Inverse of a Vandermonde Matrix

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This paper considers the problem of inverting the $n \times n$ Vandermonde matrix $M(n)$ whose (i, j) entry is i^{j-1}. For all n $(n = 1, 2, 3, \ldots)$ an exact closed-form expression for the inverse of $M(n)$ is given. It is shown that the inverse of the matrix $M(n)$ satisfies a linear recursion relation in the form of a Pascal pyramid; a two-dimensional form of the Pascal triangle. It is further shown that if the inverse of $M(n)$ is expressed as the product $U(n)T(n)$, where $T(n)$ is a lower triangular matrix whose entries consist of Stirling numbers of the first kind multiplying the entries of a Pascal triangle (binomial coefficients), then $U(n)$ obeys a very simple three-term recursion relation. In the course of this work some polynomials that are closely related to the Bernoulli polynomials were discovered. Recursion relations for these polynomials are given.

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In this paper we consider the problem of finding the inverse of the $n \times n$ Vandermonde matrix $M(n)$ whose (i, j) entry is i^{j-1}:

$$[M(n)]_{i,j} = i^{j-1}. \tag{1}$$

This problem was suggested to us by M. Golterman, who needed the result for understanding an approximate calculation in quantum field theory in which a spectral function is approximated by using poles. The problem of inverting the matrix $M(n)$ has been discussed previously by Spitzbart and Macon [1,2], who were examining numerical differentiation formulas. In this paper we present new results regarding the inverse of $M(n)$. We give an exact closed-form expression for the elements of the inverse matrix. We also display two linear recursion relations satisfied by the inverse matrix, one homogeneous and one inhomogeneous. Furthermore, we examine some polynomials related to Bernoulli polynomials that arose in the course of our study.

In general, the inverse of a matrix A is the transpose of the matrix of the minors of A divided by the determinant of A. In our case we can find the determinant of $M(n)$ in closed form because $M(n)$ is a Vandermonde matrix:

$$\det[M(n)] = 0!\,1!\,2!\,3!\,\cdots\,(n-1)!. \tag{2}$$

Since the determinant of $M(n)$ grows extremely rapidly as a function of n, one might expect that the entries of the inverse of $M(n)$ are fractions with huge denominators when n is large.

However, this is not the case. As we see below, the inverse of $M(n)$ may be written in the form

$$[M(n)]^{-1} = \frac{1}{(n-1)!} V(n), \tag{3}$$

where $V(n)$ is a matrix whose entries are all integers:

$$[M(1)]^{-1} = (1)^{-1} = \frac{1}{0!}(1), \tag{4}$$

$$[M(2)]^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{1!} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \tag{5}$$

$$[M(3)]^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}^{-1} = \frac{1}{2!} \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}, \tag{6}$$

$$[M(4)]^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix}^{-1} = \frac{1}{3!} \begin{pmatrix} 24 & -36 & 24 & -6 \\ -26 & 57 & -42 & 11 \\ 9 & -24 & 21 & -6 \\ -1 & 3 & -3 & 1 \end{pmatrix}, \tag{7}$$

$$[M(5)]^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \\ 1 & 5 & 25 & 125 & 625 \end{pmatrix}^{-1} = \frac{1}{4!} \begin{pmatrix} 120 & -240 & 240 & -120 & 24 \\ -154 & 428 & -468 & 244 & -50 \\ 71 & -236 & 294 & -164 & 35 \\ -14 & 52 & -72 & 44 & -10 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}. \tag{8}$$

One can see immediately that the elements of the matrices $V(n)$ have a checkerboard sign pattern. Also, the entry in the upper left corner is $n!$. Third, the entries in the bottom row are binomial coefficients. Fourth, the absolute values of the entries in the right column are Stirling numbers of the first kind [3]. Recall that these Stirling numbers are the absolute values of the coefficients of powers of x in the polynomials

$$\left.\begin{aligned} Q_0(x) &= 1, \\ Q_1(x) &= x-1, \\ Q_2(x) &= (x-1)(x-2) = x^2 - 3x + 2, \\ Q_3(x) &= (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6, \\ Q_4(x) &= (x-1)(x-2)(x-3)(x-4) = x^4 - 10x^3 + 35x^2 - 50x + 24. \end{aligned}\right\} \tag{9}$$

This latter observation suggests that *all* of the entries in $V(n)$ and not just those in the last column might be the coefficients of polynomials and this is indeed the case. Thus, we can write down an explicit formula for the inverse of $M(n)$. The (i,j) element of the inverse matrix is given by

$$[M(n)]^{-1}_{i,j} = \frac{(-1)^{n-j}}{(n-j)!\,(j-1)!} \left[\text{coefficient of } x^{i-1} \text{ in } \frac{(x-1)(x-2)(x-3)\cdots(x-n)}{x-j} \right]. \tag{10}$$

The inverse matrices obey an extremely simple linear recursion relation:

$$(n-1)[M(n)]^{-1}_{i,j} = n[M(n-1)]^{-1}_{i,j} - (n-1)[M(n-1)]^{-1}_{i,j-1} - [M(n-1)]^{-1}_{i-1,j} \\ +[M(n-1)]^{-1}_{i-1,j-1} + \sum_{k=1}^{n-2} [M(n-1-k)]^{-1}_{i,j-1}, \tag{11}$$

where $[M(n)]^{-1}_{i,j}$ vanishes if i or j is greater than n or less than 1. This recursion relation takes the form of a Pascal pyramid, by which we mean a two-dimensional form of the Pascal triangle. As more terms in the sum are included, more of the elements of the inverse matrix $[M(n)]^{-1}_{i,j}$ are determined exactly.

There is an even simpler recursion formula for the inverse matrices $[M(n)]^{-1}$. To construct this formula, we first observe that there is an elegant product decomposition for the matrices $V(n)$ defined in (3). We write

$$V(n) = T(n)U(n), \tag{12}$$

where $T(n)$ is an upper triangular matrix whose entries are Stirling numbers of the first kind multiplying binomial coefficients. More precisely, all of the elements below the main diagonal vanish; all of the elements on the remaining diagonals are the rows of a Pascal triangle, with each row alternating in sign and multiplied by a Stirling number of the first kind. The entries of the matrix $U(n)$ do not quite have the same checkerboard sign pattern as $V(n)$ because for $n \geq 5$ the sign pattern is broken near the top of the matrix. Also, the elements of $U(n)$ are smaller in magnitude than the corresponding elements of $V(n)$:

$$V(1) = (1) = (1)(1), \tag{13}$$

$$V(2) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \tag{14}$$

$$V(3) = \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}, \tag{15}$$

$$V(4) = \begin{pmatrix} 24 & -36 & 24 & -6 \\ -26 & 57 & -42 & 11 \\ 9 & -24 & 21 & -6 \\ -1 & 3 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 2 & -1 \\ 0 & 3 & -6 & 2 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 7 & -2 & 1 & 0 \\ -4 & 7 & -4 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 3 & -3 & 1 \end{pmatrix}, \tag{16}$$

$$V(5) = \begin{pmatrix} 120 & -240 & 240 & -120 & 24 \\ -154 & 428 & -468 & 244 & -50 \\ 71 & -236 & 294 & -164 & 35 \\ -14 & 52 & -72 & 44 & -10 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -6 & 11 & -6 & 1 \\ 0 & 4 & -18 & 22 & -6 \\ 0 & 0 & 6 & -18 & 11 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 15 & 5 & 5 & -1 & 0 \\ -8 & 13 & -9 & 5 & -1 \\ 4 & -11 & 11 & -5 & 1 \\ -2 & 7 & -9 & 5 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}. \tag{17}$$

The matrices $U(n)$ satisfy a beautifully simple recursion relation. It is easy to discover this recursion relation by observing that the elements in the bottom row of $U(n)$ are Pascal triangle entries; that is, they are the coefficients of $(x-1)^{n-1}$. This suggests an inhomogeneous recursion relation that connects $U(n-1)$ and $U(n)$:

$$[U(n)]_{i,j} = [U(n-1)]_{i-1,j-1} - [U(n-1)]_{i-1,j} + [W(n)]_{i,j}, \tag{18}$$

where for $n > 2$ all of the elements of the matrix $W(n)$ vanish except for those in the first two rows:

$$\begin{aligned} [W(n)]_{1,j} &= \sum_{k=1}^{j} \frac{(-1)^{k+j}(k+1)^{n-1} n!}{(n-j+k)!(j-k)!} + \frac{(-1)^j (n-1)!}{(j-1)!(n-j)!}, \\ [W(n)]_{2,j} &= (-1)^j \frac{(n-1)!}{(n-j)!(j-1)!}, \\ [W(n)]_{i,j} &= 0 \quad (i > 2). \end{aligned} \tag{19}$$

The first few matrices $W(n)$ are

$$W(2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{20}$$

$$W(3) = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{21}$$

$$W(4) = \begin{pmatrix} 7 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{22}$$

$$W(5) = \begin{pmatrix} 15 & 5 & 5 & -1 & 0 \\ -1 & 4 & -6 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

$$W(6) = \begin{pmatrix} 31 & 56 & 36 & -4 & 1 & 0 \\ 1 & -5 & 10 & -10 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{24}$$

Note that the vanishing of the last element in the first row of the matrix $[W(n)]_{1,n}$ is a consequence of an interesting identity,

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}(k+1)^{n-1}}{k!(n-k)!} = \frac{1}{n!}, \tag{25}$$

which is a generalization of an identity used to derive Richardson extrapolation [4].

The formula for $[W(n)]_{1,j}$ in (19) is derived as follows. First, we observe from Eqs. (20)–(24) that if the elements in the top row of the matrix $W(n-1)_{1,j}$ are subtracted and the elements $W(n-1)_{1,j-1}$ are added to the elements in the top row of $W(n)$, then we obtain a simpler set of numbers, which are all even. [For example, we subtract $(3,-1)$ from the $7,-2$ in the top row $W(4)$ and get $(4,-1,1)$. Then we add $(3,-1)$ to the last two of these numbers and obtain $(4,2,0)$. We discard the 0 and get $(4,2)$.] We call the numbers obtained this way $T_{n,j}$:

$$\begin{array}{lrrrrrr}
T_{1,j}: & 2 & 0 & 0 & 0 & 0 & 0 \\
T_{2,j}: & 4 & 2 & 0 & 0 & 0 & 0 \\
T_{3,j}: & 8 & 14 & 2 & 0 & 0 & 0 \\
T_{4,j}: & 16 & 66 & 36 & 2 & 0 & 0 \\
T_{5,j}: & 32 & 262 & 342 & 82 & 2 & 0 \\
T_{6,j}: & 64 & 946 & 2416 & 1436 & 176 & 2
\end{array} \tag{26}$$

Note that if we sum the numbers $T_{n,j}$ over j we obtain $(n+1)!$.

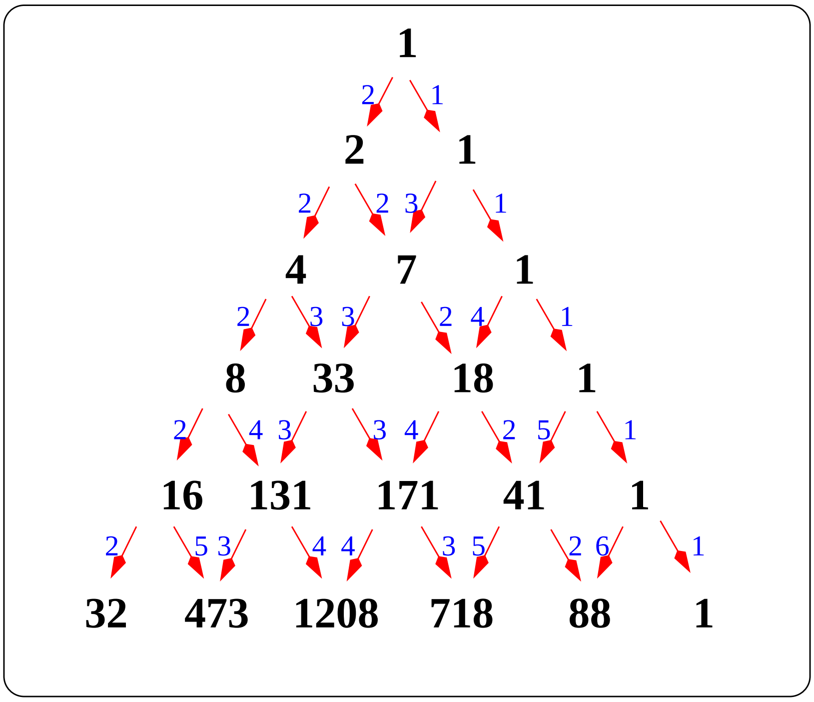

FIG. 1. Pascal triangle representation for the recursion relation satisfied by the numbers $\frac{1}{2}T_{n,j}$, where $T_{n,j}$ is given in the array (26). The boldface numbers in the rows are obtained from the numbers in the row above by multiplying by the integers in small print.

If the numbers $T_{n,j}$ are divided by two we obtain the numbers that are shown in the rows of Fig. 1. Next, we observe that the entries in Fig. 1 obey a Pascal triangle relation. That is, the numbers in every horizontal row can be obtained from the numbers in the row above by multiplying by the indicated factors. From this we obtain a formula for the entries in the figure. Specifically, the jth entry in the $(n+2)$nd row in (26) is given by

$$T_{n+2,j} = \sum_{k=1}^{j} \frac{n!\,(-1)^{k+j}(k+1)^{n-2}k}{(n-j+k)!\,(j-k)!}. \tag{27}$$

The formula for $[W(n)]_{1,j}$ in (19) is then obtained by summing this formula on n.

There is an easy and immediate generalization of our results for the matrix $M(n)$ in (1). Let us define the matrix $M(n,k)$ by

$$[M(n,k)]_{i,j} = i^{j-1+k}. \tag{28}$$

The inverse of the matrix $M(n,k)$ is very simply related to the inverse of the matrix $M(n)$; we merely divide the entries in the jth column by j^k:

$$[M(n,k)]^{-1}_{i,j} = j^{-k}[M(n)]^{-1}_{i,j}. \tag{29}$$

We conclude with some remarks about the connection between the Stirling numbers of the first kind and Bernoulli polynomials. It is quite evident that the Stirling numbers appear often in the above formulas.[1] Thus, in search of interesting identities for the inverses of the matrices $M(n)$, we were led to a more detailed study of the Stirling numbers.

Here is a matrix of the Stirling numbers of the first kind:

$$\begin{array}{lccccccc}
S_{1,k}: & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
S_{2,k}: & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
S_{3,k}: & 1 & 3 & 2 & 0 & 0 & 0 & 0 \\
S_{4,k}: & 1 & 6 & 11 & 6 & 0 & 0 & 0 \\
S_{5,k}: & 1 & 10 & 35 & 50 & 24 & 0 & 0 \\
S_{6,k}: & 1 & 15 & 85 & 225 & 274 & 120 & 0 \\
S_{7,k}: & 1 & 21 & 175 & 735 & 1624 & 1764 & 720
\end{array} \tag{30}$$

Let us construct a set of polynomials $P_k(n)$ that reproduces these Stirling numbers. For example, the numbers $(0,\ 1,\ 3,\ 6,\ 10,\ \ldots)$ in the first column in this array, as a function of the row label n, are given by the polynomial

$$\frac{1}{2^1 1!}n(n-1)P_0(n), \tag{31}$$

where

$$P_0(n) = 1. \tag{32}$$

The numbers $(0,\ 0,\ 2,\ 11,\ 35,\ \ldots)$ in the next column in this array as a function of n are given by

$$\frac{1}{2^2 2!}n(n-1)(n-2)P_1(n), \tag{33}$$

[1]Some of the connections with the Stirling numbers are not so obvious. For example, sums of the Stirling numbers are factorials. That is, $1+1=2!$, $1+3+2=3!$, $1+6+11+6=4!$, and so on. Similarly, the sums of the numbers in the top row of the matrices W are also factorials $3-1=2!$, $7-2+1=3!$, $15+5+5-1=4!$, and so on.

where

$$P_1(n) = n - \frac{1}{3}. \tag{34}$$

The numbers in the next two columns in this array are given by

$$\begin{aligned} &\frac{1}{2^3 3!} n(n-1)(n-2)(n-3)P_2(n), \\ &\frac{1}{2^4 4!} n(n-1)(n-2)(n-3)(n-4)P_3(n), \end{aligned} \tag{35}$$

where

$$\begin{aligned} P_2(n) &= n^2 - n, \\ P_3(n) &= n^3 - 2n^2 + \frac{1}{3}n + \frac{2}{15}. \end{aligned} \tag{36}$$

In general, the numbers along the kth column are expressible in the form

$$\frac{1}{2^k k!} n(n-1)\cdots(n-k)P_{k-1}(n).$$

The next seven polynomials $P_k(n)$ are

$$\begin{aligned} P_4(n) &= n^4 - \frac{10}{3}n^3 + \frac{5}{3}n^2 + \frac{2}{3}n, \\ P_5(n) &= n^5 - 5n^4 + 5n^3 + \frac{13}{9}n^2 - \frac{2}{3}n - \frac{16}{63}, \\ P_6(n) &= n^6 - 7n^5 + \frac{35}{3}n^4 + \frac{7}{9}n^3 - \frac{14}{3}n^2 - \frac{16}{9}n, \\ P_7(n) &= n^7 - \frac{28}{3}n^6 + \frac{70}{3}n^5 - \frac{56}{9}n^4 - \frac{469}{27}n^3 - 4n^2 + \frac{404}{135}n + \frac{16}{15}, \\ P_8(n) &= n^8 - 12n^7 + 42n^6 - \frac{448}{15}n^5 - \frac{133}{3}n^4 + \frac{20}{3}n^3 + \frac{404}{15}n^2 + \frac{48}{5}n, \\ P_9(n) &= n^9 - 15n^8 + 70n^7 - \frac{266}{3}n^6 - \frac{245}{3}n^5 + \frac{745}{9}n^4 + \frac{1072}{9}n^3 + \frac{188}{9}n^2 - \frac{208}{9}n - \frac{256}{33}, \\ P_{10}(n) &= n^{10} - \frac{55}{3}n^9 + 110n^8 - \frac{638}{3}n^7 - \frac{847}{9}n^6 + \frac{3179}{9}n^5 + \frac{968}{3}n^4 - \frac{1100}{9}n^3 - \frac{2288}{9}n^2 - \frac{256}{3}n. \end{aligned} \tag{37}$$

The structure of these polynomials strongly resembles that of the Bernoulli polynomials $B_m(x)$ [3]. First, note that, like the Bernoulli polynomials, every other polynomial has a constant term. Second, recall that the constant terms in the Bernoulli polynomials are Bernoulli numbers, $B_{2m}(0) = B_{2m}$, and observe that the constant terms in the polynomials $P_{2m-1}(n)$ are expressible in terms of Bernoulli numbers:

$$P_{2m-1}(0) = -\frac{1}{2m}4^m B_{2m}. \tag{38}$$

Furthermore, both the polynomials $P_m(n)$ and the Bernoulli polynomials $B_m(x)$ satisfy very similar recursion relations. The polynomials $P_m(n)$ satisfy

$$P_{2m}(n) = (2m+1)nP_{2m-1}(n) + \sum_{k=1}^{m}(-2)^k \frac{m!(2m-k+1)}{(k+1)!(m-k)!}n^{k+1}P_{2m-k-1}(n), \tag{39}$$

while the Bernoulli polynomials obey the same recursion relation with an inhomogeneous term:

$$B_{2m+1}(x) = (2m+1)xB_{2m}(x) + \sum_{k=1}^{m}(-2)^k \frac{m!(2m-k+1)}{(k+1)!(m-k)!} x^{k+1} B_{2m-k}(x) + \frac{1}{2}(-1)^{m+1}x^{2m}. \tag{40}$$

Finally, we note that $P_m(n)$ satisfies a recursion relation in terms of the coefficients of the Bernoulli polynomials:

$$\begin{aligned} P_{m+1}(n) = nP_m(n) - \frac{n}{m+2} \sum_{k=1}^{[m/2]} 4^k P_{m+1-2k}(n) \left[\text{coefficient of } x^{m+2-2k} \text{ in } B_{m+2}(x)\right] \\ -2^m \left[\frac{4}{m+2} B_{m+2}(0) - 2nB_{m+1}(0)\right]. \end{aligned} \tag{41}$$

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