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Some Characterizations of the Multivariate t Distribution*

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A multivariate t vector $\mathbf{X}$ is represented in two different forms, one associated with a normal vector and an independent chi-squared variable, and the other with a normal vector and an independent Wishart matrix. We show that $\mathbf{X}$ is multivariate t with mean $\boldsymbol{\mu}$, covariance matrix $\nu(\nu-2)^{-1}\Sigma$, $\nu > 2$ and degrees of freedom ν if and only if for any $\mathbf{a} \neq \mathbf{0}$, $(\mathbf{a}'\Sigma\mathbf{a})^{-1/2}\,\mathbf{a}'(\mathbf{X}-\boldsymbol{\mu})$ has the Student's t distribution with ν degrees of freedom under both representations. Some other characterizations are also obtained.

1. INTRODUCTION AND SUMMARY

The multivariate t distribution was first derived independently by Cornish [4] and Dunnett and Sobel [8]. It arises in multiple decision problems concerned with the selection and ranking of population means of several normal populations having a common unknown variance (see Bechhofer, Dunnett and Sobel [3]). It can be used in setting up simultaneous confidence bounds for the means of correlated normal variables, for parameters in a linear model and for future observations from a multivariate normal distribution (see John [11]). The multivariate t distribution also appears in the Bayesian multivariate analysis of variance and regression, treated by Tiao and Zellner [16], Geisser and Cornfield [9], Raiffa and Schlaifer [13], and Ando and Kaufmann [2], where the normal-Wishart distribution is considered to be the conjugate prior distribution (in the sense of Raiffa and Schlaifer) of the mean vector and covariance matrix

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of a multivariate normal distribution. However, the representation of the multivariate t vector used in the Bayesian multivariate analysis is somewhat different from that employed by Cornish [4], and Dunnett and Sobel [8].

The multivariate t can be considered as a generalization of the univariate t. Siddiqui [14] generalizes the univariate t to a bivariate t in a different direction and Dickey [7] generalizes the multivariate t to a matricvariate t. The probability integrals and the percentage points of a multivariate t have been studied by Gupta [10] and Krishnaiah and Armitage [12].

In Section 2 of this note we consider the distribution of a multivariate t vector which may be represented in two different forms, one associated with a normal vector and an independent chi-squared variable, and the other with a normal vector and an independent Wishart matrix. In Section 3, the characterization properties of a t vector are developed through multivariate normal theory.

As usual, we denote by $N(\boldsymbol{\mu}, \Sigma)$ a multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix Σ; by $W(\Sigma, n)$ a Wishart distribution with parameters n and Σ; by χ_ν^2 a chi-squared distribution with ν degrees of freedom; and by $F'_{p,\nu}(\tau^2)$ a noncentral F distribution with p and ν degrees of freedom and noncentrality τ^2. We write $F_{p,\nu}$ for $F'_{p,\nu}(0)$.

2. The Multivariate t Distribution

A p-variate random vector $\mathbf{X} = (X_1, ..., X_p)'$ is said to have a (nonsingular) multivariate t distribution with mean vector $\boldsymbol{\mu} = (\mu_1, ..., \mu_p)'$ and covariance matrix $\nu(\nu - 2)^{-1}\Sigma$, $\nu > 2$, denoted by $T_\nu(\boldsymbol{\mu}, \Sigma, p)$, if it has the probability density function (pdf) given by

$$f(\mathbf{x}) = \frac{\Gamma[\frac{1}{2}(\nu + p)]}{(\pi\nu)^{\frac{1}{2}p}\Gamma(\frac{1}{2}\nu) \mid \Sigma \mid^{\frac{1}{2}}} \left\{1 + \frac{(\mathbf{x} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})}{\nu}\right\}^{-\frac{1}{2}(\nu+p)}, \quad \nu > 2. \tag{1}$$

It is noted that $T_\nu(0, 1, 1) = t_\nu$ is the Student's t distribution with ν degrees of freedom.

Unless otherwise mentioned, it is assumed that $\mathbf{X}$, $\boldsymbol{\mu}$, $\mathbf{a}$, and $\mathbf{0}$ are $p \times 1$ vectors, V and Σ are $p \times p$ positive definite symmetric matrices, and I is the identity matrix of order p.

For the purpose of establishing the characterizations and the distribution theory in Section 3 it is convenient to represent a multivariate t vector in the following forms:

Representation A. Let $\mathbf{X} \sim T_\nu(\boldsymbol{\mu}, \Sigma, p)$. Then $\mathbf{X}$ may be written as

$$\mathbf{X} = S^{-1}\mathbf{Y} + \boldsymbol{\mu}, \tag{2}$$

where $\mathbf{Y} \sim N(\mathbf{0}, \Sigma)$ and $\nu S^2 \sim \chi_\nu^2$, independent of $\mathbf{Y}$. This implies that $\mathbf{X} \mid S^2 = s^2 \sim N(\boldsymbol{\mu}, s^{-2}\Sigma)$, where $\nu S^2 \sim \chi_\nu^2$. The Student's t random variable is then represented as $X \mid S^2 = s^2 \sim N(0, s^{-2})$, where $\nu S^2 \sim \chi_\nu^2$.

Representation B. Let $\mathbf{X} \sim T_\nu(\boldsymbol{\mu}, \Sigma, p)$. Then $\mathbf{X}$ may be written as

$$\mathbf{X} = (V^{1/2})^{-1}\mathbf{Y} + \boldsymbol{\mu}, \tag{3}$$

where $V^{1/2}$ is the symmetric square root of V, i.e.,

$$V^{1/2}V^{1/2} = V \sim W(\Sigma^{-1}, \nu + p - 1)$$

and $\mathbf{Y} \sim N(\mathbf{0}, \nu I)$, independent of V. This implies that $\mathbf{X} \mid V \sim N(\boldsymbol{\mu}, \nu V^{-1})$, where $V \sim W(\Sigma^{-1}, \nu + p - 1)$. The Student's t random variable is then represented as $X \mid V \sim N(0, \nu \mathbf{a}'V^{-1}\mathbf{a}/\mathbf{a}'\Sigma\mathbf{a})$ for any $\mathbf{a} \neq \mathbf{0}$, where

$$V \sim W(\Sigma^{-1}, \nu + p - 1).$$

It can be shown that the pdf of $\mathbf{X}$, under both Representations A and B, is given by (1).

3. Characterizations of the Multivariate t

Cornish [6] gives a characterization of the multivariate t using Representation A. His proof is lengthy and complicated. We present here the same result under Representation B and also give a simpler proof under Representation A, utilizing multivariate normal distribution theory. Two other characterizations are also obtained under Representation A.

THEOREM 1. $\mathbf{X} \sim T_\nu(\boldsymbol{\mu}, \Sigma, p)$ *if and only if, for any* $\mathbf{a} \neq \mathbf{0}$,

$$(\mathbf{a}'\Sigma\mathbf{a})^{-1/2}\mathbf{a}'(\mathbf{X} - \boldsymbol{\mu}) \sim t_\nu .$$

Proof. (i) Assume Representation A.

$$\mathbf{X} \mid S^2 = s^2 \sim N(\boldsymbol{\mu}, s^{-2}\Sigma),$$
$$\Leftrightarrow (\mathbf{a}'\Sigma\mathbf{a})^{-1/2}\mathbf{a}'(\mathbf{X} - \boldsymbol{\mu}) \mid S^2 = s^2 \sim N(0, s^{-2}), \text{ for any } \mathbf{a} \neq \mathbf{0}$$
$$\Leftrightarrow (\mathbf{a}'\Sigma\mathbf{a})^{-1/2}\mathbf{a}'(\mathbf{X} - \boldsymbol{\mu}) \sim t_\nu .$$

(ii) Assume Representation B.

$$\mathbf{X} \mid V \sim N(\boldsymbol{\mu}, \nu V^{-1})$$
$$\Leftrightarrow (\mathbf{a}'\Sigma\mathbf{a})^{-1/2}\mathbf{a}'(\mathbf{X} - \boldsymbol{\mu}) \mid V \sim N[0, \nu(\mathbf{a}'V^{-1}\mathbf{a})/(\mathbf{a}'\Sigma\mathbf{a})], \text{ for any } \mathbf{a} \neq \mathbf{0}$$
$$\Leftrightarrow (\mathbf{a}'\Sigma\mathbf{a})^{-1/2}\mathbf{a}'(\mathbf{X} - \boldsymbol{\mu}) \sim t_\nu ,$$

since $\mathbf{a}'\Sigma\mathbf{a}/\mathbf{a}'V^{-1}\mathbf{a} \sim \chi_\nu^2$. (See, for example, Stein [15, pp. 32–33].) This completes the proof of the theorem.

In order to establish some further characterizations, we need the following lemmas. The proofs are straightforward and are omitted.

LEMMA 1. *Let* $\mathbf{X} \sim T_\nu(\boldsymbol{\mu}, \Sigma, p)$, *then for any nonsingular* $p \times p$ *scalar matrix* C *and any* $\mathbf{a}$, $C\mathbf{X} + \mathbf{a} \sim T_\nu(C\boldsymbol{\mu} + \mathbf{a}, C\Sigma C', p)$.

LEMMA 2. *Let* $\mathbf{X} \sim T_\nu(\boldsymbol{\mu}, \Sigma, p)$, *then* $\mathbf{X}'\Sigma^{-1}\mathbf{X}/p \sim F'_{p,\nu}(\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}/p)$. *When* $\boldsymbol{\mu} = \mathbf{0}$, *the distribution is central* F.

LEMMA 3. *Let* Y *be a symmetrically distributed random variable. Then* $Y^2 \sim \chi_1^2$ *if, and only if* $Y \sim N(0, 1)$.

LEMMA 4. *Let* $Y_1, \ldots, Y_p$ *be independent random variables with finite variances. Assume that* Y_k, $(k = 1, \ldots, p)$, *has* pdf $h_k(y)$ *such that* $h_k(y) > 0$ *and differentiable for all* y, $-\infty < y < \infty$. *Then the joint* pdf *of* $Y_1, \ldots, Y_p$ *is a function only of* $y_1^2 + \cdots + y_p^2$ *if, and only if* $(Y_1, \ldots, Y_p)' \sim N(\mathbf{0}, \sigma^2 I)$, *for some* $\sigma^2 < \infty$.

For the remainder of this section, we will consider a multivariate t vector using Representation A.

THEOREM 2. *Let* $\nu S^2 \sim \chi_\nu^2$ *and let* $X_1, \ldots, X_p$ *be continuous random variables. Assume that* $X_1, \ldots, X_p$ *are conditionally independent, and that* X_k *is symmetrically distributed with mean* μ_k *and finite variance, given* $S^2 = s^2$. *Then*

$$\sum_{k=1}^{p} (X_k - \mu_k)^2/(p\sigma_k^2) \sim F_{p,\nu}, \tag{4}$$

if and only if $(X_1, \ldots, X_p)' \sim T_\nu(\boldsymbol{\mu}, D, p)$, *where* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)'$ *and* D *is a* $p \times p$ *diagonal matrix with its k-th diagonal element equal to* σ_k^2.

Proof. The sufficiency is a special case of Lemma 2. We shall show the necessity. Let $Y_k = (X_k - \mu_k)/\sigma_k$, $k = 1, \ldots, p$. Then the Y's also satisfy the conditions of the theorem with mean 0 and finite variance, given $S^2 = s^2$. It is well known that an $F_{p,\nu}$-distributed random variable may be represented as the ratio of two independent chi-squared random variables with p and ν degrees of freedom, repectively, multiplied by a suitable constant. Thus, letting $S_1^2 \sim \chi_p^2$ independent of S^2, we have

$$\sum_{k=1}^{p} Y_k^2 \sim S_1^2/S^2, \tag{5}$$

or

$$\sum_{k=1}^{p} (SY_k)^2 \mid S^2 \sim S_1^2 \mid S^2 \sim \chi_p^2, \tag{6}$$

which is independent of S^2. This implies that

$$(SY_k)^2 \mid S^2 \sim \chi_1^2, \qquad k = 1,..., p, \tag{7}$$

and by Lemma 3, we have $S\mathbf{Y} \mid S^2 = s^2 \sim N(\mathbf{0}, I)$, where $\mathbf{Y} = (Y_1 ,..., Y_p)'$. Thus, by Representation A, $\mathbf{Y} \sim T_\nu(\mathbf{0}, I, p)$ and by Lemma 1, $\mathbf{X} \sim T_\nu(\boldsymbol{\mu}, D, p)$.

THEOREM 3. *Let $\nu S^2 \sim \chi_\nu^2$ and let X_k, $k = 1,..., p$, be conditionally independent random variables, given $S^2 = s^2$, with* $\mathrm{Var}(X_k \mid s^2) = \sigma^2/s^2$, $\sigma^2 < \infty$. *Assume that X_k ($k = 1,..., p$), given $S^2 = s^2$, has conditional* pdf $f_k(x \mid s^2)$ *which is positive and differentiable for all x, $-\infty < x < \infty$. Then the joint* pdf *of $X_1 ,..., X_p$ is a function only of $x_1^2 + \cdots + x_p^2$ if, and only if $(X_1 ,..., X_p)' \sim T_\nu(\mathbf{0}, \sigma^2 I, p)$.*

Proof. The sufficiency is easily seen from the joint pdf of $X_1 ,..., X_p$. We shall prove the necessity. Since the joint pdf of $X_1 ,..., X_p$ is a function of $x_1^2 + \cdots + x_p^2$, it follows that the conditional joint pdf of $X_1 ,..., X_p$, given $S^2 = s^2$, is a function of $x_1^2 + \cdots + x_p^2$ and s^2 almost surely with respect to a σ-finite measure m, i.e.,

$$\prod_{k=1}^{p} f_k(x_k \mid s^2) = g_0\left(\sum_{k=1}^{p} x_k^2, s^2\right), \qquad \text{a.s. } (m). \tag{8}$$

Similarly, the conditional marginal pdf of X_k ($k = 1,..., p$), given $S^2 = s^2$, is

$$f_k(x_k \mid s^2) = g_k(x_k^2, s^2), \qquad \text{a.s. } (m), \tag{9}$$

where $g_k(t, s^2)$ ($k = 0,..., p$) is a positive and differentiable function of $t \geqslant 0$. Now, the result of Lemma 4 and the fact that $\mathrm{Var}(X_k \mid s^2) = \sigma^2/s^2$, for all $k = 1,..., p$, show that $(X_1 ,..., X_p)' \mid S^2 = s^2 \sim N(\mathbf{0}, s^{-2}\sigma^2 I)$. Therefore, $(X_1 ,..., X_p)' \sim T_\nu(\mathbf{0}, \sigma^2 I, p)$.

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