absence of polarimetric sensitivity in the case of *single polarization* SAR systems to a complete sensitivity in the case of a *fully polarimetric* SAR systems. Polarimetric sensitivity comes to a price of a more complex system that implies, on the one a hand, a heavier system and, on the other hand, the need to transmit a larger power. In addition, and due to the need to double the pulse repetition frequency to accommodate two polarizations in transmission, the radar swath is also reduced. Nevertheless, between both architectures, there exist other polarimetric radar configurations with may soften the previous limitations but at the cost to reduce the amount of acquired information.

A *single polarization* or *mono polarization* SAR system is composed by one transmission and one reception chain that operate at a fixed polarization. In most of the cases, both chains operate at the same polarization providing a *co-pol* or *co-polarized* channel. In the particular case of the linear polarization basis, these channels world correspond to σ_{hh} or σ^0_{hh} and σ_{vv} or σ^0_{vv} for the horizontal and the vertical polarization states, respectively. As indicated, these simple imaging radars deliver real SAR images, proportional to σ or σ^0, as products. One possibility to increase the amount of information is to consider a dual-polarized radar by including a second reception chain in the system, in such a way that it transmits in one polarization, for instance h, and it receives simultaneously on the same polarization h and also on the orthogonal one v, leading to one *co-pol* and the so-called *co-polarized* and the *cross-polarized* channels, respectively. A different alternative for a *dual polarized* system is to consider a transmission chain that alternates between polarizations and a single reception chain. In all these cases, the polarimetric system provides images proportional to the radar brightness.

All the previous SAR systems present the limitation that the information that may be retrieved is restricted to the information that can be extracted from the real SAR images, proportional to σ or σ^0, or their different combinations. Nevertheless, this limitation is overcome by allowing two simultaneous and coherent reception channels operating at orthogonal polarizations, making it possible to measure the relative phase between them. The coherent nature of the receiving channels allows measuring the different elements of the covariance or coherency matrix. The first option that may be considered is to assume a fixed polarization in transmission and orthogonal polarizations in reception. In the case of the transmission channel, the circular polarization and the 45° linear polarizations have been proposed, whereas for reception the horizontal and vertical linear polarizations are assumed.

This type of systems are collectively known as *compact polarized* systems as, despite they allow to measure some of the elements of the covariance and coherency matrix, they do not allow to measure the complete matrices. Finally, by allowing the system to transmit alternatively between orthogonal polarizations and to receive coherently at the same two orthogonal polarizations, a system like this is able to measure coherently the scattering matrix and to produce the covariance and coherency matrices. In the case of a bistatic configuration, without any type of assumption, these will be 4x4 complex matrices, whereas in the case of a monostatic configuration, these will be 3x3 complex matrices.

Erreur ! Source du renvoi introuvable. details the complete hierarchy of polarimetric SAR systems.

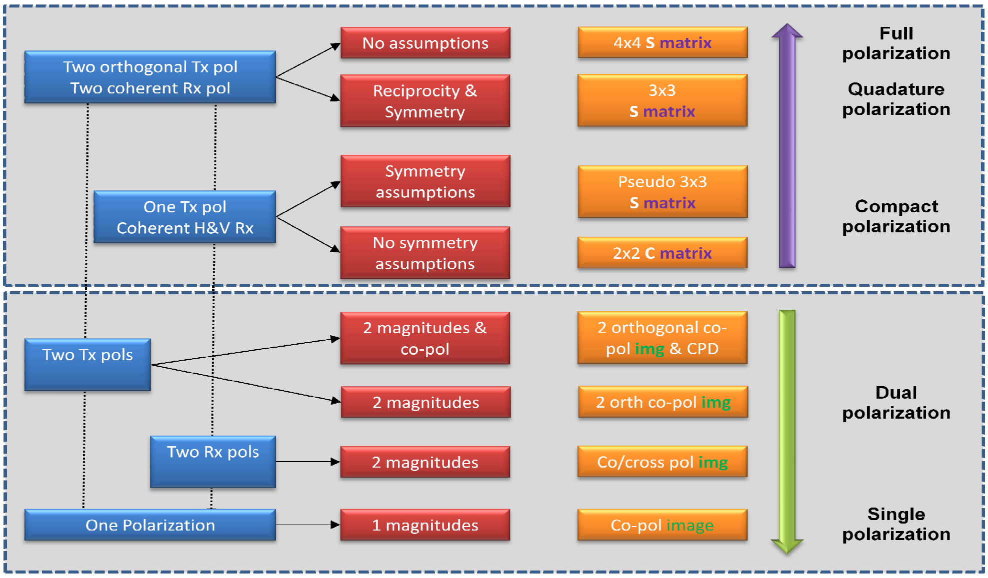


Figure 1.5 – The family of polarization diversity and polarimetric imaging radars. Courtesy of Dr. R. K. Raney.