Starting with the old assumption that al x_i are pairwise distinct, we rewrite the Vandermonde matrix with $h = x_2 - x_1$ as

$$V_h = \begin{pmatrix} & \dots & & \\ 1 & x_1 + h & (x_1 + h)^2 & (x_1 + h)^3 \\ & \dots & & \end{pmatrix}$$

ugh, another time...

— old stuff below —

General case: all x_i are pairwise distinct. Define $h = x_2 - x_1$ which is non-zero for the moment. Now write

$$\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_1 + h & (x_1+h)^2 & (x_1+h)^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_1 + h & x_1^2 + 2hx_1 + h^2 & x_1^3 + 3x_1^2 h + 3x_1 h^2 + h^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & h & h(2x_1 + h) & h(3x_1^2 + 3x_1 h + h^2) \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 - y_1 \\ y_3 \\ y_4 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & & & \\ & h & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 + h & 3x_1^2 + 3x_1 h + h^2 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & h & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \frac{y_2 - y_1}{h} \\ y_3 \\ y_4 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 + h & 3x_1^2 + 3x_1 h + h^2 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ \frac{y_2 - y_1}{h} \\ y_3 \\ y_4 \end{pmatrix}$$

As $\lim_{h \to 0} \frac{y_2 - y_1}{h} = \frac{dy}{dx} = \frac{C'(x_1)}{2y_1}$ we get the exact same system as in case 2.

– xxx – Blah, and so on, this leads to all tangential cases.

For the remaining case $\{\mathcal{Q}_1, \mathcal{Q}_2\} + \{\overline{\mathcal{Q}}_1, \mathcal{Q}_4\}$ take a more direct approach. Define $M_h = \{(x, y) \mid y - p_h(x) = 0\}$ for a p_h given above. Specifically, under the right conditions of pairwise-distinctness and $p^*(x) = \det V \cdot p(x)$,

$$p^*(x) = \begin{vmatrix} 1 & x_1 & x_1^2 & y_1 \\ 1 & x_2 & x_2^2 & y_2 \\ 1 & x_3 & x_3^2 & y_3 \\ 1 & x_4 & x_4^2 & y_4 \end{vmatrix} x^3 + \begin{vmatrix} 1 & x_1 & y_1 & x_1^3 \\ 1 & x_2 & y_2 & x_2^3 \\ 1 & x_3 & y_3 & x_3^3 \\ 1 & x_4 & y_4 & x_4^3 \end{vmatrix} x^2 + \begin{vmatrix} 1 & y_1 & x_1^2 & x_1^3 \\ 1 & y_2 & x_2^2 & x_2^3 \\ 1 & y_3 & x_3^2 & x_3^3 \\ 1 & y_4 & x_4^2 & x_4^3 \end{vmatrix} x + \begin{vmatrix} y_1 & x_1 & x_1^2 & x_1^3 \\ y_2 & x_2 & x_2^2 & x_2^3 \\ y_3 & x_3 & x_3^2 & x_3^3 \\ y_4 & x_4 & x_4^2 & x_4^3 \end{vmatrix}$$

so we look at

$$
\begin{aligned}
p_h(x) = &\begin{vmatrix} 1 & x_1 & x_1^2 & y_1 \\ 1 & x_1+h & (x_1+h)^2 & y_2 \\ 1 & x_3 & x_3^2 & y_3 \\ 1 & x_4 & x_4^2 & y_4 \end{vmatrix} x^3 + \begin{vmatrix} 1 & x_1 & y_1 & x_1^3 \\ 1 & x_1+h & y_2 & (x_1+h)^3 \\ 1 & x_3 & y_3 & x_3^3 \\ 1 & x_4 & y_4 & x_4^3 \end{vmatrix} x^2 \\
&+ \begin{vmatrix} 1 & y_1 & x_1^2 & x_1^3 \\ 1 & y_2 & (x_1+h)^2 & (x_1+h)^3 \\ 1 & y_3 & x_3^2 & x_3^3 \\ 1 & y_4 & x_4^2 & x_4^3 \end{vmatrix} x + \begin{vmatrix} y_1 & x_1 & x_1^2 & x_1^3 \\ y_2 & x_1+h & (x_1+h)^2 & (x_1+h)^3 \\ y_3 & x_3 & x_3^2 & x_3^3 \\ y_4 & x_4 & x_4^2 & x_4^3 \end{vmatrix} \\
&=
\end{aligned}
$$