# Xsuite physics manual

G. Iadarola, R. De Maria,
M. Fitterer, M. Fjellstrom, P. Hermes
CERN - Geneva, Switzerland

# Contents

# Chapter 1

# Xtrack

XTrack is a 6D single particle symplectic tracking code used to compute the trajectories of individual relativistic charged particles in circular accelerators. It has been developed based on SixTrack.
The physical models are collected from the main references [1, 2, 3, 4, 5, 6, 7], which contain more details of the derivation of the maps.

## 1.1 Notation and reference frame

The speed, momentum, energy, rest mass, charge of a particle are indicated by $v$, $P$, $E$, $m$ and $q$, respectively. These quantities are related by the following equations:

$$v = \beta c \qquad E^2 - P^2c^2 = m^2c^4 \qquad E = \gamma mc^2 \qquad Pc = \beta E \tag{1.1}$$

where $\beta$ and $\gamma$ are the relativistic factors.
In a curvilinear reference frame defined by a constant curvature $h_x$ in the $\hat{X}, \hat{Z}$ plane and parameterized by $s$, the position of the particle at a time $t$ can be written as:

$$\mathbf{Q}(t) = \mathbf{r}(s) + x\,\hat{x}(s) + y\,\hat{y}(s), \tag{1.2}$$

and therefore identified by the coordinates $s, x, y, t$ in the reference frame defined by $\hat{x}(s)$ and $\hat{y}(s)$. In particle tracking, $s$ is normally used as independent parameter and $t$ as a coordinate.
The electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be derived in a curvilinear reference frame from the potentials $V(x, y, s, t)$ and $\mathbf{A}(x, y, s, t)$, where

$$\mathbf{A}(x, y, s, t) = A_x(x, y, s, t)\hat{x}(s) + A_y(x, y, s, t)\hat{y}(s) + A_s(x, y, s, t)\hat{z}(s) \tag{1.3}$$

and for which:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\partial_x V\hat{x} - \partial_y V\hat{y} - \frac{1}{1+hx}\partial_s V\hat{z} - \partial_t \mathbf{A} \tag{1.4}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \left(\partial_y A_s - \frac{\partial_s A_y}{1+hx}\right)\hat{x} + \left(\frac{\partial_s A_x - \partial_x(1+hx)A_s}{1+hx}\right)\hat{y} \tag{1.5}$$

$$+ \quad \left(\partial_x A_y - \partial_y A_x\right)\hat{z}. \tag{1.6}$$

In this reference frame the canonical momenta are:

$$P_x = m\gamma\dot{x} + qA_x, \quad P_y = m\gamma\dot{y} + qA_y, \quad P_s = m\gamma\dot{s}(1+hx)^2 + q(1+hx)A_s. \tag{1.7}$$

and the energy of a particle and the field is

$$E = qV + c\sqrt{(mc)^2 + \frac{(P_s - qA_s(1+hx))^2}{(1+hx)^2} + (P_x - qA_x)^2 + (P_x - qA_x)^2}. \tag{1.8}$$

## 1.2 Hamiltonian and particle coordinates

If $s(t)$ is monotonically increasing, it is possible to derive the equations of motion using $s$ as the independent parameter, $(-t, E)$ as conjugate coordinates and $P_s$ as Hamiltonian.

$$P_s = (1+hx)\left(\sqrt{E^2 - (mc^2)^2 - (P_x - qA_x)^2 - (P_y - qA_y)^2} + qA_s\right) \tag{1.9}$$

Since in accelerators the orbits of the particles are often a perturbation of the reference trajectory followed by a particle with rest mass $m_0$, charge $q_0$, speed $\beta_0 c$ and momentum $P_0$, one could use the following derived quantities that usually assume small values:

$$p(x,y) = \frac{m_0}{m}\frac{P(x,y)}{P_0} \qquad \chi = \frac{q}{q_0}\frac{m_0}{m} \qquad a(x,y,s) = \frac{q_0}{P_0}A(x,y,s) \tag{1.10}$$

$$\delta = \frac{P\frac{m_0}{m} - P_0}{P_0} \qquad p_t = \frac{E\frac{m_0}{m} - E_0}{P_0 c} \qquad p_\sigma = \frac{E\frac{m_0}{m} - E_0}{\beta_0 P_0 c} \tag{1.11}$$

$$\zeta = s\frac{\beta}{\beta_0} - \beta ct \qquad \tau = \frac{s}{\beta_0} - ct \qquad \sigma = s - \beta_0 ct \tag{1.12}$$

The pairs $(\zeta, \delta)$, $(\tau, p_\tau)$, $(\sigma, p_\sigma)$ are canonical conjugate and can be generated by the following generating functions [1]

$$F_2 = xp_x + yp_y + \left(\frac{s}{\beta_0} - ct\right)\left(p_\tau + \frac{1}{\beta_0}\right) \tag{1.13}$$

$$F_2 = xp_x + yp_y + (s - ct)\left(\beta_0 p_\sigma + \frac{1}{\beta_0}\right) \tag{1.14}$$

$$F_2 = xp_x + yp_y + (-ct)\left(\sqrt{(1+\delta)^2 + m^2c^4}\right) \tag{1.15}$$

$$\tag{1.16}$$

The Hamiltonian is:

[1] $F_2(-t, p_{\text{new}}, s)$, $E/(P_0 c) = \frac{\partial F_2}{\partial(-t)}$, $q_{\text{new}} = \frac{\partial F_2}{\partial p_{\text{new}}}$, $H_{\text{new}} = H + \frac{\partial F_2}{\partial s}$,

$$H_\delta = \frac{p_t}{\beta_0} - \frac{m_0}{m}\frac{P_s}{P_0} \qquad H_\tau = \frac{p_t}{\beta_0} - \frac{m_0}{m}\frac{P_s}{P_0} \qquad H_\sigma = p_\sigma - \frac{m_0}{m}\frac{P_s}{P_0} \tag{1.17}$$

$$H_\delta = \frac{p_t}{\beta_0} - (1+hx)\left(\sqrt{(1+\delta)^2 - (p_x - \chi a_x)^2 - (p_y - \chi a_y)^2} + \chi a_s\right) \tag{1.18}$$

$$H_\tau = \frac{p_t}{\beta_0} - (1+hx)\left(\sqrt{(1+\delta)^2 - (p_x - \chi a_x)^2 - (p_y - \chi a_y)^2} + \chi a_s\right) \tag{1.19}$$

$$H_\sigma = p_\sigma - (1+hx)\left(\sqrt{(1+\delta)^2 - (p_x - \chi a_x)^2 - (p_y - \chi a_y)^2} + \chi a_s\right), \tag{1.20}$$

where

$$\delta = \frac{P\frac{m_0}{m} - P_0}{P_0} \qquad \chi = \frac{q}{q_0}\frac{m_0}{m}. \tag{1.21}$$

The following identities are useful to derive the equation of motion.

$$\delta = \sqrt{p_t^2 + 2p_t/\beta_0 + 1} - 1 \qquad \frac{d\delta}{dp_t} = \frac{p_t + 1/\beta_0}{1+\delta} = \frac{1}{\beta} \tag{1.22}$$

$$\delta = \sqrt{\beta_0^2 p_\sigma^2 + 2p_\sigma + 1} - 1 \qquad \frac{d\delta}{dp_\sigma} = \frac{p_\sigma \beta_0^2 + 1}{1+\delta} = \frac{\beta_0}{\beta} \tag{1.23}$$

## 1.3 Cavity time, energy errors and acceleration

A cavity kick depends on:

$$\sin(2\pi f T + \phi) \tag{1.24}$$

where T is laboratory time.
For the most general case:

$$\sin(2\pi f T + \phi) = \sin\left(2\pi f \frac{s-\sigma}{\beta_0 c} + \phi\right) \tag{1.25}$$

Most codes drop the term $2\pi f s/(\beta_0 c)$ that is

$$\sin(2\pi f T + \phi) \to \sin\left(-2\pi f \frac{\sigma}{\beta_0 c} + \phi\right) \tag{1.26}$$

to make sure that a particle that is syncrhonous to the reference trajectory is in phase with the cavity.

### 1.3.1 Implementing energy errors

One can define

$$\begin{aligned} s &= s_0 + n(L_0 - L) + nL \\ f_{\text{rev}} &= \beta_0 c / L \\ f &= h f_{\text{rev}} \end{aligned} \tag{1.27}$$

where $s_0$ is the path length at the cavity turn at 0, $L_0$ is the design circumference, $n$ is the turn number, $h$ is the harmonic number, L is the new path length with an energy error. Indeed one could write $L = L_0(1 + \eta\delta_s)$ where $\eta$ is a constant property of the lattice.
Multiple cavities can have their own defined $L$.
Using these definitions, then

$$\sin(2\pi f T + \phi) = \sin\left(2\pi h f_{\text{rev}} \frac{s_0 + n(L_0 - L) - \sigma}{\beta_0 c} + \phi\right) \tag{1.28}$$

$$= \sin\left(2\pi h f_{\text{rev}} \frac{n(L_0 - L) - \sigma}{\beta_0 c} + \phi'\right) \tag{1.29}$$

where $\phi' = \frac{2\pi h s_0}{L} + \phi$.
In MAD-X twiss and MAD8, indeed the longitudinal coordinates is directly $\sigma' = n(L_0 - L) - \sigma$ and the term $n(L_0 - L)$ is added smoothly in each thick element. This forces all the cavities to share the same $L$ or $f_{\text{rev}}$.
In SixTrack or MAD-X track, one could simply define a turn dependent phase

$$\phi = \phi_0 + 2\pi h f_{\text{rev}} n(L_0 - L) \tag{1.30}$$

which is very general or in alternative add a special element that perform at each turn the following transformation:

$$\sigma_{\text{new}} = (L_0 - L) - \sigma_{\text{old}} \tag{1.31}$$

### 1.3.2 Acceleration

Accelaration can be achieved by renormalized the relative variables using a new momentum reference. This has the side effect that the fields of the magnets (expressed in normalized strength) follow the energy ramp and that the cavity frequency (if expressed in terms of the harmonic number (NB we should perhaps change this in the Xtrack interface) is updated.
The re-normalization if done once at each turn is:

$$p_{x,\text{new}} = p_{x,\text{old}} \frac{P_{0,\text{old}}}{P_{0,\text{new}}} \qquad p_{y,\text{new}} = p_{y,\text{old}} \frac{P_{0,\text{old}}}{P_{0,\text{new}}} \tag{1.32}$$

$$\delta_{\text{new}} = (\delta_{\text{old}} + 1) \frac{P_{0,\text{old}}}{P_{0,\text{new}}} - 1 \qquad p_{t,\text{new}} = \frac{p_{t,\text{old}} P_{0,\text{old}} c + E_{0,\text{old}} - E_{0,\text{new}}}{P_{0,\text{new}} c} \tag{1.33}$$

$$\zeta_{\text{new}} = s\beta \left( \frac{1}{\beta_{0,\text{new}}} - \frac{1}{\beta_{0,\text{old}}} \right) - \zeta_{\text{old}} \qquad \tau_{\text{new}} = s \left( \frac{1}{\beta_{0,\text{new}}} - \frac{1}{\beta_{0,\text{old}}} \right) - \tau_{\text{old}} \tag{1.34}$$

$$\tag{1.35}$$

## 1.4 Beam elements

### 1.4.1 Drift

A drift is a straight, field-free region ($h(x, y) = 0$, $V = 0$ and $\mathbf{A} = 0$). The exact and expanded Hamiltonian for a drift space are

$$H_\tau = \frac{p_t}{\beta_0} - \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} \approx \frac{p_t}{\beta_0} - \delta + \frac{1}{2} \frac{p_x^2 + p_y^2}{1+\delta}. \tag{1.36}$$

$$H_\sigma = p_\sigma - \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} \approx p_\sigma - \delta + \frac{1}{2} \frac{p_x^2 + p_y^2}{1+\delta}. \tag{1.37}$$

#### 1.4.1.1 Expanded Drift

The map relative to the expanded Hamiltonian is (eq. 3.49 in [3])

$$x_p = \frac{p_x}{1+\delta} \qquad y_p = \frac{p_y}{1+\delta} \tag{1.38}$$

$$x \leftarrow x + x_p l \qquad y \leftarrow y + y_p l \tag{1.39}$$

$$\tau \leftarrow \tau + \frac{l}{\beta_0} - \frac{l}{\beta} - \frac{l}{\beta} \frac{x_p^2 + y_p^2}{2} = \tau + l \left( \frac{\delta}{\beta_0} - \frac{p_t}{1+\delta} - \frac{x_p^2 + y_p^2}{2\beta} \right) \tag{1.40}$$

$$\sigma \leftarrow \sigma + l - \frac{\beta_0}{\beta} l - \frac{\beta_0}{\beta} l \frac{x_p^2 + y_p^2}{2} = \sigma + l \left( 1 - \frac{\beta_0}{\beta} \left( 1 + \frac{x_p^2 + y_p^2}{2} \right) \right) \tag{1.41}$$

#### 1.4.1.2 Exact Drift

The map relative to the exact Hamiltonian is (eq. 3.49 in [3])

$$p_z = \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} \tag{1.42}$$

$$\frac{dp_z}{dp_t} = \frac{p_t + 1/\beta_0}{p_z} = \frac{1}{\beta_z} \tag{1.43}$$

$$\frac{dp_z}{dp_\sigma} = \frac{\beta_0^2 p_\sigma + 1}{p_z} = \frac{\beta_0}{\beta_z} \tag{1.44}$$

$$x \leftarrow x + \frac{p_x}{p_z} l \qquad\qquad y \leftarrow y + \frac{p_x}{p_z} l \tag{1.45}$$

$$\tau \leftarrow \tau + \frac{l}{\beta_0} - \frac{l}{\beta_z} = l\left(\frac{1}{\beta_0} - \frac{p_t + 1/\beta_0}{p_z}\right) \tag{1.46}$$

$$\sigma \leftarrow \sigma + l - l\frac{\beta_0}{\beta_z} = l\left(1 - \frac{\beta_0^2 p_\sigma + 1}{p_z}\right) \tag{1.47}$$

#### 1.4.1.3 Polar Drift

It is possible to define a "polar" drift that has the effect of rotating the reference frame [8] for instance in the $x$-$z$ plane

$$p_x \leftarrow p_x \cos\theta + p_z \sin\theta \qquad p_z \leftarrow -p_x \sin\theta + p_z \cos\theta \tag{1.48}$$

$$z = -x \sin\theta \qquad x' = p_x/p_z \qquad y' = p_y/p_z \tag{1.49}$$

$$x \leftarrow x\cos\theta - x'z \qquad y \leftarrow y - x'z \qquad \tau \leftarrow \tau + z/\beta_z. \tag{1.50}$$

where $\theta$ is the angle bringing the new $\hat{x}$ towards the old $\hat{z}$. The map can be also generated by combining a rotation with a $-x\sin(\theta)$-length drift. In case of an $\hat{x}$ rotation the role of $x$ and $y$ are interchanged.

### 1.4.2 Dipole

In a curvilinear reference system with a constant curvature $h$ in the horizontal plane a uniform magnetic field can be derived by the vector potential:

$$A_x = 0, \qquad A_y = 0, \qquad A_s = -B_y\left(x - \frac{hx^2}{2(1+hx)}\right). \tag{1.51}$$

With the following normalization $k_0 = \frac{q_0}{p} B_y$ is the inverse of the bending radius of the reference particle.
The exact and expanded Hamiltonian for a horizontal bending magnet is (eq. 2.12 in [2])

$$H = \frac{p_t}{\beta_0} - (1+hx)\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} + \chi k_0\left(x + \frac{hx^2}{2}\right) \tag{1.52}$$

$$\simeq \frac{p_t}{\beta_0} + \frac{1}{2}\frac{p_x^2 + p_y^2}{1+\delta} - (1+hx)(1+\delta) + \chi k_0\left(x + \frac{hx^2}{2}\right) \tag{1.53}$$

#### 1.4.2.1 Thin dipole

The map for a thin dipole kick (horizontal or vertical) from the expanded Hamiltonian is (eq. 4.12 in [4]):

$$p_x \leftarrow p_x + (h_x l - \chi k_0 l) + h_x l\delta - \chi k_0 l h_x x \tag{1.54}$$
$$p_y \leftarrow p_y - (h_y l - \chi \hat{k}_0 l) - h_y l\delta + \chi k_0 l h_y y \tag{1.55}$$
$$\tau \leftarrow \tau - \frac{h_x x - h_y y}{\beta} l. \tag{1.56}$$

#### 1.4.2.2 mad thin dipole

to be checked

$$p_x \leftarrow p_x + (h_x l - \chi k_0 l) + h_x l(p_z - 1) - \chi k_0 l h_x x \tag{1.57}$$
$$p_y \leftarrow p_y - (h_y l - \chi \hat{k}_0 l) - h_y l(p_z - 1) + \chi k_0 l h_y y \tag{1.58}$$
$$\tau \leftarrow \tau - \frac{h_x x - h_y y}{\beta_z} l. \tag{1.59}$$

#### 1.4.2.3 Thick dipole

Defining the following quantities,

$$G_x = \chi \frac{k_0 h_x}{1+\delta}, \qquad G_y = \chi \frac{\hat{k}_0 h_y}{1+\delta} \tag{1.60}$$
$$C_{x,y} = \cos(\sqrt{G_{x,y}} L), \qquad S_{x,y} = \sin(\sqrt{G_{x,y}} L) \tag{1.61}$$

the map relative to the expanded Hamiltonian is (eq. 4.11 in [2])

$$x \leftarrow C_x \cdot x + \frac{S_x}{\sqrt{G_x}} \frac{1}{1+\delta} \cdot p_x + \frac{\delta}{h_x}(1 - C_x) \tag{1.62}$$

$$p_x \leftarrow -\sqrt{G_x}(1+\delta) \cdot S_x \cdot x + C_x \cdot p_x + \delta\sqrt{1+\delta} \cdot S_x \tag{1.63}$$

$$y \leftarrow C_y \cdot y + \frac{S_y}{\sqrt{G_y}} \frac{1}{1+\delta} \cdot p_y + \frac{\delta}{h_y}(1 - C_y) \tag{1.64}$$

$$p_y \leftarrow -\sqrt{G_y}(1+\delta) \cdot S_y \cdot y + C_y \cdot p_y + \delta\sqrt{1+\delta} \cdot S_y \tag{1.65}$$

$$\sigma \leftarrow \sigma + L\left(1 - \frac{\beta_0}{\beta}\right) \tag{1.66}$$

$$- \frac{\beta_0}{\beta}\left[\frac{h_x S_x}{\sqrt{G_x}} \cdot x + \frac{1 - C_x}{h_x} \cdot p_x + \frac{h_y S_y}{\sqrt{G_y}} \cdot y + \frac{1 - C_y}{h_y} \cdot p_y + \delta\left(2L - \frac{S_x}{\sqrt{G_x}} - \frac{S_y}{\sqrt{G_y}}\right)\right] \tag{1.67}$$

$$- \frac{1}{4}\frac{\beta_0}{\beta}\left[G_x\left(L - \frac{C_x S_x}{\sqrt{G_x}}\right)\left(x - \frac{\delta}{h_x}\right)^2 + \left(L + \frac{C_x S_x}{\sqrt{G_x}}\right)\frac{p_x^2}{(1+\delta)^2} - \left(x - \frac{\delta}{h_x}\right)\frac{2S_x^2}{1+\delta} \cdot p_x\right. \tag{1.68}$$

$$\left. + G_y\left(L - \frac{C_y S_y}{\sqrt{G_y}}\right)\left(y - \frac{\delta}{h_y}\right)^2 + \left(L + \frac{C_y S_y}{\sqrt{G_y}}\right)\frac{p_y^2}{(1+\delta)^2} - \left(y - \frac{\delta}{h_y}\right)\frac{2S_y^2}{1+\delta} \cdot p_y\right]. \tag{1.69}$$

#### 1.4.2.4 Dipole Edge effects

Considering the dipole edge effects from a dipole of length $L$ and bending angle $\theta$, the map is

$$p_x \to p_x + \frac{1+\delta}{\rho}\tan(\alpha) \cdot x$$
$$p_y \to p_y - \frac{1+\delta}{\rho}\tan(\alpha) \cdot y,$$

where the bending radius $\rho$ and $\alpha$ are defined as

$$\rho^{-1} = \frac{h_x}{\sqrt{1+\delta}} \qquad\qquad \alpha = \frac{1}{2}\frac{L}{\rho} = \frac{\theta}{2}.$$

### 1.4.3 Combined dipole quadrupole

The following vector potential in curvilinear coordinates

$$A_s = -\frac{g}{1+hx}\left(\frac{x^2}{2} - \frac{y^2}{2} + \frac{hx^3}{3}\right) \tag{1.70}$$

produce a field

$$B_x = g\left(y + \frac{hxy}{1+hx}\right) \qquad B_y = gx \tag{1.71}$$

The following vector potential in curvilinear coordinates

$$A_s = -\frac{g}{1+hx}\left(\frac{x^2}{2} - \frac{y^2}{2} + \frac{hx^3}{3} - \frac{hxy^2}{2}\right) \tag{1.72}$$

produce a field

$$B_x = gy \qquad B_y = g\left(x + \frac{hy^2}{2+2hx}\right) \tag{1.73}$$

### 1.4.4 Thin Multipole

The effect of a thin multipole can be approximated by the following Hamiltonian
A longitudinally uniform static magnetic field can be described by the following equations

$$B_y + iB_x = \sum_{n=1} \frac{B_n + iA_n}{r_0^{n-1}}(x+iy)^{n-1} \tag{1.74}$$

$$= B_N \sum_{n=N} \frac{b_n + ia_n}{r_0^{n-1}}(x+iy)^{n-1}. \tag{1.75}$$

Usually multipole are expressed as relative to
A thin multiple idealize the effect of the field by taking the limit of the integration length going to zero while keeping constant the integrated strength. The Hamiltonian is:

$$H = -\delta(s)\chi L \Re\left[\sum_{n=0}(k_n + i\hat{k}_n)(x+iy)^{n+1}\right]. \tag{1.76}$$

where

$$k_n = n!\frac{q_0}{p_0}\frac{B_{n+1}}{r_0^n} \qquad \hat{k}_n = n!\frac{q_0}{p_0}\frac{A_{n+1}}{r_0^n}. \tag{1.77}$$

The corresponding map is:

$$p_x \leftarrow p_x - \chi L \cdot \Re\left[\sum_{n=0}\frac{1}{n!}(k_n + i\hat{k}_n)(x+iy)^n\right], \tag{1.78}$$

$$p_y \leftarrow p_y + \chi L \cdot \Im\left[\sum_{n=0}\frac{1}{n!}(k_n + i\hat{k}_n)(x+iy)^n\right], \tag{1.79}$$

In case a curvature $h$ is the vector potential become:

$$f(x,y) = \int B_x(x,y)dy \tag{1.80}$$

$$g(x,y) = \int \partial x B_x(x,y)dy \tag{1.81}$$

$$a_s(x,y) = \frac{c_1}{1+hx} + f(x,y) - \frac{\int_1^x (1+h\xi)(g(\xi,y)+\xi) + hf(x,y)\,d\xi}{1+hx} \tag{1.82}$$

$$\frac{\int_1^x \left(-h\xi\,(g(x,y)) - \int \mathrm{bx}^{(1,0)}(\xi,y)\,dy - h\int \mathrm{bx}(\xi,y)\,dy - h\xi\mathrm{by}(\xi,y) - \mathrm{by}(\xi,y)\right)\,d\xi}{hx+1} \tag{1.83}$$

### 1.4.5 Accelerating Cavity

The approximated energy gain of a particle passing through an electric field of frequency $f = \frac{k}{2\pi c}$ for which:

$$V\sin(\phi - k\tau) = \int_{-l/2}^{l/2} E_s(0,0,t,s)\,\mathrm{d}s. \tag{1.84}$$

An equivalent vector potential can be derived and normalized as

$$A_s = -\frac{V}{\omega}\cos(\phi - k\tau) \qquad\qquad V_n = \frac{q_0}{P_0 c}V \tag{1.85}$$

from which one can derive the following map

$$p_t \leftarrow p_t + \chi V_n \sin(\phi - k\tau + k\frac{s-s_0}{\beta_0}), \tag{1.86}$$

where the additional terms in the phase is added in case harmonic number is not exactly integer and the phase is unlocked phase ). The new $\delta$ can be updated from the new $p_t$.

### 1.4.6 RF-Multipole

The RF-multipole generalizes the interaction of a particle with an electromagnetic field by assuming that

$$\Delta E(x,y,\tau) = q\int_{-L/2}^{L/2} E_z(x,y,t)\,\mathrm{d}s \tag{1.87}$$

$$\Delta P_x(x,y,\tau) = q\int_{-L/2}^{L/2} E_x(x,y,t) + \beta c B_y(x,y,t)\,\mathrm{d}s \tag{1.88}$$

$$\Delta P_y(x,y,\tau) = q\int_{-L/2}^{L/2} E_y(x,y,t) - \beta c B_x(x,y,t)\,\mathrm{d}s. \tag{1.89}$$

are harmonic in $x, y$ and periodic in $\tau$ of frequency $f = \frac{k}{2\pi c}$ such that:

$$a_s(x, y, \tau) = \Re\left[\sum_{n=1}^{N}\left(k_n \cos(\phi_n - k\tau) + i\hat{k}_n \cos(\hat{\phi}_n - k\tau)\right)(x + iy)^n\right], \tag{1.90}$$

The map then follows:

$$\Delta p_x = -\sum_{n=1}^{N}\frac{\chi}{n!}\Re\left[(k_n C_n + i\hat{k}_n\hat{C}_n)(x + iy)^{(n-1)}\right], \tag{1.91}$$

$$\Delta p_y = \sum_{n=1}^{N}\frac{\chi}{n!}\Im\left[(k_n C_n + i\hat{k}_n\hat{C}_n)(x + iy)^{(n-1)}\right], \tag{1.92}$$

$$\Delta p_t = -\chi k\sum_{n=1}^{N}\Re\left[(k_n S_n + ik_n\hat{S}_n)(x + iy)^n\right], \tag{1.93}$$

where

$$C_n = \cos(\phi_n - \omega\Delta t) \qquad \hat{C}_n = \cos(\hat{\phi}_n - \omega\Delta t) \tag{1.94}$$

$$S_n = \sin(\phi_n - \omega\Delta t) \qquad \hat{S}_n = \sin(\hat{\phi}_n - \omega\Delta t). \tag{1.95}$$

### 1.4.7 Solenoid

The expanded Hamiltonian for a particle in a solenoid is

$$H = p_\sigma + \frac{1}{2}\frac{(p_x + R\cdot y)^2 + (p_y - R\cdot x)}{1+\delta},$$

where $R = \frac{1}{2}\frac{q}{P_0 c}\mathbf{B}(0, 0, s)$. The map for a solenoid of length $L$ in the thin lens approximation with the expanded Hamiltonian (eq. 4.35 in [4])

$$\begin{aligned}
x &\to \quad C\cdot x + S\cdot y\\
p_x &\to -\theta R\cdot C\cdot x + C\cdot p_x - \theta R\cdot S\cdot y + S\cdot p_y\\
y &\to -S\cdot x + C\cdot y\\
p_y &\to \quad \theta R\cdot S\cdot x - S\cdot p_x - \theta R\cdot C\cdot y + C\cdot p_y\\
\sigma &\to \sigma - \frac{\beta_0}{\beta}\frac{\theta}{1+\delta}\left(\frac{1}{2}R(x^2+y^2) + (p_x y - p_y x)\right)
\end{aligned}$$

where $R \equiv R(s_0)$, $\theta = \frac{R}{1+\delta}$, $C = \cos(\theta)$ and $S = \sin(\theta)$.

The map for a thick solenoid is (eq. 3.47, 3.48 in [1])

$$
\begin{aligned}
x &\to C^2 \cdot x + \frac{1}{R} \cdot S \cdot C \cdot p_x + S \cdot C \cdot y + \frac{1}{R} \cdot S^2 \cdot p_y \\
p_x &\to -R \cdot S \cdot C \cdot x + C^2 \cdot p_x - R \cdot S^2 \cdot y + S \cdot C \cdot p_y \\
y &\to -S \cdot C \cdot x - \frac{1}{R} \cdot S^2 \cdot p_x + C^2 \cdot y + \frac{1}{R} \cdot S \cdot C \cdot p_y \\
p_y &\to R \cdot S^2 \cdot x - S \cdot C \cdot p_x - R \cdot S \cdot C \cdot y + C^2 \cdot p_y \\
\sigma &\to \sigma - \frac{L}{2} \left[ R^2 (x^2 + y^2) + 2R \left( \frac{p_x}{1+\delta} y - \frac{p_y}{1+\delta} x \right) + \frac{p_x^2 + p_y^2}{(1+\delta)^2} \right]
\end{aligned}
$$

where $\theta = R \cdot L$, $C = \cos\theta$ and $S = \sin\theta$.

### 1.4.8 AC-dipole

The excitation amplitude of the AC-dipole is denoted by $A$ [Tm], the excitation frequency by $q_d$ [$2\pi$] and the phase of the excitation by $\phi$. The map presented here is for a purely horizontal dipole, the map for a vertical dipole is obtained by replacing $p_x \to p_y$.
The effect of the AC-dipole is split into four stages. The turn number is denoted by $n$.

1. A number of free turns $n_{\text{free}}$, in which the AC-dipole has no effect on the motion.
2. Ramp-up of the voltage from 0 to the excitation amplitude $A$ for $n_{\text{ramp-up}}$ turns.

$$
\begin{aligned}
n' &= \frac{n - n_{\text{free}}}{n_{\text{ramp-up}}} \\
p_x &\to p_x + n' \cdot \frac{A}{pc} \cdot (1+\delta) \sin\left(2\pi q_d \cdot (n - n_{\text{free}}) + \phi\right)
\end{aligned}
$$

3. Constant excitation amplitude for $n_{\text{flat}}$ turns.

$$
p_x \to p_x + \frac{A}{pc} \cdot (1+\delta) \sin\left(2\pi q_d \cdot (n - n_{\text{free}}) + \phi\right)
$$

4. Ramp-down of the voltage from the excitation amplitude $A$ to 0 for $n_{\text{ramp-down}}$ turns.

$$
\begin{aligned}
n' &= \frac{n - n_{\text{free}} - n_{\text{ramp-up}} - n_{\text{flat}} - n_{\text{ramp-down}}}{n_{\text{ramp-down}}} \\
p_x &\to p_x + n' \cdot \frac{A}{p} \cdot (1+\delta) \sin\left(2\pi q_d \cdot (n - n_{\text{free}}) + \phi\right)
\end{aligned}
$$

### 1.4.9 Wire

For each part we define $p_z = \sqrt{(1+\delta)^2 - x'^2 - y'^2}$, using the current values for $x'$ and $y'$.

Step 1. Initial backwards drift of length $L = \frac{embl}{2}$.

$$x \to x - L \cdot \frac{x'}{p_z}$$
$$y \to y - L \cdot \frac{y'}{p_z}$$

Step 2.

$$y \to y - \frac{x \cdot \sin(t_x)}{\cos\left(\arctan\left(\frac{x'}{p_z}\right) - t_x\right)} \cdot \frac{y'}{\sqrt{(1+\delta)^2 - y'^2}}$$
$$x \to x \cdot \left[\cos(t_x) - \sin(t_x) \cdot \tan\left(\arctan\left(\frac{x'}{p_z}\right) - t_x\right)\right]$$
$$x' \to \sqrt{(1+\delta)^2 - y'^2} \cdot \sin\left(\arctan\left(\frac{x'}{p_z}\right) - t_x\right)$$
$$x \to x - \frac{y \cdot \sin(t_y)}{\cos\left(\arctan\left(\frac{y'}{p_z}\right) - t_y\right)} \cdot \frac{x'}{\sqrt{(1+\delta)^2 - x'^2}}$$
$$y \to y \left[\cos(t_y) - \sin(t_y) \cdot \tan\left(\arctan\left(\frac{y'}{p_z}\right) - t_y\right)\right]$$
$$y' \to \sqrt{(1+\delta)^2 - x'^2} \sin\left(\arctan\left(\frac{y'}{p_z}\right) - t_y\right)$$

Step 3. Drift part of length $L = lin$.

$$x \to x + L \cdot \frac{x'}{p_z}$$
$$y \to y + L \cdot \frac{y'}{p_z}$$

Step 4. Here $x_i = x - r_x$ and $y = y - r_y$.

$$x' \to x' - \frac{\frac{cur \cdot 10^{-7}}{chi}}{x_i^2 + y_i^2} \cdot x_i \left[\sqrt{(lin + l)^2 + x_i^2 + y_i^2} - \sqrt{(lin - l)^2 + x_i^2 + y_i^2}\right]$$
$$y' \to y' - \frac{\frac{cur \cdot 10^{-7}}{chi}}{x_i^2 + y_i^2} \cdot y_i \left[\sqrt{(lin + l)^2 + x_i^2 + y_i^2} - \sqrt{(lin - l)^2 + x_i^2 + y_i^2}\right]$$

Step 5. Drift of length $L = leff - lin$.

$$x \to x + L\frac{x'}{p_z}$$
$$y \to y + L\frac{y'}{p_z}$$

Step 6.

$$
\begin{aligned}
x &\to x - \frac{y \cdot \sin(-t_y)}{\cos\left(\arctan\left(\frac{y'}{p_z}\right) + t_y\right)} \cdot \frac{x'}{\sqrt{(1+\delta)^2 - x'^2}} \\
y &\to y \cdot \left[\cos(-t_y) - \sin(-t_y) \cdot \tan\left(\arctan\left(\frac{y'}{p_z}\right) + t_y\right)\right] \\
y' &\to \sqrt{(1+\delta)^2 - x'^2} \cdot \sin\left(\arctan\left(\frac{y'}{p_z}\right) + t_y\right) \\
y &\to y - \frac{x \cdot \sin(-t_x)}{\cos\left(\arctan\left(\frac{x'}{p_z}\right) + t_x\right)} \cdot \frac{y'}{\sqrt{(1+\delta)^2 - y'^2}} \\
x &\to x \left[\cos(-t_x) - \sin(-t_x) \cdot \tan\left(\arctan\left(\frac{x'}{p_z}\right) + t_x\right)\right] \\
x' &\to \sqrt{(1+\delta)^2 - y'^2} \cdot \sin\left(\arctan\left(\frac{x'}{p_z}\right) + t_x\right)
\end{aligned}
$$

Step 7. Shift.

$$
\begin{aligned}
x &\to x + embl \cdot \tan(t_x) \\
y &\to y + embl \cdot \frac{\tan(t_y)}{\cos(t_x)}
\end{aligned}
$$

Step 8. Negative drift of length $L = \frac{embl}{2}$.

$$
\begin{aligned}
x &\to x - L \cdot \frac{x'}{p_z} \\
y &\to y - L \cdot \frac{y'}{p_z}
\end{aligned}
$$

### 1.4.10 Misalignment

Misalignments of elements affects the coordinates at the entrance of an element as follows

$$
\begin{aligned}
x &\to (x - x_s) \cdot t_c + (y - y_s) \cdot t_s \\
y &\to -(x - x_s) \cdot t_s + (y - y_s) \cdot t_c,
\end{aligned}
$$

where $x_s$ and $y_s$ are the displacements in the horizontal and vertical directions, respectively. $t_c$ and $t_s$ are the cosine and sine of the tilt angle for the element.

### 1.4.11 Electron Lens

#### 1.4.11.1 Hollow electron lens - uniform annular profile

For a uniform distribution of the electron beam between $R_1$ and $R_2$, the radial kick can be described by a shape function $f(r)$ and a maximum kick strength $\theta_{\max}$:

$$
\theta(r) = \frac{f(r)}{(r/R_2)} \cdot \theta_{\max} \tag{1.96}
$$

with $r = \sqrt{x^2 + y^2}$ and $\theta_{\max}$ independent of $r$. The shape function $f(r)$ is defined as

$$f(r) = \frac{I(r)}{I_T} = \frac{2\pi}{I_T} \int_0^r r\rho(r)dr \tag{1.97}$$

where $I_T$ is the total electron beam current, $I(r)$ is the current enclosed in a radius $r$ and $\rho(r)$ is the electron beam density distribution.
For a uniform profile one then obtains:

$$\begin{cases} 0 & , \quad r < R_1 \\ \frac{r^2 - R_1^2}{R_2^2 - R_1^2} & , \quad R_1 \leq r < R_2 \\ 1 & , \quad R_2 \leq r \end{cases} \tag{1.98}$$

and

$$\theta_{\max} = \theta(R_2) = \frac{2LI_T(1 \pm \beta_e\beta_p)}{4\pi\epsilon_0 \left(B\rho\right)_p \beta_e\beta_p c^2} \cdot \frac{1}{R_2} \tag{1.99}$$

where $L$ is the length of the e-lens, $I_T$ the total electron beam current, $\beta_{e/p}$ the relativistic $\beta$ of electron/proton beam, $B\rho$ the magnetic rigidity, $c$ the speed of light and $\epsilon_0$ the vacuum permittivity. The $\pm$-sign represents the two cases of the electron beam traveling in the direction of the proton beam $(+)$ or in the opposite direction $(-)$. For hollow electron beam collimation, electron and proton beam travel in the same direction.
The kick in $(x', y')$ can then be expressed as (note $\frac{x}{r} = \cos(\phi)$, $\frac{y}{r} = \sin(\phi)$):

$$x' = x' - \theta_{\max} \cdot \frac{r_2}{r^2} \cdot f(r) \cdot x \tag{1.100}$$

$$y' = y' - \theta_{\max} \cdot \frac{r_2}{r^2} \cdot f(r) \cdot y \tag{1.101}$$

If the electron lens is offset by $(x_{\text{offset}}, y_{\text{offset}})$, the coordinates $(x, y)$ are simply transfered to:

$$\tilde{x} = x + x_{\text{offset}} \tag{1.102}$$

$$\tilde{y} = y + y_{\text{offset}} \tag{1.103}$$

$$\tilde{r} = \sqrt{\tilde{x}^2 + \tilde{y}^2} \tag{1.104}$$

and the kick is then given by:

$$x' = x' - \theta_{\max} \cdot \frac{r_2}{\tilde{r}^2} \cdot f(\tilde{r}) \cdot \tilde{x} \tag{1.105}$$

$$y' = y' - \theta_{\max} \cdot \frac{r_2}{\tilde{r}^2} \cdot f(\tilde{r}) \cdot \tilde{y} \tag{1.106}$$

## 1.5 Linear optics calculations

Optics calculation are needed to study the motion around the closed orbit. By defining $z$ as the vector of $2k$ coordinates,

$$z = (z_1, \ldots, z_{2k})^T = (x - x_0, p_x - p_{x0}, y - y_0, p_y - p_{y0}, \tau - \tau_0, p_t - p_{t0})^T \tag{1.107}$$

one can define linear transfer maps (e.g. $M_{1\to 2}$ that propagates coordinates between two points $s_1, s_2$) and the one-turn map (e.g. $M_1$ that combines the effects for one turn starting from $s_1$):

$$z(s_2) = M_{1\to 2} z(s_1) \qquad\qquad z(C + s_1) = M_1 z(s_1). \tag{1.108}$$

In the following we will describe the optics calculation based on the Ripken formalism described in [9]. A good summary is also given in the MAD8 physics manual [10].

### 1.5.1 Diagonalisation of one-turn matrix

Since the matrices derive from symplectic maps, the eigenvalue spectrum of the one-turn map $M$ consists of $2k$ distinct eigenvalues and linearly independent eigenvectors. In addition, for the motion to be stable the eigenvalues $\lambda_k^{\pm}$ with eigenvectors $v_k^{\pm}$ have to be complex [9]:

$$M v_k^{\pm} = \lambda_k^{\pm} v_k^{\pm},\ k = 1, \ldots, 3 \tag{1.109}$$

$$v_k^{+} = (v_k^{-})^*, \quad \lambda_k^{+} = (\lambda_k^{-})^*, \quad |\lambda_k^{\pm}| = 1 \tag{1.110}$$

As the eigenvectors are linearly independent $M$ can be diagonalized with

$$M = V \Lambda V^{-1}, \tag{1.111}$$

where $V$ consists of the eigenvectors and $\Lambda$ of the eigenvalues:

$$V = \begin{pmatrix} v_{1,1}^{+} & v_{1,1}^{-} & \cdots & v_{3,1}^{-} \\ v_{1,2}^{+} & v_{1,2}^{-} & \cdots & v_{3,2}^{-} \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \qquad \Lambda = \begin{pmatrix} \lambda_1^{+} & & & \\ & \lambda_1^{-} & & \\ & & \ddots & \\ & & & \lambda_3^{-} \end{pmatrix} \tag{1.112}$$

for which $v_{i,j}^{\pm}$ is the component $j$ of eigenvector $v_i^{\pm}$.
The same calculation can be carried out with real numbers by the following definitions:

$$v_k^{\pm} = a_k \pm i b_k, \qquad \lambda_k^{\pm} = \cos\mu_k \pm i \sin\mu_k, \qquad \mu_k, a_k, b_k \in \mathbb{R} \tag{1.113}$$

such that:

$$M = W R W^{-1} \tag{1.114}$$

with

$$R = R(\mu_k) = \begin{pmatrix} \cos\mu_1 & \sin\mu_1 & & & & \\ -\sin\mu_1 & \cos\mu_1 & & & & \\ & & \ddots & & & \\ & & & \cos\mu_3 & \sin\mu_3 & \\ & & & -\sin\mu_3 & \cos\mu_3 & \end{pmatrix}, \tag{1.115}$$

$$W = \begin{pmatrix} a_{1,1} & b_{1,1} & \cdots & a_{3,1} & b_{3,1} \\ a_{1,2} & b_{1,2} & \cdots & a_{3,2} & b_{3,2} \\ \vdots & \vdots & \vdots & \vdots & \\ a_{1,6} & b_{1,6} & \cdots & a_{3,6} & b_{3,6} \end{pmatrix} \tag{1.116}$$

Usually $\mu_k$ is written as $\mu_k = 2\pi Q_k$, where $Q_k$ is then the tune of the mode $k$.

### 1.5.2 Normalisation of eigenvectors

By convention, the eigenvectors and values are normalized, sorted and rotated so that the following three conditions are fulfilled:

1. Plane 1 is associated with the horizontal, plane 2 with the vertical and plane 3 with the longitudinal plane. This is achieved by first normalizing the eigenvectors $v_k^{\pm}$ and then sorting them so that:

$$|v^+_{j,2j-1}| = |v^-_{j,2j-1}| = \max_{k=1,2,3} v_{k,j}, \quad j = 1,\ldots,3 \tag{1.117}$$

2. The eigenvectors are then rotated with a phase term $\psi_k$

$$v_k \to v_k \exp(i\psi_k) \tag{1.118}$$

such that

$$\text{angle}(v^+_{k,2k-1}) = 0 \leftrightarrow \psi_k = -\text{angle}(v^+_{k,2k-1}) \tag{1.119}$$

In real space, Eqn. 1.117 and 1.119 then become equivalent to:

$$|a_{j,2j-1}| = \max_{k=1,2,3} |a_{k,j}|, \qquad b_{j,2j-1} = 0, \qquad j = 1,\ldots,3 \tag{1.120}$$

This has the effect that a particle with $x = 0$ is transformed to $\tilde{x}$ in the normalized phase space.

3. The sign of $b_{k,j}$ is fixed by the symplectic condition on $W$

$$W^T S W = S \tag{1.121}$$

with $S$ defined as

$$S = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & \ddots \end{pmatrix} \tag{1.122}$$

which is equivalent to:

$$\begin{aligned} &a_k^T \cdot S \cdot b_k = 1, \quad b_k^T \cdot S \cdot a_k = -1, && \text{for } k = l \\ &a_k^T \cdot S \cdot b_l = 0, && \text{for } k \neq l \\ &a_k^T \cdot S \cdot a_l = 0, \quad b_k^T \cdot S \cdot b_l = 0, && k, l = 1, \ldots, 3 \end{aligned} \tag{1.123}$$

Eqn. 1.123 yields that in phase space $a_k$ is thus obtained by an anticlockwise rotation of $b_k$ by $\pi/2$ and a scaling of its length with $|a_k| = \frac{1}{|b_k|}$.

### 1.5.3 Conversion to normalized coordinates

We will show in the following that in the normalized phase space the propagation of particle coordinates $z(s)$ from $s_1$ to $s_2$ is just a rotation by an angle $\phi_k$ in the $k = 1, \ldots, 3$ planes, while the amplitude $I_k$ and initial phase $\phi_{k,0}$ stay constant, explicitly $z(s)$ is then given by:

$$z(s) = \sum_{k=1}^{3} \sqrt{2I_k} \left( a_k(s) \cos\left(\phi_{k,0} + \phi_k(s)\right) - b_k(s) \sin\left(\phi_{k,0} + \phi_k(s)\right)\right) \tag{1.124}$$

and

$$\begin{aligned} z(s_2) = W(s_2) R(\phi_k) W(s_1)^{-1} z(s_1), \\ \text{with } \phi_k = \phi_k(s_2) - \phi_k(s_1) \end{aligned} \tag{1.125}$$

This implies that one turn is simply a rotation by $\phi_k = 2\pi Q_k$ where $Q_k$ is the tune of the mode $k$. In the transverse plane the tune ($Q_{I,II}$) is usually positive and the particles rotate clockwise, while in the longitudinal plane the tune ($Q_{III}$) is negative above $\gamma_T$ leading to an anticlockwise rotation.
For the derivation the following steps are needed:

1. The effect of one turn on the normalized variable $\tilde{z}(s) = W^{-1}(s) z(s)$ is a rotation:

$$\tilde{z}(C + s) = W^{-1} z(s + C) \overset{\text{(Eqn.1.114)}}{=} W^{-1} W R W^{-1} z(s) = R\tilde{z}(s), \tag{1.126}$$

As $M$ and $R$ are symplectic also $W$ is symplectic, and its inverse is thus given by

$S^{-1}W^TS$, explicitly:

$$W^{-1} = \begin{pmatrix} b_{12} & -b_{11} & b_{14} & -b_{13} & b_{16} & -b_{15} \\ -a_{12} & a_{11} & -a_{14} & a_{13} & -a_{16} & a_{15} \\ b_{22} & -b_{21} & b_{24} & -b_{23} & b_{26} & -b_{25} \\ -a_{22} & a_{21} & -a_{24} & a_{23} & -a_{26} & a_{25} \\ b_{32} & -b_{31} & b_{34} & -b_{33} & b_{36} & -b_{35} \\ -a_{32} & a_{31} & -a_{34} & a_{33} & -a_{36} & a_{35} \end{pmatrix} \tag{1.127}$$

2. The one-turn map and $W$-matrix can be propagated from $s_1$ to $s_2$ by

$$M_2 = M_{1\to 2} M_1 M_{1\to 2}^{-1} \qquad\qquad W_2 = M_{1\to 2} W_1 \tag{1.128}$$

As Eqn. 1.126 represents a similarity transformation, the eigenvalues are thus independent of the position $s$ and as the rotation matrix $R$ consists of the eigenvalues of $M$, the angle of the rotation $\mu_k = 2\pi Q_k$ is thus also independent of $s$.

3. As Eqn. 1.114 represents a basis transformation from the standard $\mathbb{R}^2$ basis to the eigenvector basis, the vectors $a_k$ and $b_k$ are projected onto (Eqn. 1.123):

$$\begin{aligned} \tilde{a}_1 = W^{-1}a_1 &= -SW^TSa_1 \\ &= -S(a_1Sa_1, b_1Sa_1, \ldots, b_3Sa_1)^T = (1,0,\ldots,0) \\ \tilde{b}_1 = W^{-1}b_1 &= -SW^TSb_1 \\ &= -S(a_1Sb_1, b_1Sb_1, \ldots, b_3Sb_1)^T = (0,1,\ldots,0) \\ &\ldots \\ \tilde{b}_3 = W^{-1}b_3 &= -SW^TSb_3 \\ &= -S(a_1Sb_3, b_1Sb_3, \ldots, b_3Sb_3)^T = (0,0,\ldots,1) \end{aligned} \tag{1.129}$$

in the normalized phase space.

4. From Eqn. 1.126 it follows that the amplitude $I_k$ and initial phase $\phi_{k0}$ of $\tilde{z} = W^{-1}z = (\tilde{z}_{a_1}, \tilde{z}_{b_1}, \ldots, \tilde{z}_{b_3})$

$$I_k = \frac{(\tilde{z}_{a_k})^2 + (\tilde{z}_{b_k})^2}{2}, \quad k = 1,\ldots,3 \tag{1.130}$$

$$\tan\phi_{k0} = -\frac{\tilde{z}_{b_k}}{\tilde{z}_{a_k}} \tag{1.131}$$

are constants of the motion. The initial phase is defined with a minus sign in view of the definition of the Twiss parameters, where the initial phase is then added (and not subtracted) to the phase advance. The components of $\tilde{z}$ are then

explicitly given by:

$$\tilde{z}_{a_k} = \sum_{j=1}^{3} b_{k,2j} z_{2j-1} - b_{k,2j-1} z_{2j}, \quad k = 1, \ldots, 3 \tag{1.132}$$

$$\tilde{z}_{b_k} = \sum_{j=1}^{3} a_{k,2j-1} z_{2j} - a_{k,2j} z_{2j-1}, \quad k = 1, \ldots, 3. \tag{1.133}$$

An arbitrary vector $z(s)$ can thus be written in the following form:

$$\begin{aligned} z(s) &= W(s)\tilde{z}(s) \\ &= W(s) \left( \sum_{k=1}^{3} \tilde{z}_{a_k} \tilde{a}_k + \tilde{z}_{b_k} \tilde{b}_k \right) \\ &= \sum_{k=1}^{3} \tilde{z}_{a_k} W(s) \tilde{a}_k + \tilde{z}_{b_k} W(s) \tilde{b}_k \overset{\text{Eqn. 1.129}}{=} \sum_{k=1}^{3} \tilde{z}_{a_k} a_k + \tilde{z}_{b_k} b_k \\ &\overset{\text{Eqns. 1.130,1.131}}{=} \sum_{k=1}^{3} \sqrt{2I_k} \left( a_k \cos \phi_{k0} - b_k \sin \phi_{k0} \right) \end{aligned} \tag{1.134}$$

### 1.5.4 Twiss parameters

In the following the parameter $k$ will always be used for the mode $k$ and the parameter $j = 1, 2, 3$ for the horizontal $(x, p_x)$, vertical $(y, p_y)$ and longitudinal plane $(\sigma, \delta)$ in the phase space. $z_{2j-1}$ then stands for the coordinates $(x, y, \sigma)$ and $z_{2j}$ for $(p_{x,y}, \delta)$.
The Twiss parameters can be introduced by writing the components of the eigenvector basis $(a_k(s), b_k(s))$ as the product of two envelope functions $\sqrt{\beta_{k,j}(s)}$, $\sqrt{\gamma_{k,j}(s)}$ and phase functions $\phi_{k,j}(s)$, $\bar{\phi}_{k,j}(s)$, also called Twiss parameters or lattice functions, with

$$\begin{aligned} a_{k,2j-1}(s) &= \sqrt{\beta_{k,j}(s)} \cos \phi_{k,j}(s), \\ b_{k,2j-1}(s) &= \sqrt{\beta_{k,j}(s)} \sin \phi_{k,j}(s), \; k, j = 1, \ldots, 3, \end{aligned} \tag{1.135}$$

$$\begin{aligned} a_{k,2j}(s) &= \sqrt{\gamma_{k,j}(s)} \cos \bar{\phi}_{k,j}(s), \\ b_{k,2j}(s) &= \sqrt{\gamma_{k,j}(s)} \sin \bar{\phi}_{k,j}(s), \; k, j = 1, \ldots, 3 \end{aligned} \tag{1.136}$$

where $\beta_{k,j}(s), \alpha_{k,j}(s), \gamma_{k,j}(s)$ represent the projection of the ellipse of mode $k$ on the plane of coordinates $z_{2k-1} - z_{2k}$.
Using Eqns. 1.124, 1.135, 1.136 and $\cos(x + y) = \cos x \cos y - \sin x \sin y$, the coordinates $z(s)$ can be expressed by:

$$z_{2j-1}(s) = \sum_{k=1}^{3} \sqrt{2I_k \beta_{k,j}(s)} \cos \left( \phi_{k,j}(s) + \phi_{k,0} \right) \tag{1.137}$$

$$z_{2j}(s) = \sum_{k=1}^{3} \sqrt{2I_k \gamma_{k,j}(s)} \cos \left( \bar{\phi}_{k,j}(s) + \phi_{k,0} \right), \; j = 1, \ldots, 3 \tag{1.138}$$

Conversely the lattice functions can also be expressed by $a_k$ and $b_k$ with

$$\beta_{k,j}(s) = a_{k,2j-1}(s)^2 + b_{k,2j-1}(s)^2 \tag{1.139}$$
$$\alpha_{k,j}(s) = -a_{k,2j-1}(s)a_{k,2j}(s) - b_{k,2j-1}(s)b_{k,2j}(s) \tag{1.140}$$
$$\gamma_{k,j}(s) = a_{k,2j}(s)^2 + b_{k,2j}(s)^2, \tag{1.141}$$

The well known relations between the lattice functions

$$\sum_{j=1}^{3} \beta_{k,j}\phi'_{k,j} = 1 \tag{1.142}$$
$$\gamma_{k,j} = \frac{\beta_{k,j}^2\phi_{k,j}'^2 + \alpha_{k,j}^2}{\beta_{k,j}}, \text{ with} \tag{1.143}$$
$$\alpha_{k,j} := -\frac{1}{2}\beta'_{k,j} \tag{1.144}$$

can then be derived with the help of the normalization condition (Eqn. 1.123)

$$a_k^T S b_k = 1 \tag{1.145}$$

by the following steps:

1. As $x' = \frac{dx}{ds}$, $y' = \frac{dy}{ds}$ and $\delta = \frac{d\sigma}{ds}$ the following relations hold also for $a_k$ and $b_k$:

$$a_{k,2j} = a'_{k,2j-1} = \frac{d}{ds}(a_{k,2j-1}), \tag{1.146}$$
$$b_{k,2j} = b'_{k,2j-1} = \frac{d}{ds}(b_{k,2j-1}),\ k,j = 1,\ldots,3 \tag{1.147}$$

2. The normalization condition Eqn. 1.123 can then be written as

$$\begin{aligned} a_k^T S b_k &= \sum_{j=1}^{3} \sqrt{\beta_{k,j}}\cos\phi_{k,j}\left(\sqrt{\beta_{k,j}}\sin\phi_{k,j}\right)' \\ &\quad - \left(\sqrt{\beta_{k,j}}\cos\phi_{k,j}\right)'\sqrt{\beta_{k,j}}\sin\phi_{k,j} \\ &= \sum_{j=1}^{3} \beta_{k,j}\phi'_{k,j} \\ &= 1 \end{aligned} \tag{1.148}$$

Note that Eqn. 1.148 yields the the following relation between the phase advance $\phi$ and $\beta$ in 2D:

$$\phi(s) = \phi(0) + \int_{s_0}^{s} \frac{1}{\beta(\bar{s})} d\bar{s} \tag{1.149}$$

3. Using the abbreviation $\alpha_{k,j} := -\frac{1}{2}\beta_{k,j}$, one finds for each mode $k$ and plane $j$

$$\sqrt{\gamma_{k,j}}\cos\phi_{k,j} = a_{k,2j} = a'_{k,2j-1} = (\sqrt{\beta_{k,j}}\cos\phi_{k,j})' \tag{1}$$

$$\sqrt{\gamma_{k,j}}\sin\phi_{k,j} = b_{k,2j} = b'_{k,2j-1} = (\sqrt{\beta_{k,j}}\sin\phi_{k,j})' \tag{2}$$

$$\overset{(1)^2+(2)^2}{\Rightarrow} \gamma_{k,j} = \frac{\beta_{k,j}^2\phi_{k,j}'^2 + \alpha_{k,j}^2}{\beta_{k,j}}, \quad k,j = 1,\ldots,3 \tag{1.150}$$

which simplifies in the 2D case to:

$$\gamma \overset{\text{Eqn. 1.148}}{=} \frac{1+\alpha^2}{\beta} \tag{1.151}$$

# Chapter 2

# Xfields

## 2.1 Fields generated by a bunch of particles

We assume that the bunch travels rigidly along $s$ with velocity $\beta_0 c$:

$$\rho\left(x,y,s,t\right) = \rho_0\left(x,y,s-\beta_0 ct\right) \tag{2.1}$$
$$\mathbf{J}\left(x,y,s,t\right) = \beta_0 c\,\rho_0\left(x,y,s-\beta_0 ct\right)\hat{\mathbf{i}}_s \tag{2.2}$$

We define an auxiliary variable $\zeta$ as the position along the bunch:

$$\zeta = s - \beta_0 ct\,. \tag{2.3}$$

We call $K$ the lab reference frame in which we have defined all equations above, and we introduce a boosted frame $K'$ moving rigidly with the reference particle. The coordinates in the two systems are related by a Lorentz transformation [**?**]:

$$ct' = \gamma_0\left(ct - \beta_0 s\right) \tag{2.4}$$
$$x' = x \tag{2.5}$$
$$y' = y \tag{2.6}$$
$$s' = \gamma_0\left(s - \beta_0 ct\right) = \gamma_0\zeta \tag{2.7}$$

The corresponding inverse transformation is:

$$ct = \gamma_0\left(ct' + \beta_0 s'\right) \tag{2.8}$$
$$x = x' \tag{2.9}$$
$$y = y' \tag{2.10}$$
$$s = \gamma_0\left(s' + \beta_0 ct'\right) \tag{2.11}$$

The quantities $\left(c\rho, J_x, J_y, J_s\right)$ form a Lorentz 4-vector and therefore they are transformed between $K$ and $K'$ by relationships similar to the Eqs. 2.4-2.6 [**?**]:

$$c\rho'\left(\mathbf{r'},t'\right) = \gamma_0\left[c\rho\left(\mathbf{r}\left(\mathbf{r'},t'\right),t\left(\mathbf{r'},t'\right)\right) - \beta_0 J_s\left(\mathbf{r}\left(\mathbf{r'},t'\right),t\left(\mathbf{r'},t'\right)\right)\right] \tag{2.12}$$
$$J_s'\left(\mathbf{r'},t'\right) = \gamma_0\left[J_s\left(\mathbf{r}\left(\mathbf{r'},t'\right),t\left(\mathbf{r'},t'\right)\right) - \beta_0 c\rho\left(\mathbf{r}\left(\mathbf{r'},t'\right),t\left(\mathbf{r'},t'\right)\right)\right] \tag{2.13}$$

where the transformations $\mathbf{r}\left(\mathbf{r'}, t'\right)$ and $t\left(\mathbf{r'}, t'\right)$ are defined by Eqs. 2.8 and 2.11 respectively. The transverse components $J_x$ and $J_y$ of the current vector are invariant for our transformation, and are anyhow zero in our case.
Using Eq. 2.2 these become:

$$\rho'\left(\mathbf{r'}, t'\right) = \frac{1}{\gamma_0}\rho\left(\mathbf{r}\left(\mathbf{r'}, t'\right), t\left(\mathbf{r'}, t'\right)\right) \tag{2.14}$$

$$J'_s\left(\mathbf{r'}, t'\right) = 0 \tag{2.15}$$

Using Eqs. 2.1 and 2.8-2.10, we obtain:

$$\rho\left(x', y', s(s', t'), t(s', t')\right) = \rho_0\left(x', y', s(s', t') - \beta_0 c\, t(s', t')\right) \tag{2.16}$$

From Eq. 2.7 we get:

$$s(s', t') - \beta_0 c\, t(s', t') = \frac{s'}{\gamma_0} \tag{2.17}$$

where the coordinate $t'$ has disappeared.
We can therefore write:

$$\rho'\left(x', y', s', t'\right) = \frac{1}{\gamma_0}\rho_0\left(x', y', \frac{s'}{\gamma_0}\right) \tag{2.18}$$

The electric potential in the bunch frame is solution of Poisson's equation:

$$\frac{\partial^2\phi'}{\partial x'^2} + \frac{\partial^2\phi'}{\partial y'^2} + \frac{\partial^2\phi'}{\partial s'^2} = -\frac{\rho'(x', y', s')}{\varepsilon_0} \tag{2.19}$$

From Eq. 2.18 we can write:

$$\frac{\partial^2\phi'}{\partial x'^2} + \frac{\partial^2\phi'}{\partial y'^2} + \frac{\partial^2\phi'}{\partial s'^2} = -\frac{1}{\gamma_0\varepsilon_0}\rho_0\left(x', y', \frac{s'}{\gamma_0}\right) \tag{2.20}$$

We now make the substitution:

$$\zeta = \frac{s'}{\gamma_0} \tag{2.21}$$

obtained from Eq. 2.7, which allows to rewrite Eq. 2.20 as:

$$\frac{\partial^2\phi'}{\partial x^2} + \frac{\partial^2\phi'}{\partial y^2} + \frac{1}{\gamma_0^2}\frac{\partial^2\phi'}{\partial \zeta^2} = -\frac{1}{\gamma_0\varepsilon_0}\rho_0\left(x, y, \zeta\right) \tag{2.22}$$

Here we have dropped the "'" sign from $x$ and $y$ as these coordinates are unaffected by the Lorentz boost.
The quantities $\left(\frac{\phi}{c}, A_x, A_y, A_s\right)$ form a Lorentz 4-vector, so we can write:

$$\phi = \gamma_0\left(\phi' + \beta_0 c A'_s\right) \tag{2.23}$$

$$A_s = A'_s + \beta_0\frac{\phi'}{c} \tag{2.24}$$

In the bunch frame the charges are at rest therefore $A'_x = A'_y = A'_s = 0$ therefore:

$$\phi = \gamma_0 \phi' \tag{2.25}$$

$$A_s = \beta_0 \frac{\phi'}{c} = \frac{\beta_0}{\gamma_0 c} \phi \tag{2.26}$$

Combining Eq. 2.25 with Eq. 2.22 we obtain the equation in $\phi$:

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{1}{\gamma_0^2} \frac{\partial^2 \phi}{\partial \zeta^2} = -\frac{1}{\varepsilon_0} \rho_0 \left(x, y, \zeta\right)} \tag{2.27}$$

### 2.1.1 2.5D approximation

For large enough values of $\gamma_0$, Eq. 2.22 can be approximated by:

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\frac{1}{\varepsilon_0} \rho_0 \left(x, y, \zeta\right)} \tag{2.28}$$

which means that we can solve a simple 2D problem for each beam slice (identified by its coordinate $\zeta$).

### 2.1.2 Modulated 2D

Often the beam distribution can be factorized as:

$$\rho_0(x, y, \zeta) = q_0 \lambda_0(\zeta) \rho_\perp(x, y) \tag{2.29}$$

where:

$$\int \rho_\perp(x, y)\, dx\, dy = 1 \tag{2.30}$$

and $\lambda_0(z)$ is therefore the bunch line density.
For a bunched beam:

$$\int \lambda_0(z)\, dz = N \tag{2.31}$$

where $N$ is the bunch population.
In this case the potential can be factorized as:

$$\phi(x, y, \zeta) = q_0 \lambda(\zeta) \phi_\perp(x, y) \tag{2.32}$$

where $\phi_\perp(x, y)$ is the solution of the following 2D Poisson equation:

$$\frac{\partial^2 \phi_\perp}{\partial x^2} + \frac{\partial^2 \phi_\perp}{\partial y^2} = -\frac{1}{\varepsilon_0} \rho_\perp \left(x, y\right) \tag{2.33}$$

## 2.2 Lorentz force

We now compute the Lorentz force on the particles moving in the longitudinal directions, including particles of the bunch itself (space charge forces) and particles of a colliding bunch moving in the opposite directions (beam-beam forces). The angles of such test particles are neglected as done in the usual thin-lens approximation. Therefore the velocity of a test particle can be written as:

$$\mathbf{v} = \beta c\, \hat{\mathbf{i}}_s \tag{2.34}$$

The Lorenz force can be written as:

$$\begin{aligned} \mathbf{F} &= q\left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \beta c\, \hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A})\right) \\ &= q\left(-\nabla\phi - \frac{\beta_0}{\gamma_0 c}\frac{\partial \phi}{\partial t}\hat{\mathbf{i}}_s + \beta c\, \hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A})\right) \end{aligned} \tag{2.35}$$

We compute the vector product:

$$\begin{aligned} \hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A}) &= \left(\frac{\partial A_s}{\partial x} - \frac{\partial A_x}{\partial s}\right)\hat{\mathbf{i}}_x + \left(\frac{\partial A_s}{\partial y} - \frac{\partial A_y}{\partial s}\right)\hat{\mathbf{i}}_y \\ &= \left(\frac{\partial A_s}{\partial x} - \frac{\partial A_x}{\partial s}\right)\hat{\mathbf{i}}_x + \left(\frac{\partial A_s}{\partial y} - \frac{\partial A_y}{\partial s}\right)\hat{\mathbf{i}}_y + \underbrace{\left(\frac{\partial A_s}{\partial s} - \frac{\partial A_s}{\partial s}\right)}_{=0}\hat{\mathbf{i}}_s \\ &= \nabla A_s - \frac{\partial \mathbf{A}}{\partial s} \end{aligned} \tag{2.36}$$

We replace:

$$\mathbf{F} = q\left(-\nabla\phi - \frac{\beta_0}{\gamma_0 c}\frac{\partial \phi}{\partial t}\hat{\mathbf{i}}_s + \beta\beta_0\nabla\phi - \frac{\beta\beta_0}{\gamma_0}\frac{\partial \phi}{\partial s}\hat{\mathbf{i}}_s\right) \tag{2.37}$$

The potentials will have the same form as the sources (this can be shown explicitly using the Lorentz transformations):

$$\phi(x, y, s, t) = \phi\left(x, y, t - \frac{s}{\beta_0 c}\right) \tag{2.38}$$

For a function in this form we can write:

$$\frac{\partial \phi}{\partial s} = \frac{\partial}{\partial \zeta} = -\frac{1}{\beta_0 c}\frac{\partial \phi}{\partial t} \tag{2.39}$$

obtaining:

$$\mathbf{F} = q\left(-\nabla\phi + \frac{\beta_0^2}{\gamma_0}\frac{\partial \phi}{\partial \zeta}\hat{\mathbf{i}}_s + \beta\beta_0\nabla\phi - \frac{\beta\beta_0}{\gamma_0}\frac{\partial \phi}{\partial \zeta}\hat{\mathbf{i}}_s\right) \tag{2.40}$$

Reorganizing:

$$\mathbf{F} = -q(1 - \beta\beta_0)\nabla\phi - \frac{\beta_0(\beta - \beta_0)}{\gamma_0}\frac{\partial \phi}{\partial \zeta}\hat{\mathbf{i}}_s \tag{2.41}$$

Writing the dependencies explicitly:

$$F_x(x,y,\zeta(t)) = -q(1-\beta\beta_0)\frac{\partial\phi}{\partial x}(x,y,\zeta(t)) \tag{2.42}$$

$$F_y(x,y,\zeta(t)) = -q(1-\beta\beta_0)\frac{\partial\phi}{\partial y}(x,y,\zeta(t)) \tag{2.43}$$

$$F_z(x,y,\zeta(t)) = -q\left(1-\beta\beta_0-\frac{\beta_0(\beta-\beta_0)}{\gamma_0}\right)\frac{\partial\phi}{\partial\zeta}(x,y,\zeta(t)) \tag{2.44}$$

where $\zeta(t)$ is the position of the particle within the bunch.

## 2.3 Space charge

Over the single interaction we neglect the particle slippage[1]:

$$\beta = \beta_0 \tag{2.45}$$

$$\zeta(t) = \zeta \tag{2.46}$$

This gives the following simplification of Eqs. (2.42) - (2.44):

$$F_x(x,y,\zeta) = -q(1-\beta_0^2)\frac{\partial\phi}{\partial x}(x,y,\zeta) \tag{2.47}$$

$$F_y(x,y,\zeta) = -q(1-\beta_0^2)\frac{\partial\phi}{\partial y}(x,y,\zeta) \tag{2.48}$$

$$F_z(x,y,\zeta) = -q(1-\beta_0^2)\frac{\partial\phi}{\partial\zeta}(x,y,\zeta) \tag{2.49}$$

In this way the force over the single interaction becomes independent on time and therefore we can compute the kicks simply as:

$$\Delta\mathbf{P} = \frac{L}{\beta_0 c}\mathbf{F} \tag{2.50}$$

where $L$ is the portion of the machine on which we want to compute the e-cloud interaction.
The kicks on the normalized momenta can be expressed as (recalling that $P_0 = m_0\beta_0\gamma_0 c$):

$$\boxed{\Delta p_x = \frac{m_0}{m}\frac{\Delta P_x}{P_0} = -\frac{qL(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\frac{\partial\phi}{\partial x}(x,y,\zeta)} \tag{2.51}$$

$$\boxed{\Delta p_y = \frac{m_0}{m}\frac{\Delta P_y}{P_0} = -\frac{qL(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\frac{\partial\phi}{\partial y}(x,y,\zeta)} \tag{2.52}$$

$$\boxed{\Delta\delta \simeq \Delta p_z = \frac{m_0}{m}\frac{\Delta P_z}{P_0} = -\frac{qL(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\frac{\partial\phi}{\partial\zeta}(x,y,\zeta)} \tag{2.53}$$

[1] In any case one would need to take into account also the dispersion in order to have the right slippage.

If the beam includes particles of different species (tracking of fragments), note that here $q$ and $m$ refer to the individual particle while $m_0$ is the mass of the reference particle.
In the modulated 2D case (see Sec. 2.1.2 and in particular Eq. 2.32), the kick can be expressed as:

$$\boxed{\Delta p_x = \frac{m_0}{m}\frac{\Delta P_x}{P_0} = -\frac{qq_0L(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\,\lambda_0(\zeta)\,\frac{\partial\phi_\perp}{\partial x}\,(x,y)} \tag{2.54}$$

$$\boxed{\Delta p_y = \frac{m_0}{m}\frac{\Delta P_y}{P_0} = -\frac{qq_0L(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\,\lambda_0(\zeta)\,\frac{\partial\phi_\perp}{\partial y}\,(x,y)} \tag{2.55}$$

$$\boxed{\Delta\delta \simeq \Delta p_z = \frac{m_0}{m}\frac{\Delta P_z}{P_0} = -\frac{qq_0L(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\,\frac{d\lambda_0}{d\zeta}(\zeta)\,\phi_\perp\,(x,y)} \tag{2.56}$$

## 2.4 Beam-beam interaction

We consider a test particle moving in the opposite direction with velocity:

$$\mathbf{v}_W = -\beta_{0W}c\,\hat{\mathbf{i}}_s \tag{2.57}$$
$$s_W(t) = -\beta_{0W}ct \tag{2.58}$$

Equations (2.42) - (2.44) become:

$$F_x(x,y,\zeta_W(t)) = -q(1+\beta_{0W}\beta_{0s})\frac{\partial\phi}{\partial x}(x,y,\zeta_W(t)) \tag{2.59}$$
$$F_y(x,y,\zeta_W(t)) = -q(1+\beta_{0W}\beta_{0S})\frac{\partial\phi}{\partial y}(x,y,\zeta_W(t)) \tag{2.60}$$
$$F_z(x,y,\zeta_W(t)) = -q\left(1+\beta_{0W}\beta_{0S}-\frac{\beta_{0S}(\beta_{0W}+\beta_{0S})}{\gamma_0}\right)\frac{\partial\phi}{\partial\zeta}(x,y,\zeta_W(t)) \tag{2.61}$$

where we have used the the subscript $S$ (strong) for the bunch generating the fields, and the subscript $W$ (weak) for the test particle.
$\zeta_W(t)$ is the position of the test particle within the bunch generating the fields:

$$\zeta_W(t) = s_W(t) - \beta_{0S}ct = -(\beta_{0W}+\beta_{0S})ct \tag{2.62}$$

In modulated-2D case (Eq. 2.32), Eqs. (2.59) - (2.60) become:

$$F_x(x,y,\zeta_W(t)) = -qq_{0S}(1+\beta_{0W}\beta_{0s})\lambda_{0S}(\zeta_W(t))\frac{\partial\phi_\perp}{\partial x}(x,y) \tag{2.63}$$
$$F_y(x,y,\zeta_W(t)) = -qq_{0S}(1+\beta_{0W}\beta_{0s})\lambda_{0S}(\zeta_W(t))\frac{\partial\phi_\perp}{\partial y}(x,y) \tag{2.64}$$
$$F_z(x,y,\zeta_W(t)) = -qq_{0S}\left(1+\beta_{0W}\beta_{0S}-\frac{\beta_{0S}(\beta_{0W}+\beta_{0S})}{\gamma_0}\right)\frac{d\lambda_{0S}}{d\zeta}(\zeta_W(t))\phi_\perp(x,y) \tag{2.65}$$

The change in momentum for the test particle is given by:

$$\Delta \mathbf{P} = \int_{-\infty}^{+\infty} \mathbf{F}(t)\, dt \tag{2.66}$$

Therefore:

$$\Delta P_x(x,y,\zeta_W(t)) = -qq_{0S}N_S(1+\beta_{0W}\beta_{0s})\frac{\partial \phi_\perp}{\partial x}(x,y)\int_{-\infty}^{+\infty}\lambda_{0S}(\zeta_W(t))\, dt \tag{2.67}$$

$$\Delta P_y(x,y,\zeta_W(t)) = -qq_{0S}N_S(1+\beta_{0W}\beta_{0s})\frac{\partial \phi_\perp}{\partial y}(x,y)\int_{-\infty}^{+\infty}\lambda_{0S}(\zeta_W(t))\, dt \tag{2.68}$$

$$\Delta P_z(x,y,\zeta_W(t)) = -qq_{0S}\left(1+\beta_{0W}\beta_{0S}-\frac{\beta_{0S}(\beta_{0W}+\beta_{0S})}{\gamma_0}\right)\phi_\perp(x,y)\int_{-\infty}^{+\infty}\frac{d\lambda_{0S}}{d\zeta}(\zeta_W(t))\, dt \tag{2.69}$$

Using Eq. (2.62) and Eq. (2.31) we can write:

$$\int_{-\infty}^{+\infty}\lambda_{0S}(\zeta_W(t))\, dt = \frac{1}{(\beta_{0W}+\beta_{0S})c}\int_{-\infty}^{+\infty}\lambda_{0S}(\zeta)\, d\zeta = \frac{N_S}{(\beta_{0W}+\beta_{0S})c} \tag{2.70}$$

Similarly, for a bunched beam:

$$\int_{-\infty}^{+\infty}\frac{d\lambda_{0S}}{d\zeta}(\zeta_W(t))\, dt = \frac{1}{(\beta_{0W}+\beta_{0S})c}\int_{-\infty}^{+\infty}\frac{d\lambda_{0S}}{d\zeta}\, d\zeta = \frac{\lambda_{0S}(+\infty)-\lambda_{0S}(-\infty)}{(\beta_{0W}+\beta_{0S})c} = 0 \tag{2.71}$$

From which we can write:

$$\boxed{\Delta p_x = \frac{m_0}{m}\frac{\Delta P_x}{P_0} = -\frac{qq_{0S}N_S}{m\beta_{0W}\gamma_{0W}c^2}\frac{(1+\beta_{0W}\beta_{0s})}{(\beta_{0W}+\beta_{0S})}\frac{\partial \phi_\perp}{\partial x}(x,y)} \tag{2.72}$$

$$\boxed{\Delta p_y = \frac{m_0}{m}\frac{\Delta P_y}{P_0} = -\frac{qq_{0S}N_S}{m\beta_{0W}\gamma_{0W}c^2}\frac{(1+\beta_{0W}\beta_{0s})}{(\beta_{0W}+\beta_{0S})}\frac{\partial \phi_\perp}{\partial y}(x,y)} \tag{2.73}$$

$$\boxed{\Delta p_z = \frac{m_0}{m}\frac{\Delta P_z}{P_0} = 0} \tag{2.74}$$

## 2.5 Electron lens

$I_{\text{elens}}$ is the current, $L$ is the length, $\beta_e$ the relativistic beta of the electrons.

$$\rho_e(x,y) = \lambda_e \rho_\perp(x,y) \tag{2.75}$$

$$I_{\text{elens}} = \int J_e(x,y)dA = \beta_e c\int \rho_e(x,y)dA = \beta_e\lambda_e c\int \rho_\perp(x,y)dA \tag{2.76}$$

$$I_{\text{elens}} = \beta_e\lambda_e c \tag{2.77}$$

$$\lambda_e = \frac{I_{\text{elens}}}{\beta_e c} \tag{2.78}$$

$$\rho_e(x,y) = \frac{I_{\text{elens}}}{\beta_e c}\rho_\perp(x,y) \tag{2.79}$$

Hence:

$$\phi_e(x,y) = \frac{I_{\text{elens}}}{\beta_e c}\phi_\perp(x,y) \tag{2.80}$$

$$F_x(x,y) = -q_0(1-\beta_0\beta_e)\frac{\partial \phi_e}{\partial x}(x,y \tag{2.81}$$

$$F_y(x,y) = -q_0(1-\beta_0\beta_e)\frac{\partial \phi_e}{\partial y}(x,y) \tag{2.82}$$

$$\tag{2.83}$$

$$F_x(x,y) = -q_0\frac{I_{\text{elens}}}{\beta_e c}(1-\beta_0\beta_e)\frac{\partial \phi_\perp}{\partial x}(x,y) \tag{2.84}$$

$$F_y(x,y) = -q_0\frac{I_{\text{elens}}}{\beta_e c}(1-\beta_0\beta_e)\frac{\partial \phi_\perp}{\partial y}(x,y) \tag{2.85}$$

$$\tag{2.86}$$

$$\boxed{\Delta p_x = \frac{1}{P_0\beta_0 c}F_x(x,y)L = -\frac{I_{\text{elens}}L}{m_0\gamma_0\beta_0^2 c^3}q_0\frac{(1-\beta_0\beta_e)}{\beta_e}\frac{\partial \phi_\perp}{\partial x}(x,y)} \tag{2.87}$$

## 2.6 Longitudinal profiles

### 2.6.1 Gaussian profile

The profile is in the form:

$$\lambda_0(z) = \frac{N}{\sqrt{2\pi}\sigma}e^{-\frac{(z-z_0)^2}{2\sigma^2}} \tag{2.88}$$

### 2.6.2 q-Gaussian

The profile is in the form:

$$\lambda_0(z) = \frac{N\sqrt{\beta}}{C_q}e_q\left(-\beta(z-z_0)^2\right) \tag{2.89}$$

where $e_q$ is the q-exponential function:

$$e_q(x) = [1+(1-q)x]_+^{\frac{1}{1-q}} \tag{2.90}$$

$C_q$ is a normalization factor dependent on $q$ alone:

$$C_q = \frac{\sqrt{\pi}\Gamma\left(\frac{3-q}{2(q-1)}\right)}{\sqrt{q-1}\Gamma\left(\frac{1}{q-1}\right)} \tag{2.91}$$

The parameter beta defines the standard deviation of the distribution:

$$\sigma = \sqrt{\frac{1}{\beta(5-3q)}} \iff \beta = \frac{1}{\sigma^2(5-3q)} \tag{2.92}$$

These expressions are valid for values of the parameter $q$ is the range of interest:

$$1 < q < \frac{5}{3} \tag{2.93}$$

In general the q-Gaussian is defined outside this range, but for smaller values it has a limited support (not of interest) and for larger values has a not defined standard deviation.

## 2.7 FFT Poisson solver

### 2.7.1 Notation for Discrete Fourier Transform

We will use the following notation for the Discrete Fourier Transform of a sequence of length $M$:

$$\hat{a}_k = \mathrm{DFT}_M(a_m) = \sum_{m=0}^{M-1} a_m \, e^{-j2\pi\frac{km}{M}} \quad \text{for } k \in 0, ..., M \tag{2.94}$$

The corresponding inverse transform is defined as:

$$a_n = \mathrm{DFT}_M^{-1}(\hat{a}_k) = \frac{1}{M}\sum_{k=0}^{M-1} \hat{a}_k \, e^{j2\pi\frac{km}{M}} \quad \text{for } m \in 0, ..., M \tag{2.95}$$

Multidimensional Discrete Fourier Transforms are obtained by applying sequentially 1D DFTs.. For example, in two dimensions:

$$\begin{aligned} \hat{a}_{k_x k_y} &= \mathrm{DFT}_{M_x M_y}\left\{a_{m_x m_y}\right\} = \mathrm{DFT}_{M_y}\left\{\mathrm{DFT}_{M_x}\left\{a_{m_x m_y}\right\}\right\} \\ &= \sum_{m_x=0}^{M_x-1} e^{-j2\pi\frac{k_x m_x}{M_x}} \sum_{m_y=0}^{M_y-1} e^{-j2\pi\frac{k_y m_y}{M_y}} a_{m_x m_y} \end{aligned} \tag{2.96}$$

$$\begin{aligned} a_{n_x n_y} &= \mathrm{DFT}^{-1}_{M_x M_y}\left\{a_{k_x k_y}\right\} = \mathrm{DFT}^{-1}_{M_y}\left\{\mathrm{DFT}^{-1}_{M_x}\left\{\hat{a}_{k_x k_y}\right\}\right\} \\ &= \frac{1}{M_x M_y}\sum_{k_x=0}^{M_x-1} e^{j2\pi\frac{k_x m_x}{M_x}} \sum_{k_y=0}^{M_y-1} e^{j2\pi\frac{k_y m_y}{M_y}} \hat{a}_{k_x k_y} \end{aligned} \tag{2.97}$$

### 2.7.2 FFT convolution - 1D case

The potential can be written as the convolution of a Green function with the charge distribution:

$$\phi(x) = \int_{-\infty}^{+\infty} \rho(x')\, G(x - x')dx' \tag{2.98}$$

We assume that the source is limited to the region $[0, L]$:

$$\rho(x) = \rho(x)\, \Pi_{[0,L]}\,(x) \tag{2.99}$$

where $\Pi_{[a,b]}(x)$ is a rectangular window function defined as:

$$\Pi_{[a,b]}(x) = \begin{cases} 1 & \text{for } x \in [a, b] \\ 0 & \text{elsewhere} \end{cases} \tag{2.100}$$

We are interested in the electric potential only the region occupied by the sources, so we can compute:

$$\phi_L(x) = \phi(x)\Pi_{[0,L]}\left(\frac{x}{L}\right) \tag{2.101}$$

We replace Eq. (2.99) and Eq. (2.101) into Eq.(2.98), obtaining:

$$\phi_L(x) = \Pi_{[0,L]}\,(x) \int_{-\infty}^{+\infty} \Pi_{[0,L]}\,(x')\, \rho(x')\, G(x - x')dx' \tag{2.102}$$

We apply the change of variable $x'' = x - x'$:

$$\phi_L(x) = \int_{-\infty}^{+\infty} \Pi_{[0,L]}\,(x)\, \Pi_{[0,L]}\,(x - x'')\, \rho(x - x'')\, G(x'')\, dx'' \tag{2.103}$$

The integrand vanishes outside the set of the $(x, x'')$ defined by:

$$\begin{cases} 0 < x < L \\ 0 < (x - x'') < L \end{cases} \tag{2.104}$$

We flip the signs in the second equation, obtaining:

$$\begin{cases} 0 < x < L \\ -L < (x'' - x) < 0 \end{cases} \tag{2.105}$$

Combining the two equations we obtain:

$$-L < -L + x < x'' < x < L \tag{2.106}$$

i.e. the integrand is zero for $-L < x'' < L$. Therefore in Eq. (2.103) we can replace $G(x'')$ with its truncated version:

$$G_{2L}(x'') = G(x'')\, \Pi_{[-L,L]}\,(x'') \tag{2.107}$$

obtaining:

$$\phi_L(x) = \int_{-\infty}^{+\infty} \Pi_{[0,L]}\left(\frac{x}{L}\right) \Pi_{[0,L]}\left(\frac{x-x''}{L}\right) \rho(x-x'')\, G_{2L}(x'')dx'' \tag{2.108}$$

Since the two window functions force the integrand to zero outside the region $|x''| < L$ we can replace $G_{2L}(x'')$ with its replicated version:

$$G_{2LR}(x'') = \sum_{n=-\infty}^{+\infty} G_{2L}(x''-2nL) = \sum_{n=-\infty}^{+\infty} G(x''-2nL)\, \Pi_{[-L,L]}\left(\frac{x''-2nL}{2L}\right) \tag{2.109}$$

obtaining:

$$\phi_L(x) = \int_{-\infty}^{+\infty} \Pi_{[0,L]}\left(\frac{x}{L}\right) \Pi_{[0,L]}\left(\frac{x-x''}{L}\right) \rho(x-x'')\, G_{2LR}(x'')dx'' \tag{2.110}$$

We can go back to the initial coordinate by substituting $x'' = x - x'$:

$$\phi_L(x) = \Pi_{[0,L]}\left(\frac{x}{L}\right) \int_{-\infty}^{+\infty} \rho(x')\, G_{2LR}(x-x')dx' \tag{2.111}$$

This is a cyclic convolution, so we can proceed as follows. We split the integral:

$$\phi_L(x) = \Pi_{[0,L]}\left(\frac{x}{L}\right) \sum_{n=-\infty}^{+\infty} \int_{2nL}^{2(n+1)L} \rho(x')\, G_{2LR}(x-x')\, dx' \tag{2.112}$$

In each term we replace $x''' = x' + 2nL$:

$$\phi_L(x) = \Pi_{[0,L]}\left(\frac{x}{L}\right) \sum_{n=-\infty}^{+\infty} \int_{0}^{2L} \rho(x'''-2nL)\, G_{2LR}(x-x'''-2nL)\, dx''' \tag{2.113}$$

We use the fact that $G_{2LR}(x)$ is periodic:

$$\begin{aligned} \phi_L(x) &= \Pi_{[0,L]}\left(\frac{x}{L}\right) \sum_{n=-\infty}^{+\infty} \int_{0}^{2L} \rho(x'''-2nL)\, G_{2LR}(x-x''')dx''' \\ &= \Pi_{[0,L]}\left(\frac{x}{L}\right) \int_{0}^{2L} \sum_{n=-\infty}^{+\infty} \rho(x'''-2nL)\, G_{2LR}(x-x''')dx''' \end{aligned} \tag{2.114}$$

We can define a replicated version of $\rho(x)$:

$$\rho_{2LR}(x) = \sum_{n=-\infty}^{+\infty} \rho(x-2nL) \tag{2.115}$$

noting that this implies:

$$\rho_{2LR}(x) = 0 \quad \text{for } x \in [L, 2L] \tag{2.116}$$

We obtain:

$$\phi_L(x) = \Pi_{[0,L]}\left(\frac{x}{L}\right) \int_{0}^{2L} \rho_{2LR}(x')\, G_{2LR}(x-x')dx' \tag{2.117}$$

The function:

$$\phi_{2LR}(x) = \int_0^{2L} \rho_{2LR}(x')\, G_{2LR}(x - x')dx' \tag{2.118}$$

is periodic of period $2L$. From it the potential of interest can be simply calculated by selecting the first half period $[0, L]$:

$$\phi_L(x) = \Pi_{[0,L]}\left(\frac{x}{L}\right) \phi_{2LR}(x) \tag{2.119}$$

To compute the convolution in Eq. 2.118 we expand $\phi_{2LR}(x)$ in Fourier series:

$$\phi_{2LR}(x) = \sum_{k=-\infty}^{+\infty} \tilde{\phi}_k\, e^{j2\pi k \frac{x}{2L}} \tag{2.120}$$

where the Fourier coefficients are given by:

$$\tilde{\phi}_k = \frac{1}{2L} \int_0^{2L} \phi_{2LR}(x)\, e^{-j2\pi k \frac{x}{2L}}\, dx \tag{2.121}$$

We replace Eq. (2.118) into Eq. (2.121) obtaining:

$$\hat{\phi}_k = \frac{1}{2L} \int_0^{2L} \int_0^{2L} \rho_{2LR}(x')\, G_{2LR}(x - x')\, e^{-j2\pi k \frac{x}{2L}}\, dx'\, dx \tag{2.122}$$

With the change of variable $x'' = x - x'$ we obtain:

$$\tilde{\phi}_k = \frac{1}{2L} \int_0^{2L} \rho_{2LR}(x') e^{-j2\pi k \frac{x'}{2L}} dx' \int_0^{2L} G_{2LR}(x'') e^{-j2\pi k \frac{x''}{2L}}\, dx'' \tag{2.123}$$

where we recognize the Fourier coefficients of $\rho_{2LR}(x)$ and $G_{2LR}(x)$:

$$\tilde{\rho}_k = \frac{1}{2L} \int_0^{2L} \rho_{2LR}(x)\, e^{-j2\pi k \frac{x}{2L}}\, dx \tag{2.124}$$

$$\tilde{G}_k = \frac{1}{2L} \int_0^{2L} G_{2LR}(x)\, e^{-j2\pi k \frac{x}{2L}}\, dx \tag{2.125}$$

obtaining simply:

$$\hat{\phi}_k = 2L\, \hat{G}_k\, \hat{\rho}_k \tag{2.126}$$

I assume to have the functions $\rho_{2LR}(x)$ and $G_{2LR}(x)$ sampled (or averaged) with step:

$$h_x = \frac{2L}{M} = \frac{L}{N} \tag{2.127}$$

I can approximate the integrals in Eqs. (2.124) and (2.125) as:

$$\tilde{\rho}_k = \frac{1}{M} \sum_{n=0}^{M-1} \rho_{2LR}(x_n)\, e^{-j2\pi \frac{kn}{M}} = \frac{1}{M} \hat{\rho}_k \tag{2.128}$$

$$\tilde{G}_k = \frac{1}{M} \sum_{n=0}^{M-1} G_{2LR}(x_n)\, e^{-j2\pi \frac{kn}{M}} = \frac{1}{M} \hat{G}_k \tag{2.129}$$

where we recognize the Discrete Fourier Transforms:

$$\hat{\rho}_k = \mathrm{DFT}_M \left\{ \rho_{2LR}(x_n) \right\} \tag{2.130}$$

$$\hat{G}_k = \mathrm{DFT}_M \left\{ G_{2LR}(x_n) \right\} \tag{2.131}$$

Using Eq. (2.120) we can obtain a sampled version of $\phi(x)$:

$$\phi_{2LR}(x_n) = \sum_{n=0}^{M-1} \tilde{\phi}_k \, e^{j2\pi \frac{kn}{M}} \tag{2.132}$$

where we have assumed that $\phi(x)$ is sufficiently smooth to allow truncating the sum. Using Eqs. (2.126), (2.128) and (2.129) we obtain:

$$\phi_{2LR}(x_n) = 2L \sum_{n=0}^{M-1} \tilde{G}_k \, \tilde{\rho}_k \, e^{j2\pi \frac{kn}{M}} = \frac{2L}{M^2} \sum_{n=0}^{M-1} \hat{G}_k \, \hat{\rho}_k \, e^{j2\pi \frac{kn}{M}} \tag{2.133}$$

This can be rewritten as:

$$\phi_{2LR}(x_n) = \frac{1}{M} \sum_{n=0}^{M-1} (h_x \hat{G}_k) \, \hat{\rho}_k \, e^{j2\pi \frac{kn}{M}} = \mathrm{DFT}_M^{-1} \left\{ \phi_k \right\} \tag{2.134}$$

where

$$\hat{\phi}_k = h_x \hat{G}_k \, \hat{\rho}_k \tag{2.135}$$

We call "Integrated Green Function" the quantity:

$$G_{2LR}(x_n) = h_x G_{2LR}(x_n) \tag{2.136}$$

we introduce the corresponding Fourier transform:

$$\hat{G}_k^{\mathrm{int}} = \mathrm{DFT}_M \left\{ G_{2LR}^{\mathrm{int}}(x_n) \right\} \tag{2.137}$$

Eq. (2.135) can be rewritten as:

$$\boxed{\hat{\phi}_k = \hat{G}_k^{\mathrm{int}} \, \hat{\rho}_k} \tag{2.138}$$

In summary the potential at the grid nodes can be computed as follows:

1. We compute the Integrated Green function at the grid points in the range $[0, L]$:

$$G_{2LR}^{\mathrm{int}}(x_n) = \int_{x_n - \frac{h_x}{2}}^{x_n + \frac{h_x}{2}} G(x) dx \tag{2.139}$$

2. We extend to the interval $[L, 2L]$ using the fact that in this interval:

$$G_{2LR}^{\mathrm{int}}(x_n) = G_{2LR}^{\mathrm{int}}(x_n - 2L) = G_{2LR}^{\mathrm{int}}(2L - x_n) \tag{2.140}$$

where the first equality comes from the periodicity of $G_{2LR}^{\mathrm{int}}(x)$ and the second from the fact that $G(x)$ is an even function (i.e. $G(x) = G(-x)$). Note that for $x_n \in [L, 2L]$ we have that $2L - x_n \in [0, L]$ so we can reuse the values computed at the previous step.

3. We transform it:
$$\hat{G}_k^{\text{int}} = \text{DFT}_{2N}\left\{G_{2LR}(x_n)\right\} \tag{2.141}$$

4. We assume that we are given $\rho(x_n)$ in the interval $[0, L]$. From this we can obtain $\rho_{2LR}(x_n)$ over the interval $[0, 2L]$ simply extending the sequence with zeros (see Eq. (2.116)).

5. We transform it:
$$\hat{\rho}_k = \text{DFT}_{2N}\left\{\rho_{2LR}(x_n)\right\} \tag{2.142}$$

6. We compute the potential in the transformed domain:
$$\hat{\phi}_k = \hat{G}_k^{\text{int}}\hat{\rho}_k \quad \text{for } k \in [0, 2N] \tag{2.143}$$

7. We inverse-transform:
$$\phi_{2LR}(x_n) = \text{DFT}_{2N}^{-1}\left\{\hat{\phi}_k\right\} \tag{2.144}$$
which provides the physical potential in the range $[0, L]$:
$$\phi(x_n) = \phi_{2LR}(x_n) \quad \text{for } x_n \in [0, L] \tag{2.145}$$

### 2.7.3 Extension to multiple dimensionss

The procedure described above can be extended to multiple dimensions by applying the same reasoning for all coordinates. This gives the following procedure:

1. We compute the Integrated Green function at the grid points in the volume $[0, L_x] \times [0, L_y] \times [0, L_z]$:
$$G_{2LR}^{\text{int}}(x_{n_x}, y_{n_y}, z_{n_z}) = \int_{x_{n_x}-\frac{h_x}{2}}^{x_{n_x}+\frac{h_x}{2}} dx \int_{y_{n_y}-\frac{h_y}{2}}^{y_{n_y}+\frac{h_y}{2}} dy \int_{z_{n_z}-\frac{h_z}{2}}^{z_{n_z}+\frac{h_z}{2}} dz\, G(x, y, z) \tag{2.146}$$

2. We extend to the region $[0, 2L_x] \times [0, 2L_y] \times [0, 2L_z]$ using the fact that:
$$G_{2LR}^{\text{int}}(x_n, y_n, z_n) = G_{2LR}^{\text{int}}(x_n - 2L_x, y_n, z_n) = G_{2LR}^{\text{int}}(2L_x - x_n, y_n, z_n) \quad \text{for } x_n \in [L_x, 2L_x], y_n \in [0, 2L_y], z_n \in [0, 2L_z] \tag{2.147}$$

$$G_{2LR}^{\text{int}}(x_n, y_n, z_n) = G_{2LR}^{\text{int}}(x_n, y_n - 2L_y, z_n) = G_{2LR}^{\text{int}}(x_n, 2L_y - y_n, z_n) \quad \text{for } y_n \in [L_y, 2L_y], x_n \in [0, 2L_x], z_n \in [0, 2L_z] \tag{2.148}$$

$$G_{2LR}^{\text{int}}(x_n, y_n, z_n) = G_{2LR}^{\text{int}}(x_n, y_n, z_n - 2L_z) = G_{2LR}^{\text{int}}(x_n, y_n, 2L_z - z_n) \quad \text{for } z_n \in [L_z, 2L_z], x_n \in [0, 2L_x], y_n \in [0, 2L_y] \tag{2.149}$$

This allows reusing the values computed at the previous step.

3. We transform it:

$$\hat{G}^{\text{int}}_{k_x k_y k_z} = \text{DFT}_{2N_x 2N_y 2N_z} \{G_{2LR}(x_n, y_n, z_n)\} \tag{2.150}$$

4. We assume that we are given $\rho(x_n, y_n, z_n)$ in the region $[0, L_x] \times [0, L_y] \times [0, L_z]$. From this we can obtain $\rho_{2LR}(x_n)$ over the region $[0, 2L_x] \times [0, 2L_y] \times [0, 2L_z]$ simply extending the matrix with zeros (see Eq. (2.116)).

5. We transform it:

$$\hat{\rho}^{\text{int}}_{k_x k_y k_z} = \text{DFT}_{2N_x 2N_y 2N_z} \{\rho_{2LR}(x_n, y_n, z_n)\} \tag{2.151}$$

6. We compute the potential in the transformed domain:

$$\hat{\phi}_{k_x k_y k_z} = \hat{G}^{\text{int}}_{k_x k_y k_z} \hat{\rho}_{k_x k_y k_z} \quad \text{for } k_{x/y/z} \in [0, 2N_{x/y/z}] \tag{2.152}$$

7. We inverse-transform:

$$\phi_{2LR}(x_n, y_n, z_n) = \text{DFT}^{-1}_{2N_x 2N_y 2N_z} \left\{ \hat{\phi}_{k_x k_y k_z} \right\} \tag{2.153}$$

which provides the physical potential in the region $[0, L_x] \times [0, L_y] \times [0, L_z]$:

$$\phi(x_n, y_n, z_n) = \phi_{2LR}(x_n, y_n, z_n) \text{ for } (x_n, y_n, z_n) \in [0, L_x] \times [0, L_y] \times [0, L_z] \tag{2.154}$$

## 2.7.4 Green functions for 2D and 3D Poisson problems

### 2.7.4.1 3D Poisson problem, free space boundary conditions

For the equation:

$$\nabla^2 \phi(x, y, z) = -\frac{1}{\varepsilon_0} \rho(x, y, z) \tag{2.155}$$

where:

$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \tag{2.156}$$

the solution can be written as

$$\phi(x, y, z) = \iiint_{-\infty}^{+\infty} \rho(x', y', z')\, G(x - x', y - y', z - z')\, dx'\, dy'\, dz' \tag{2.157}$$

where:

$$G(x, y, z) = \frac{1}{4\pi\varepsilon_0} \frac{1}{\sqrt{x^2 + y^2 + z^2}} \tag{2.158}$$

The corresponding integrated Green function can be written as:

$$G^{\text{int}}_{2LR}(x_{n_x}, y_{n_y}, z_{n_z}) = \int_{x_{n_x}-\frac{h_x}{2}}^{x_{n_x}+\frac{h_x}{2}} dx \int_{y_{n_y}-\frac{h_y}{2}}^{y_{n_y}+\frac{h_y}{2}} dy \int_{z_{n_z}-\frac{h_z}{2}}^{x_{n_z}+\frac{h_z}{2}} dz\, G(x,y,z) \tag{2.159}$$

$$= +F\left(x_{n_x}+\frac{h_x}{2}, y_{n_x}+\frac{h_y}{2}, z_{n_x}+\frac{h_z}{2}\right) \tag{2.160}$$

$$-F\left(x_{n_x}+\frac{h_x}{2}, y_{n_x}+\frac{h_y}{2}, z_{n_x}-\frac{h_z}{2}\right) \tag{2.161}$$

$$-F\left(x_{n_x}+\frac{h_x}{2}, y_{n_x}-\frac{h_y}{2}, z_{n_x}+\frac{h_z}{2}\right) \tag{2.162}$$

$$+F\left(x_{n_x}+\frac{h_x}{2}, y_{n_x}-\frac{h_y}{2}, z_{n_x}-\frac{h_z}{2}\right) \tag{2.163}$$

$$-F\left(x_{n_x}-\frac{h_x}{2}, y_{n_x}+\frac{h_y}{2}, z_{n_x}+\frac{h_z}{2}\right) \tag{2.164}$$

$$+F\left(x_{n_x}-\frac{h_x}{2}, y_{n_x}+\frac{h_y}{2}, z_{n_x}-\frac{h_z}{2}\right) \tag{2.165}$$

$$+F\left(x_{n_x}-\frac{h_x}{2}, y_{n_x}-\frac{h_y}{2}, z_{n_x}+\frac{h_z}{2}\right) \tag{2.166}$$

$$-F\left(x_{n_x}-\frac{h_x}{2}, y_{n_x}-\frac{h_y}{2}, z_{n_x}-\frac{h_z}{2}\right) \tag{2.167}$$

where $F(x,y,z)$ is a primitive of $G(x,y,z)$, which can be obtained as:

$$F(x,y,z) = \int_{x_0}^{x} dx \int_{y_0}^{y} dy \int_{z_0}^{x} dz\, G(x,y,z) \tag{2.168}$$

with $(x_0, y_0, z_0)$ being an arbitrary starting point.
An expression for $F(x,y,z)$ is the following

$$F(x,y,z) = \frac{1}{4\pi\varepsilon_0} \iiint \frac{1}{\sqrt{x^2+y^2+z^2}} dxdydz \tag{2.169}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[-\frac{z^2}{2}\arctan\left(\frac{xy}{z\sqrt{x^2+y^2+z^2}}\right) - \frac{y^2}{2}\arctan\left(\frac{xz}{y\sqrt{x^2+y^2+z^2}}\right)\right. \tag{2.170}$$

$$-\frac{x^2}{2}\arctan\left(\frac{yz}{x\sqrt{x^2+y^2+z^2}}\right) + yz\ln\left(x+\sqrt{x^2+y^2+z^2}\right) \tag{2.171}$$

$$\left. +xz\ln\left(y+\sqrt{x^2+y^2+z^2}\right) + xy\ln\left(z+\sqrt{x^2+y^2+z^2}\right)\right] \tag{2.172}$$

Note that we need to choose the first cell center to be in (0,0,0) for evaluation of the integrated Green function. Therefore the cell edges have non zero coordinates and the denominators in the formula will always be non-vanishing.

#### 2.7.4.2 2D Poisson problem, free space boundary conditions

For the equation:

$$\nabla_{\perp}^2 \phi(x,y) = -\frac{1}{\varepsilon_0}\rho(x,y) \tag{2.173}$$

where:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \tag{2.174}$$

the solution can be written as

$$\phi(x,y) = \iint_{-\infty}^{+\infty} \rho(x',y')\, G(x-x', y-y')\, dx'\, dy' \tag{2.175}$$

where:

$$G(x,y) = -\frac{1}{4\pi\varepsilon_0} \log\left(\frac{x^2+y^2}{r_0^2}\right) \tag{2.176}$$

where $r_0$ is arbitrary constant which has no effect on the evaluated fields (changes the potential by an additive constant).
The corresponding integrated Green function can be written as:

$$G_{2LR}^{\text{int}}(x_{n_x}, y_{n_y}) = \int_{x_{n_x}-\frac{h_x}{2}}^{x_{n_x}+\frac{h_x}{2}} dx \int_{y_{n_y}-\frac{h_y}{2}}^{y_{n_y}+\frac{h_y}{2}} dy\, G(x,y,z) \tag{2.177}$$

$$= + F\left(x_{n_x} + \frac{h_x}{2}, y_{n_x} + \frac{h_y}{2}\right) \tag{2.178}$$

$$- F\left(x_{n_x} + \frac{h_x}{2}, y_{n_x} - \frac{h_y}{2}\right) \tag{2.179}$$

$$- F\left(x_{n_x} - \frac{h_x}{2}, y_{n_x} + \frac{h_y}{2}\right) \tag{2.180}$$

$$+ F\left(x_{n_x} - \frac{h_x}{2}, y_{n_x} - \frac{h_y}{2}\right) \tag{2.181}$$

$$\tag{2.182}$$

where $F(x,y)$ is a primitive of $G(x,y)$, which can be obtained as:

$$F(x,y) = \int_{x_0}^{x} dx \int_{y_0}^{y} dy\, G(x,y) \tag{2.183}$$

where $(x_0, y_0)$ is an arbitrary starting point.
An expression for $F(x,y,z)$ is the following (where we have chosen $r_0 = 1$:

$$F(x,y,z) = -\frac{1}{4\pi\varepsilon_0} \iint \ln\left(x^2+y^2\right) dx/, dy \tag{2.184}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[3xy - x^2 \arctan(y/x) - y^2 \arctan(x/y) - xy \ln\left(x^2+y^2\right)\right] \tag{2.185}$$

Note that we need to choose the first cell center to be in (0,0) for evaluation of the integrated Green function. Therefore the cell edges have non zero coordinates and the denominators in the formula will always be non-vanishing.

# Bibliography


[1] G. Ripken. Non-linear canonical equations of coupled synchro-betatron motion and their solutions within the framework of a non-linear six-dimensional (symplectic) tracking program for ultrarelativistic protons. Technical Report 85-084, DESY, 1985.

[2] D.P. Barber, G. Ripken, and F. Schmidt. A non-linear canonical formalism for the coupled synchro-betatron motion of protons with arbitrary energy. Technical Report 87-36, DESY, 1987.

[3] G. Ripken and F. Schmidt. A symplectic six-dimensional thin-lens formalism for tracking. Technical Report DESY 95-63 and CERN/SL/95-12(AP), DESY, CERN, 1995.

[4] K. Heinemann, G. Ripken, and F. Schmidt. Construction of nonlinear symplectic six-dimensional thin-lens maps by exponentiation. Technical Report 95-189, DESY, 1995.

[5] D.P. Barber, K. Heinemann, G. Ripken, and F. Schmidt. Symplectic thin-lens transfer maps for SIXTRACK: Treatment of bending magnets in terms of the exact hamiltonian. Technical Report 96-156, DESY, 1996.

[6] L.H.A. Leunissen, F. Schmidt, and G. Ripken. 6D beam-beam kick including coupled motion. Technical report, 2001.

[7] A. Latina and R. De Maria. RF multipole implementation. Technical report, CERN-ATS. 2012-088.

[8] E. Forest. *Beam Dynamics: A New Attitude and Framework*. Harcourt Academic Publisher, 1999.

[9] F. Willeke and G. Ripken. Methods of beam optics. Technical Report 88-114, DESY, 1988.

[10] F. C. Iselin. The mad program (methodical accelerator design) - physics methods manual. Technical Report CERN/SL/92-?? (AP), CERN, 1992.