Then

$$\phi(r) \sum_{\substack{d\mid r\\(d,n)=1}} d\mu\left(\frac{r}{d}\right)\frac{1}{\phi(d)} = \phi(r)\left[\sum_{d\mid r} - \sum_{q\mid(r,n)} \sum_{\substack{d\mid r\\q\mid d}} + \cdots\right]. \tag{6}$$

By (4)

$$\phi(r) \sum_{\substack{d\mid r\\c\mid d}} d\mu\left(\frac{r}{d}\right)\frac{1}{\phi(d)} = \phi(r) \sum_{d\mid(r/c)} cd\mu\left(\frac{r}{cd}\right)\frac{1}{\phi(c)\phi(d)} = c;$$

hence, by (6)

$$\phi(r) \sum_{\substack{d\mid r\\(d,n)=1}} d\mu\left(\frac{r}{d}\right)\frac{1}{\phi(d)} = 1 - q_1 - q_2 - \cdots - q_k + q_1q_2 + q_1q_3 + \cdots$$

$$= \prod_{j=1}^{k}(1 - q_j),$$

which is equal to the right side of (1) and our proof is complete.

**Reference**

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## CLASSROOM NOTES

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### THE VANDERMONDE MATRIX

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**1.** The explicit representation of the inverse to Vandermonde's matrix seems unavailable in the mathematical literature and generally inaccessible, though simple in form. Inverting this matrix is necessary in many areas of numerical analysis (in exponential approximation and quadrature sums, in addition to the applications we will present). The material which follows outlines a derivation for the inverse and relates matrices of Vandermonde-form to polynomial interpolation and least-square fitting of data, in a way suitable for classroom exercises.

The $n\times n$ Vandermonde matrix $V$ is given by

$$V = \begin{bmatrix} 1 & t_1 & t_1^2 \cdots t_1^{n-1} \\ \cdot & \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ 1 & t_n & t_n^2 \cdots t_n^{n-1} \end{bmatrix} = [t_i^{j-1}]. \tag{1}$$

There are two well-known facts concerning $V$ which appear in texts on matrix theory ([7] pp. 78 and 80; [1] p. 186):

(2) $$V \text{ is nonsingular if } \{t_i\}_{i=1}^{n} \text{ are distinct;}$$

(3) $$\det V = \prod_{n \geqq j > i \geqq 1} (t_j - t_i).$$

As a consequence of (2), an equation with $m$ distinct specified roots, $\{r_i\}_{i=1}^{m}$, can be written

(4) $$\det \begin{bmatrix} 1 & r_1 & r_1^2 & \cdots & r_1^m \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & r_m & r_m^2 & \cdots & r_m^m \\ 1 & x & x^2 & \cdots & x^m \end{bmatrix} = 0.$$

**2. Inverse of the Vandermonde matrix.** The determinant formed from an $n \times n$ Vandermonde determinant, det $V$, by deleting the $(k+1)$-th column ($k$th powers) and adjoining as a new $n$th column the $n$th powers of the $t_i$, det $V_k$, satisfies

(5) $$\det V_k = (a_{n-k})(\det V),$$

where $a_{n-k}$, is the $(n-k)$-th elementary symmetric function of the $n$ values of $t_i$ ([2] pp. 144–145; [8] p. 99). $a_r$ is the coefficient of $y^{n-r}$ in the expansion of $\prod_{s=1}^{n}(y+t_s)$; thus

$$a_r = \sum_{1 \leqq s_1 < s_2 < \cdots < s_r \leqq n} \prod_{q=1}^{r} t_{s_q}.$$

Let $a_{n-k}^{i}$ be the $(n-k)$-th elementary symmetric function of the $(n-1)$ arguments $t_l$, $l=1, 2, \cdots, n$, $l \neq i$.

Making use of (3), (5), and noting that the cofactors appearing in the expression for the inverse matrix are $(n-1)\times(n-1)$ determinants of the same form as det $V_k$, leads to

(6) $$V^{-1} = \left[ \frac{(-1)^{i+j} a_{n-j}^{i}}{\prod_{k=i+1}^{n} (t_k - t_i) \prod_{l=1}^{i-1} (t_i - t_l)} \right]^T,$$

where $T$ denotes transpose. With $\sum_{i=1}^{nj} a_i$ for $\sum_{i=1, i\neq j}^{n} a_i$ and $\prod_{i=1}^{nj} a_i$ for $\prod_{i=1, i \neq j}^{n} a_i$, equation (6) can easily be rewritten as

(7) $$V^{-1} = \left[ \frac{(-1)^{i+1} \sum_{1 \leqq s_1 < s_2 < \cdots < s_{n-i} \leqq n}^{j} \prod_{q=1}^{n-i} t_{s_q}}{\prod_{k=1}^{nj} (t_k - t_j)} \right].$$

(This result appears, without proof, in [4] p. 312.) The inverse for the special case $t_i = x + ih$, $i=0, 1, \cdots, n-1$, and an outline of an alternate derivation of the inverse for the general Vandermonde matrix (but not the explicit representation of $V^{-1}$) may be found in [5].

**3. Rectangular Vandermonde matrices.** The concept of Vandermonde matrices generalizes to the rectangular case. Thus, we define

$$V_{nm} = \begin{bmatrix} 1 & t_1 \cdots t_1^{m-1} \\ 1 & t_2 \cdots t_2^{m-1} \\ \cdot & \cdot \cdot \cdot \cdot \cdot \\ 1 & t_n \cdots t_n^{m-1} \end{bmatrix} = [\mathbf{T}_1\mathbf{T}_2 \cdots \mathbf{T}_n]^T, \tag{8}$$

where $n \neq m$ and $\mathbf{T}_i$ is the vector

$$\mathbf{T}_i = [1, t_i, t_i^2, \cdots, t_i^{m-1}]^T, \qquad i = 1, \cdots, n. \tag{9}$$

Now put $\mathbf{U} = [u_i] = [u(t_i)]$ for brevity. If we seek an $(m-1)$-th degree polynomial which yields a least squares fit to $\{u_i\}_{i=1}^n$, we want an $m$-dimensional constant vector $\mathbf{C}$, such that $\mathbf{C}$ attains

$$\underset{\mathbf{C}}{\text{Min}} \sum_{i=1}^{n} (u_i - \mathbf{C}^T\mathbf{T}_i)^2. \tag{10}$$

Then, the usual procedure of differentiation with respect to the components of $\mathbf{C}$, and equating the resulting expressions to zero to obtain the normal equations ([6] pp. 343–355), yields a result which may easily be expressed by means of a rectangular Vandermonde matrix

$$V_{nm}^T V_{nm}\mathbf{C} = V_{nm}^T\mathbf{U}. \tag{11}$$

**4. Interpolating polynomials.** The interpolating polynomial through $(t_i, u(t_i))$, $i = 1, 2, \cdots, n$ is ([3] p. 193):

$$P(t) = \sum_{i=1}^{n} \frac{\omega(t)}{(t - t_i)\omega'(t_i)} u(t_i) = \sum_{i=1}^{n} u(t_i) \prod_{j=1}^{n_i} \frac{t - t_j}{t_i - t_j} = \mathbf{L}^T\mathbf{U}, \tag{12}$$

where prime signifies derivative with respect to $t$,

$$\mathbf{L} = \left[ \prod_{j=1}^{n_i} \frac{t - t_j}{t_i - t_j} \right]^T, \text{ and } \omega(t) = \prod_{j=1}^{n} (t - t_j).$$

By setting $m = n$ in (11) we can obtain a matrix expression for this polynomial:

$$P(t) = [V_{nn}^{-1}\mathbf{U}]^T\mathbf{T} = \mathbf{T}^T V_{nn}^{-1}\mathbf{U}. \tag{13}$$

That $\mathbf{L}^T \equiv \mathbf{T}^T V_{nn}^{-1}$ is easily established by use of (6) or (7).

Any views expressed in this paper are those of the author. They should not be interpreted as reflecting the views of the RAND Corporation or the official opinion or policy of any of its governmental or private research sponsors.

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## A GENERALIZATION OF HJEMSLEV'S THEOREM

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Hjemslev's Theorem [1, p. 47; 2, p. 268] states that, in the Euclidean plane, when all the points $P$ on one line are related by an isometry to all the points $P'$ on another, the midpoints of the segments $PP'$ are distinct and collinear or they all coincide.

A natural generalization is the following

THEOREM. *If $S$, $S'$ are two figures in a plane, which are congruent (related by an isometry) and $X$ is the midpoint of the segment joining any point $P$ in $S$ to the corresponding point $P'$ in $S'$, then the set of points $X$ either all coincide or are collinear or they form a figure similar to $S$ and $S'$.*

The proof is an application of Coxeter [1, p. 46, 3.51]: Any direct isometry is either a translation or a rotation. Any opposite isometry is either a reflection or a glide reflection.

Suppose, first, that the figures $S$, $S'$ are two triangles $ABC$, $A'B'C'$.

(1) If the isometry is opposite, the midpoints $X$ all lie on the line which is the mirror of the reflection or glide reflection.

(2) If the isometry is direct and is a translation, it is immediately evident that the figure formed by the points $X$ is congruent (therefore certainly similar) to $S$, being related to $S$ by a translation through half the displacement of the original translation.

(3) The only remaining possibility is a rotation about some point $O$ through an angle $\alpha$. If $\alpha=\pi$, the points $X$ all coincide at $O$. Otherwise, we can choose the sense of rotation suitably so that $\alpha<\pi$.

If $X$, $Y$ are the midpoints of $AA'$, $BB'$, it is easy to see that $OX=OA\cos(\alpha/2)$, $OY=OB\cos(\alpha/2)$, $\angle XOY=\angle AOB$. It follows that $XY=AB\cos(\alpha/2)$.

This relation holds for any two of the points $A$, $B$, $C$ and leads to the conclusion that the triangles $XYZ$ and $ABC$ are similar.