INVERSES OF VANDERMONDE MATRICES

N. MACON AND A. SPITZBART,* General Electric Company, Evendale

1. Introduction. In a recent paper [1], one finds explicit formulas for the derivatives of a polynomial $y=f(x)$ of degree n in terms of its values $y_i=f(x_i)$ at $n+1$ points defined by $x_i=x_0+ih$, $(i=0, \cdots, n)$. These results are used here to invert the Vandermonde matrix

$$V(x_1, \cdots, x_n) = \begin{bmatrix} 1 & 1 & \cdots 1 \\ x_1 & x_2 & \cdots x_n \\ \cdot & \cdot & \cdots \cdot \\ x_1^{n-1} & x_2^{n-1} & \cdots x_n^{n-1} \end{bmatrix},$$

where the x_i are distinct, different from zero, but otherwise arbitrary.

The paper is in three parts. In the first we outline the results from [1] required later. In the second, these formulas are applied to the special Vandermonde matrix

$$V_{n+1}(x_0, h) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_0 + h & \cdots & x_0 + nh \\ \cdot & \cdot & \cdots & \cdot \\ x_0^n & (x_0 + h)^n & & (x_0 + nh)^n \end{bmatrix};$$

and the elements of $V_{n+1}^{-1}(x_0, h)$ are obtained in terms of Stirling numbers. Finally, the methods of [1] are extended in such a way that the elements of $V^{-1}(x_1, \cdots, x_n)$ can be expressed in terms of the elementary symmetric functions of the x's. This last result is offered as an alternative to the derivation one would obtain from the classical formula for the values of Vandermonde determinants with missing powers ([2], p. 99).

2. Preliminary results. Let $y=f(x)$ be a polynomial of degree n, and $y_i=f(x_i)$, $(i=0, 1, \cdots, n)$, where $x_i=x_0+ih$. It was shown in [1] that if we write

$$h^k f^{(k)}(x) = \sum_{i=0}^{n} A_{mi}^{k} y_i, \tag{1}$$

then

$$A_{mi}^{k} = \sum_{j=k}^{n} \frac{(-1)^{i+j}\binom{j}{i} C_{mj}^{k}}{j!}, \tag{2}$$

where

* Now at Alabama Polytechnic Institute and University of Wisconsin-Milwaukee, respectively.

$$\binom{j}{i} = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i > j, \end{cases}$$

$$C_{mj}^{k} = \sum_{r=k}^{j} p_k(r)\ S_j^r m^{r-k}, \tag{3}$$

m is any real number, and $x = x_0 + mh$. In the above, $p_k(r)$ denotes the factorial polynomial of degree k, and the S_k^j are the Stirling numbers of the first kind. Thus, we have $p_k(x) = \sum_{j=1}^{k} S_k^j x^j$. A wide variety of classical numerical differentiation formulas are special cases of (1).

It was shown further that these results enable one to invert the Vandermonde matrix

$$M(m) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -m & 1-m & \cdots & n-m \\ \cdot & \cdot & \cdots & \cdot \\ (-m)^n & (1-m)^n & \cdots & (n-m)^n \end{bmatrix}.$$

If we write $M^{-1}(m) = \{a_{\lambda,\mu}^m\}$, $(\lambda, \mu = 1, \cdots, n+1)$, then

$$a_{\lambda,\mu}^{m} = \frac{A_{m,\lambda-1}^{\mu-1}}{(\mu-1)!}, \tag{4}$$

where $A_{m,\lambda-1}^{\mu-1}$ is given by (2).

3. Vandermonde Matrices for equally spaced points. It is easy to show that

$$\begin{bmatrix} 1 & 0 & 0 \cdots 0 \\ 0 & h & 0 \cdots 0 \\ \cdot & \cdot & \cdot\cdot\cdot\cdot\cdot \\ 0 & 0 & 0 \cdots h^n \end{bmatrix} M(m) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -mh & -mh+h & \cdots & -mh+nh \\ \cdot & \cdot & \cdots & \cdot \\ (-mh)^n & (-mh+h)^n & \cdots & (-mh+nh)^n \end{bmatrix}$$

If we write $x_0 = -mh$, it follows that

$$V_{n+1}(x_0, h) \equiv \begin{bmatrix} 1 & 1 & & 1 \\ x_0 & x_0+h & \cdots & x_0+nh \\ \cdot & \cdot & \cdots & \cdot \\ x_0^n & (x_0+h)^n & \cdots & (x_0+nh)^n \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 0 & 0 \cdots 0 \\ 0 & h & 0 \cdots 0 \\ \cdot & \cdot & \cdot\cdot\cdot\cdot\cdot \\ 0 & 0 & 0 \cdots h^n \end{bmatrix} M(-x_0/h),$$

and so

$$V_{n+1}^{-1}(x_0, h) = M^{-1}(-x_0/h)\begin{bmatrix} 1 & 0 & 0 \cdots 0 \\ 0 & h^{-1} & 0 \cdots 0 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ 0 & 0 & 0 \cdots h^{-n} \end{bmatrix}.$$

Thus, by (4) the element in the λth row and μth column of $V_{n+1}^{-1}(x_0, h)$ is precisely

$$\nu_{\lambda\mu} = \frac{1}{h^{\mu-1}(\mu-1)!} A^{\mu-1}_{-x_0/h,\lambda-1}.$$

It is interesting to note that even though $V_{n+1}^{-1}(1, 1)$ can be obtained directly from the above, a slight modification enables one to express the elements of this inverse very compactly. For simplicity of notation, we apply the method to obtain $V_n^{-1}(1, 1)$.

Let us write

$$M(0) = \begin{bmatrix} 1 & Q_{1n} \\ R_{n1} & P_{nn} \end{bmatrix}, \qquad M^{-1}(0) = \begin{bmatrix} b_{11} & S_{1n} \\ T_{n1} & U_{nn} \end{bmatrix},$$

where $Q_{1n} = (1, \cdots, 1)$, R_{n1} is a column vector of zeros, and

$$P_{nn} = \begin{bmatrix} 1 & 2 & \cdots & n \\ 1 & 4 & \cdots & n^2 \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ 1 & 2^n & \cdots & n^n \end{bmatrix}.$$

It suffices to invert the matrix P_{nn}, since its columns are scalar multiples of those of $V_n(1, 1)$. Now

$$M(0)M^{-1}(0) = \begin{bmatrix} b_{11} + Q_{1n}T_{n1} & S_{1n} + Q_{1n}U_{nn} \\ R_{n1}b_{11} + P_{nn}T_{n1} & R_{n1}S_{1n} + P_{nn}U_{nn} \end{bmatrix} = I,$$

and so $I_n = R_{n1}S_{1n} + P_{nn}U_{nn} = P_{nn}U_{nn}$. Hence $U_{nn} = P_{nn}^{-1}$. Denote the elements of U_{nn} by μ_{ik}. Since b_{11} contains only a single element, it follows that $\mu_{ik} = a^0_{i+1,k+1}$, and in turn, by (4), $\mu_{ik} = A^k_{0i}/k!$. We have from (2) that

$$A^k_{0,i} = \sum_{j=k}^{n} \frac{(-1)^{j+i}\binom{j}{i} C^k_{0j}}{j!} \qquad (i, k = 1, \cdots, n).$$

From (3) $C^k_{0j} = p_k(k)S^k_j = k!S^k_j$. Finally, we have

$$A_{0i}^{k} = \sum_{j=k}^{n} \frac{(-1)^{j+i}\binom{j}{i} k! S_j^k}{j!},$$

and so, by (4),

$$\mu_{ik} = \sum_{j=k}^{n} \frac{(-1)^{j+i}\binom{j}{i} S_j^k}{j!}. \tag{5}$$

Since the kth column of V_{nn} (1, 1) is k^{-1} times the kth column of P_{nn}, $(k=1, \cdots, n)$, it follows that $V_n^{-1}(1, 1) = \{i\mu_{ik}\}$, $(i, k=1, \cdots, n)$, where μ_{ik} is given by (5). As an illustration, the following inverses are given:

$$P_{44}^{-1} = \begin{bmatrix} 4 & -\frac{13}{3} & \frac{3}{2} & -\frac{1}{6} \\ -3 & \frac{19}{4} & -2 & \frac{1}{4} \\ \frac{4}{3} & -\frac{7}{3} & \frac{7}{6} & -\frac{1}{6} \\ -\frac{1}{4} & \frac{11}{24} & -\frac{1}{4} & \frac{1}{24} \end{bmatrix}, \quad V_4^{-1}(1, 1) = \begin{bmatrix} 4 & -\frac{13}{3} & \frac{3}{2} & -\frac{1}{6} \\ -6 & \frac{19}{2} & -4 & \frac{1}{2} \\ 4 & -7 & \frac{7}{2} & -\frac{1}{2} \\ -1 & \frac{11}{6} & -1 & \frac{1}{6} \end{bmatrix}.$$

These inverses may be checked by multiplication with the corresponding direct matrix. P_{44}^{-1} was obtained directly, by use of (5).

4. The inverse of the general Vandermonde matrix. Let $x_0=0, x_1, \cdots, x_n$ be the given distinct numbers. The polynomial $y=y(x)$ of degree n assuming $n+1$ arbitrary values $y_i=y(x_i)$, $i=0, \cdots, n$, can be written, by Lagrange's interpolation formula, as

$$y = \sum_{i=0}^{n} A_{x,i} y_i. \tag{6}$$

If we write the kth derivative of $y(x)$ as

$$y^{(k)} = \sum_{i=0}^{n} A_{xi}^{k} y_i \qquad (k = 1, \cdots, n), \tag{7}$$

where

$$A_{x,i}^{k} = \frac{d^k}{dx^k} A_{x\,i}.$$

and set $x=x_0=0$, we have

$$y_0^{(k)} = \sum_{i=0}^{n} A_{0,i}^{k} y_i. \tag{8}$$

In the following, we obtain explicit expressions for the $A_{0,i}^k$, and then show that the $A_{0,i}^k$ satisfy systems of linear equations having the same coefficient matrix. Thus, we obtain the inverse of this matrix and, in turn, the inverse of $V(x_1, \cdots, x_n)$ in terms of the $A_{0,i}^k$.

The functions A_{xi}, as given by the Lagrange interpolation formula, are

$$\begin{aligned} A_{xi} &= \frac{1}{p_n{}'_{+1}(x_i)} \cdot \frac{p_{n+1}(x)}{(x - x_i)} \\ &= \frac{1}{p_n{}'_{+1}(x_i)} (x - x_0) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n); \end{aligned}$$

and so

$$\begin{aligned} A_{xi} = \frac{1}{p_n{}'_{+1}(x_i)} \big[x^n &- \overset{i}{\sigma}_{1,n-1} x^{n-1} + \overset{i}{\sigma}_{2,n-1} x^{n-2} - \cdots \\ &+ (-1)^{n-2} \overset{i}{\sigma}_{n-2,n-1} x^2 + (-1)^{n-1} \overset{i}{\sigma}_{n-1,n-1} x\big], \end{aligned}$$

where $\sigma_{j,n-1}^i$ is the sum of all products of j of the numbers $x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$ without permutations or repetitions ($\sigma_{0,n-1}^i \equiv 1$). If we differentiate the above k times, set $x = 0$, and notice that

$$p'_{n+1}(x_i) = \prod_{j=0}^{n}{}' (x_i - x_j),$$

where the dash indicates that $i \neq j$, we obtain*

$$A_{0i}^{k} = \frac{(-1)^{n-k} \cdot k!}{\prod_{j=0}^{n}{}' (x_i - x_j)} \cdot \overset{i}{\sigma}_{n-k,n-1}. \tag{9}$$

In order to exhibit the linear systems mentioned above, we expand $y(x)$ about $x_0 = 0$, and substitute $x = x_i$ to get

$$y_i = \sum_{\mu=0}^{n} \frac{1}{\mu!} y_0^{(\mu)} x_i^{\mu}.$$

* More generally, if we do not substitute $x = 0$ after the above differentiation, and substitute the result in (7), we obtain

$$y^{(k)}(x) = \sum_{i=0}^{n} y_i \sum_{j=k}^{n} \frac{(-1)^{n-j} \overset{i}{\sigma}_{n-j,n-1}}{p_{n+1}(x_i)} j(j-1) \cdots (j-k+1) x^{j-k},$$

which is a formula for an arbitrary derivative of $y(x)$ at an arbitrary x.

Inserting this into (8) and rearranging, we get

$$y_0^{(k)} = \sum_{\mu=0}^{n} \frac{1}{\mu!} y_0^{(\mu)} \sum_{i=0}^{n} x_i^{\mu} A_{0i}^{k} \qquad (k = 1, \cdots, n).$$

Since these are identities in $y_0^{(k)}$, we must have

$$\sum_{i=0}^{n} x_i^{\mu} A_{0i}^{k} = \delta_{\mu k} k! \qquad (\mu = 0, 1, \cdots, n),$$

where $\delta_{\mu k}$ is the Kronecker delta. Since $x_0 = 0$, it follows that

$$\sum_{i=1}^{n} x_i^{\mu} A_{0i}^{k} = \delta_{\mu k} \cdot k! \qquad (\mu = 1, \cdots, n).$$

For each k ($k = 1, \cdots, n$), this is a system of n linear equations in the unknowns $A_{01}^{k}, A_{02}^{k}, \cdots, A_{0n}^{k}$. These n systems can be combined into the matrix equation

$$\{x_k^{i}\} \cdot \{A_{0i}^{k}\} = \begin{bmatrix} 1! & 0 & 0 \cdots 0 \\ 0 & 2! & 0 \cdots 0 \\ \cdot & \cdot & \cdot \cdots \cdot \\ 0 & 0 & 0 \cdots n! \end{bmatrix}$$

where A_{0i}^{k} is the element of $\{A_{0i}^{k}\}$ in the ith row and kth column, and similarly for the element x_k^{i} of $\{x_k^{i}\}$. (In x_k^{i} the i denotes an actual exponent.) Denoting the right member of the above equation by $D(n)$, we may write $\{A_{0i}^{k}\} = \{x_k^{i}\}^{-1} \cdot D(n)$. Thus, the element $b_{\lambda\mu}$ in the λth row and μth column of $\{x_k^{i}\}^{-1}$ ($\lambda, \mu = 1, \cdots, n$) is given by

$$b_{\lambda\mu} = \frac{1}{\mu!} A_{0\lambda}^{\mu}, \tag{10}$$

where the $A_{0\lambda}^{\mu}$ may be obtained from (9).

Finally, if $V^{-1}(x_1, \cdots, x_n) \equiv \{v_{\lambda\mu}\}$, a method similar to that used in Section 3 yields $v_{\lambda\mu} = x_\lambda b_{\lambda\mu}$, ($\lambda, \mu = 1, \cdots, n$), which, together with (10), gives an explicit representation for the inverse of the general Vandermonde matrix.

References

1. A. Spitzbart and N. Macon, Numerical differentiation formulas, this MONTHLY, vol. 64, 1957, pp. 721–723.
2. G. Pólya and G. Szegö, Aufgaben und Lehrsätze aus der Analysis, vol. II, New York, 1945.