0 Introduction

Hyperelliptic curves of genus $g \geq 1$ over a field K are equations of the form $y^2 = C(x)$ where $C(x) \in K[x]$ is monic of degree $2g{+}1$, $\mathrm{char}(K) \neq 2$. To include fields of characteristic 2, the equation would be $y^2 + yH(x) = C(x)$ where $H(x) \in K[x]$ has degree less or equal g. Elliptic curves have genus 1.

This paper sets out to imitate the approach and techniques used on elliptic curves in a series of lectures given by Prof. David Masser between 2009 and 2010 at the University of Basel [4]. The aim is to define a group on curves of genus 2, where the bulk of the work will be going into proving associativity.

We will derive the addition law in the most explicit way possible. Behind this lies the desire to not only construct a group over the curve, but to do so using elementary tools without the need to delve into algebraic geometry.

xx An additional advantage is that can work over any field of characteristic not 2 or 3 so we don't restrict ourselves to $\mathbb{C}$ as done in earlier literature [3].

The reason for choosing curves of genus 2 is twofold. First, elliptc curves are vastly well-known, whereas many things about hyperelliptics are not. Second, it is our opinion that 2 is the highest genus for whom it is still practical to derive the group law in such an explicit way. For any higher genus, the list of case-distinctions would simply be too high, although a priori it looks at least theoretically feasible.

The are however a few caveats to our explicit ansatz. For one, the fickleness of distinguishing between six cases is certainly not desirable in the sense of mathematical elegance. But there is also the question of applicability. As it stands, we have to exclude fields of characteristic 2 (and 3) and we need to work within the closure $\overline{K}$, both of which are not optimal from a computational standpoint. In addition to that, the field extension sabotages the the original goal of finding rational solutions to diophantine equations.

More broadly represented in literature is the construction of the Jacobian out of formal divisors on the function field of the curve [1, 2]. As a comparison, we first derive the addition law and will then prove the existence of a rational function with a divisor depending on a given sum.

Work has been done to make this [6] and as it has a lot of uses and potential uses, it would be interesting to answer the question of equivalence between these two approaches.xxx

The next logical step for our approach would be to study torsion points, and while there is no Lutz-Nagell for hyperelliptic curves, work has been done on an equivalent [5].

1 Addition Law

1.1 Definitions and Notation

Definition 1: Let K be a field with $\mathrm{char}(K) \neq 2, 3$ and $\overline{K}$ its algebraic closure. Define the hyperelliptic curve of genus two $H_0(K)$ as the set of solutions in K^2 to the equation $y^2 = C(x)$ where

$$C(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$$

is a polynomial over K. Similarly, the set of solutions in the closure would be denoted $H_0(\overline{K})$. Define $H(K)$ as $H_0(K) \cup \{\infty\}$.

Note that we could obtain a more reduced form of $C(x)$, eliminating a by shifting x to $x - a/5$. However, since this would rob us of the possibility of $\mathrm{char}(K) = 5$ without simplifying our coming calculations in any significant manner, we shall be reluctant towards using this trick.

For the purpose of clarity, let points on the hyperelliptic curve — in the sense of solutions to $y^2 = C(x)$ — be designated by the calligraphic letter $\mathcal{Q} = (x, y) \in H_0(\overline{K})$. The point opposite to $\mathcal{Q}$ will be written $\overline{\mathcal{Q}} = (x, -y)$ and by symmetry of the curve in y also belongs to $H_0(\overline{K})$. In the case where $\mathcal{Q} = \infty$, define $\overline{\mathcal{Q}} = \infty$.

We then want to consider the set of all pairs $(\mathcal{Q}_1, \mathcal{Q}_2)$ and tame it with an equivalence relation with the goal of obtaining an additive group:

Definition 2: Define $\mathbf{J}$ to be the set $\mathcal{J}/\sim$ where $\mathcal{J} = H(\overline{K}) \times H(\overline{K})$. It is called the "Jacobian" and the equivalence relation is defined by

$$\begin{aligned} (\mathcal{Q}_1, \mathcal{Q}_2) &\sim (\mathcal{Q}_2, \mathcal{Q}_1) \\ \text{and} \quad (\mathcal{Q}, \overline{\mathcal{Q}}) &\sim (\infty, \infty). \end{aligned}$$

Write $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ from now on and let bold letters denote points on the curve in the sense of classes of unordered pairs $\mathbf{P} = \{\mathcal{Q}_1, \mathcal{Q}_2\} \in \mathbf{J}$. The point $\{\overline{\mathcal{Q}}_1, \overline{\mathcal{Q}}_2\}$ will be called $\overline{\mathbf{P}}$ for now but can already tentatively be thought of as $-\mathbf{P}$. Call $\{\infty, \infty\}$ the zero of our set. Call $\mathcal{Q}_i$ a point-component.

A point $\mathcal{Q} = (x_0, y_0)$ is called singular if it fulfills both $y_0 = 0$ and $C'(x_0) = 0$. A curve is called singular if and only if it has a singular point. We consider only non-singular hyperelliptics from here on; this amounts to $C(x)$ having no repeated factors over $\overline{K}$ ("squarefree").

1.2 The General Case

Let's start with $\mathbf{P}_1 = \{\mathcal{Q}_1, \mathcal{Q}_2\}$, $\mathbf{P}_2 = \{\mathcal{Q}_3, \mathcal{Q}_4\}$ with $\mathcal{Q}_i = (x_i, y_i) \in H_0(K)$ or $\mathcal{Q}_i = \infty$. To define $\mathbf{P}_3 = \mathbf{P}_1 + \mathbf{P}_2$ we distinguish between one general case and a number of special cases and first derive the results of the former before enumerating the latter ones.

Case 1, Four Distinct Point-Components: Let $\mathcal{Q}_i \in H_0(K)$ and $\mathbf{P}_i$ be defined as above with $x_i \neq x_j$ whenever $i \neq j$.

The overarching idea is to obtain a fifth and sixth x-coordinate and the corresponding y-coordinates by passing a polynomial of degree three through the four points $\mathcal{Q}_i$. Ideally this should give us two additional intersections with the curve which we then use as the components of our point $\mathbf{P}_1 + \mathbf{P}_2$.

Step 1: It is known that the Vandermonde matrix

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}$$

has determinant $\prod_{i<j}(x_i - x_j)$ which is conveniently non-zero if and only if the x_i are pairwise distinct. Let $P(x) = p_3x^3 + p_2x^2 + p_1x + p_0 \in \overline{K}[x]$ be the polynomial in unknown coefficients that we are looking for. With $\mathbf{y} = (y_1\ y_2\ y_3\ y_4)^t$ and $\mathbf{p} = (p_0\ p_1\ p_2\ p_3)^t$, the problem of determining $P(x)$ can be rewritten as

$$V \cdot \mathbf{p} = \mathbf{y}$$

which by invertibility of V has a unique solution for $\mathbf{p}$ with $p_i \in K$.

Note that the leading coefficient of $P(x)$ is

$$p_3 = \frac{1}{\det(V)} \begin{vmatrix} 1 & x_1 & x_1^2 & y_1 \\ 1 & x_2 & x_2^2 & y_2 \\ 1 & x_3 & x_3^2 & y_3 \\ 1 & x_4 & x_4^2 & y_4 \end{vmatrix}$$

and the next step will depend on whether $P(x)$ is truly of degree 3 or not.

Define $$D(x) = C(x) - (P(x))^2 . \qquad (*)$$

Step 2a: Knowing the coefficients p_i we first assume $p_3 \neq 0$ so we can proceed to look for the additional solutions of the sextic equation $D(x) = 0$. Observe that this vanishes at x_1, x_2, x_3 and x_4, so write the lefthand side as

$$D(x) = -p_3^2(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)(x - x_6) \qquad (*_1)$$

for x_5 and x_6 unknowns in $\overline{K}$. Comparing the coefficients at x^4 and x^5 yields

$$\sum_{\substack{i,j=1\\i<j}}^{6} x_i x_j = T_4$$

and

$$\sum_{i=1}^{6} x_i = T_5$$

where $T_4 = \frac{p_2^2+2p_1p_3-a}{p_3^2}$ and $T_5 = \frac{1-2p_2p_3}{p_3^2}$. The first expression gives

$$x_6 \sum_{i=1}^{5} x_i + \sum_{\substack{i,j=1\\i<j}}^{5} x_i x_j = T_4.$$

Replacing x_6 by $T_4 - \sum_{\substack{i,j=1\\i<j}}^{5} x_i x_j$ gives the tidy quadratic equation

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$$x^2 - \left(T_5 - \sum_{i=1}^{4} x_i\right) \cdot x + \left(T_4 - T_5 \sum_{i=1}^{4} x_i + \sum_{\substack{i,j=1\\i\leq j}}^{4} x_i x_j\right) = 0 \qquad (\dagger)$$

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of which x_5 is one solution and — by symmetry of the above steps — x_6 the other one. Compute $y_i = P(x_i)$ for $i = 5, 6$ to obtain $\mathcal{Q}_5 = \{x_5, y_5\}$ and $\mathcal{Q}_6 = \{x_6, y_6\}$, at which point it becomes clear that the worst-case scenario for our field extension to accommodate the new coordinates is to be quadratic. Finally we define $\mathbf{P}_1 + \mathbf{P}_2$ to be equal to $\mathbf{P}_3 = \{\overline{\mathcal{Q}}_5, \overline{\mathcal{Q}}_6\}$.

Step 2b: If p_3 were zero, the equation $(*)$ would be quintic instead. We may therefore write the lefthand side factorized, so we get a different $D(x)$,

$$D(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5), \qquad (*_1)$$

again for x_5 somewhere in $\overline{K}$. We used the tag $(*_1)$ twice to refer to both equations. Defining $T_{4\infty} = p_2^2 - a$ and comparing the coefficients at x^4 gives

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$$x_5 = T_{4\infty} - \sum_{i=1}^{4} x_i \qquad (\ddagger)$$

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and we may rejoice in the implication of x_5 staying in K.

Compute $y_5 = P(x_5)$ and define $\mathbf{P}_1 + \mathbf{P}_2$ to be the point $\mathbf{P}_3 = \{(x_5, -y_5), \infty\}$.

Step $1'$: To extend our construction from $H_0(K)$ to $H(K)$ we now consider $\mathcal{Q}_4 = \infty$ with the other $\mathcal{Q}_i = (x_i, y_i)$ as before. This can be seen as the x_i still being pairwise distinct, only that one of them is allowed to be ∞ here.

There is no coordinate x_4 this time, so we pass a quadratic polynomial $P(x)$ through the remaining three points (x_i, y_i). This means that we solve the linear system $\widetilde{V}\cdot\mathbf{p} = \widetilde{\mathbf{y}}$ where $\widetilde{V}$ is the Vandermonde matrix for x_i, $i = 1, .., 3$,

$$\widetilde{V} = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}$$

As before, the leading coefficient of $P(x)$ might or might not be zero, but $(*)$ will be quintic in either case, so we only have to worry about one step 2.

Step $2'$: Doing a coefficient comparison at x^3 and at x^4 in $(*)$ through

$$D(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_5)(x - x_6), \qquad (*_1')$$

gives
$$\sum_{\substack{i,j=1 \\ i<j}}^{3} x_i x_j + x_5 \sum_{i=1}^{3} x_i + x_6 \sum_{i=1}^{3} x_i + x_5 x_6 = b - 2p_1 p_2$$

and
$$\sum_{i=1}^{3} x_i + x_5 + x_6 = p_2^2 - a.$$

Call the righthand terms $T_{3\infty}$ and $T_{4\infty}$, combine both equations and obtain

$$x^2 - \left(T_{4\infty} - \sum_{i=1}^{3} x_i\right) \cdot x + \left(T_{3\infty} - T_{4\infty} \sum_{i=1}^{3} x_i + \sum_{\substack{i,j=1 \\ i\leq j}}^{3} x_i x_j\right) \qquad (\dagger')$$

Solve, call the two solutions x_5 and x_6, compute y_5 and y_6 through $P(x_5)$ and $P(x_6)$ and define $\mathbf{P}_1 + \mathbf{P}_2$ to be $\mathbf{P}_3 = \{(x_5, -y_5), (x_6, -y_6)\} = \{\overline{\mathcal{Q}}_5, \overline{\mathcal{Q}}_6\}$.

Remark 1: Up to this point, we have given the general-case rule only for elements in $H(K) \times H(K)$. Extending to $\mathcal{J} = H(\overline{K}) \times H(\overline{K})$ is trivial as we can simply choose $\overline{K}$ as the new K. The final step is to note that everything until now is well-defined on $\mathbf{J}$. The equivalence relation from Definition 2 corresponds to interchanging $\mathcal{Q}_1$ and $\mathcal{Q}_2$ or $\mathcal{Q}_3$ and $\mathcal{Q}_4$ or both. To this end we can check that interchanging (x_i, y_i) and (x_j, y_j) does nothing to the resulting $\mathcal{Q}_5$ and $\mathcal{Q}_6$. This is the case because $P(x)$ depends only on the solution of the linear system $V \cdot \mathbf{p} = \mathbf{y}$ on which the aforementioned permutation has no effect. With p invariant, the subsequent steps remain unchanged as well.

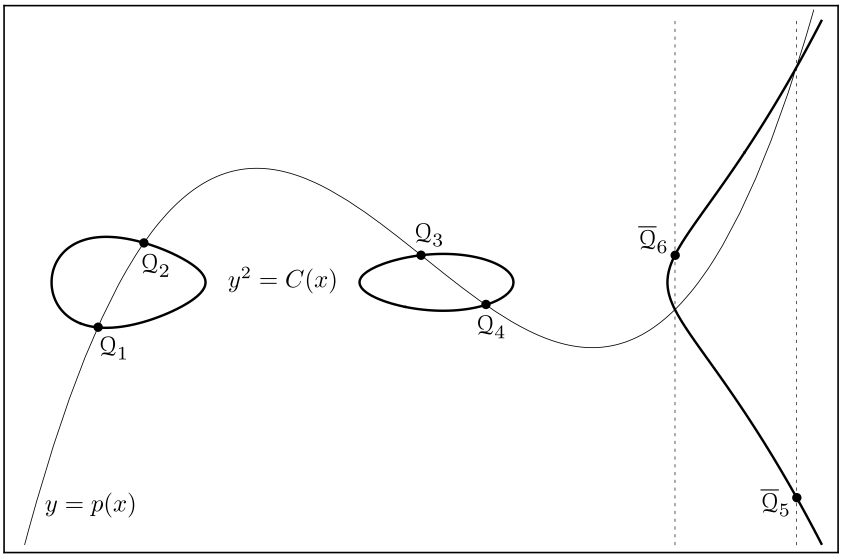
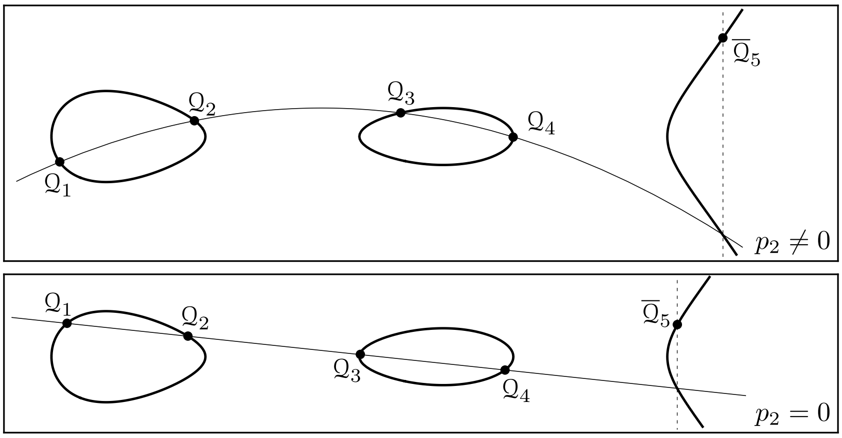


FIGURE 1A: The general case for the addition law in $\mathbb{R}^2$ where $p_3 \neq 0$.



FIGURES 1B and 1C: If $p_3 = 0$ we have exactly five finite intersections.

Finally, the restriction to K instead of $\overline{K}$ was to showcase the degree of the required field extension, a feat we will now leave aside by always using $\overline{K}$.

Before we begin listing the special cases, we impose the following property: The sum of any $\mathbf{P}_1 = \{\mathcal{Q}_1, \mathcal{Q}_2\}$ and $\mathbf{P}_2 = \{\mathcal{Q}_3, \mathcal{Q}_4\}$ in $\mathbf{J}$ fulfills the equality

$$\mathbf{P}_1 + \mathbf{P}_2 = \{\mathcal{Q}_3, \mathcal{Q}_2\} + \{\mathcal{Q}_1, \mathcal{Q}_4\}. \tag{$\diamondsuit$}$$

Note that the general-case addition we just defined already fulfills this, as we noted before that interchanging any of the (x_i, y_i) does not impact the solution $\mathbf{p}$ of the linear system nor any of the dagger equations.

Remark 2: The property ($\diamondsuit$) corresponds to the permutation $(1\,3)$ on the point-components $\mathcal{Q}_i$. With Definition 2 allowing the permutations $(1\,2)$ and $(3\,4)$ this naturally leads to the observation that in fact all permutations of point-components must now leave the sum unchanged. In fact it is easy to check that the three transpositions generate all of S_4 and the remark above already provides compatibility with the construction of the general case.

A substantial bonus of this is the fact that commutativity corresponds to the permutation $(1\,3)(2\,4)$ which by the above we now obtained for free.

1.3 Complete List of Cases

We may now impose conditions on the relations between the $\mathcal{Q}_i$ without mentioning whether they belong to $\mathbf{P}_1$ or $\mathbf{P}_2$. As a result, the list of special cases can be written in a significantly more concise manner.

As we strive to define $\{\mathcal{Q}_1, \mathcal{Q}_2\} + \{\mathcal{Q}_3, \mathcal{Q}_4\}$, for any $\mathcal{Q}_i = (x_i, y_i) \in H_0(\overline{K})$ or $\mathcal{Q}_i = \infty \in H(\overline{K})$ we observe that the following simplification takes place:

Let $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Whenever $\mathcal{Q}_i = \overline{\mathcal{Q}}_j$ for $i \neq j$ we have

$$\begin{aligned} \{\mathcal{Q}_1, \mathcal{Q}_2\} + \{\mathcal{Q}_3, \mathcal{Q}_4\} &= \{\mathcal{Q}_i, \mathcal{Q}_j\} + \{\mathcal{Q}_k, \mathcal{Q}_l\} \\ &= \{\mathcal{Q}_k, \mathcal{Q}_l\} + \{\mathcal{Q}_j, \overline{\mathcal{Q}}_j\} \\ &= \{\mathcal{Q}_k, \mathcal{Q}_l\} + \{\infty, \infty\} = \{\mathcal{Q}_k, \mathcal{Q}_l\} \end{aligned} \tag{0}$$

The justification for this follows directly from Remark 2 and Definition 2. It is easily checked that this sum remains well defined if the choice of i and j is not unique. This allows us to treat the above as a separate case in order to characterize the other cases solely through the x-coordinates of the $\mathcal{Q}_i$, so we may assume

$$x_i = x_j \iff \mathcal{Q}_i = \mathcal{Q}_j$$

for any $\mathcal{Q}_i, \mathcal{Q}_j \in H(\overline{K})$, provided we assign $\mathcal{Q}_i = \infty$ the x-coordinate $x_i = \infty$. There is a possibility for ambiguity of "∞" here, which we will however avoid later on by making it obvious which infinity we are working with.

We may now write the distinction between the remaining cases as follows:

1. The general case where x_1, x_2, x_3, x_4 are pairwise distinct.
2. The case where exactly two of x_1, x_2, x_3, x_4 are equal.
3. The case where x_1, x_2, x_3, x_4 are equal in pairs.
4. The case where exactly three of x_1, x_2, x_3, x_4 are equal.
5. The case where where all of x_1, x_2, x_3, x_4 are equal.

Since we are allowed to permute point-components anyway, we may fix which of the $\mathcal{Q}_i$ are equal and write everything out for the complete and final list.

Given the addition $\{\mathcal{Q}_1, \mathcal{Q}_2\} + \{\mathcal{Q}_3, \mathcal{Q}_4\}$, distinguish between the following:

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0. The addition with zero, i.e. $\mathcal{Q}_1 = \overline{\mathcal{Q}}_2$ with $\mathcal{Q}_i \in H(\overline{K})$ for $i = 1, .., 4$.
1. The general case where $\mathcal{Q}_i \neq \mathcal{Q}_j$ and $\mathcal{Q}_i \neq \overline{\mathcal{Q}}_j$ for $i \neq j$.
2. The single tangential case where all $\mathcal{Q}_i$ as in case 1 with the exception of $\mathcal{Q}_1 = \mathcal{Q}_2 \in H_0(\overline{K})$ with $y_1 \neq 0$.
3. The double tangential case where all $\mathcal{Q}_i$ as in case 1 with the exception of $\mathcal{Q}_1 = \mathcal{Q}_2 \in H_0(\overline{K})$ with $y_1 \neq 0$ and $\mathcal{Q}_3 = \mathcal{Q}_4 \in H_0(\overline{K})$ with $y_3 \neq 0$.
4. The triple point case wherein $\mathcal{Q}_1 = \mathcal{Q}_2 = \mathcal{Q}_3 \in H_0(\overline{K})$ with $y_1 \neq 0$ and $\mathcal{Q}_4 \in H(\overline{K})$ differs from both $\mathcal{Q}_1$ and $\overline{\mathcal{Q}}_1$.
5. The quadruple point case where all $\mathcal{Q}_i \in H_0(\overline{K})$ are equal with $y_1 \neq 0$.

Remark 3:

(i) In both lists, the cases do not overlap.

(ii) For the list to be complete, we must allow for $\mathcal{Q}_i = \infty$ for some i. Note that one is sufficient, since if two or more $\mathcal{Q}_i$ were to be ∞, we would be back at the case 0 by virtue of ($\diamondsuit$), bringing the two infinities together in one point. As we did for case 1, we consider $\mathcal{Q}_i \in H_0(\overline{K})$ and $\mathcal{Q}_i \in H(\overline{K})$ in two separate cases and postpone handling the latter.

(iii) As for the first case, the constructions will a priori only be made on $\mathcal{J}$. It remains to check that this makes indeed for a well-defined addition on $\mathbf{J}$ by being invariant under interchanges of $\mathcal{Q}_i$ and $\mathcal{Q}_j$ for any i, j.

1.4 Addition Law for the Special Cases

Before we handle the next construction steps we need an intermediate result.

Lemma 1: If the derivatives $D(\tau) = D'(\tau) = \cdots = D^{(k-1)}(\tau) = 0$ for a polynomial D over a field of characteristic either 0 or at least k, then

$$(t - \tau)^k \mid D(t).$$

Proof. Assume $\tau = 0$ by shifting t to $t + \tau$. With $D(t) = \sum_{i=0}^{d} a_i t^i$ we have

$$a_i = \frac{1}{i!} D^{(i)}(0) = 0 \qquad\qquad i = 0, .., k-1$$

as $i! \neq 0$ by the condition on the characteristic, so t^k divides $D(t)$. □

We will now use this property on $D(x) = C(x) - (P(x))^2$ when factoring $(*)$.

CASE 0, ADDITION WITH ZERO: As anticipated, (0) gives $\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_1$ for every $\mathbf{P}_1 \in \mathbf{J}$ if $\mathbf{P}_2 = \{\mathcal{Q}, \overline{\mathcal{Q}}\} = 0$. Similarly $\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_2$ if $\mathbf{P}_1 = 0$.

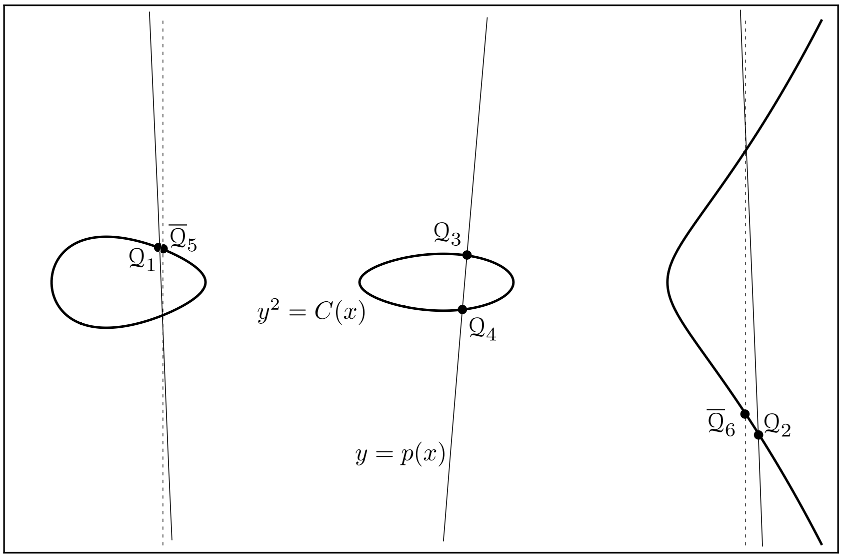

FIGURE 0A: Illustrating a limit argument where $\mathcal{Q}_3$ is very close to $\overline{\mathcal{Q}}_4$.

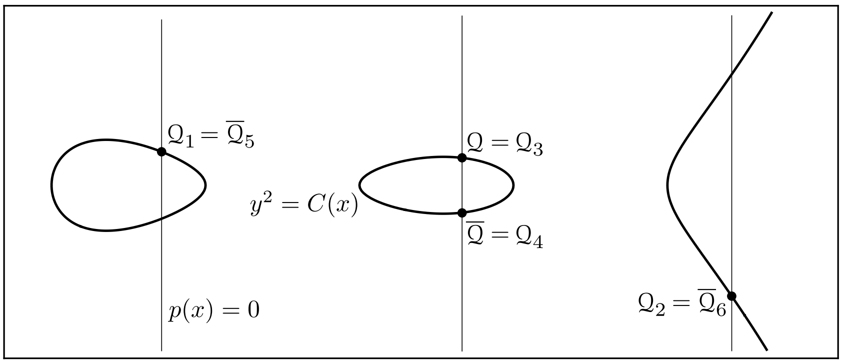

FIGURE 0B The addition with zero and justifying $\{\mathcal{Q}, \overline{\mathcal{Q}}\} \sim 0$.

CASE 2, TANGENTIAL: Let $\mathcal{Q}_i \in H_0(\overline{K}), \mathcal{Q}_1 = \mathcal{Q}_2$, $y_1 \neq 0$ but x_1, x_3 and x_4 are pairwise distinct. We cannot use the Vandermonde matrix in this case because it won't possess maximal rank, consequently being non-invertible. We can however obtain an additional equation by demanding that our polynomial $P(x)$ be tangential to the curve at $\mathcal{Q}_1$. This gives

$$2y\frac{dy}{dx} = 5x^4 + 4ax^3 + 3bx^2 + 2cx + d$$

and
$$\frac{dy}{dx} = 3p_3x^2 + 2p_2x + p_1$$

meaning that the system to solve for $\mathbf{p}$ is now $V_1 \cdot \mathbf{p} = \mathbf{y}_1$ with

$$V_1 = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}.$$

The subscript indicates at which points the intersections have higher order. Here $\mathbf{y}_1$ is defined as $\mathbf{y}$ with y_2 replaced by $y_2' = \frac{C'(x_1)}{2y_1}$ where $C'(x)$ is the derivative of C and $2y_1 \neq 0$ because $\mathrm{char}(\overline{K}) \neq 2$ and $y_1 \neq 0$. Since $\det(V_1) = (x_4 - x_1)^2(x_3 - x_1)^2(x_4 - x_3)$ this fits neatly into our constraints by being non-zero exactly in the case where x_1, x_3 and x_4 are pairwise distinct.

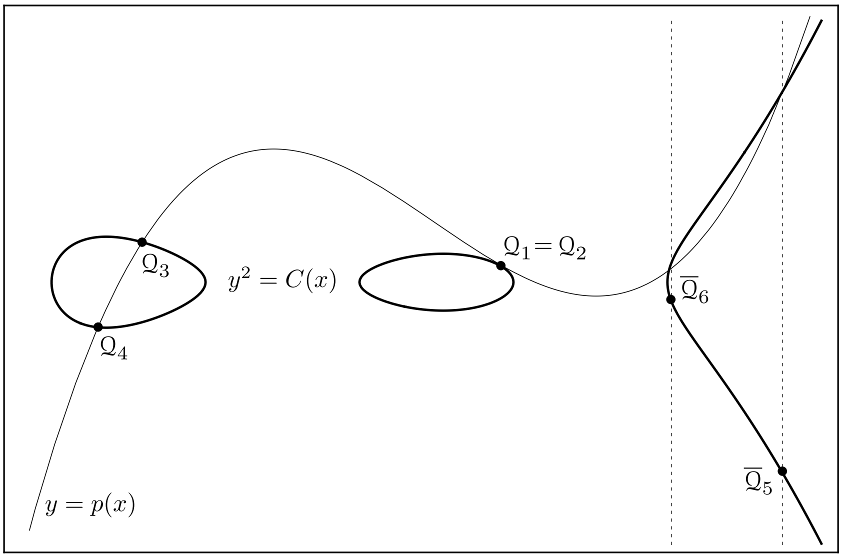

FIGURE 2: The case of a tangential intersection at $\mathcal{Q}_1$.

Once the polynomial is determined, we note that the $p^2(x)$ shares a tangent with $C(x)$ at x_1 on purpose, specifically

$$\begin{aligned} D'(x_1) &= C'(x_1) - 2P(x_1)p'(x_1) \\ &= C'(x_1) - 2y_1y_2' = 0 \end{aligned}$$

so we use Lemma 1 to see that the lefthand side of $(*)$ either factors as

$$D(x) = -p_3^2(x - x_1)^2 \prod_{i=3}^{6}(x - x_i)$$

or

$$D(x) = (x - x_1)^2 \prod_{i=3}^{5}(x - x_i). \tag{$*_2$}$$

Now step two will be entirely identical to that of the general case and we can again solve (‡) or (†) for x_5 or x_5 and x_6.

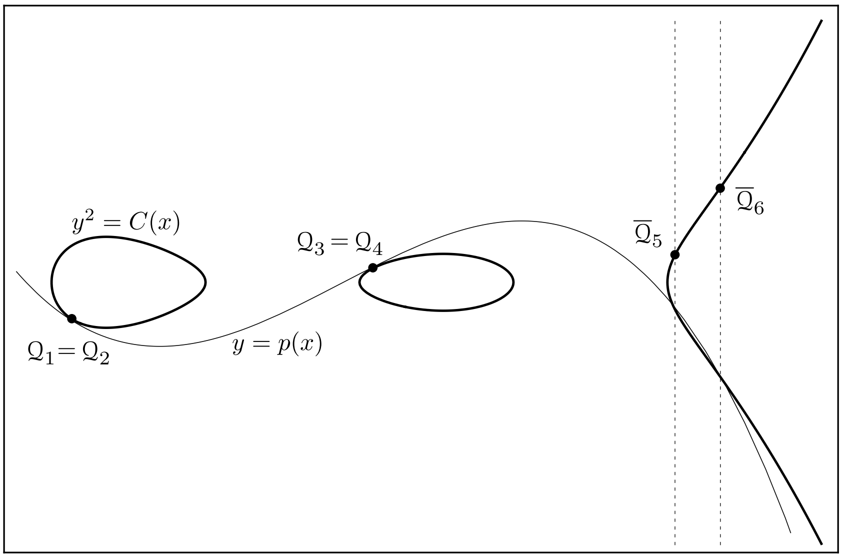

FIGURE 3: Two tangential intersection at $\mathcal{Q}_1$ and $\mathcal{Q}_3$ respectively.

CASE 3, DOUBLE TANGENTIAL: Let $\mathcal{Q}_i \in H_0(\overline{K})$, $\mathcal{Q}_1 = \mathcal{Q}_2$ and $\mathcal{Q}_3 = \mathcal{Q}_4$ but $x_1 \neq x_3$ and $\mathcal{Q}_i \neq \overline{\mathcal{Q}}_i$ meaning that neither y_1 nor y_3 will be zero. As before, we lack equations for our linear system, requiring the use of a second tangential constraint. Replace the fourth row of V_1 and $\mathbf{y}_1$ exactly like we did for the second one: $y_4' = \frac{C'(x_3)}{2y_3}$ and

$$V_{13} = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 0 & 1 & 2x_3 & 3x_3^2 \end{pmatrix}.$$

Now $\det(V_{13}) = (x_3 - x_1)^4$ and this is again different from zero precisely whenever $x_3 \neq x_1$, so as before solve $V_{13} \cdot \mathbf{p} = \mathbf{y}_{13}$ for $\mathbf{p}$, then note that

$D(x)$ has double zeroes at x_1 and x_3 by the same argument as in case 2. By Lemma 1 we know that $(*)$ has a factor $(x - x_1)^2(x - x_3)^2$ so we use

$$\begin{aligned} & D(x) = -p_3^2(x - x_1)^2(x - x_3)^2(x - x_5)(x - x_6) \\ \text{or} \quad & D(x) = (x - x_1)^2(x - x_3)^2(x - x_5). \end{aligned} \tag{$*_3$}$$

and the procedure of case 1 with $x_2 = x_1$ and $x_4 = x_3$ to solve (†) or (‡).

Case 4, Triple Point: Let $\mathcal{Q}_i \in H_0(\overline{K}), \mathcal{Q}_1 = \mathcal{Q}_2 = \mathcal{Q}_3$ but $x_1 \neq x_4$ and $y_1 \neq 0$. We can thus see this as a third-order intersection and demand that the curve and the polynomial share a second-order derivative at $\mathcal{Q}_1$:

$$V_{11} = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_1 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}.$$

Here $\det(V_{11}) = 2(x_4 - x_1)^3$ and with this, define $\mathbf{y}_{11}$ by taking $\mathbf{y}_1$ and replacing the third coordinate by $y_3'' = \frac{C''(x_1)}{2y_1} - \frac{(C'(x_1))^2}{4y_1^3}$ where $C''(x)$ is the second-order derivative of C.

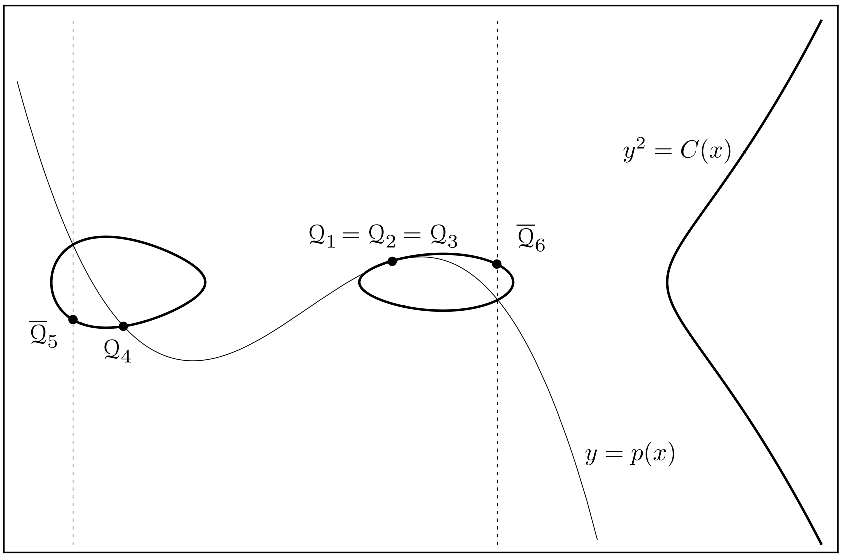

Figure 4: An intersection of order three at $\mathcal{Q}_1$.

Apply Lemma 1 as in case 2, only this time we also have

$$\begin{aligned} D''(x_1) &= C''(x_1) - 2P(x_1)p''(x_1) - 2(p'(x_1))^2 \\ &= C''(x_1) - 2y_1 y_3'' - 2(y_2')^2 = 0 \end{aligned}$$

as can be checked by glancing at the definition of y_3''. With $(*)$ factoring as

$$D(x) = -p_3^2(x - x_1)^3 \prod_{i=4}^{6}(x - x_i)$$

or

$$D(x) = (x - x_1)^3 \prod_{i=4}^{5}(x - x_i), \qquad (*_4)$$

this allows us to continue with one of the second steps of case 1 to find $\mathbf{P}_3$.

Case 5, Quadruple Point: Given the situation where $\mathcal{Q}_i = \mathcal{Q}_1 \in H_0(\overline{K})$ for every i with $y_1 \neq 0$, we use

$$V_{111} = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_1 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

which is invertible in all fields but those of characteristic 2 and 3.

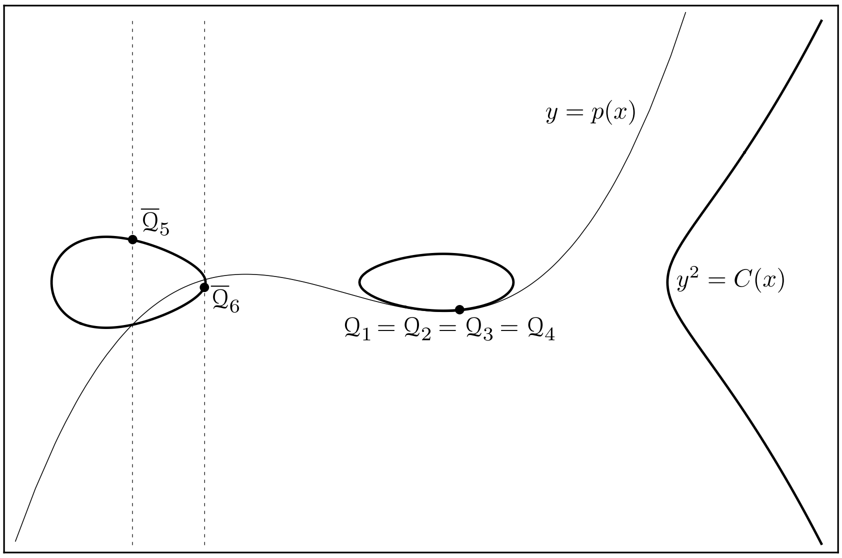

Figure 5: An intersection of order four at $\mathcal{Q}_1$.

Here $\mathbf{y}_{111}$ is the same as $\mathbf{y}_{11}$ except for the last coordinate which should read

$$y_4''' = \frac{C'''(x_1)}{2y_1} - \frac{3C'(x_1)C''(x_1)}{4y_1^3} + \frac{3(C'(x_1))^3}{8y_1^5}$$

where $C'''(x)$ is the third-order derivative of C. Once more, we solve the linear system $V_{111} \cdot \mathbf{p} = \mathbf{y}_{111}$. In addition to $D^{(i)}(x_1) = 0$, $i = 0, .., 2$ we have

$$D'''(x_1) = C'''(x_1) - 2y_1 y_4''' - 6y_2' y_3'' = 0$$

$$\begin{array}{lll} \text{so either} & D(x) = -p_3^2(x - x_1)^4(x - x_5)(x - x_6) & \\ \text{or} & D(x) = (x - x_1)^4(x - x_5) & \end{array} \qquad (*_5)$$

which we subsequently use to solve (†) or (‡) and we're done.

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Finally, as noted in Remark 3, (ii) we have yet to extend our definition from $H_0(\overline{K})$ to $H(\overline{K})$. Observe that this is relevant only for cases number two and four where we now consider $\mathcal{Q}_4 = \infty$ as we did in Step $1'$ of the general case. As an analogue to this, the relevant matrices $\widetilde{V}_1$ and $\widetilde{V}_{11}$ will be the upper-left 3×3 sub-matrices of their $H_0(\overline{K})$-counterparts V_1 and V_{11}.

In both cases we obtain a linear system of the form $\widetilde{V}_* \cdot \mathbf{p} = \widetilde{\mathbf{y}}_*$ for a three-element vector $\mathbf{p}$ and the vectors $\widetilde{\mathbf{y}}_1$ and $\widetilde{\mathbf{y}}_{11}$ are defined like their 4-element counterparts $\mathbf{y}_1$ and $\mathbf{y}_{11}$ with the last coordinate omitted.

Both matrices $\widetilde{V}_*$ are invertible and we therefore get a unique polynomial $P(x)$ wich we use to solve (†′), obtaining x_5, x_6, $y_5 = P(x_5)$ and $y_6 = P(x_6)$ in $\overline{K}$ and we define $\{\mathcal{Q}_1, \mathcal{Q}_2\} + \{\mathcal{Q}_3, \infty\} = \{\overline{\mathcal{Q}}_5, \overline{\mathcal{Q}}_6\} = \{(x_5, -y_5), (x_6, -y_6)\}$.

The counterparts to $(*_2)$ and $(*_4)$ in these cases are

$$D(x) = (x - x_1)^2(x - x_3)(x - x_5)(x - x_6) \qquad (*_2')$$

and $$D(x) = (x - x_1)^3(x - x_5)(x - x_6) \qquad (*_4')$$

1.5 Well-Definedness of the Addition Law

To check whether our addition is well defined on $\mathbf{J}$ in each of the cases, we have to consider the permutation of point-components $\mathcal{Q}_i$ under the equivalence relation from Definition 2. Furthermore, to claim that all possible cases are all covered, it is necessary to check the permutations under (♢). Both cases can be combined into a single one by the following statement:

Lemma 2: In each given definition of "+", the result $\{\mathcal{Q}_5, \mathcal{Q}_6\}$ is invariant under the interchange of $\mathcal{Q}_i$ and $\mathcal{Q}_j$ for any $i, j \in \{1, .., 4\}$.

Proof. Case 0 was done at (0). In cases 1–5, interchanging x_i with x_j and y_i with y_j in our linear systems $V_* \cdot \mathbf{p} = \mathbf{y}_*$ has the effect of permuting rows of V_* and $\mathbf{y}_*$ and relabeling (x_k, y_k) whenever $\mathcal{Q}_k$ was equal to $\mathcal{Q}_i$ or $\mathcal{Q}_j$.

Consequently the resulting $P(x)$ doesn't change, so neither do any of the terms T_*. The three dagger equations remain unchanged as well, as can easily be checked at (†), (‡) and (†′) which are invariant under the interchange of x_i and x_j. Finally, the version of Step 2 we fall into remains the same, since it is only imposed by the distinction of p_3 being zero or not. $\square$

1.6 Summary and Conclusions

For the convenience of the reader we attempt to summarize our approach and give some preliminary implications. First, whenever we refer to $\mathcal{Q}_i$ in $H(\overline{K})$, $\mathcal{Q}_i = (x_i, y_i)$ for $i = 1, .., 6$, from now on it is implicitly understood to be in the context of the sum

$$\{\mathcal{Q}_1, \mathcal{Q}_2\} + \{\mathcal{Q}_3, \mathcal{Q}_4\} = \{\overline{\mathcal{Q}}_5, \overline{\mathcal{Q}}_6\}$$

where our goal was to define $\mathcal{Q}_5$ and $\mathcal{Q}_6$. We started with x_1, x_2, x_3, x_4 all finite and different and used the graph of a cubic polynomial $P(x)$ which may however degenerate into a quadratic, linear or even constant one. The two new intersections of the graph of the polynomial with the curve would then determine the result of our sum. For this, we used the polynomial $D(x) = C(x) - (P(x))^2$ (see Remark 4). If there is only one new intersection, we call the other one ∞. There is at least one new intersection as $D(x)$ had degree six or five which can have no less than five roots in $\overline{K}$.

As a first generalization we allowed for $\mathcal{Q}_4 = \infty$ with x_1, x_2, x_3 still all finite and different. In that case, $D(x)$ has degree five, so x_5, x_6 have to be finite.

This completed case 1 "all x_i different" because (♢) allowed interchanging $\mathcal{Q}_4$ with $\mathcal{Q}_1, \mathcal{Q}_2$ or $\mathcal{Q}_3$. This symmetry was consistent with the addition rule so far and it furthermore allowed us to say that, if $\mathcal{Q}_i = \overline{\mathcal{Q}}_j$ for $i \neq j$, $\mathcal{Q}_i \neq \mathcal{Q}_j$, then we are in the zero case. We used this to write the complete list of cases, considering four cases of $x_1, .., x_4$ finite but not different. We completed each case with $x_1, .., x_3$ finite but $x_4 = \infty$ as we did for case 1.

We can extend the distinction "$\mathcal{Q}_i \neq \overline{\mathcal{Q}}_j$" between zero and non-zero cases.

Lemma 3: Provided we are not in case 0, then $\mathcal{Q}_i \neq \overline{\mathcal{Q}}_j$ for $i \neq j$, $\mathcal{Q}_i \neq \mathcal{Q}_j$ is not only true for $i, j = 1, .., 4$ but for $i, j = 1, .., 6$ as well.

Proof. All points $\mathcal{Q}_i = (x_i, y_i) \neq \infty$ lie on the graph of P, so no two points can have the same x-coordinate but different y-coordinates. $\square$

A very useful special case of "$\mathcal{Q}_i \neq \overline{\mathcal{Q}}_j$" is that no more that one of the six components $\mathcal{Q}_1, .., \mathcal{Q}_6$ may equal ∞ in non-zero cases.

The symmetry we defined at ($\diamondsuit$) will turn out to be a provable property, but we can already broaden it in the next Lemma.

Lemma 4: The symmetry imposed on the sum in $\mathcal{Q}_1,..,\mathcal{Q}_4$ now extends to symmetry in $\mathcal{Q}_1,..,\mathcal{Q}_6$. For instance, once we explicitely derive the sum $\{\mathcal{Q}_1,\mathcal{Q}_2\}+\{\mathcal{Q}_3,\mathcal{Q}_4\} = \{\overline{\mathcal{Q}}_5,\overline{\mathcal{Q}}_6\}$, we know that $\{\mathcal{Q}_5,\mathcal{Q}_2\}+\{\mathcal{Q}_3,\mathcal{Q}_4\} = \{\overline{\mathcal{Q}}_1,\overline{\mathcal{Q}}_6\}$.

To see why this is obvious, we first summarize the various $D(x)$ we derived at $(*_1)$ through $(*_5)$ and $(*_1')$, $(*_2')$ and $(*_4')$ for cases 1–5 and give the simplest general expression possible.

Remark 4: In every non-zero case we have a polynomial P whose graph we let intersect the curve such that the new intersections give $\mathcal{Q}_5$ and $\mathcal{Q}_6$ where they are finite. This is done by finding x_5 and x_6 if they exist, which in turn was done by solving $D(x) = 0$ where we defined $D(x) = C(x) - (P(x))^2$.

In all non-zero cases we manually derived the expression for the various $D(x)$ at $(*_i)$ and $(*_i')$ for cases 1–5, which all come down to

$$D(x) = -p_3^2 \prod_{i=1}^{6} (x - x_i)$$

if $p_3 \neq 0$. If $p_3 = 0$ and therefore $\mathcal{Q}_j = \infty$ for exactly one $j \in \{1,..,6\}$ then

$$D(x) = \prod_{\substack{i=1 \\ i\neq j}}^{6} (x - x_i).$$

Remember that at most one of the $\mathcal{Q}_i$ can be ∞, covering all non-zero cases.

For later use we can even write the more compact product

$$D(x) = \delta \prod_{\mathcal{Q}} \big(x - x(\mathcal{Q})\big)^{e(\mathcal{Q})}$$

taken over all $\mathcal{Q} \in H_0(\overline{K})$ where $e(\mathcal{Q})$ is the number among $\mathcal{Q}_1,..,\mathcal{Q}_6$ that equal $\mathcal{Q} = (x(\mathcal{Q}), y(\mathcal{Q}))$. This works because by Lemma 3, if $x_i = x_j$ for $i \neq j$, then $\mathcal{Q}_i = \mathcal{Q}_j$.

Proof of Lemma 4. Both expressions above remain unchanged under permutation on the x_i. And since $\mathcal{Q}_i = (x_i, y_i)$ with $y_i = P(x_i)$, we're done.

Last but not least, this Lemma remains universally valid, as the property can be checked for the zero case as well through a quick glance at (0). $\square$

2 Rational Functions on Hyperelliptics

The goal of this chapter is to look at rational functions on the hyperelliptic curve $y^2 = C(x)$ and to define a notion of the order of a function in a point on the curve. We then give some basic properties for this order function before we introduce divisors and proceed to look at functions with a specified number of poles at ∞ as a preparation for the proof of associativity on $\mathbf{J}$.

2.1 Function Field and Order of Rational Functions

Definition 3: Define the ring of rational functions on the curve as

$$\mathcal{F} = K(x)[y] \Big/ \big(y^2 - C(x)\big).$$

As $y^2 - C(x)$ is irreducible in $K(x)[y]$, $\mathcal{F}$ is a field and $\mathcal{F} = K(x) + K(x)y$, so write elements $f \in \mathcal{F}$ as $f = g + hy$ where $g, h \in K(x)$. Define $\overline{f} = g - hy$.

We will occasionally write things like $K[x, y] \subset \mathcal{F}$ but this is always implicitly understood to be in conjunction with $y^2 = C(x)$.

Reminders: A formal Laurent series written $\tau = \gamma t^e(1 + \cdots) \in K(\!(t)\!)$ with $\gamma \in K^*$, $e \in \mathbb{Z}$ has a a unique inverse $\tau^{-1} = \gamma^{-1}t^{-e}(1 + \cdots) \in K(\!(t)\!)$.

From now on we will use an ellipsis to denote any terms of ascending order everywhere where we are not interested in the specifics.

If $\mathrm{char}(K) \neq 2$ and τ is a series of the form $\tau = 1 + \sum_{i=1}^{\infty} a_i t^i$ then it has a unique square root of the form $\sigma = 1 + \sum_{i=1}^{\infty} b_i t^i$ in $\overline{K}(\!(t)\!)$ meaning $\sigma^2 = \tau$. Write $\sigma = \sqrt{\tau} = 1 + \cdots$.

$\sim$

In order to define the order of a function f in a point $\mathcal{Q}$, $\mathrm{ord}_{\mathcal{Q}}(f) \in \mathbb{Z}$ for $f \in \mathcal{F}^*$ and $\mathcal{Q} \in H(\overline{K})$ we first construct a K-homomorphism

$$\lambda_{\mathcal{Q}} : \mathcal{F} \to \overline{K}(\!(t)\!)$$

Because $C\big(\lambda_{\mathcal{Q}}(x)\big) = \big(\lambda_{\mathcal{Q}}(y)\big)^2$ has to be fulfilled, we decide on $\lambda_{\mathcal{Q}}(x)$ and deduce $\lambda_{\mathcal{Q}}(y)$. For this, we distinguish between three cases for $\mathcal{Q} \in H(\overline{K})$.

Definition 4: 1. Let $\mathcal{Q} = (x_0, y_0) \in H_0(\overline{K})$ with $y_0 \neq 0$ and consequently $C(x_0) \neq 0$.

Define $\lambda_{\mathcal{Q}}(x) = x_0 + t$. Now

$$\begin{aligned} C(\lambda_{\mathcal{Q}}(x)) &= C(x_0 + t) \\ &= C(x_0) + \cdots + t^5 \\ &= C(x_0)(1 + \cdots) \\ &= y_0^2 \tau_1 \qquad \text{with } \tau_1 \in K(\!(t)\!). \end{aligned}$$

Define $\lambda_{\mathcal{Q}}(y) = y_0 \sqrt{\tau_1}$.

2. Let $\mathcal{Q} = (x_0, y_0) \in H_0(\overline{K})$ with $y_0 = 0$. Points with this property are called "Weierstrass Points" and as $C(x_0) = 0$ there are at most five of them. Write $C(x) = \prod_{i=1}^{5}(x - \alpha_i)$ for $\alpha_i \in \overline{K}$ and for instance $\alpha_1 = x_0$.

Define $\lambda_{\mathcal{Q}}(x) = x_0 + t^2$. Now

$$\begin{aligned} C(\lambda_{\mathcal{Q}}(x)) &= t^2 \prod_{i=2}^{5}(x_0 - \alpha_i + t^2) \\ &= \mu t^2 (1 + \cdots) \end{aligned}$$

with $\mu \in \overline{K}$ being $\mu = \prod_{i=2}^{5}(x_0 - \alpha_i) = C'(x_0)$ which is non-zero because our curve is non-singular. Write therefore $C(\lambda_{\mathcal{Q}}(x)) = \mu t^2 \tau_2$ with τ_2 of the form $1 + \cdots$ and define $\lambda_{\mathcal{Q}}(y) = \nu t \sqrt{\tau_2}$ with $\nu^2 = \mu$, $\nu \in \overline{K}$.

Note that we may choose one out of two square roots for ν so whichever we take we demand that we stay consistent in our choice for now. It will however turn out that this choice bears no effect on definition 5.

3. Let $\mathcal{Q} = \infty$. Define $\lambda_{\mathcal{Q}}(x) = t^{-2}$. It follows that

$$\begin{aligned} C(\lambda_{\mathcal{Q}}(x)) &= t^{-10} + at^{-8} + bt^{-6} + ct^{-4} + dt^{-2} + e \\ &= t^{-10}\sqrt{\tau_3} \end{aligned}$$

so define $\lambda_{\mathcal{Q}}(y) = t^{-5}\sqrt{\tau_3}$.

Definition 5: For $\mathcal{Q} \in H(\overline{K})$ and $f \in \mathcal{F}^*$ define $\operatorname{ord}_{\mathcal{Q}}(f) = \operatorname{ord} \lambda_{\mathcal{Q}}(f)$.

Lemma 5: Let $f \in K[x, y] \subset \mathcal{F}$, $f \neq 0$ and $\mathcal{Q} \in H_0(\overline{K})$, $\mathcal{Q} = (x_0, y_0)$. Then

(a) $\operatorname{ord}_{\mathcal{Q}} f \geq 0$ and

(b) if $f(\mathcal{Q}) = 0$ then $\operatorname{ord}_{\mathcal{Q}} f \geq 1$.

Proof.

(a) Because $\mathcal{Q} \in H_0(\overline{K})$ we have $\lambda_{\mathcal{Q}}(x)$, $\lambda_{\mathcal{Q}}(y) \in \overline{K}[[t]]$ so with $f \in K[x, y]$ we have $\mathrm{ord}_{\mathcal{Q}}(f) \in \mathbb{N} \cup \{0\}$.

(b) Write $f = A(x, y)$, $A(X, Y) \in K[X, Y]$. First, let $y_0 \neq 0$.

$$\begin{aligned}\mathrm{ord}_{\mathcal{Q}} f &= \mathrm{ord}\, A(\lambda_{\mathcal{Q}}(x), \lambda_{\mathcal{Q}}(y)) \\ &= \mathrm{ord}\, A(x_0 + t, y_0\sqrt{\tau_1}).\end{aligned}$$

But $A(x_0 + t, y_0\sqrt{\tau_1}) = \sum_{i=0}^{\infty} a_i t^i$ so with $t = 0$ we get $A(x_0, y_0) = a_0$ but the former is $f(\mathcal{Q})$ which is 0, so $a_0 = 0$ and the claim follows.

If $y_0 = 0$ we would have $\mathrm{ord}_{\mathcal{Q}} f = \mathrm{ord}\, A(x_0 + t^2, \nu t\sqrt{\tau_2})$ instead. But like before, this means $0 = f(\mathcal{Q}) = A(x_0, 0) = a_0$ so again $\mathrm{ord}_{\mathcal{Q}}(f) \geq 1$.

□

Lemma 6: Let $f \in K(x) \subset \mathcal{F}$, $f \neq 0$, $\mathcal{Q} \in H(\overline{K})$. Then

(a) $\mathrm{ord}_{\mathcal{Q}}(f) = \mathrm{ord}_{x=x_0} f(x)$ if $\mathcal{Q} = (x_0, y_0) \in H_0(\overline{K})$ with $y_0 \neq 0$.

(b) $\mathrm{ord}_{\mathcal{Q}}(f) = 2\mathrm{ord}_{x=x_0} f(x)$ if $\mathcal{Q} = (x_0, 0) \in H_0(\overline{K})$.

(c) $\mathrm{ord}_{\infty}(f) = 2\mathrm{ord}_{x=\infty} f(x) = 2\mathrm{ord}_{x=0} f(\frac{1}{x})$.

Note that the right-hand sides of the equalities refer to the usual definition of the order of a rational function in a point $x_0 \in \overline{K} \cup \{\infty\}$.

Proof. Take $f \in K[x]$ and define $e = \mathrm{ord}_{x=x_0} f \in \mathbb{N}$ so $f(x) = (x - x_0)^e g(x)$ with $g \in \overline{K}[x]$ and $g(x_0) \neq 0$.

(a) If $\mathcal{Q} = (x_0, y_0)$, $y_0 \neq 0$ then $\lambda_{\mathcal{Q}}(f) = f(x_0 + t) = t^e g(x_0 + t)$ and so $\mathrm{ord}\, \lambda_{\mathcal{Q}}(f) = e$ because $g(x_0 + t) = g(x_0) + \cdots$ with $g(x_0) \neq 0$.

(b) Here $\lambda_{\mathcal{Q}}(f) = f(x_0 + t^2) = t^{2e} g(x_0 + t^2)$ and again $g(x_0 + t^2) = g(x_0) + \cdots$ so $\mathrm{ord}\, \lambda_{\mathcal{Q}}(f) = 2e$.

(c) For $\mathcal{Q} = \infty$ we have $\lambda_{\mathcal{Q}}(f) = f(t^{-2}) = \kappa t^{-2d} + \cdots$ with $\kappa \neq 0$ if $d = \deg f$ so $\mathrm{ord}_{x=0} f(\frac{1}{x}) = -d$ and so $\mathrm{ord}\, \lambda_{\mathcal{Q}}(f) = 2\mathrm{ord}_{x=0} f(\frac{1}{x})$.

Generally, if $f \in K(x)$ we can write $f = \frac{r}{q}$ for $r, q \in K[x]$ and apply $\mathrm{ord}_{x=x_0} f = \mathrm{ord}_{x=x_0} r - \mathrm{ord}_{x=x_0} q$ in order to use the above on r and q. □

Lemma 7: For $f \in \mathcal{F}^*$ and $\mathcal{Q} \in H(\overline{K})$ the order satisfies $\mathrm{ord}_{\mathcal{Q}}(\overline{f}) = \mathrm{ord}_{\overline{\mathcal{Q}}}(f)$

Proof. Split into three possible cases:

1. For $\mathcal{Q} \in H_0(\overline{K})$ with $y_0 \neq 0$ we've got $\lambda_{\mathcal{Q}}(x) = x_0 + t$ and $\lambda_{\mathcal{Q}}(y) = y_0\sqrt{\tau_1}$.

Therefore $\lambda_{\overline{\mathcal{Q}}}(x) = x_0 + t = \lambda_{\mathcal{Q}}(\overline{x})$ and $\lambda_{\overline{\mathcal{Q}}}(y) = -y_0\sqrt{\tau}_1 = \lambda_{\mathcal{Q}}(\overline{y})$ so

$$\begin{aligned}\lambda_{\overline{\mathcal{Q}}}(f(x,y)) &= f(\lambda_{\overline{\mathcal{Q}}}(x), \lambda_{\overline{\mathcal{Q}}}(y))\\ &= \lambda_{\mathcal{Q}}(f(\overline{x}, \overline{y}))\\ &= \lambda_{\mathcal{Q}}(\overline{f}(x,y)).\end{aligned}$$

2. If $\mathcal{Q} \in H_0(\overline{K})$ with $y_0 = 0$ then $\overline{\mathcal{Q}} = \mathcal{Q}$. Write $f(x) = g(x) + h(x)y$, so

$$\lambda_{\mathcal{Q}}(\overline{f}) = g(x_0 + t^2) - h(x_0 + t^2)\nu t\sqrt{\tau_2}.$$

   Calling this $l(t) = \lambda_{\mathcal{Q}}(\overline{f})$ and looking at the construction of $\lambda_{\mathcal{Q}}$ we see that τ_2 sports only even powers of t so we see above that $l(-t) = \lambda_{\mathcal{Q}}(f)$. As interchanging t with $-t$ doesn't change the order, we're done.

3. For $\mathcal{Q} = \infty$, $\tau_3 = 1 + at^2 + bt^4 + ct^6 + dt^8 + et^{10}$ features only even powers as well, so again $l(-t) = \lambda_{\mathcal{Q}}(f)$ for $l(t) = \lambda_{\mathcal{Q}}(\overline{f}) = g(t^{-2}) - h(t^{-2})t^{-5}\sqrt{\tau_3}$. Finally, $\lambda_{\mathcal{Q}}(f)$ is equal to $\lambda_{\overline{\mathcal{Q}}}(f)$ since $\infty = \overline{\infty}$. Again $\operatorname{ord} l(t) = \operatorname{ord} l(-t)$.

□

Remark 5: We now see why the choice of the square root in definition 4 has no effect on $\operatorname{ord}_{\mathcal{Q}} f$. If $\lambda'_{\mathcal{Q}}$ were to correspond to the other choice, we would have $\lambda'_{\mathcal{Q}}(x) = \lambda_{\mathcal{Q}}(x)$ and $\lambda'_{\mathcal{Q}}(y) = -\lambda_{\mathcal{Q}}(y)$, so $\lambda'_{\mathcal{Q}}(f) = \lambda_{\mathcal{Q}}(\overline{f})$. As this was under item 2 where $y_0 = 0$, meaning $\overline{\mathcal{Q}} = \mathcal{Q}$, we have $\operatorname{ord} \lambda_{\mathcal{Q}}(\overline{f}) = \operatorname{ord} \lambda_{\mathcal{Q}}(f)$.

Lemma 8: If $f \in \mathcal{F}^*$ then the set $\left\{\mathcal{Q} \in H(\overline{K}) \;\middle|\; \operatorname{ord}_{\mathcal{Q}} f \neq 0\right\}$ is finite and

$$\sum_{\mathcal{Q} \in H(\overline{K})} \operatorname{ord}_{\mathcal{Q}} f = 0.$$

Proof. First take $f \in K[x,y]$, $f \neq 0$ and let $\mathcal{Q} = (x_0, y_0) \in H_0(\overline{K})$, $y_0 \neq 0$, with $\operatorname{ord}_{\mathcal{Q}} f \neq 0$. By Lemma 5 (a) we know that $\operatorname{ord}_{\mathcal{Q}} f \geq 1$. It follows that $\operatorname{ord}_{\mathcal{Q}}(f\overline{f}) = \operatorname{ord}_{\mathcal{Q}} f + \operatorname{ord}_{\mathcal{Q}} \overline{f} \geq 1$. Now since $f = g + hy$, $f\overline{f} = g^2 - h^2C(x)$ which lies in $K[x]$, so by Lemma 6 we have $\operatorname{ord}_{x=x_0}(f\overline{f}) = \operatorname{ord}_{\mathcal{Q}}(f\overline{f}) \geq 1$. But there are only finitely many such x_0 and so only finitely many such $y_0 = \pm\sqrt{C(x_0)}$.

If $f \in K(x,y)$, $f = \frac{r}{q}$, $r, q \in K[x,y]$ then $\operatorname{ord}_{x=x_0} f = \operatorname{ord}_{x=x_0} r - \operatorname{ord}_{x=x_0} q$, and again only finitely many x_0 exist for which this differs from zero.

For the second claim, give our sum the name

$$s(f) = \sum_{\mathcal{Q} \in H(\overline{K})} \operatorname{ord}_{\mathcal{Q}} f$$

and note that $s(f) = s(\overline{f})$ due to Lemma 7 and the fact that we take the sum over all $\mathcal{Q}$. Because $\operatorname{ord}_{\mathcal{Q}}(f\overline{f}) = \operatorname{ord}_{\mathcal{Q}}(f) + \operatorname{ord}_{\mathcal{Q}}(\overline{f})$ we can see that

$$s(f\overline{f}) = s(f) + s(\overline{f}) = 2s(f).$$

But because $f\overline{f} \in K(x)$ we can use Lemma 6 to write this out as

$$\begin{aligned}
2s(f) &= \sum_{\mathcal{Q} \in H(\overline{K})} \operatorname{ord}_{\mathcal{Q}} f\overline{f} \\
&= 2 \sum_{\substack{x_0 \neq \infty \\ C(x_0) \neq 0}} \operatorname{ord}_{x=x_0} f\overline{f} + 2 \sum_{\substack{x_0 \neq \infty \\ C(x_0) = 0}} \operatorname{ord}_{x=x_0} f\overline{f} + 2 \sum_{x_0 = \infty} \operatorname{ord}_{x=x_0} f\overline{f} \\
&= 2 \sum_{x_0 \in \overline{K} \cup \{\infty\}} \operatorname{ord}_{x=x_0} f\overline{f}.
\end{aligned}$$

Since
$$\sum_{x_0 \in \overline{K} \cup \{\infty\}} \operatorname{ord}_{x=x_0} g = 0$$
for any $g \in K(x)$ we have $s(f) = 0$ in $\mathbb{Z}$. $\square$

2.2 Divisors and Lemmas

Definition 6: The divisor of a function $f \in \mathcal{F}^*$ is the formal sum

$$(f) = \sum_{\mathcal{Q} \in H(\overline{K})} \operatorname{ord}_{\mathcal{Q}} f \cdot \mathcal{Q}$$

Thanks to Lemma 8 the sum is finite and the sum of all coefficients is 0.

Points $\mathcal{Q} \in H(\overline{K})$ with a positive coefficient in (f) are called zeroes of f while those with a negative coefficient are called poles.

Lemma 9: If $f \in \mathcal{F}^*$ has no poles on $H(\overline{K})$, i.e. if $\operatorname{ord}_{\mathcal{Q}} f \geq 0$ for all $\mathcal{Q}$, then f is constant.

Proof. With $f = g + hy$, $\operatorname{ord}_{\mathcal{Q}} f \geq 0$ for every $\mathcal{Q} \in H(\overline{K})$ we take a look at $f + \overline{f} = 2g \in K(x)$ and $f\overline{f} = g^2 - h^2C \in K(x)$ and observe that by well-known properties of orders

$$\begin{aligned}
\operatorname{ord}_{\mathcal{Q}}(f + \overline{f}) &\geq \min\{\operatorname{ord}_{\mathcal{Q}} f, \operatorname{ord}_{\mathcal{Q}} \overline{f}\} \\
&= \min\{\operatorname{ord}_{\mathcal{Q}} f, \operatorname{ord}_{\overline{\mathcal{Q}}} f\} \geq 0.
\end{aligned}$$

with Lemma 7 and similarly

$$\mathrm{ord}_{\mathcal{Q}}(f\overline{f}) = \mathrm{ord}_{\mathcal{Q}} f + \mathrm{ord}_{\overline{\mathcal{Q}}} f \geq 0.$$

With the help of Lemma 6 we conclude that $\mathrm{ord}_{x=x_0}(f+\overline{f}) \geq 0$ and $\mathrm{ord}_{x=x_0}(f\overline{f}) \geq 0$ respectively.

But in $\overline{K}(x)$, a function q with $\mathrm{ord}_{x=x_0} q \geq 0$ for every $x_0 \in \overline{K} \cup \{\infty\}$ must be constant, so both $f+\overline{f}$ and $f\overline{f}$ are constant functions. Since f is a root of $(T-f)(T-\overline{f}) = T^2 - (f+\overline{f})T + f\overline{f} \in \overline{K}[T]$, f lies in $\overline{K}$. □

Theorem 1: *(Proof in appendix)* Let $x, y \in K(t)$ be rational functions that fulfill $y^2 = C(x)$. Then x and y are actually constants, i.e. $x, y \in K$.

Lemma 10: Suppose $f \in \mathcal{F}^*$ has a pole of order at most one at ∞ i.e. $\mathrm{ord}_\infty f \geq -1$ and no pole at any other $\mathcal{Q} \in H_0(\overline{K})$. Then f is a constant.

Proof. By Lemma 9 we can assume $\mathrm{ord}_\infty f = -1$ and $\mathrm{ord}_{\mathcal{Q}} f \geq 0$ for every $\mathcal{Q} \in H_0(\overline{K})$. Now $f = g + hy$ and $\lambda_\infty(f) = \kappa t^{-1} + \cdots$ with $\kappa \neq 0$ in $\overline{K}$. Since we are only interested in the order, replace f with $\kappa^{-1} f$ so $\lambda_\infty(f) = t^{-1} + \cdots$.

Now remember that $\lambda_\infty(x) = t^{-2}$ so $\lambda_\infty(x - f^2) = \alpha t^{-1} + \cdots$, $\alpha \in \overline{K}$ and finally we have a regular power series $\lambda_\infty(x - f^2 - \alpha f)$ so

$$\mathrm{ord}_\infty(x - f^2 - \alpha f) \geq 0.$$

Also for any other $\mathcal{Q} \in H_0(\overline{K})$ we have

$$\mathrm{ord}_{\mathcal{Q}}(x - f^2 - \alpha f) \geq \min\{\mathrm{ord}_{\mathcal{Q}}(x), \mathrm{ord}_{\mathcal{Q}}(f^2), \mathrm{ord}_{\mathcal{Q}} f\}$$

which is non-negative by virtue of the prerequisite on f and $\lambda_{\mathcal{Q}}(x) = x_0 + \cdots$. The previous Lemma now implies $x - f^2 - \alpha f \in \overline{K}$ so we see $x = f^2 + \alpha f + \beta$ as a polynomial $x = X(f)$ with $X(T) \in \overline{K}[T] \setminus \overline{K}$.

Do the same thing with $\lambda_\infty(y) = t^{-5}\sqrt{\tau_3} = \quad t^{-5} + \cdots$

so $\lambda_\infty(y - f^5) = \beta_4 t^{-4} + \cdots$

and $\lambda_\infty(y - f^5 - \beta_4 f^4) = \beta_3 t^{-3} + \cdots$

and so on, so $\lambda_\infty(y - f^5 - \beta_4 f^4 - \beta_3 f^3 - \beta_2 f^2 - \beta_1 f)$ is a power series as well. Again $y = f^5 + \beta_4 f^4 + \beta_3 f^3 + \beta_2 f^2 + \beta_1 f + \beta_0 = Y(f)$ with $Y(T) \in \overline{K}[T] \setminus \overline{K}$.

Combined we have the polynomial equation $Y(f)^2 - C(X(f)) = 0$ over $\overline{K}$ whose algebraic closure dictates either $f \in \overline{K}$ or $Y(T)^2 = C(X(T))$. The latter can't be true by Theorem 1 and the former is in contradiction to $\mathrm{ord}_\infty f = -1$. □

Lemma 11: If $f \in \mathcal{F}^*$ has a pole of order at most two at ∞ and $\operatorname{ord}_{\mathcal{Q}} f \geq 0$ for every $\mathcal{Q} \in H_0(\overline{K})$ then f is of the form $f = \alpha + \beta x$ with $\alpha, \beta \in K$.

Proof. Because $\lambda_\infty(f) = \beta t^{-2} + \cdots$ where $\beta \in \overline{K}$ we have $\lambda_\infty(f - \beta x) = \gamma t^{-1} + \cdots$ and thanks to Lemma 10 we know this to imply $f - \beta x = \alpha$. □

Note that it is easy to see that in this case $(f) = \mathcal{A} + \overline{\mathcal{A}} - 2\infty$ for $\mathcal{A} = (-\frac{\alpha}{\beta}, *)$

Lemma 12: If $f \in \mathcal{F}^*$ has a pole of order at most three at ∞ and $\operatorname{ord}_{\mathcal{Q}} f \geq 0$ for every $\mathcal{Q} \in H_0(\overline{K})$ then f is of the form $f = \alpha + \beta x$ with $\alpha, \beta \in K$.

Proof. From $\lambda_\infty(f) = \gamma t^{-3} + \cdots$ and $f + \overline{f} \in K(x)$ we get that the order of the latter must be even by Lemma 6 (c), so $\operatorname{ord}_\infty(f + \overline{f}) \geq -2$. Like f, $f + \overline{f}$, has no poles on $H_0(\overline{K})$ so Lemma 11 implies $f + \overline{f} = \alpha + \beta x$. Writing $f = g + hy$ we therefore see that $g = \frac{\alpha + \beta x}{2}$ and $f - g = hy$. The latter also has a pole of order three at ∞ and none on $H_0(\overline{K})$ so we are effectively reduced to studying functions f of the form $f = hy$ for $h \in K(x)$, $h \neq 0$.

Lemma 11 lets us assume $\operatorname{ord}_\infty f = -3$. Now with $f = hy$ we have that $-3 = \operatorname{ord}_\infty h + \operatorname{ord}_\infty y = \operatorname{ord}_\infty h - 5$ so $\operatorname{ord}_\infty h = 2$. Writing $h = \frac{r}{q}$ for $r, q \in K[x]$ coprime, this means that q cannot be constant. Use this to pick a $\xi \in \overline{K}$ with $q(\xi) = 0$, implying $r(\xi) \neq 0$ by coprimality.

Pick a point $\mathcal{R} = (\xi, \eta)$ in $H_0(\overline{K})$. If $C(\xi) \neq 0$ then $\operatorname{ord}_{\mathcal{R}} y = 0$ and $\operatorname{ord}_{\mathcal{R}} q \geq 1$ from which we deduce $\operatorname{ord}_{\mathcal{R}} f = \operatorname{ord}_{\mathcal{R}} \frac{y}{q} \leq -1$, a contradiction. In case ξ is a Weierstrass Point, $C(\xi) = 0$ then $\operatorname{ord}_{\mathcal{R}} y = 1$ and $\operatorname{ord}_{\mathcal{R}} q \geq 2$ and so we get the same contradiction. This finishes the proof. □

Lemma 13: If $f \in \mathcal{F}^*$ has a pole of order at most four at ∞ and $\operatorname{ord}_{\mathcal{Q}} f \geq 0$ for every $\mathcal{Q} \in H_0(\overline{K})$ then f is of the form $f = \alpha + \beta x + \gamma x^2$ with $\alpha, \beta, \gamma \in K$.

Proof. With $\lambda_\infty(f) = \gamma t^{-4} + \cdots$ we know that $\lambda_\infty(f - \gamma x^2) = \delta t^{-3} + \cdots$ and we use the previous Lemma to see that $f - \gamma x^2 = \alpha + \beta x$. □

3 Associativity

We return to the context of the sum defined in Chapter 1. Suppose we have $\mathcal{Q}_i \in H(\overline{K})$, $i = 1, .., 6$ that fulfill

$$\{\mathcal{Q}_1, \mathcal{Q}_2\} + \{\mathcal{Q}_3, \mathcal{Q}_4\} = \{\overline{\mathcal{Q}}_5, \overline{\mathcal{Q}}_6\}.$$

If we are in a non-zero case, then we had the graph of a polynomial P passing through all $\mathcal{Q}_i \neq \infty$, $i = 1, .., 6$. Remember that then, by Lemma 3, at most one of the $\mathcal{Q}_i$ can be ∞. We defined $D(x) = C(x) - (P(x))^2$.

Lemma 14: If the sum falls into one of the cases 1–5, pick any $\mathcal{Q}_k \neq \infty$ for $k = 1, .., 6$, so $\mathcal{Q}_k = (x_k, y_k) \in H_0(\overline{K})$. Now it can be said that $\mathcal{Q}_k$ occurs exactly e_k times in the expression of the sum, e_k being between 1 and 6. Then

$$\mathrm{ord}_{x=x_k} D(x) = e_k.$$

Proof. With Remark 4 we know $D(x) = (x - x_k)^{e_k} g(x)$ with $g_k(x_k) \neq 0$. □

Now write $\mathbf{P}_1 = \{\mathcal{Q}_1, \mathcal{Q}_2\}, \mathbf{P}_2 = \{\mathcal{Q}_3, \mathcal{Q}_4\}, \mathbf{P}_3 = \{\overline{\mathcal{Q}}_5, \overline{\mathcal{Q}}_6\}$ and keep in mind that we will now allow for all six cases 0–5.

Theorem 2: If we have $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbf{J}$ with

$$\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_3,$$

then there exists an $f \in \mathcal{F}^*$ with divisor

$$(f) = \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 + \mathcal{Q}_4 - \overline{\mathcal{Q}}_5 - \overline{\mathcal{Q}}_6 - 2\infty.$$

Proof. First part

Begin with the observation that in case 0 we are looking for a function f with divisor $(f) = \mathcal{Q}_1 + \overline{\mathcal{Q}}_1 - 2\infty$. If $\mathcal{Q}_1 \neq \infty$, we may take $f = x - x_1$, otherwise $f = 1$ fills the criterion, so we are completely done with case 0.

Exclude the zero case from now on, so for the remaining cases we have the polynomial $P(x)$ at our disposal. As $y_i = P(x_i)$ whenever $\mathcal{Q}_i \neq \infty$, we define

$$q = y - P(x) \in K[x, y] \subset \mathcal{F}^*$$

with the intention of showing that $(q) = \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 + \mathcal{Q}_4 + \mathcal{Q}_5 + \mathcal{Q}_6 - 6\infty$.

By Lemma 5 (b) we know that $\operatorname{ord}_{\mathcal{Q}_i} q \geq 1$ for those $\mathcal{Q}_i$ that are not ∞.

Let's first assume that all $\mathcal{Q}_i \in H_0(\overline{K})$. This means that we are in the case where $P(x)$ is of degree 3, so as $\lambda_\infty(x) = t^{-2}$ we see that $\operatorname{ord}_\infty q = -6$.

Use Lemma 14 in conjunction with Lemma 6 to see that for each of the $\mathcal{Q}_i$

$$\begin{aligned} \operatorname{ord}_{\mathcal{Q}_i} q + \operatorname{ord}_{\mathcal{Q}_i} \overline{q} &= \operatorname{ord}_{\mathcal{Q}_i} q\overline{q} \\ &= \operatorname{ord}_{x=x_i} D(x) \\ &= e_i. \end{aligned}$$

As in Lemma 14, e_i denotes the exact number of occurrences of $\mathcal{Q}_i$. Also note $\operatorname{ord}_{\mathcal{Q}_i} \overline{q} = 0$, otherwise $q - \overline{q} = 2y$ would imply $y_i = 0$ which we excluded.

Since $q \in K[x, y]$ implies the only negative order may be at ∞ and $\sum e_i = 6$, $\operatorname{ord}_\infty q = -6$, we have $\operatorname{ord}_{\mathcal{Q}} q = 0$ for all $\mathcal{Q} \in H_0(\overline{K})$ that are not some $\mathcal{Q}_i$.

In the case where $y_i = 0$, the same lemmas imply

$$\operatorname{ord}_{\mathcal{Q}_i} q + \operatorname{ord}_{\mathcal{Q}_i} \overline{q} = 2e_i$$

and since $\overline{\mathcal{Q}}_i = \mathcal{Q}_i$, Lemma 7 gives $\operatorname{ord}_{\mathcal{Q}_i} \overline{q} = \operatorname{ord}_{\mathcal{Q}_i} q$ so again $\operatorname{ord}_{\mathcal{Q}_i} q = e_i$.

So in both cases, if $\mathcal{Q}_i$ appears e_i times in the sum $\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_3$ then it appears the same number of times in (q) and all in all

$$(q) = \sum_{\{\mathcal{Q}_i\}} e_i \mathcal{Q}_i - 6\infty = \sum_{i=1}^{6} \mathcal{Q}_i - 6\infty.$$

Now consider the case where $\mathcal{Q}_k = \infty$ for some k. Because $P(x)$ is now of degree two or less and $\lambda_\infty(y) = t^{-5}\sqrt{\tau}$ we have $\operatorname{ord}_\infty q = -5$. This leads to

$$(q) = \sum_{\substack{i=1 \\ i \neq k}}^{6} \mathcal{Q}_i - 5\infty = \sum_{i=1}^{6} \mathcal{Q}_i - 6\infty$$

completing the first part of our proof, as no more than one $\mathcal{Q}_k$ may be ∞.

Second Part

The divisor of $x - x_i$ is $(x - x_i) = \mathcal{Q}_i + \overline{\mathcal{Q}}_i - 2\infty$ both for $\mathcal{Q}_i = \overline{\mathcal{Q}}_i$ and $\mathcal{Q}_i \neq \overline{\mathcal{Q}}_i$.

Without loss of generality let $\mathcal{Q}_5 \neq \infty$ and distinguish between $\mathcal{Q}_6 \neq \infty$ and $\mathcal{Q}_6 = \infty$. In the first case it is easy to check that for $f = q(x - x_5)^{-1}(x - x_6)^{-1}$

$$\begin{aligned}(f) &= \sum_{i=1}^{6} \mathcal{Q}_i - 6\infty - \mathcal{Q}_5 - \overline{\mathcal{Q}}_5 - \mathcal{Q}_6 - \overline{\mathcal{Q}}_6 + 4\infty \\ &= \sum_{i=1}^{4} \mathcal{Q}_i - \overline{\mathcal{Q}}_5 - \overline{\mathcal{Q}}_6 - 2\infty\end{aligned}$$

which corresponds precisely to the statement we aim to prove. If $\mathcal{Q}_6 = \infty$, we can define $f = q(x - x_5)^{-1}$ and check that

$$\begin{aligned}(f) &= \sum_{i=1}^{5} \mathcal{Q}_i - 5\infty - \mathcal{Q}_5 - \overline{\mathcal{Q}}_5 + 2\infty \\ &= \sum_{i=1}^{4} \mathcal{Q}_i - \overline{\mathcal{Q}}_5 - \overline{\infty} - 2\infty.\end{aligned}$$

So the function f that we were looking for is

$$f = \frac{y - P(x)}{(x - x_5)(x - x_6)}$$

if $\mathcal{Q}_5$ and $\mathcal{Q}_6$ are both different from ∞ and otherwise if $\mathcal{Q}_6 = \infty$ then

$$f = \frac{y - P(x)}{x - x_5}.$$

$\square$

Theorem 3: *Associativity.* The sums

$$\begin{aligned} &\underbrace{(\mathbf{P} + \mathbf{Q})}_{=\mathbf{T}} + \mathbf{R} = \mathbf{S} \\ \text{and} \quad &\mathbf{P} + \underbrace{(\mathbf{Q} + \mathbf{R})}_{=\mathbf{W}} = \mathbf{S}' \end{aligned}$$

give the same result, namely $\mathbf{S} = \mathbf{S}'$.

Proof. We will use the following notation:

$$\begin{gathered}\mathbf{P} = \{\mathcal{P}_1, \mathcal{P}_2\}, \mathbf{Q} = \{\mathcal{Q}_1, \mathcal{Q}_2\}, \mathbf{R} = \{\mathcal{R}_1, \mathcal{R}_2\}, \mathbf{T} = \{\mathcal{T}_1, \mathcal{T}_2\}, \mathbf{W} = \{\mathcal{W}_1, \mathcal{W}_2\}, \\ \mathbf{S} = \{\mathcal{A}, \mathcal{B}\}, \mathbf{S}' = \{\mathcal{U}, \mathcal{V}\}, \\ \mathcal{A} = (a, *), \mathcal{B} = (b, *).\end{gathered}$$

Thanks to the previous theorem we have functions $f_{\mathbf{PQ}}$, $f_{\mathbf{TR}}$, $f_{\mathbf{PW}}$ and $f_{\mathbf{QR}}$ in $\mathcal{F}^*$ such that the divisor of

$$f = \frac{f_{\mathbf{PQ}} f_{\mathbf{TR}}}{f_{\mathbf{PW}} f_{\mathbf{QR}}}$$

is

$$\begin{aligned}(f) &= \mathcal{P}_1+\mathcal{P}_2+\mathcal{Q}_1+\mathcal{Q}_2-\mathcal{T}_1-\mathcal{T}_2-2\infty\\ &+ \mathcal{T}_1+\mathcal{T}_2+\mathcal{R}_1+\mathcal{R}_2-\mathcal{A}-\mathcal{B}-2\infty\\ &- \mathcal{P}_1-\mathcal{P}_2-\mathcal{W}_1-\mathcal{W}_2+\mathcal{U}+\mathcal{V}+2\infty\\ &- \mathcal{Q}_1-\mathcal{Q}_2-\mathcal{R}_1-\mathcal{R}_2+\mathcal{W}_1+\mathcal{W}_2+2\infty\\ &= \mathcal{U}+\mathcal{V}-\mathcal{A}-\mathcal{B}.\end{aligned}$$

This leaves us with three cases: both $\mathcal{A}$ and $\mathcal{B}$ being ∞, only $\mathcal{B}$ being ∞ and neither being ∞. In the first case Lemma 11 tells us $\mathcal{U} = \overline{\mathcal{V}}$ so $\mathbf{S}' = 0 = \mathbf{S}$.

In the second case, $a \neq \infty$ so we can look at $\widetilde{f} = (x-a)f$which now has divisor $(\widetilde{f}) = \mathcal{U} + \mathcal{V} + \overline{\mathcal{B}} - 3\infty$ but only two real poles at ∞ according to Lemma 12. This means that either $\mathcal{U} = \infty$ or $\mathcal{V} = \infty$. Either way, the other one has to be equal to $\mathcal{B}$, again as can be seen in Lemma 11, and so $\mathbf{S} = \mathbf{S}'$

This leaves $\widetilde{f} = (x-a)(x-b)f$ which has divisor $(\widetilde{f}) = \overline{\mathcal{A}}+\overline{\mathcal{B}}+\mathcal{U}+\mathcal{V}-4\infty$. By Lemma 13, $\widetilde{f}$ must be of the form $\kappa(x-\varrho)(x-\varsigma) \in \overline{K}[x]$, $\kappa \neq 0$ and so

$$f = \frac{\kappa(x-\varrho)(x-\varsigma)}{(x-a)(x-b)}.$$

But this has divisor of the form

$$(f) = \mathcal{M} + \overline{\mathcal{M}} + \mathcal{N} + \overline{\mathcal{N}} - \mathcal{A} - \overline{\mathcal{A}} - \mathcal{B} - \overline{\mathcal{B}}.$$

So, without loss of generality, either $\mathcal{A} = \mathcal{M}$ and $\mathcal{B} = \overline{\mathcal{M}}$ or $\mathcal{A} = \mathcal{M}$ and $\mathcal{B} = \mathcal{N}$. First case implies $\mathbf{S}$ and $\mathbf{S}'$ equals 0, second case implies that $(f) = \mathcal{A} + \mathcal{B} - \mathcal{A} - \mathcal{B}$ and in both cases we're done. $\square$

Remark 6: We can finally prove property ($\diamondsuit$) we imposed on page 6 with the exact same technique of the theorem above. Let's have the two sums

$$\{\mathcal{Q}_1, \mathcal{Q}_2\} + \{\mathcal{Q}_3, \mathcal{Q}_4\} = \{\mathcal{Q}_5, \mathcal{Q}_6\}$$

and

$$\{\mathcal{Q}_3, \mathcal{Q}_2\} + \{\mathcal{Q}_1, \mathcal{Q}_4\} = \{\mathcal{Q}'_5, \mathcal{Q}'_6\}.$$

Use Theorem 2 again to get two functions f_1 and f_2 corresponding to the two sums. Let's just look at the third case of the proof above. Defining the function $f = \frac{f_1}{f_2}$ we conclude that (f) must equal to both $\mathcal{Q}'_5 + \mathcal{Q}'_6 - \mathcal{Q}_5 - \mathcal{Q}_6$ and $\mathcal{M}+\overline{\mathcal{M}}+\mathcal{N}+\overline{\mathcal{N}}-\mathcal{Q}_5-\overline{\mathcal{Q}}_5-\mathcal{Q}_6-\overline{\mathcal{Q}}_6$ for some $\mathcal{M}$, $\mathcal{N} \in H(\overline{K})$. As above, this means either $\mathcal{Q}'_5 = \mathcal{Q}_5$ and $\mathcal{Q}'_6 = \mathcal{Q}_6$ or the other way around.

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Appendix

Theorem 1: Let x, $y \in K(t)$ be rational functions that fulfill $y^2 = C(x)$. Then x and y are actually constants, i.e. x, $y \in K$.

Proof. For a rational function $f = \frac{g}{h} \in K(t)$ with $f, g \in K[t]$ coprime we define $\deg f = \max\{\deg g, \deg h\}$. Suppose we have a counterexample (x, y) with $\deg x > 0$ but with minimal degree. Out of $y^2 = C(x)$ we conclude that $2yy' = x'C'(x)$ and we look at $f = \frac{x'}{y} = \frac{2y'}{C'(x)} \in K(t)$. We first want to show that $f = 0$ and conclude from that that x must be constant.

First part

In case $f \neq 0$, we would like to show that $\mathrm{ord}_\tau f \geq 0$ for all $\tau \in \overline{K}$. Have a look at $\mathrm{ord}_\tau x$ first. There are two possibilities for a given τ.

(i) Let $\mathrm{ord}_\tau x \geq 0$. Now we have an $x_0 = x(\tau)$ and $x'(\tau)$, $y'(\tau)$.

If $C(x_0) \neq 0$ then $f^2 = \frac{x'^2}{y'^2} = \frac{x'^2}{C(x)}$ so f^2 and f don't have a pole at $t - \tau$.

If $C(x_0) = 0$ then $C'(x_0) \neq 0$ and so $f = \frac{2y'}{C'(x)}$ gives $\mathrm{ord}_\tau f \geq 0$.

(ii) Let $\mathrm{ord}_\tau x = -m$ with $m \in \mathbb{N}$. Then $\mathrm{ord}_\tau(x^5 + \ldots + e) = -5m$. But this is equal to $\mathrm{ord}_\tau y^2$ so $2\mathrm{ord}_\tau y = -5m$. So $m = 2n$ is even, $n \in \mathbb{N}$. Therefore $\mathrm{ord}_\tau x = -2n$, $\mathrm{ord}_\tau y = -5n$ and $\mathrm{ord}_\tau x' \geq -2n - 1$.

Since $f = \frac{x'}{y}$ we have $\mathrm{ord}_\tau f = \mathrm{ord}_\tau x' - \mathrm{ord}_\tau y \geq -2n - 1 + 5n = 3n - 1 \geq 0$. This concludes $\mathrm{ord}_\tau f \geq 0$ for all $\tau \in \overline{K}$, so $f(t) \in K[t]$.

Now look at $x_1(t) = x(\frac{1}{t})$, $y_1(t) = y(\frac{1}{t})$. Of course $y_1^2 = C(x_1)$ and so we have $f_1(t) \in K[t]$ with $f_1 = \frac{x_1'}{y_1}$.

But

$$f_1(t) = \frac{-\frac{1}{t^2}x'(\frac{1}{t})}{y(\frac{1}{t})} = -\frac{1}{t^2} \in \frac{1}{t^2}K[\frac{1}{t}]$$

so $f_1 \in K[t] \cap \frac{1}{t^2}K[\frac{1}{t}] = \{0\}$ and so $f = 0$ as well.

Second part

With $f = 0$ we can immediately follow $x' = 0$ and thus $x \in K$ if we are in characteristic 0. However, if $\mathrm{char}(K) = p > 0$ we need to show that from $x' = 0$ follows $x(t) = x_p(t^p)$ with $x_p(s) \in K(s)$. See Lemma 15 for that.

Similarly, we follow $y' = 0$ from $f = 0$ for $p \neq 2$. Again, $y(t) = y_p(t^p)$ with $y_p(s) \in K(s)$. Now for $y_p^2 - C(x_p) = z$ follows $z(t^p) = y^2 - C(x) = 0$ and so $z(t) = 0$. This means (x_p, y_p) is a counterexample to the theorem as well $((x, y)$ being the original one) but we also have $\deg x = p \deg x_p > \deg x_p$ which strictly contradicts the minimality of the degree of x. $\square$

Lemma 15: Let $x(t) \in K(t)$ with $x'(t) = 0$ for $\mathrm{char}(K) = p$. Then we have $x(t) = x_p(t^p)$ with $x_p(s) \in K(s)$.

Proof. First let $x(t) \in K[t]$ with $x(t) = \sum_{i=0}^{n} a_i t^i$ so that $x'(t) = \sum_{i=1}^{n} i a_i t^{i-1}$. Since $x'(t) = 0$, wherever $a_i \neq 0$ we have $i \mid p$ so $x \in K[t^p]$.

If $x = \frac{g}{h} \in K(t)$, $g, h \in K[t]$, we simply apply the above to g and h, so $g, h \in K[t^p]$ and thus $x \in K(t^p)$. $\square$