1 Addition Law

1.1 Definitions and Notation

Definition 1: Let K be a field with $\mathrm{char}(K) \neq 2, 3$ and $\overline{K}$ its algebraic closure. Define the hyperelliptic curve of genus two $H_0(K)$ as the set of solutions in K^2 to the equation $y^2 = C(x)$ where $C(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$ is a polynomial over K. Similarly, the set of solutions in the closure would be denoted $H_0(\overline{K})$. Define $H(K)$ as $H_0(K) \cup \{\infty\}$.

Note that we could obtain a more reduced form of $C(x)$, eliminating a by shifting x to $x - a/5$. However, since this would rob us of the possibility of $\mathrm{char}(K) = 5$ without simplifying our coming calculations in any significant manner, we shall be reluctant towards using this trick.

For the purpose of clarity, let points on the hyperelliptic curve — in the sense of solutions to $y^2 = C(x)$ — be designated by the calligraphic letter $\mathcal{Q} = (x, y) \in H_0(\overline{K})$. The point opposite to $\mathcal{Q}$ will be written $\overline{\mathcal{Q}} = (x, -y)$ and by symmetry of the curve in y also belongs to $H_0(\overline{K})$. In the case where $\mathcal{Q} = \infty$, define $\overline{\mathcal{Q}} = \infty$. We allow ourselves to write $\pm\mathcal{Q}$ whenever we mean in fact 'either $\mathcal{Q}$ or $\overline{\mathcal{Q}}$'.

We want to consider the set of all pairs $(\mathcal{Q}_1, \mathcal{Q}_2)$ and tame it with an equivalence relation with the goal of obtaining an additive group:

Definition 2: Define $\mathbf{J}$ to be the set $\mathcal{J}/\sim$ where $\mathcal{J} = H(\overline{K}) \times H(\overline{K})$. $\mathbf{J}$ is called the 'Jacobian' and the equivalence relation fullfills

$$\begin{aligned} (\mathcal{Q}_1, \mathcal{Q}_2) &\sim (\mathcal{Q}_2, \mathcal{Q}_1) \\ \text{and} \quad (\mathcal{Q}, \overline{\mathcal{Q}}) &\sim (\infty, \infty). \end{aligned}$$

Write $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ from now on and let bold letters denote points on the curve in the sense of classes of unordered pairs $\mathbf{P} = \{\mathcal{Q}_1, \mathcal{Q}_2\} \in \mathbf{J}$. The point $\{\overline{\mathcal{Q}}_1, \overline{\mathcal{Q}}_2\}$ will be called $\overline{\mathbf{P}}$ for now but can already tentatively be thought of as $-\mathbf{P}$. Call $\{\infty, \infty\}$ the zero of our set. We will also permit ourselves the notation $\{\mathcal{Q}, \overline{\mathcal{Q}}\} = 0$ and we refrain from explicitly stating that $\mathbf{P}$ is in fact an equivalence class.

A point $\mathcal{Q} = (x_0, y_0)$ is called singular if it fulfills both $y_0 = 0$ and $C'(x_0) = 0$. A curve is called singular if and only if it has a singular point. We consider only non-singular hyperelliptics from here on.

1.2 The General Case

Let $\mathbf{P}_1 = \{\mathcal{Q}_1, \mathcal{Q}_2\}$, $\mathbf{P}_2 = \{\mathcal{Q}_3, \mathcal{Q}_4\}$ with $\mathcal{Q}_i = (x_i, y_i) \in H_0(K)$ or $\mathcal{Q}_i = \infty$. To define $\mathbf{P}_3 = \mathbf{P}_1 + \mathbf{P}_2$ we distinguish between one general case and a number of special cases and first derive the results of the former before enumerating the latter ones.

Case 1, Four Distinct Component-Points: Let first $\mathcal{Q}_i \in H_0(K)$ and $\mathbf{P}_i$ be defined as above with $x_i \neq x_j$ whenever $i \neq j$.

The overarching idea is to obtain a fifth and sixth x-coordinate and the corresponding y-coordinates by passing a polynomial of degree three through the four points $\mathcal{Q}_i$. Ideally this should give us two additional intersections with the curve which we then use as the components of our point $\mathbf{P}_1 + \mathbf{P}_2$.

Step 1: It is known that the Vandermonde matrix

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}$$

has determinant $\prod_{i<j}(x_i - x_j)$ which is conveniently non-zero if and only if the x_i are pairwise distinct. Let $p(x) = p_3x^3 + p_2x^2 + p_1x + p_0 \in \overline{K}[x]$ be the polynomial in unknown coefficients that we are looking for. With $\mathbf{y} = (y_1\ y_2\ y_3\ y_4)^t$ and $\mathbf{p} = (p_0\ p_1\ p_2\ p_3)^t$, the problem of determining $p(x)$ can be rewritten as

$$V \cdot \mathbf{p} = \mathbf{y}$$

which by invertibility of V has a unique solution for $\mathbf{p}$ with $p_i \in K$.

Note that the leading coefficient of $p(x)$ is

$$p_3 = \frac{1}{\det(V)} \begin{vmatrix} 1 & x_1 & x_1^2 & y_1 \\ 1 & x_2 & x_2^2 & y_2 \\ 1 & x_3 & x_3^2 & y_3 \\ 1 & x_4 & x_4^2 & y_4 \end{vmatrix}$$

and the next step will depend on whether $p(x)$ is truly of degree 3 or not.

Step 2a: Knowing the coefficients p_i of $p(x)$ we first assume that $p_3 \neq 0$, so can proceed to look for the two additional solutions of the sextic equation

$$C(x) - (p(x))^2 = 0. \tag{$*$}$$

Observe that this vanishes at x_1, x_2, x_3 and x_4, so write the lefthand side as $-p_3^2(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)(x - x_6)$ for x_5 and x_6 in $\overline{K}$.

Comparing the coefficients of both expressions at x^4 and x^5 yields

$$\sum_{\substack{i,j=1\\ i<j}}^{6} x_i x_j = T_4 \tag{4}$$

$$\text{and} \qquad \sum_{i=1}^{6} x_i = T_5 \tag{5}$$

where $T_5 = \frac{1-2p_2p_3}{p_3^2}$ and $T_4 = \frac{p_2^2+2p_1p_3-a}{p_3^2}$. The first expression gives

$$x_6 \sum_{i=1}^{5} x_i + \sum_{\substack{i,j=1\\ i<j}}^{5} x_i x_j = T_4.$$

Doing this twice and replacing x_6 with the information from (5) gives the tidy quadratic equation

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$$x^2 - \left(T_5 - \sum_{i=1}^{4} x_i\right) \cdot x + \left(T_4 - T_5 \sum_{i=1}^{4} x_i + \sum_{\substack{i,j=1\\ i\leq j}}^{4} x_i x_j\right) = 0 \tag{†}$$

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of which x_5 is one solution and — by symmetry of the above steps — x_6 the other one. Compute $y_i = p(x_i)$, $i = 5, 6$ to obtain $\mathcal{Q}_5 = \{x_5, -y_5\}$ and $\mathcal{Q}_6 = \{x_6, -y_6\}$, at which point it becomes clear that the worst-case scenario for our field extension to accommodate the new coordinates is to be quadratic. Finally we define $\mathbf{P}_1 + \mathbf{P}_2$ to be equal to $\mathbf{P}_3 = \{\mathcal{Q}_5, \mathcal{Q}_6\}$.

Step 2b: If p_3 were zero, the equation $(*)$ would be quintic instead. We may therefore write the lefthand side as $(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)$, again for x_5 somewhere in $\overline{K}$. Defining $T_{4\infty} = p_2^2 - a$ and comparing the coefficients at x^4 gives

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$$x_5 = T_{4\infty} - \sum_{i=1}^{4} x_i \tag{‡}$$

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and we may rejoice in the implication of x_5 staying in K.

Compute $y_5 = p(x_5)$ and define $\mathbf{P}_1 + \mathbf{P}_2$ to be the point $\mathbf{P}_3 = \{(x_5, y_5), \infty\}$.

STEP $1'$: To extend our construction from $H_0(K)$ to $H(K)$ we now consider $\mathcal{Q}_4 = \infty$ with the other $\mathcal{Q}_i = (x_i, y_i)$ as before. We will later reason about why this is sufficient to cover the general case.

There is no coordinate x_4 this time, so we pass a quadratic polynomial $p(x)$ through the remaining three points (x_i, y_i). This means that we solve the linear system $\widetilde{V} \cdot \mathbf{p} = \widetilde{\mathbf{y}}$ where $\widetilde{V}$ is the Vandermonde matrix for x_i, $i = 1, .., 3$ which incidentally is the upper-left $3{\times}3$ sub-matrix of V. Here $\mathbf{p} = (p_0\ p_1\ p_2)^t$ and $\widetilde{\mathbf{y}} = (y_1\ y_2\ y_3)^t$ are defined as expected.

As before, the leading coefficient of $p(x)$ might or might not be zero, but $(*)$ will be quintic in either case, so we only have to worry about one step two.

STEP $2'$: Doing a coefficient comparison at x^3 and at x^4 in $(*)$ gives

$$\sum_{\substack{i,j=1\\ i<j}}^{3} x_i x_j + x_5 \sum_{i=1}^{3} x_i + x_5 x_6 = b - 2p_1 p_2 \tag{3'}$$

$$\text{and} \qquad \sum_{i=1}^{3} x_i + x_5 + x_6 = p_2^2 - a. \tag{4'}$$

Call the righthand terms $T_{3\infty}$ and $T_{4\infty}$, combine both equations and obtain

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$$x^2 - T_{4\infty} \cdot x + \left(T_{3\infty} - \sum_{\substack{i,j=1\\ i<j}}^{3} x_i x_j \right) = 0. \tag{$\dagger'$}$$

Solve, call the two solutions x_5 and x_6, compute y_5 and y_6 through $p(x_5)$ and $p(x_6)$ and define $\mathbf{P}_1 + \mathbf{P}_2$ to be $\mathbf{P}_3 = \{(x_5, -y_5), (x_6, -y_6)\} = \{\mathcal{Q}_5, \mathcal{Q}_6\}$.

Remark 1: Note that an interesting consequence is — if the above does truly complete the general case — that at most one of the $\mathcal{Q}_i$ for $i = 1, .., 6$ can be the point at infinity, provided that the $\mathcal{Q}_i$ for $i = 1, .., 4$ are all pairwise distinct.

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Before we begin listing the special cases, we impose the following property on the addition of any two points:

The sum of any $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ and $\{\mathcal{Q}_3, \mathcal{Q}_4\}$ in $\mathbf{J}$ fulfills the following equality:

$$\{\mathcal{Q}_1, \mathcal{Q}_2\} + \{\mathcal{Q}_3, \mathcal{Q}_4\} = \{\mathcal{Q}_1, \mathcal{Q}_3\} + \{\mathcal{Q}_2, \mathcal{Q}_4\}. \tag{$\diamondsuit$}$$

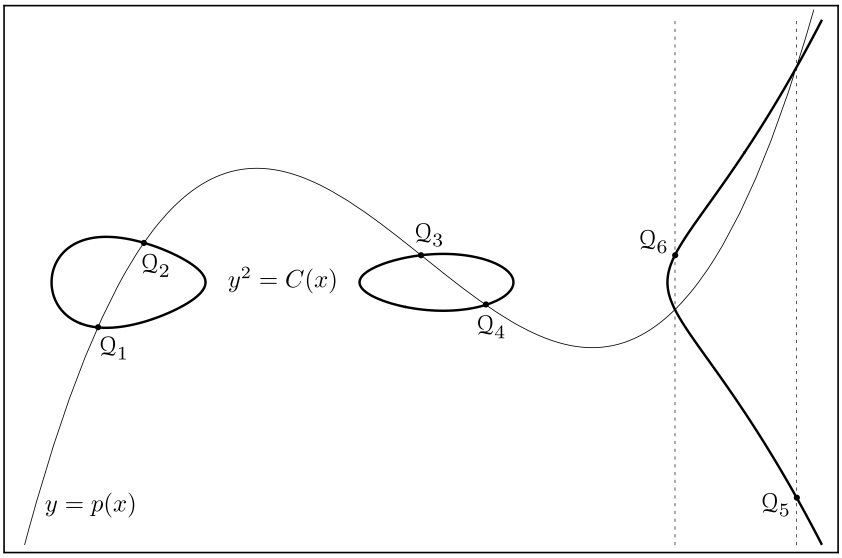
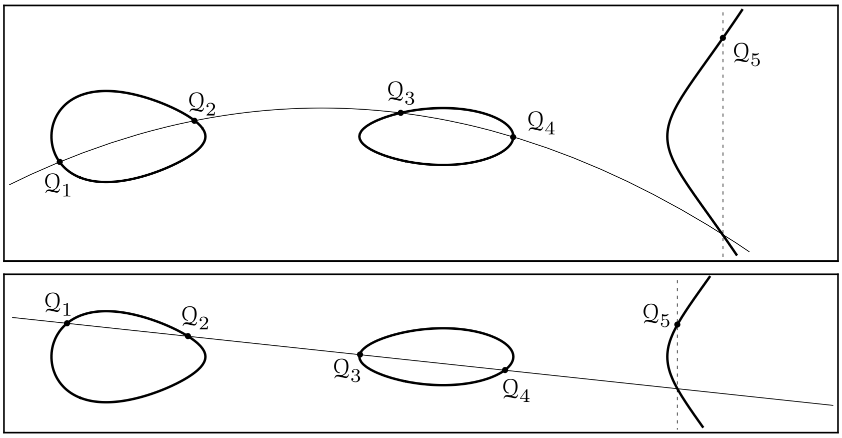


FIGURE 1A: The general case for the addition law in $\mathbb{R}^2$ where $p_3 \neq 0$.



FIGURES 1B and 1C: If $p_3 = 0$ we have at most five intersections.

Remark 2: We will use this property extensively from now on and it will be worth the additional trouble of having to check well-definedness because

- (i) Since $(\diamondsuit)$ implies that we can interchange *any* two point-components in a given sum, we may now impose conditions on the $\mathcal{Q}_i$ without mentioning whether they belong to $\mathbf{P}_1$ or $\mathbf{P}_2$.
- (ii) As a result, the list of special cases can be written in a significantly more concise manner.
- (iii) It is immediately clear now that if we have a well-defined addition, then $-\mathbf{P} = \overline{\mathbf{P}}$ because $\{\mathcal{Q}_1, \mathcal{Q}_2\} + \{\overline{\mathcal{Q}}_1, \overline{\mathcal{Q}}_2\} = \{\mathcal{Q}_1, \overline{\mathcal{Q}}_1\} + \{\mathcal{Q}_2, \overline{\mathcal{Q}}_2\} = 0$.
- (iv) The property even gives us commutativity for free.

1.3 List of Special Cases

Let's first list all configurations that can be taken by the $(x_i,\, y_i) \in H_0(K)$:

A. All x_i are pairwise distinct.

B. Exactly two of the x_i are equal, for instance $x_1 = x_2$ and
   - a. $y_1 = y_2 \neq 0$.
   - b. $y_1 = -y_2 \neq 0$. $\star$
   - c. $y_1 = y_2 = 0$. $\star$

C. Exactly three of the x_i are equal, e.g. $x_1 = x_2 = x_3$ and
   - a. All three y-coordinates are equal: $y_1 = y_2 = y_3 \neq 0$.
   - b. Only two y-coordinates are equal: $y_1 = y_2 \neq 0$ so $y_3 = -y_1$. $\star$
   - c. All three $y_1 = y_2 = y_3 = 0$. $\star$

D. All four x_i are equal and
   - a. All four y_i are equal but different from zero.
   - b. The y_i are equal two by two, e.g. $y_1 = y_2 \neq 0$ and $y_3 = y_4 = -y_1$. $\star$
   - c. Exactly three of the y_i are equal, e.g. $y_1 = y_2 = y_3 = -y_4 \neq 0$. $\star$
   - d. All four $y_i = 0$. $\star$

E. The x_i are equal two-by-two: $x_1 = x_2$ and $x_3 = x_4$ but $x_1 \neq x_3$ and
   - a. $y_1 = y_2 \neq 0$ and $y_3 = y_4 \neq 0$.
   - b. $y_1 = -y_2$ and $y_3 = y_4 \neq 0$. $\star$
   - c. $y_1 = -y_2$ and $y_3 = -y_4$. $\star$

Due to ($\diamondsuit$) and the property that $\{\mathcal{Q}_i, \mathcal{Q}_j\} \sim 0$ whenever $x_i = x_j$ and $y_i = -y_j$, every case marked with $\star$ comes down to the addition with 0. Considering the possibility of $\mathcal{Q}_i = \infty$ and reordering the above thus leads to the following condensed and complete list of cases for the addition law:

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0. The addition with zero, i.e. $\mathbf{P} + \{\infty, \infty\}$ or in short $\mathbf{P} + 0$, $\mathbf{P} \in \mathbf{J}$.
1. The general case where all $\mathcal{Q}_i$ in $H(K)$ are pairwise distinct from $\pm\mathcal{Q}_j$.
2. The single tangential case where $\mathcal{Q}_1 = \mathcal{Q}_2 \in H_0(K)$ with $y_1 \neq 0$ and the remaining $\mathcal{Q}_3$, $\mathcal{Q}_4 \in H(K)$ are both distinct from each other as well as from $\pm\mathcal{Q}_1, \overline{\mathcal{Q}}_3$ and $\overline{\mathcal{Q}}_4$.
3. The double tangential case where $\mathcal{Q}_1 = \mathcal{Q}_2 \in H_0(K)$ with $y_1 \neq 0$ and $\mathcal{Q}_3 = \mathcal{Q}_4 \in H_0(K)$ with $y_3 \neq 0$ and $\mathcal{Q}_1 \neq \pm\mathcal{Q}_3$.
4. The triple-point tangential case wherein $\mathcal{Q}_1 = \mathcal{Q}_2 = \mathcal{Q}_3 \in H_0(K)$ with $y_1 \neq 0$ and $\mathcal{Q}_4 \in H(K)$ differs from both $\pm\mathcal{Q}_1$.
5. The quadruple case where all $\mathcal{Q}_i \in H_0(K)$ are equal with $y_i \neq 0$.

Remark 3:

(i) In both lists, the cases do not overlap.

(ii) For the list to be complete, we must allow for $\mathcal{Q}_i = \infty$ for some i. Note that one is sufficient, since if two or more $\mathcal{Q}_i$ were to be ∞, we would be back at the zero case by virtue of ($\diamondsuit$), bringing the two infinities together in one point. As for the first case, we consider $\mathcal{Q}_i \in H_0(K)$ and $\mathcal{Q}_i \in H(K)$ in two seperate cases and postpone handling the latter.

(iii) As for the first case, the constructions will a priori only be made on $\mathcal{J}$. It remains to check that this makes indeed for a well-defined addition on $\mathbf{J}$ by being invariant under the permutations $\mathcal{Q}_1 \rightleftharpoons \mathcal{Q}_2$ and $\mathcal{Q}_3 \rightleftharpoons \mathcal{Q}_4$.

1.4 Addition Law for the Special Cases

Case 0, Addition with Zero: As one might have anticipated, if $\mathbf{P}_2 = 0$ we define $\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_1$ for every $\mathbf{P}_1 \in \mathbf{J}$.

Case 2, Tangential: Let $\mathcal{Q}_i \in H_0(K), \mathcal{Q}_1 = \mathcal{Q}_2$, $y_1 \neq 0$ but x_1, x_3 and x_4 are pairwise distinct. We cannot use the Vandermonde matrix in this case because it won't possess maximal rank, consequently being non-invertible. We can however obtain an additional equation by demanding that our polynomial $p(x)$ be tangential to the curve at $\mathcal{Q}_1$. This gives

$$\begin{aligned} 2y\frac{dy}{dx} &= 5x^4 + 4ax^3 + 3bx^2 + 2cx + d \\ \text{and} \qquad \frac{dy}{dx} &= 3p_3x^2 + 2p_2x + p_1 \end{aligned}$$

meaning that the system to solve for $\mathbf{p}$ is now $V_1 \cdot \mathbf{p} = \mathbf{y}_1$ with

$$V_1 = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}.$$

The subscript indicates at which points the intersections have higher order. Here $\mathbf{y}_1$ is defined as $\mathbf{y}$ with y_2 replaced by $y_2' = \frac{C'(x_1)}{2y_1}$ where $C'(x)$ is the derivative of C and $2y_1 \neq 0$ because $\mathrm{char}(K) \neq 2$ and $y_1 \neq 0$. Since $\det(V_1) = (x_4 - x_1)^2(x_3 - x_1)^2(x_4 - x_3)$ this fits neatly into our constraints by being non-zero exactly in the case where x_1, x_3 and x_4 are pairwise distinct.

Once $p(x)$ is determined, step two will be entirely identical to the general case and we can again solve (‡) or (†) for x_5 or x_5 and x_6.

Case 3, Double Tangential: Let $\mathcal{Q}_i \in H_0(K)$, $\mathcal{Q}_1 = \mathcal{Q}_2$ and $\mathcal{Q}_3 = \mathcal{Q}_4$ but $x_1 \neq x_3$ and $\mathcal{Q}_i \neq \pm\mathcal{Q}_i$ meaning that neither y_1 nor y_3 will be zero. As before, we lack equations for our linear system, requiring the use of a second tangential constraint. Replace the fourth row of V_1 and $\mathbf{y}_1$ exactly like we did for the second one: $y_4' = \frac{C'(x_3)}{2y_3}$ and

$$V_{13} = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 0 & 1 & 2x_3 & 3x_3^2 \end{pmatrix}.$$

Now $\det(V_{13}) = (x_3 - x_1)^4$ and this is again different from zero precisely whenever $x_3 \neq x_1$, so as before solve $V_{13} \cdot \mathbf{p} = \mathbf{y}_{13}$ for $\mathbf{p}$, then (†) or (‡).

Case 4, Second Order Tangential: Let $\mathcal{Q}_i \in H_0(K), \mathcal{Q}_1 = \mathcal{Q}_2 = \mathcal{Q}_3$ but $x_1 \neq x_4$ and $y_1 \neq 0$. We can thus see this as a third-order intersection and demand that the curve and the polynomial share a second-order derivative at $\mathcal{Q}_1$:

$$V_{11} = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_1 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}.$$

Here $\det(V_{11}) = 2(x_4 - x_1)^3$ and with this, define $\mathbf{y}_{11}$ by taking $\mathbf{y}_1$ and replacing the third coordinate by $y_3'' = \frac{C''(x_1)}{2y_1} - \frac{(C'(x_1))^2}{4y_1^3}$ where $C''(x)$ is the second-order derivative of C. Again, apply the same procedure of the second steps of case 1 to find $\mathbf{P}_3$.

Case 5, Third Order Tangential: Given the quadruple situation where $\mathcal{Q}_i = \mathcal{Q}_1 \in H_0(K)$ for every i with $y_1 \neq 0$, we use

$$V_{111} = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_1 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

which is invertible in all fields but those of characteristic 2 and 3.

Here $\mathbf{y}_{111}$ is the same as $\mathbf{y}_{11}$ except for the last coordinate which should read

$$y_4''' = \frac{C'''(x_1)}{2y_1} - \frac{3C'(x_1)C''(x_1)}{4y_1^3} + \frac{3(C'(x_1))^3}{8y_1^5}$$

where $C'''(x)$ is the third-order derivative of C. Once more, we solve the linear system $V_{111} \cdot \mathbf{p} = \mathbf{y}_{111}$ and subsequently (†) or (‡) and we're done.

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Finally, as noted in Remark 3, (ii) we have yet to extend our definition from $H_0(K)$ to $H(K)$. Observe that this is only relevant for cases number two and four where we now consider $\mathcal{Q}_4 = \infty$ as we did in Step $1'$ of the general case. As an analogue to this, the relevant matrices $\widetilde{V_1}$ and $\widetilde{V_{11}}$ will be the upper-left 3×3 sub-matrices of their $H_0(K)$-counterparts V_1 and V_{11}.

In both cases we obtain a linear system of the form $\widetilde{V_*} \cdot \mathbf{p} = \widetilde{\mathbf{y}}_*$ for a three-element vector $\mathbf{p}$ and the vectors $\widetilde{\mathbf{y}}_1$ and $\widetilde{\mathbf{y}}_{11}$ are defined like their 4-element counterparts $\mathbf{y}_1$ and $\mathbf{y}_{11}$ with the last coordinate omitted.

Both matrices $\widetilde{V_*}$ are invertible and we therefore get a unique polynomial $p(x)$ wich we use to solve ($\dagger'$), obtaining x_5, x_6, $y_5 = p(x_5)$ and $y_6 = p(x_6)$ in $\overline{K}$ and we define $\{\mathcal{Q}_1, \mathcal{Q}_2\} + \{\mathcal{Q}_3, \infty\} = \{\mathcal{Q}_5, \mathcal{Q}_6\} = \{(x_5, -y_5), (x_6, -y_6)\}$.

1.5 Well-Definedness of the Addition Law

To check whether our addition is well defined on $\mathbf{J}$ in each of the cases, we have to consider the permutation of point-components $\mathcal{Q}_i$ under the equivalence relation from Definition 2. Furthermore, to claim that all possible cases are all covered, it is necessary to check the permutations under ($\diamondsuit$). Both cases can be combined into a single one by the following statement:

Lemma: In each given definition of '+', the result is invariant under the permutation $\mathcal{Q}_i \rightleftharpoons \mathcal{Q}_j$.

Proof. Simultaneously interchanging $x_i \rightleftharpoons x_j$ and $y_i \rightleftharpoons y_j$ in our linear systems $V_* \cdot \mathbf{p} = \mathbf{y}_*$ has the effect of permuting the rows of V_* and $\mathbf{y}_*$ and relabeling (x_k, y_k) whenever $\mathcal{Q}_k$ was equal to $\mathcal{Q}_i$ or $\mathcal{Q}_j$.

Consequently the resulting $p(x)$ doesn't change, so neither do any of the terms T_*. The three dagger equations remain unchanged as well, as can easily be checked at ($\dagger$), ($\ddagger$) and ($\dagger'$) which are invariant under $x_i \rightleftharpoons x_j$.

Finally, the version of Step 2 we fall into remains the same, since it is only imposed by the distinction of p_3 being zero or not. $\square$

2 Rational Functions on Hyperelliptics

2.1 Function Field and Order of Rational Functions

Definition 3: Define the ring of rational functions on the curve as

$$\mathcal{F} = K(x)[y] \Big/ \big(y^2 - C(x)\big).$$

As $y^2 - C(x)$ is irreducible in $K(x)[y]$, this is a field and $\mathcal{F} = K(x) + K(x)y$, so write elements $f \in \mathcal{F}$ as $f = g + hy$ where $g, h \in K(x)$. Define $\overline{f} = g - hy$.

We will occasionally write things like $K[x, y] \subset \mathcal{F}$ but xxx we always implicitly mean this in conjunction with $y^2 = C(x)$.

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We now wish to define the order of a function f in a point $\mathcal{Q}$, $\mathrm{ord}_{\mathcal{Q}}(f) \in \mathbb{Z}$ for $f \in \mathcal{F}^*$ and $\mathcal{Q} \in H(\overline{K})$. To this end we first construct a K-homomorphism

$$\lambda_{\mathcal{Q}} : \mathcal{F} \to \overline{K}(\!(t)\!)$$

with the intent of defining $\mathrm{ord}_{\mathcal{Q}}(f) = \mathrm{ord}\, \lambda_{\mathcal{Q}}(f)$.

Reminder: If τ is a formal Laurent series of the form $\tau = 1 + \sum_{i=1}^{\infty} a_i t^i$ then it has a unique squareroot of the form $\sigma = 1 + \sum_{i=1}^{\infty} b_i t^i$ meaning $\sigma^2 = \tau$. Write $\sigma = \sqrt{\tau} = 1 + \dots$.

From now on we will use an ellipsis to denote terms of ascending order everywhere where we are not interested in the specifics.

Definition 4: Because $C\big(\lambda_{\mathcal{Q}}(x)\big) = \big(\lambda_{\mathcal{Q}}(y)\big)^2$ has to be fulfilled, we first decide on $\lambda_{\mathcal{Q}}(x)$ and deduce $\lambda_{\mathcal{Q}}(y)$. For this, we distinguish between three cases for $\mathcal{Q} \in H(\overline{K})$:

1. Let $\mathcal{Q} = (x_0, y_0) \in H_0(\overline{K})$ with $y_0 \neq 0$ and consequently $C(x_0) \neq 0$.

   Define $\lambda_{\mathcal{Q}}(x) = x_0 + t$. Now

   $$\begin{aligned} C(\lambda_{\mathcal{Q}}(x)) &= C(x_0 + t) \\ &= C(x_0) + \dots + t^5 \\ &= C(x_0)(1 + \dots) \\ &= y_0^2 \tau_1 \qquad \text{with } \tau_1 \in K(\!(t)\!). \end{aligned}$$

   Define $\lambda_{\mathcal{Q}}(y) = y_0 \sigma_1$ where $\sigma_1 = \sqrt{\tau_1}$.

2. Let $\mathcal{Q} = (x_0, y_0) \in H_0(\overline{K})$ with $y_0 = 0$. Note that $C(x_0) = 0$ and write $C(x) = \prod_{i=1}^{5}(x - \alpha_i)$ for $\alpha_i \in \overline{K}$ and for instance $\alpha_1 = x_0$.

Define $\lambda_{\mathcal{Q}}(x) = x_0 + t^2$. Now

$$\begin{aligned} C(\lambda_{\mathcal{Q}}(x)) &= t^2 \prod_{i=2}^{5} (x_0 - \alpha_i + t^2) \\ &= \mu t^2 (1 + \dots) \end{aligned}$$

with $\mu \in \overline{K}$ being $\prod_{i=2}^{5}(x_0 - \alpha_i) = C'(x_0)$ which is non-zero because our curve is non-singular. Write therefore $C(\lambda_{\mathcal{Q}}(x)) = \mu t^2 \tau_2$ with $\tau_2 = 1 + \dots$ and define $\lambda_{\mathcal{Q}}(y) = \nu t \sigma_2$ with $\sigma_2 = \sqrt{\tau_2}$ and $\nu^2 = \mu$.

3. Let $\mathcal{Q} = \infty$. Define $\lambda_{\mathcal{Q}}(x) = \frac{1}{t^2}$. It follows that

$$\begin{aligned} C(\frac{1}{t^2}) &= \frac{1}{t^{10}} + \frac{a}{t^8} + \frac{b}{t^6} + \frac{c}{t^4} + \frac{d}{t^2} + e \\ &= \frac{1}{t^{10}}(1 + \dots) \end{aligned}$$

So define $\lambda_{\mathcal{Q}}(y) = \frac{1}{t^5}\sigma_3$ with the notation $\sigma_3 = \sqrt{\tau_3}$ as before.

Definition 5: For $\mathcal{Q} \in H(\overline{K})$ and $f \in \mathcal{F}^*$ define $\operatorname{ord}_{\mathcal{Q}}(f) = \operatorname{ord} \lambda_{\mathcal{Q}}(f)$.

$\sim$

Lemma 1: Let $f\ inK[x, y] \subset \mathcal{F}$, $f \neq 0$ and $\mathcal{Q} \in H_0(\overline{K})$, $\mathcal{Q} = (x_0, y_0)$. Then

(a) $\operatorname{ord}_{\mathcal{Q}} f \geq 0$ and

(b) if $f(\mathcal{Q}) = 0$ then $\operatorname{ord}_{\mathcal{Q}} f \geq 1$.

Proof.

(a) Because $\mathcal{Q} \in H_0(\overline{K})$ we have $\lambda_{\mathcal{Q}}(x)$, $\lambda_{\mathcal{Q}}(y) \in \overline{K}[[t]]$ and with $f \in K[x, y]$ we have $\operatorname{ord}_{\mathcal{Q}}(f) \geq 0$.

(b) Write $f = A(x, y)$, $A(X, Y) \in K[X, Y]$. First, let $y_0 \neq 0$.

$$\begin{aligned} \operatorname{ord}_{\mathcal{Q}} f &= \operatorname{ord} A(\lambda_{\mathcal{Q}}(x), \lambda_{\mathcal{Q}}(y)) \\ &= \operatorname{ord} A(x_0 + t, y_0 \sigma_1). \end{aligned}$$

But $A(x_0 + t, y_0\sigma_1) = \sum_{i=0}^{\infty} a_i t^i$ so with $t = 0$ we get $A(x_0, y_0) = a_0$ but the former is $f(\mathcal{Q})$ which is 0, so $a_0 = 0$ and the claim follows.

If $y_0 = 0$ we would have $\operatorname{ord}_{\mathcal{Q}} f = \operatorname{ord} A(x_0 + t^2, \sqrt{\mu} t \sigma_2)$ instead. But like before, this means $0 = f(\mathcal{Q}) = A(x_0, 0) = a_0$ so again $\operatorname{ord}_{\mathcal{Q}}(f) \geq 1$.

□

Lemma 2: Let $f \in K(x)$, $f \neq 0$, $\mathcal{Q} \in H(\overline{K})$. Then

(a) $\operatorname{ord}_{\mathcal{Q}}(f) = \operatorname{ord}_{x_0} f(x)$ if $\mathcal{Q} = (x_0, y_0) \in H_0(\overline{K})$ with $y_0 \neq 0$.

(b) $\mathrm{ord}_{\mathcal{Q}}(f) = 2\mathrm{ord}_{x_0} f(x)$ if $\mathcal{Q} = (x_0, 0) \in H_0(\overline{K})$.

(c) $\mathrm{ord}_{\infty}(f) = 2\mathrm{ord}_0\, f(\frac{1}{x})$.

Note that the righthand side of the equalities refer to the usual definition of the order of a rational function in a point $x_0 \in \overline{K}$.

Proof. First take $f \in K[x]$ and define $e = \mathrm{ord}_{x_0} f \in \mathbb{N}$ so $f = (x - x_0)^e g(x)$ with $g \in \overline{K}[x]$ and $g(x_0) \neq 0$.

(a) If $\mathcal{Q} = (x_0, y_0)$, $y_0 \neq 0$ then $\lambda_{\mathcal{Q}}(f) = f(x_0 + t) = t^e g(x_0 + t)$ and so $\mathrm{ord}\,\lambda_{\mathcal{Q}}(f) = e$ because $g(x_0 + t) = g(x_0) + \ldots$ with $g(x_0) \neq 0$.

(b) Here $\lambda_{\mathcal{Q}}(f) = f(x_0 + t^2) = t^{2e} g(x_0 + t^2)$ and again $g(x_0 + t^2) = g(x_0) + \ldots$ so $\mathrm{ord}\,\lambda_{\mathcal{Q}}(f) = 2e$.

(c) For $\mathcal{Q} = \infty$ we have $\lambda_{\mathcal{Q}}(f) = f(\frac{1}{t^2}) = \frac{\alpha}{t^{2d}} + \ldots$ with $\alpha \neq 0$ if $d = \deg f$. Obviously, $\mathrm{ord}_0\, f(\frac{1}{x}) = \mathrm{ord}_0 \frac{\alpha + \ldots}{x^d} = -d$ so $\mathrm{ord}\,\lambda_{\mathcal{Q}}(f) = 2\mathrm{ord}_0\, f(\frac{1}{x})$.

Generally, if $f \in K(x)$ we can write $f = \frac{p}{q}$ for $p, q \in K[x]$ and we apply the above to p and q, subtracting $\mathrm{ord}_{\mathcal{Q}} q$ from $\mathrm{ord}_{\mathcal{Q}} p$. □

Lemma 3: For $f \in \mathcal{F}^*$ and $\mathcal{Q} \in H(\overline{K})$ the order satisfies $\mathrm{ord}_{\mathcal{Q}}(\overline{f}) = \mathrm{ord}_{\overline{\mathcal{Q}}}(f)$

Proof.

1. For $\mathcal{Q} \in H_0(\overline{K})$ with $y_0 \neq 0$ we've got $\lambda_{\mathcal{Q}}(x) = x_0 + t$ and $\lambda_{\mathcal{Q}}(y) = y_0\sqrt{\tau}$. Therefore $\lambda_{\overline{\mathcal{Q}}}(x) = x_0 + t = \lambda_{\mathcal{Q}}(\overline{x})$ and $\lambda_{\overline{\mathcal{Q}}}(y) = -y_0\sqrt{\tau} = \lambda_{\mathcal{Q}}(\overline{y})$ so

$$\begin{aligned} \lambda_{\overline{\mathcal{Q}}}(f(x,y)) &= f(\lambda_{\overline{\mathcal{Q}}}(x), \lambda_{\overline{\mathcal{Q}}}(y)) \\ &= \lambda_{\mathcal{Q}}(f(\overline{x}, \overline{y})) \\ &= \lambda_{\mathcal{Q}}(\overline{f}(x,y)). \end{aligned}$$

2. If $\mathcal{Q} \in H_0(\overline{K})$ with $y_0 = 0$ then $\overline{\mathcal{Q}} = \mathcal{Q}$. Write $f = g(x) + h(x)y$ so

$$\lambda_{\mathcal{Q}}(\overline{f}) = g(x_0 + t^2) - h(x_0 + t^2)\nu t\sqrt{\tau}.$$

   Calling this $l(t) = \lambda_{\mathcal{Q}}(\overline{f})$ and looking at the construction of $\lambda_{\mathcal{Q}}$ we see that τ sports only even powers of t so we see above that $l(-t) = \lambda_{\mathcal{Q}}(f)$. As interchanging t with $-t$ doesn't change the order, we're done.

3. For $\mathcal{Q} = \infty$, $\tau = 1 + at^2 + bt^4 + ct^6 + dt^8 + ct^{10}$ features only even powers as well, so again $l(-t) = \lambda_{\mathcal{Q}}(f)$ for $l(t) = \lambda_{\mathcal{Q}}(\overline{f}) = g(\frac{1}{t^2}) - h(\frac{1}{t^2})\frac{1}{t^5}\sqrt{\tau}$. Finally, $\lambda_{\mathcal{Q}}(f)$ is equal to $\lambda_{\overline{\mathcal{Q}}}(f)$ since $\infty = \overline{\infty}$. Again $\mathrm{ord}\,l(t) = \mathrm{ord}\,l(-t)$.

□

Lemma 4: If $f \in \mathcal{F}^*$ then the set $\{\mathcal{Q} \in H(\overline{K}) \mid \text{ord}_{\mathcal{Q}} f \neq 0\}$ is finite and

$$\sum_{\mathcal{Q} \in H(\overline{K})} \text{ord}_{\mathcal{Q}} f = 0.$$

Proof. First take $f \in K[x,y]$, $f \neq 0$ and let $\mathcal{Q} = (x_0, y_0) \in H_0(\overline{K})$ with $\text{ord}_{\mathcal{Q}} f \neq 0$. By Lemma 1 (a) we know that $\text{ord}_{\mathcal{Q}} f \geq 1$. It follows that $\text{ord}_{\mathcal{Q}}(f\overline{f}) = \text{ord}_{\mathcal{Q}} f + \text{ord}_{\mathcal{Q}} \overline{f} \geq 1$. Now since $f = g + hy$, $f\overline{f} = g^2 - h^2 C(x)$ which lies in $K[x]$, so by Lemma 2 we have $\text{ord}_{x_0}(f\overline{f}) > 0$. But there are only finitely many such x_0 and so only finitely many $y_0 = \pm\sqrt{C(x_0)}$.

For $f \in K(x,y)$, $f = \frac{f_1}{f_2}$, $f_1, f_2 \in K[x,y]$ we have $\text{ord}_{x_0} f = \text{ord}_{x_0} f_1 - \text{ord}_{x_0} f_2$ so there are also only finitely many x_0 for which this differs from zero.

Give a name to our sum

$$s(f) = \sum_{\mathcal{Q} \in H(\overline{K})} \text{ord}_{\mathcal{Q}} f$$

and note that $s(f) = s(\overline{f})$ due to Lemma 3 and the fact that we take the sum over all $\mathcal{Q}$. Because $\text{ord}_{\mathcal{Q}}(f\overline{f}) = \text{ord}_{\mathcal{Q}}(f) + \text{ord}_{\mathcal{Q}}(\overline{f})$ we can see that

$$s(f\overline{f}) = s(f) + s(\overline{f}) = 2s(f).$$

But because $f\overline{f} \in K(x)$ we can use Lemma 2 to write this out as

$$\begin{aligned}
2s(f) &= \sum_{\mathcal{Q} \in H(\overline{K})} \text{ord}_{\mathcal{Q}} f\overline{f} \\
&= 2 \sum_{\substack{x_0 \neq \infty \\ x_0 \neq \alpha_i}} \text{ord}_{x_0} f\overline{f} + 2 \sum_{x_0 = \alpha_i} \text{ord}_{x_0} f\overline{f} + 2 \sum_{x_0 = \infty} \text{ord}_{x_0} f\overline{f} \\
&= 2 \sum_{x_0 \in \overline{K} \cup \{\infty\}} \text{ord}_{x_0} f\overline{f}.
\end{aligned}$$

Here α_i are the points on which $C(x)$ vanishes and since

$$\sum_{x_0 \in \overline{K} \cup \{\infty\}} \text{ord}_{x_0} g = 0$$

for any $g \in K(x)$ we have $2s(f) = 0$ meaning $s(f) = 0$ in $\mathbb{Z}$. □

2.2 Divisors and Lemmas

Before we continue with the next row of Lemmas, we introduce

Definition 6: The divisor of a function $f \in \mathcal{F}$ is the formal sum

$$(f) = \sum_{\mathcal{Q} \in H(\overline{K})} \mathrm{ord}_{\mathcal{Q}} f \cdot \mathcal{Q}$$

Lemma 5: If f has no poles, f is a constant.

Proof. □

Lemma 6: If f has at most one pole at ∞, f is a constant.

Proof. □

Lemma 7: If f has at most two poles at ∞, f is of the form $f = \alpha + \beta x$ with $\alpha, \beta \in K$.

Proof. □

Lemma 8: If f has at most three poles at ∞, f is of the form $f = \alpha + \beta x$ with $\alpha, \beta \in K$.

Proof. □

Lemma 9: If f has at most four poles at ∞, f is of the form $f = \alpha + \beta x + \gamma x^2$ with $\alpha, \beta, \gamma \in K$.

Proof. □

Minilemma 1: If f is such that $\lambda_\infty(f) = \gamma t^e + \dots$ then $\lambda_\infty(\overline{f}) = -\gamma t^e + \dots$

Proof. Let $\lambda_\infty(\overline{f}) = \widetilde{\gamma} t^e + \dots$. Since $f + \overline{f} = 2p$ and $\mathrm{ord}_\infty(p)$ is always even for $p \in K(x)$, we must have $(\gamma + \widetilde{\gamma})t^{-3} + \dots = \sigma t^{-2} + \dots$ so $\gamma = -\widetilde{\gamma}$. □

Lemma 2: Let $f \in \mathcal{F}^*$ with $(f) = \frac{***}{3\infty}$. Then $f = \alpha + \beta x$.

Proof. Let $f = p + qy$ with $p, q \in K(t)$. We have $\lambda_\infty(f + \overline{f}) = \sigma t^{-2} + \dots$ so $p = \frac{1}{2}(f + \overline{f}) = \alpha + \beta x$.

Now, consider $f - \overline{f} = 2qy$. We get $\lambda_\infty(f - \overline{f}) = 2\lambda_\infty(q) t^{-5} \sqrt{\tau} = 2\gamma t^{-3} + \dots$ and since $\sqrt{\tau}$ is a a series of the form $1 + \dots$ we must have $\lambda_\infty(q) = \gamma t^2 + \dots$

Suppose $q \neq 0$, this implies $\frac{1}{q} = \widetilde{\delta} + \widetilde{\epsilon} x$. Rewriting to accomodate for $q = 0$ gives $q = \frac{\epsilon}{x - \delta}$ with $\delta, \epsilon \in K$ as well so

$$f = \alpha + \beta x + \frac{\epsilon}{x - \delta} y$$

However, now $\lambda_{\mathcal{Q}}(f) = \alpha + \beta(t^2 + \delta) + \epsilon\sqrt{\mu}\frac{t\sqrt{\tau}}{t^2}$ for $\mathcal{Q} = (\delta, 0)$ which is equal to $\epsilon\sqrt{\mu}t^{-1} + \ldots$. If $\epsilon \neq 0$ (μ is non zero anyway) this means that $\mathcal{Q}$ is a new (true) pole for f which contradicts $(f) = \frac{***}{3\infty}$, so q must be 0. $\square$

Lemma 3: Let $f \in \mathcal{F}^*$ with $(f) = \frac{****}{4\infty}$. Then $f = \alpha + \beta x + \gamma x^2$.

Proof. (different function f): With $\lambda_\infty(f) = \gamma t^{-4} + \ldots$ so $\lambda_\infty(f - \gamma x^2)$ being of the form $\sigma t^{-3} + \ldots$ we fall into the case above and the claim follows. $\square$