1 Addition Law

1.1 Definitions and Notation

Definition 1: Let K be a field with $\mathrm{char}(K) \neq 2, 3$ and $\overline{K}$ its algebraic closure. Define the hyperelliptic curve of genus two $H_0(K)$ as the set of solutions in K^2 to the equation $y^2 = C(x)$ where

$$C(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$$

is a polynomial over K. Similarly, the set of solutions in the closure would be denoted $H_0(\overline{K})$. Define $H(K)$ as $H_0(\overline{K}) \cup \{\infty\}$.

Note that we could obtain a more reduced form of $C(x)$, eliminating a by shifting x to $x - a/5$. However, since this would rob us of the possibility of $\mathrm{char}(K) = 5$ without simplifying our coming calculations in any significant manner, we shall be reluctant towards using this trick.

For the purpose of clarity, let points on the hyperelliptic curve — in the sense of solutions to $y^2 = C(x)$ — be designated by the calligraphic letter $\mathcal{Q} = (x, y) \in H_0(\overline{K})$. The point opposite to $\mathcal{Q}$ will be written $\overline{\mathcal{Q}} = (x, -y)$ and by symmetry of the curve in y also belongs to $H_0(\overline{K})$. In the case where $\mathcal{Q} = \infty$, define $\overline{\mathcal{Q}} = \infty$.

We want to consider the set of all pairs $(\mathcal{Q}_1, \mathcal{Q}_2)$ and tame it with an equivalence relation with the goal of obtaining an additive group:

Definition 2: Define $\mathbf{J}$ to be the set $\mathcal{J}/\sim$ where $\mathcal{J} = H(\overline{K}) \times H(\overline{K})$. It is called the "Jacobian" and the equivalence relation is defined by

$$\begin{aligned} (\mathcal{Q}_1, \mathcal{Q}_2) &\sim (\mathcal{Q}_2, \mathcal{Q}_1) \\ \text{and} \quad (\mathcal{Q}, \overline{\mathcal{Q}}) &\sim (\infty, \infty). \end{aligned}$$

Write $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ from now on and let bold letters denote points on the curve in the sense of classes of unordered pairs $\mathbf{P} = \{\mathcal{Q}_1, \mathcal{Q}_2\} \in \mathbf{J}$. The point $\{\overline{\mathcal{Q}}_1, \overline{\mathcal{Q}}_2\}$ will be called $\overline{\mathbf{P}}$ for now but can already tentatively be thought of as $-\mathbf{P}$. Call $\{\infty, \infty\}$ the zero of our set. Call $\mathcal{Q}_i$ a point-component.

A point $\mathcal{Q} = (x_0, y_0)$ is called singular if it fulfills both $y_0 = 0$ and $C'(x_0) = 0$. A curve is called singular if and only if it has a singular point. We consider only non-singular hyperelliptics from here on; this amounts to $C(x)$ having no repeated factors over $\overline{K}$ ("squarefree").

1.2 The General Case

Let's start with $\mathbf{P}_1 = \{\mathcal{Q}_1, \mathcal{Q}_2\}$, $\mathbf{P}_2 = \{\mathcal{Q}_3, \mathcal{Q}_4\}$ with $\mathcal{Q}_i = (x_i, y_i) \in H_0(K)$ or $\mathcal{Q}_i = \infty$. To define $\mathbf{P}_3 = \mathbf{P}_1 + \mathbf{P}_2$ we distinguish between one general case and a number of special cases and first derive the results of the former before enumerating the latter ones.

Case 1, Four Distinct Point-Components: Let $\mathcal{Q}_i \in H_0(K)$ and $\mathbf{P}_i$ be defined as above with $x_i \neq x_j$ whenever $i \neq j$.

The overarching idea is to obtain a fifth and sixth x-coordinate and the corresponding y-coordinates by passing a polynomial of degree three through the four points $\mathcal{Q}_i$. Ideally this should give us two additional intersections with the curve which we then use as the components of our point $\mathbf{P}_1 + \mathbf{P}_2$.

Step 1: It is known that the Vandermonde matrix

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}$$

has determinant $\prod_{i<j}(x_i - x_j)$ which is conveniently non-zero if and only if the x_i are pairwise distinct. Let $p(x) = p_3x^3 + p_2x^2 + p_1x + p_0 \in \overline{K}[x]$ be the polynomial in unknown coefficients that we are looking for. With $\mathbf{y} = (y_1\ y_2\ y_3\ y_4)^t$ and $\mathbf{p} = (p_0\ p_1\ p_2\ p_3)^t$, the problem of determining $p(x)$ can be rewritten as

$$V \cdot \mathbf{p} = \mathbf{y}$$

which by invertibility of V has a unique solution for $\mathbf{p}$ with $p_i \in K$.

Note that the leading coefficient of $p(x)$ is

$$p_3 = \frac{1}{\det(V)} \begin{vmatrix} 1 & x_1 & x_1^2 & y_1 \\ 1 & x_2 & x_2^2 & y_2 \\ 1 & x_3 & x_3^2 & y_3 \\ 1 & x_4 & x_4^2 & y_4 \end{vmatrix}$$

and the next step will depend on whether $p(x)$ is truly of degree 3 or not.

Step 2a: Knowing the coefficients p_i of $p(x)$ we first assume that $p_3 \neq 0$ so we can proceed to look for the two additional solutions of the sextic equation

$$C(x) - (p(x))^2 = 0. \qquad (*)$$

Observe that this vanishes at x_1, x_2, x_3 and x_4, so write the lefthand side as

$$C(x) - (p(x))^2 = -p_3^2(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)(x - x_6) \qquad (1)$$

for x_5 and x_6 in $\overline{K}$. Comparing the coefficients of both expressions at x^4 and x^5 yields

$$\sum_{\substack{i,j=1\\i<j}}^{6} x_i x_j = T_4 \tag{4}$$

$$\text{and} \quad \sum_{i=1}^{6} x_i = T_5 \tag{5}$$

where $T_4 = \frac{p_2^2+2p_1p_3-a}{p_3^2}$ and $T_5 = \frac{1-2p_2p_3}{p_3^2}$. The first expression gives

$$x_6 \sum_{i=1}^{5} x_i + \sum_{\substack{i,j=1\\i<j}}^{5} x_i x_j = T_4.$$

Doing this twice and replacing x_6 with the information from (5) gives the tidy quadratic equation

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$$x^2 - \left(T_5 - \sum_{i=1}^{4} x_i\right) \cdot x + \left(T_4 - T_5 \sum_{i=1}^{4} x_i + \sum_{\substack{i,j=1\\i\leq j}}^{4} x_i x_j\right) = 0 \tag{$\dagger$}$$

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of which x_5 is one solution and — by symmetry of the above steps — x_6 the other one. Compute $y_i = p(x_i)$, $i = 5,6$ to obtain $\mathcal{Q}_5 = \{x_5, -y_5\}$ and $\mathcal{Q}_6 = \{x_6, -y_6\}$, at which point it becomes clear that the worst-case scenario for our field extension to accommodate the new coordinates is to be quadratic. Finally we define $\mathbf{P}_1 + \mathbf{P}_2$ to be equal to $\mathbf{P}_3 = \{\mathcal{Q}_5, \mathcal{Q}_6\}$.

Step 2b: If p_3 were zero, the equation $(*)$ would be quintic instead. We may therefore write the lefthand side factorized, so

$$(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5), \tag{2}$$

again for x_5 somewhere in $\overline{K}$. Defining $T_{4\infty} = p_2^2 - a$ and comparing the coefficients at x^4 gives

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$$x_5 = T_{4\infty} - \sum_{i=1}^{4} x_i \tag{$\ddagger$}$$

---

and we may rejoice in the implication of x_5 staying in K.

Compute $y_5 = p(x_5)$ and define $\mathbf{P}_1 + \mathbf{P}_2$ to be the point $\mathbf{P}_3 = \{(x_5, y_5), \infty\}$.

Step $1'$: To extend our construction from $H_0(K)$ to $H(K)$ we now consider $\mathcal{Q}_4 = \infty$ with the other $\mathcal{Q}_i = (x_i, y_i)$ as before. This can be seen as the x_i still being pairwise distinct, only that one of them is allowed to be ∞ here.

There is no coordinate x_4 this time, so we pass a quadratic polynomial $p(x)$ through the remaining three points (x_i, y_i). This means that we solve the linear system $\widetilde{V} \cdot \mathbf{p} = \widetilde{\mathbf{y}}$ where $\widetilde{V}$ is the Vandermonde matrix for x_i, $i = 1, .., 3$ which incidentally is the upper-left 3×3 sub-matrix of V. Here $\mathbf{p} = (p_0\ p_1\ p_2)^t$ and $\widetilde{\mathbf{y}} = (y_1\ y_2\ y_3)^t$ are defined as expected.

As before, the leading coefficient of $p(x)$ might or might not be zero, but $(*)$ will be quintic in either case, so we only have to worry about one step 2.

Step $2'$: Doing a coefficient comparison at x^3 and at x^4 in $(*)$ gives

$$\sum_{\substack{i,j=1\\ i<j}}^{3} x_i x_j + x_5 \sum_{i=1}^{3} x_i + x_5 x_6 = b - 2p_1 p_2 \tag{3'}$$

$$\text{and} \qquad \sum_{i=1}^{3} x_i + x_5 + x_6 = p_2^2 - a. \tag{4'}$$

Call the righthand terms $T_{3\infty}$ and $T_{4\infty}$, combine both equations and obtain

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$$x^2 - \left(T_{4\infty} \sum_{i=1}^{3} x_i \right) \cdot x + \left(T_{3\infty} - T_{4\infty} \sum_{i=1}^{3} x_i + \sum_{\substack{i,j=1\\ i\le j}}^{3} x_i x_j \right) \tag{$\dagger'$}$$

Solve, call the two solutions x_5 and x_6, compute y_5 and y_6 through $p(x_5)$ and $p(x_6)$ and define $\mathbf{P}_1 + \mathbf{P}_2$ to be $\mathbf{P}_3 = \{(x_5, -y_5), (x_6, -y_6)\} = \{\mathcal{Q}_5, \mathcal{Q}_6\}$.

$\sim$

Remark 1: Up to this point, we have given the general-case rule only for elements in $H(K) \times H(K)$. Extending to $\mathcal{J} = H(\overline{K}) \times H(\overline{K})$ is trivial as we can simply choose $\overline{K}$ as the new K. The final step is to note that everything until now is well-defined on $\mathbf{J}$. The equivalence relation from Definition 2 corresponds to interchanging $\mathcal{Q}_1$ and $\mathcal{Q}_2$ or $\mathcal{Q}_3$ and $\mathcal{Q}_4$ or both. To this end we can check that interchanging (x_i, y_i) and (x_j, y_j) does nothing to the resulting $\mathcal{Q}_5$ and $\mathcal{Q}_6$. This is the case because $p(x)$ depends only on the solution of the linear system $V \cdot \mathbf{p} = \mathbf{y}$ on which the aforementioned

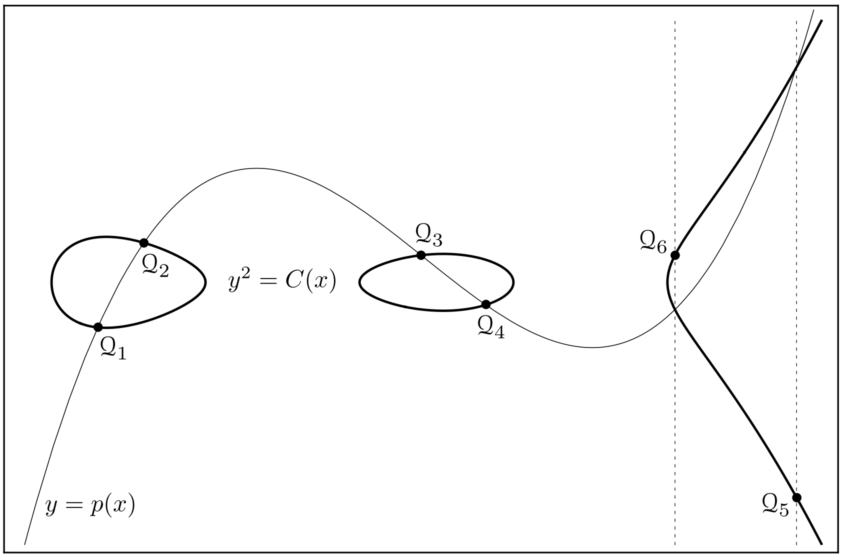
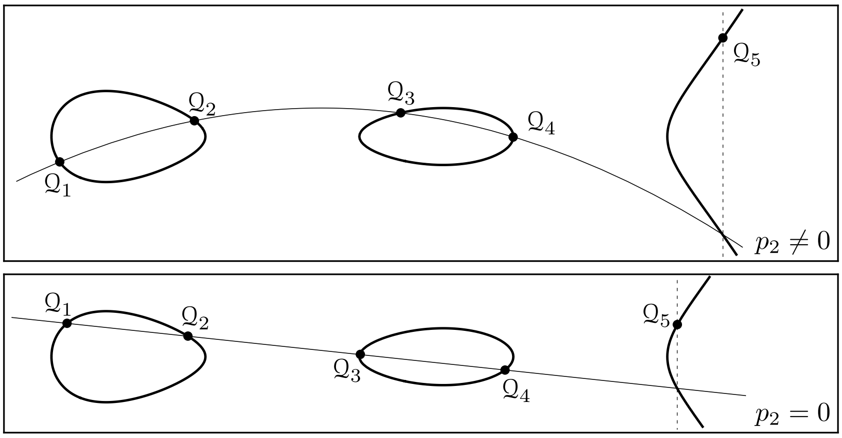


FIGURE 1A: The general case for the addition law in $\mathbb{R}^2$ where $p_3 \neq 0$.



FIGURES 1B and 1C: If $p_3 = 0$ we have at most five intersections.

permutation has no effect. With p invariant, the subsequent steps remain unchanged as well.

Finally, the restriction to K instead of $\overline{K}$ was to showcase the degree of the required field extension, a feat we will now leave aside by always using $\overline{K}$.

Before we begin listing the special cases, we impose the following property: The sum of any $\mathbf{P}_1 = \{\mathcal{Q}_1, \mathcal{Q}_2\}$ and $\mathbf{P}_2 = \{\mathcal{Q}_3, \mathcal{Q}_4\}$ in $\mathbf{J}$ fulfills the equality

$$\mathbf{P}_1 + \mathbf{P}_2 = \{\mathcal{Q}_3, \mathcal{Q}_2\} + \{\mathcal{Q}_1, \mathcal{Q}_4\}. \tag{$\diamondsuit$}$$

Note that the general-case addition we just defined already fulfills this, as we noted before that interchanging any of the (x_i, y_i) does not impact the solution $\mathbf{p}$ of the linear system nor any of the dagger equations.

Remark 2: The property $(\diamondsuit)$ corresponds to the permutation $(1\,3)$ on the point-components $\mathcal{Q}_i$. With Definition 2 allowing the permutations $(1\,2)$ and $(3\,4)$ this naturally leads to the observation that in fact all permutations of point-components must now leave the sum unchanged. In fact it is easy to check that the three transpositions generate all of S_4 and the remark above already provides compatibility with the construction of the general case.

A substantial bonus of this is the fact that commutativity corresponds to the permutation $(1\,3)(2\,4)$ which by the above we now obtained for free.

1.3 Complete List of Cases

We may now impose conditions on the relations between the $\mathcal{Q}_i$ without mentioning whether they belong to $\mathbf{P}_1$ or $\mathbf{P}_2$. As a result, the list of special cases can be written in a significantly more concise manner.

As we strive to define $\{\mathcal{Q}_1, \mathcal{Q}_2\} + \{\mathcal{Q}_3, \mathcal{Q}_4\}$, for any $\mathcal{Q}_i = (x_i, y_i) \in H_0(\overline{K})$ or $\mathcal{Q}_i = \infty \in H(\overline{K})$ we observe that the following simplification takes place:

Let $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Whenever $\mathcal{Q}_i = \overline{\mathcal{Q}}_j$ for $i \neq j$ we have

$$\begin{aligned} \{\mathcal{Q}_1, \mathcal{Q}_2\} + \{\mathcal{Q}_3, \mathcal{Q}_4\} &= \{\mathcal{Q}_i, \mathcal{Q}_j\} + \{\mathcal{Q}_k, \mathcal{Q}_l\} \\ &= \{\mathcal{Q}_k, \mathcal{Q}_l\} + \{\mathcal{Q}_j, \overline{\mathcal{Q}}_j\} \\ &= \{\mathcal{Q}_k, \mathcal{Q}_l\} + \{\infty, \infty\} \end{aligned}$$

so define $$\{\mathcal{Q}_1, \mathcal{Q}_2\} + \{\mathcal{Q}_3, \mathcal{Q}_4\} = \{\mathcal{Q}_k, \mathcal{Q}_l\}.$$

The justification for this follows directly from Remark 2 and Definition 2. It is easily checked that this sum remains well defined if the choice of i and j is not unique. This allows us to treat the above as a separate case in order to characterize the other cases solely through the x-coordinates of the $\mathcal{Q}_i$, so we may assume

$$x_i = x_j \iff \mathcal{Q}_i = \mathcal{Q}_j$$

for any $\mathcal{Q}_i, \mathcal{Q}_j \in H(\overline{K})$, provided we assign $\mathcal{Q}_i = \infty$ the x-coordinate $x_i = \infty$. There is a possibility for ambiguity of "∞" here, which we will however avoid later on by making it obvious which infinity we are working with.

We may now write the distinction between the remaining cases as follows:

1. The general case where all $x_1, .., x_4$ are pairwise distinct.

2. The case $x_i = x_j$ but x_i, x_k and x_l are pairwise distinct.
3. The case where $x_i = x_j$ and $x_k = x_l$ but $x_i \neq x_k$.
4. The case $x_i = x_j = x_k$ but $x_l \neq x_i$.
5. The case where $x_1 = x_2 = x_3 = x_4$.

Since we are allowed to permute point-components anyway, we may fix which of the $\mathcal{Q}_i$ are equal and write everything out for the complete and final list:

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0. The addition with zero, i.e. $\{\mathcal{Q}_1, \mathcal{Q}_2\} + \{\infty, \infty\}$, $\mathcal{Q}_1$, $\mathcal{Q}_2 \in H(\overline{K})$.
1. The general case where $\mathcal{Q}_i \neq \mathcal{Q}_j$ and $\mathcal{Q}_i \neq \overline{\mathcal{Q}}_j$ for $i \neq j$.
2. The single tangential case where all $\mathcal{Q}_i$ as in case 1 with the exception of $\mathcal{Q}_1 = \mathcal{Q}_2 \in H_0(\overline{K})$ with $y_1 \neq 0$.
3. The double tangential case where all $\mathcal{Q}_i$ as in case 1 with the exception of $\mathcal{Q}_1 = \mathcal{Q}_2 \in H_0(\overline{K})$ with $y_1 \neq 0$ and $\mathcal{Q}_3 = \mathcal{Q}_4 \in H_0(\overline{K})$ with $y_3 \neq 0$.
4. The triple point case wherein $\mathcal{Q}_1 = \mathcal{Q}_2 = \mathcal{Q}_3 \in H_0(\overline{K})$ with $y_1 \neq 0$ and $\mathcal{Q}_4 \in H(\overline{K})$ differs from both $\mathcal{Q}_1$ and $\overline{\mathcal{Q}}_1$.
5. The quadruple point case where all $\mathcal{Q}_i \in H_0(\overline{K})$ are equal with $y_1 \neq 0$.

Remark 3:

(i) In both lists, the cases do not overlap.

(ii) For the list to be complete, we must allow for $\mathcal{Q}_i = \infty$ for some i. Note that one is sufficient, since if two or more $\mathcal{Q}_i$ were to be ∞, we would be back at the case 0 by virtue of ($\diamondsuit$), bringing the two infinities together in one point. As for case 1, we consider $\mathcal{Q}_i \in H_0(\overline{K})$ and $\mathcal{Q}_i \in H(\overline{K})$ in two separate cases and postpone handling the latter.

(iii) As for the first case, the constructions will a priori only be made on $\mathcal{J}$. It remains to check that this makes indeed for a well-defined addition on $\mathbf{J}$ by being invariant under interchanges of $\mathcal{Q}_i$ and $\mathcal{Q}_j$ for any i, j.

1.4 Addition Law for the Special Cases

Before we handle the next construction steps we need an intermediate result.

Lemma 1: If the derivatives $D(\tau) = D'(\tau) = \cdots = D^{(k-1)}(\tau) = 0$ for a polynomial D over a field of characteristic 0 or $p \geq k$ then

$$(t - \tau)^k \mid D(t).$$

Proof. Assume $\tau = 0$ by shifting t to $t + \tau$. With $D(t) = \sum_{i=0}^{d} a_i t^i$ we have

$$a_i = \frac{1}{i!} D^{(i)}(0) = 0 \qquad\qquad i = 0, .., k-1$$

so t^k divides $D(t)$. $\square$

We will now use this property for $D(x) = C(x) - p^2(x)$ when factoring $(*)$.

Case 0, Addition with Zero: As one might have anticipated we define $\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_1$ for every $\mathbf{P}_1 \in \mathbf{J}$ if $\mathbf{P}_2 = \{\infty, \infty\} = 0$.

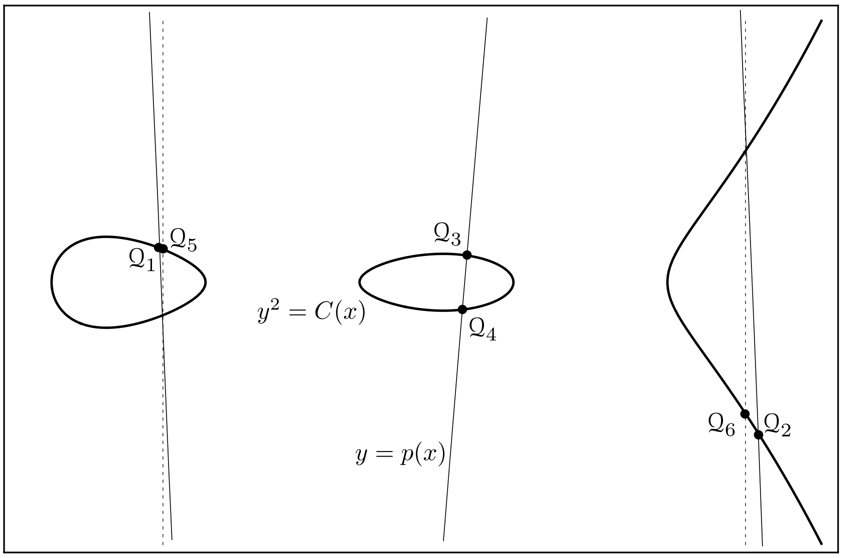

Figure 0a: Illustrating a limit argument where $\mathcal{Q}_3$ is very close to $\overline{\mathcal{Q}}_4$.

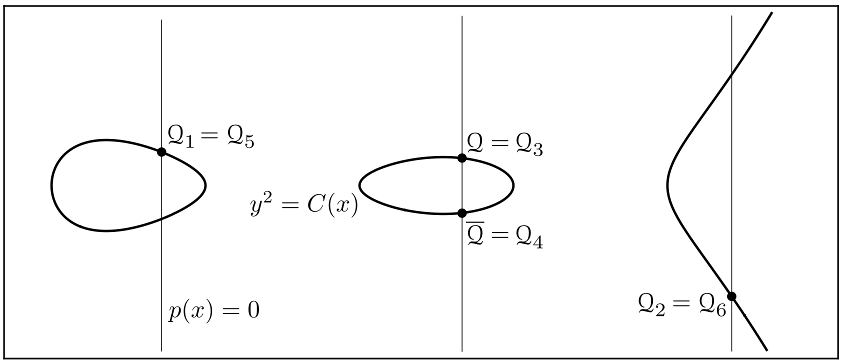

Figure 0b The addition with zero and justifying $\{\mathcal{Q}, \overline{\mathcal{Q}}\} \sim 0$.

Case 2, Tangential: Let $\mathcal{Q}_i \in H_0(\overline{K}), \mathcal{Q}_1 = \mathcal{Q}_2$, $y_1 \neq 0$ but x_1, x_3 and x_4 are pairwise distinct. We cannot use the Vandermonde matrix in this case because it won't possess maximal rank, consequently being non-invertible. We can however obtain an additional equation by demanding that our polynomial $p(x)$ be tangential to the curve at $\mathcal{Q}_1$. This gives

$$
\begin{aligned}
2y\frac{dy}{dx} &= 5x^4 + 4ax^3 + 3bx^2 + 2cx + d \\
\text{and} \qquad \frac{dy}{dx} &= 3p_3x^2 + 2p_2x + p_1
\end{aligned}
$$

meaning that the system to solve for $\mathbf{p}$ is now $V_1 \cdot \mathbf{p} = \mathbf{y}_1$ with

$$
V_1 = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}.
$$

The subscript indicates at which points the intersections have higher order. Here $\mathbf{y}_1$ is defined as $\mathbf{y}$ with y_2 replaced by $y_2' = \frac{C'(x_1)}{2y_1}$ where $C'(x)$ is the derivative of C and $2y_1 \neq 0$ because $\operatorname{char}(\overline{K}) \neq 2$ and $y_1 \neq 0$. Since $\det(V_1) = (x_4 - x_1)^2(x_3 - x_1)^2(x_4 - x_3)$ this fits neatly into our constraints by being non-zero exactly in the case where x_1, x_3 and x_4 are pairwise distinct.

Once the polynomial is determined, we note that the $p^2(x)$ shares a tangent with $C(x)$ at x_1 on purpose, specifically

$$
\begin{aligned}
D'(x_1) &= C'(x_1) - 2p(x_1)p'(x_1) \\
&= C'(x_1) - 2y_1y_2' = 0
\end{aligned}
$$

so we use Lemma 1 to see that the lefthand side of $(*)$ either factors as

$$
-p_3^2(x - x_1)^2 \prod_{i=3}^{6}(x - x_i)
$$

or

$$
(x - x_1)^2 \prod_{i=3}^{5}(x - x_i).
$$

Now step two will be entirely identical to that of the general case and we can again solve $(\ddagger)$ or $(\dagger)$ for x_5 or x_5 and x_6.

Case 3, Double Tangential: Let $\mathcal{Q}_i \in H_0(\overline{K})$, $\mathcal{Q}_1 = \mathcal{Q}_2$ and $\mathcal{Q}_3 = \mathcal{Q}_4$ but $x_1 \neq x_3$ and $\mathcal{Q}_i \neq \overline{\mathcal{Q}}_i$ meaning that neither y_1 nor y_3 will be zero. As before, we lack equations for our linear system, requiring the use of a second tangential constraint. Replace the fourth row of V_1 and $\mathbf{y}_1$ exactly like we did for the second one: $y_4' = \frac{C'(x_3)}{2y_3}$ and

$$
V_{13} = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 0 & 1 & 2x_3 & 3x_3^2 \end{pmatrix}.
$$

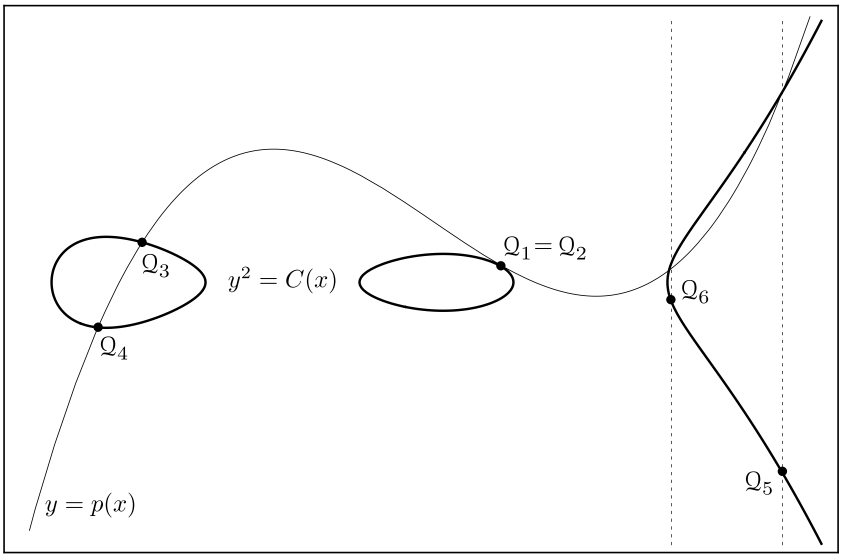

FIGURE 2: The case of a tangential intersection at $\mathcal{Q}_1$.

Now $\det(V_{13}) = (x_3 - x_1)^4$ and this is again different from zero precisely whenever $x_3 \neq x_1$, so as before solve $V_{13} \cdot \mathbf{p} = \mathbf{y}_{13}$ for $\mathbf{p}$, then note that $D(x)$ has double zeroes at x_1 and x_3 by the same argument as in case 2. By Lemma 1 we know that $(*)$ has a factor $(x - x_1)^2(x - x_3)^2$ so we use the same procedure of case 1 with $x_2 = x_1$ and $x_4 = x_3$ to solve (†) or (‡).

CASE 4, TRIPLE POINT: Let $\mathcal{Q}_i \in H_0(\overline{K}), \mathcal{Q}_1 = \mathcal{Q}_2 = \mathcal{Q}_3$ but $x_1 \neq x_4$ and $y_1 \neq 0$. We can thus see this as a third-order intersection and demand that the curve and the polynomial share a second-order derivative at $\mathcal{Q}_1$:

$$V_{11} = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_1 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}.$$

Here $\det(V_{11}) = 2(x_4 - x_1)^3$ and with this, define $\mathbf{y}_{11}$ by taking $\mathbf{y}_1$ and replacing the third coordinate by $y_3'' = \frac{C''(x_1)}{2y_1} - \frac{(C'(x_1))^2}{4y_1^3}$ where $C''(x)$ is the second-order derivative of C.

Apply Lemma 1 as in case 2, only this time we also have

$$\begin{aligned} D''(x_1) &= C''(x_1) - 2p(x_1)p''(x_1) - 2(p'(x_1))^2 \\ &= C''(x_1) - 2y_1y_3'' - 2(y_2')^2 = 0 \end{aligned}$$

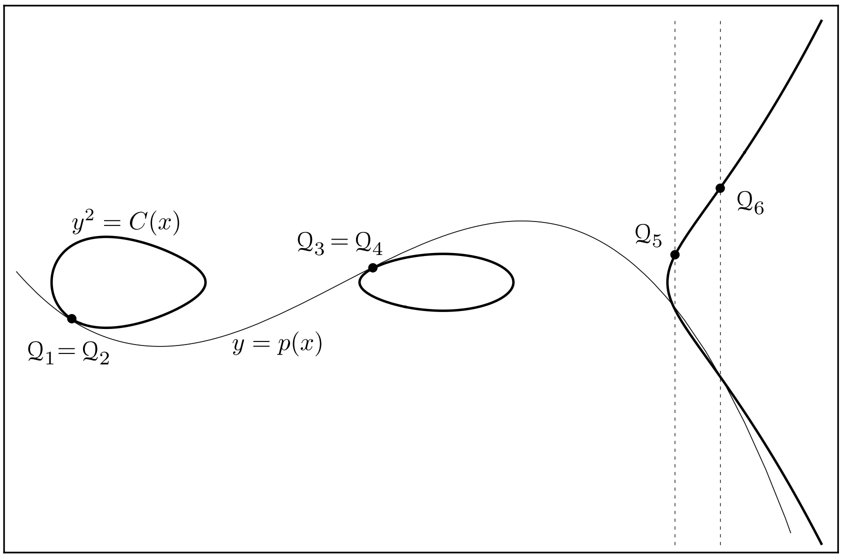

FIGURE 3: Two tangential intersection at $\mathcal{Q}_1$ and $\mathcal{Q}_3$ respectively.

as can be checked by glancing at the definition of y_3''. With $(*)$ factoring as

$$-p_3^2(x - x_1)^3 \prod_{i=4}^{6}(x - x_i)$$

or

$$(x - x_1)^3 \prod_{i=4}^{5}(x - x_i),$$

this again allows us to continue with one of the second steps of case 1 to find $\mathbf{P}_3$.

CASE 5, QUADRUPLE POINT: Given the situation where $\mathcal{Q}_i = \mathcal{Q}_1 \in H_0(\overline{K})$ for every i with $y_1 \neq 0$, we use

$$V_{111} = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_1 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

which is invertible in all fields but those of characteristic 2 and 3.

Here $\mathbf{y}_{111}$ is the same as $\mathbf{y}_{11}$ except for the last coordinate which should read

$$y_4''' = \frac{C'''(x_1)}{2y_1} - \frac{3C'(x_1)C''(x_1)}{4y_1^3} + \frac{3(C'(x_1))^3}{8y_1^5}$$

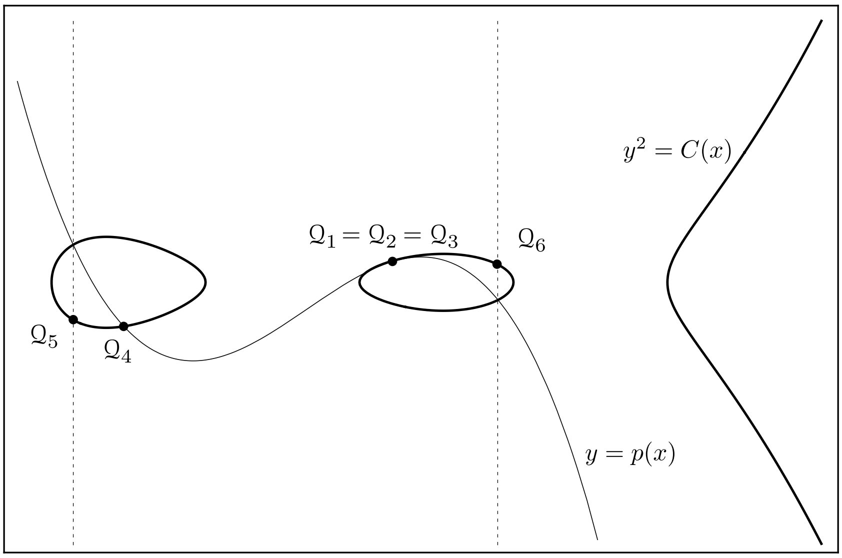

FIGURE 4: An intersection of order three at $\mathcal{Q}_1$.

where $C'''(x)$ is the third-order derivative of C. Once more, we solve the linear system $V_{111} \cdot \mathbf{p} = \mathbf{y}_{111}$. In addition to $D^{(i)}(x_1) = 0$, $i = 0, .., 2$ we have

$$D'''(x_1) = C'''(x_1) - 2y_1y_4''' - 6y_2'y_3'' = 0$$

so either $\quad D(x) = -p_3^2(x - x_1)^4(x - x_5)(x - x_6)$

or $\quad D(x) = (x - x_1)^4(x - x_5)$

which we subsequently use to solve (†) or (‡) and we're done.

$\sim$

Finally, as noted in Remark 3, (ii) we have yet to extend our definition from $H_0(\overline{K})$ to $H(\overline{K})$. Observe that this is only relevant for cases number two and four where we now consider $\mathcal{Q}_4 = \infty$ as we did in STEP $1'$ of the general case. As an analogue to this, the relevant matrices $\widetilde{V}_1$ and $\widetilde{V}_{11}$ will be the upper-left 3×3 sub-matrices of their $H_0(\overline{K})$-counterparts V_1 and V_{11}.

In both cases we obtain a linear system of the form $\widetilde{V}_* \cdot \mathbf{p} = \widetilde{\mathbf{y}}_*$ for a three-element vector $\mathbf{p}$ and the vectors $\widetilde{\mathbf{y}}_1$ and $\widetilde{\mathbf{y}}_{11}$ are defined like their 4-element counterparts $\mathbf{y}_1$ and $\mathbf{y}_{11}$ with the last coordinate omitted.

Both matrices $\widetilde{V}_*$ are invertible and we therefore get a unique polynomial $p(x)$ wich we use to solve (†′), obtaining x_5, x_6, $y_5 = p(x_5)$ and $y_6 = p(x_6)$ in $\overline{K}$ and we define $\{\mathcal{Q}_1, \mathcal{Q}_2\} + \{\mathcal{Q}_3, \infty\} = \{\mathcal{Q}_5, \mathcal{Q}_6\} = \{(x_5, -y_5), (x_6, -y_6)\}$.

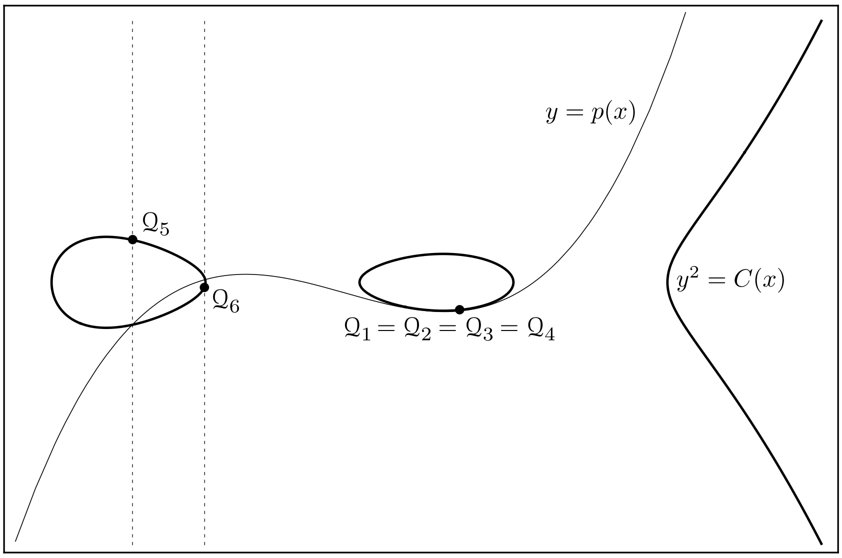

FIGURE 5: An intersection of order four at $\mathcal{Q}_1$.

1.5 Well-Definedness of the Addition Law

To check whether our addition is well defined on $\mathbf{J}$ in each of the cases, we have to consider the permutation of point-components $\mathcal{Q}_i$ under the equivalence relation from Definition 2. Furthermore, to claim that all possible cases are all covered, it is necessary to check the permutations under ($\diamondsuit$). Both cases can be combined into a single one by the following statement:

Lemma 2: In each given definition of '+', the result is invariant under the interchange of $\mathcal{Q}_i$ and $\mathcal{Q}_j$.

Proof. Simultaneously interchanging $x_i \rightleftharpoons x_j$ and $y_i \rightleftharpoons y_j$ in our linear systems $V_* \cdot \mathbf{p} = \mathbf{y}_*$ has the effect of permuting the rows of V_* and $\mathbf{y}_*$ and relabeling (x_k, y_k) whenever $\mathcal{Q}_k$ was equal to $\mathcal{Q}_i$ or $\mathcal{Q}_j$.

Consequently the resulting $p(x)$ doesn't change, so neither do any of the terms T_*. The three dagger equations remain unchanged as well, as can easily be checked at (†), (‡) and (†′) which are invariant under $x_i \rightleftharpoons x_j$.

Finally, the version of Step 2 we fall into remains the same, since it is only imposed by the distinction of p_3 being zero or not. $\square$

2 Rational Functions on Hyperelliptics

The goal of this chapter is to look at rational functions on the hyperelliptic curve $y^2 = C(x)$ and to define a notion of the order of a function in a point on the curve. We then give some basic properties for this order function before we introduce divisors and proceed to look at functions with a specified number of poles at ∞ as a preparation for the proof of associativity on $\mathbf{J}$.

2.1 Function Field and Order of Rational Functions

Definition 3: Define the ring of rational functions on the curve as

$$\mathcal{F} = K(x)[y] \Big/ \big(y^2 - C(x)\big) \cdot$$

As $y^2 - C(x)$ is irreducible in $K(x)[y]$, $\mathcal{F}$ is a field and $\mathcal{F} = K(x) + K(x)y$, so write elements $f \in \mathcal{F}$ as $f = g + hy$ where $g, h \in K(x)$. Define $\overline{f} = g - hy$.

We will occasionally write things like $K[x, y] \subset \mathcal{F}$ but this is always implicitly understood to be in conjunction with $y^2 = C(x)$.

Reminders: A formal Laurent series written $\tau = \gamma t^e(1 + \cdots) \in K(\!(t)\!)$ with $\gamma \in K^*$, $e \in \mathbb{Z}$ has a a unique inverse $\tau^{-1} = \gamma^{-1} t^{-e}(1 + \cdots) \in K(\!(t)\!)$.

From now on we will use an ellipsis to denote any terms of ascending order everywhere where we are not interested in the specifics.

If $\mathrm{char}(K) \neq 2$ and τ is a series of the form $\tau = 1 + \sum_{i=1}^{\infty} a_i t^i$ then it has a unique squareroot of the form $\sigma = 1 + \sum_{i=1}^{\infty} b_i t^i$ in $\overline{K}(\!(t)\!)$ meaning $\sigma^2 = \tau$. Write $\sigma = \sqrt{\tau} = 1 + \cdots$.

$\sim$

In order to define the order of a function f in a point $\mathcal{Q}$, $\mathrm{ord}_{\mathcal{Q}}(f) \in \mathbb{Z}$ for $f \in \mathcal{F}^*$ and $\mathcal{Q} \in H(\overline{K})$ we first construct a K-homomorphism

$$\lambda_{\mathcal{Q}} : \mathcal{F} \to \overline{K}(\!(t)\!)$$

Because $C\big(\lambda_{\mathcal{Q}}(x)\big) = \big(\lambda_{\mathcal{Q}}(y)\big)^2$ has to be fulfilled, we decide on $\lambda_{\mathcal{Q}}(x)$ and deduce $\lambda_{\mathcal{Q}}(y)$. For this, we distinguish between three cases for $\mathcal{Q} \in H(\overline{K})$.

Definition 4: 1. Let $\mathcal{Q} = (x_0, y_0) \in H_0(\overline{K})$ with $y_0 \neq 0$ and consequently $C(x_0) \neq 0$.

Define $\lambda_{\mathcal{Q}}(x) = x_0 + t$. Now

$$\begin{aligned} C(\lambda_{\mathcal{Q}}(x)) &= C(x_0 + t) \\ &= C(x_0) + \cdots + t^5 \\ &= C(x_0)(1 + \cdots) \\ &= y_0^2 \tau_1 \qquad \text{with } \tau_1 \in K(\!(t)\!). \end{aligned}$$

Define $\lambda_{\mathcal{Q}}(y) = y_0\sqrt{\tau_1}$.

2. Let $\mathcal{Q} = (x_0, y_0) \in H_0(\overline{K})$ with $y_0 = 0$. Points with this property are called "Weierstrass Points" and as $C(x_0) = 0$ there are at most five of them. Write $C(x) = \prod_{i=1}^{5}(x - \alpha_i)$ for $\alpha_i \in \overline{K}$ and for instance $\alpha_1 = x_0$.

   Define $\lambda_{\mathcal{Q}}(x) = x_0 + t^2$. Now

$$\begin{aligned} C(\lambda_{\mathcal{Q}}(x)) &= t^2 \prod_{i=2}^{5}(x_0 - \alpha_i + t^2) \\ &= \mu t^2 (1 + \cdots) \end{aligned}$$

   with $\mu \in \overline{K}$ being $\mu = \prod_{i=2}^{5}(x_0 - \alpha_i) = C'(x_0)$ which is non-zero because our curve is non-singular. Write therefore $C(\lambda_{\mathcal{Q}}(x)) = \mu t^2 \tau_2$ with $\tau_2 = 1 + \cdots$ and define $\lambda_{\mathcal{Q}}(y) = \nu t\sqrt{\tau_2}$ with $\nu^2 = \mu$.

   Note that we may choose one out of two squareroots for ν so whichever we take we naturally demand that we stay consistent in our choice.

3. Let $\mathcal{Q} = \infty$. Define $\lambda_{\mathcal{Q}}(x) = t^{-2}$. It follows that

$$\begin{aligned} C(\lambda_{\mathcal{Q}}(x)) &= t^{-10} + at^{-8} + bt^{-6} + ct^{-4} + dt^{-2} + e \\ &= t^{-10}\sqrt{\tau_3} \end{aligned}$$

   so define $\lambda_{\mathcal{Q}}(y) = t^{-5}\sqrt{\tau_3}$.

Definition 5: For $\mathcal{Q} \in H(\overline{K})$ and $f \in \mathcal{F}^*$ define $\operatorname{ord}_{\mathcal{Q}}(f) = \operatorname{ord} \lambda_{\mathcal{Q}}(f)$.

Lemma 3: Let $f \in K[x, y] \subset \mathcal{F}$, $f \neq 0$ and $\mathcal{Q} \in H_0(\overline{K})$, $\mathcal{Q} = (x_0, y_0)$. Then

(a) $\operatorname{ord}_{\mathcal{Q}} f \geq 0$ and

(b) if $f(\mathcal{Q}) = 0$ then $\operatorname{ord}_{\mathcal{Q}} f \geq 1$.

Proof.

(a) Because $\mathcal{Q} \in H_0(\overline{K})$ we have $\lambda_{\mathcal{Q}}(x)$, $\lambda_{\mathcal{Q}}(y) \in \overline{K}[[t]]$ so with $f \in K[x, y]$ we have $\operatorname{ord}_{\mathcal{Q}}(f) \in \mathbb{N}$.

(b) Write $f = A(x, y)$, $A(X, Y) \in K[X, Y]$. First, let $y_0 \neq 0$.

$$\begin{aligned}
\operatorname{ord}_{\mathcal{Q}} f &= \operatorname{ord} A(\lambda_{\mathcal{Q}}(x), \lambda_{\mathcal{Q}}(y)) \\
&= \operatorname{ord} A(x_0 + t, y_0\sqrt{\tau_1}).
\end{aligned}$$

But $A(x_0 + t, y_0\sqrt{\tau_1}) = \sum_{i=0}^{\infty} a_i t^i$ so with $t = 0$ we get $A(x_0, y_0) = a_0$ but the former is $f(\mathcal{Q})$ which is 0, so $a_0 = 0$ and the claim follows.

If $y_0 = 0$ we would have $\operatorname{ord}_{\mathcal{Q}} f = \operatorname{ord} A(x_0 + t^2, \nu t\sqrt{\tau_2})$ instead. But like before, this means $0 = f(\mathcal{Q}) = A(x_0, 0) = a_0$ so again $\operatorname{ord}_{\mathcal{Q}}(f) \geq 1$.

□

Lemma 4: Let $f \in K(x) \subset \mathcal{F}$, $f \neq 0$, $\mathcal{Q} \in H(\overline{K})$. Then

(a) $\operatorname{ord}_{\mathcal{Q}}(f) = \operatorname{ord}_{x=x_0} f(x)$ if $\mathcal{Q} = (x_0, y_0) \in H_0(\overline{K})$ with $y_0 \neq 0$.

(b) $\operatorname{ord}_{\mathcal{Q}}(f) = 2\operatorname{ord}_{x=x_0} f(x)$ if $\mathcal{Q} = (x_0, 0) \in H_0(\overline{K})$.

(c) $\operatorname{ord}_{\infty}(f) = 2\operatorname{ord}_{x=\infty} f(x) = 2\operatorname{ord}_{x=0} f(\frac{1}{x})$.

Note that the right-hand sides of the equalities refer to the usual definition of the order of a rational function in a point $x_0 \in \overline{K} \cup \{\infty\}$.

Proof. Take $f \in K[x]$ and define $e = \operatorname{ord}_{x=x_0} f \in \mathbb{N}$ so $f(x) = (x - x_0)^e g(x)$ with $g \in \overline{K}[x]$ and $g(x_0) \neq 0$.

(a) If $\mathcal{Q} = (x_0, y_0)$, $y_0 \neq 0$ then $\lambda_{\mathcal{Q}}(f) = f(x_0 + t) = t^e g(x_0 + t)$ and so $\operatorname{ord} \lambda_{\mathcal{Q}}(f) = e$ because $g(x_0 + t) = g(x_0) + \cdots$ with $g(x_0) \neq 0$.

(b) Here $\lambda_{\mathcal{Q}}(f) = f(x_0 + t^2) = t^{2e} g(x_0 + t^2)$ and again $g(x_0 + t^2) = g(x_0) + \cdots$ so $\operatorname{ord} \lambda_{\mathcal{Q}}(f) = 2e$.

(c) For $\mathcal{Q} = \infty$ we have $\lambda_{\mathcal{Q}}(f) = f(t^{-2}) = \kappa t^{-2d} + \cdots$ with $\kappa \neq 0$ if $d = \deg f$ so $\operatorname{ord}_{x=0} f(\frac{1}{x}) = -d$ and so $\operatorname{ord} \lambda_{\mathcal{Q}}(f) = 2\operatorname{ord}_{x=0} f(\frac{1}{x})$.

Generally, if $f \in K(x)$ we can write $f = \frac{r}{q}$ for $r, q \in K[x]$ and apply $\operatorname{ord}_{x=x_0} f = \operatorname{ord}_{x=x_0} r - \operatorname{ord}_{x=x_0} q$ in order to use the above on r and q. □

Lemma 5: For $f \in \mathcal{F}^*$ and $\mathcal{Q} \in H(\overline{K})$ the order satisfies $\operatorname{ord}_{\mathcal{Q}}(\overline{f}) = \operatorname{ord}_{\overline{\mathcal{Q}}}(f)$

Proof. Split into three possible cases:

1. For $\mathcal{Q} \in H_0(\overline{K})$ with $y_0 \neq 0$ we've got $\lambda_{\mathcal{Q}}(x) = x_0 + t$ and $\lambda_{\mathcal{Q}}(y) = y_0\sqrt{\tau_1}$. Therefore $\lambda_{\overline{\mathcal{Q}}}(x) = x_0 + t = \lambda_{\mathcal{Q}}(\overline{x})$ and $\lambda_{\overline{\mathcal{Q}}}(y) = -y_0\sqrt{\tau_1} = \lambda_{\mathcal{Q}}(\overline{y})$ so

$$\begin{aligned}
\lambda_{\overline{\mathcal{Q}}}(f(x, y)) &= f(\lambda_{\overline{\mathcal{Q}}}(x), \lambda_{\overline{\mathcal{Q}}}(y)) \\
&= \lambda_{\mathcal{Q}}(f(\overline{x}, \overline{y})) \\
&= \lambda_{\mathcal{Q}}(\overline{f}(x, y)).
\end{aligned}$$

2. If $\mathcal{Q} \in H_0(\overline{K})$ with $y_0 = 0$ then $\overline{\mathcal{Q}} = \mathcal{Q}$. Write $f(x) = g(x) + h(x)y$, so
$$\lambda_{\mathcal{Q}}(\overline{f}) = g(x_0 + t^2) - h(x_0 + t^2)\nu t\sqrt{\tau_2}.$$
Calling this $l(t) = \lambda_{\mathcal{Q}}(\overline{f})$ and looking at the construction of $\lambda_{\mathcal{Q}}$ we see that τ_2 sports only even powers of t so we see above that $l(-t) = \lambda_{\mathcal{Q}}(f)$. As interchanging t with $-t$ doesn't change the order, we're done.

3. For $\mathcal{Q} = \infty$, $\tau_3 = 1 + at^2 + bt^4 + ct^6 + dt^8 + et^{10}$ features only even powers as well, so again $l(-t) = \lambda_{\mathcal{Q}}(f)$ for $l(t) = \lambda_{\mathcal{Q}}(\overline{f}) = g(t^{-2}) - h(t^{-2})t^{-5}\sqrt{\tau_3}$. Finally, $\lambda_{\mathcal{Q}}(f)$ is equal to $\lambda_{\overline{\mathcal{Q}}}(f)$ since $\infty = \overline{\infty}$. Again $\operatorname{ord} l(t) = \operatorname{ord} l(-t)$.

$\square$

Lemma 6: If $f \in \mathcal{F}^*$ then the set $\left\{\mathcal{Q} \in H(\overline{K}) \,\middle|\, \operatorname{ord}_{\mathcal{Q}} f \neq 0\right\}$ is finite and
$$\sum_{\mathcal{Q} \in H(\overline{K})} \operatorname{ord}_{\mathcal{Q}} f = 0.$$

Proof. First take $f \in K[x,y]$, $f \neq 0$ and let $\mathcal{Q} = (x_0, y_0) \in H_0(\overline{K})$ with $\operatorname{ord}_{\mathcal{Q}} f \neq 0$. By Lemma 3 (a) we know that $\operatorname{ord}_{\mathcal{Q}} f \geq 1$. It follows that $\operatorname{ord}_{\mathcal{Q}} (f\overline{f}) = \operatorname{ord}_{\mathcal{Q}} f + \operatorname{ord}_{\mathcal{Q}} \overline{f} \geq 1$. Now since $f = g + hy$, $f\overline{f} = g^2 - h^2C(x)$ which lies in $K[x]$, so by Lemma 4 we have $\operatorname{ord}_{x=x_0} (f\overline{f}) = \operatorname{ord}_{\mathcal{Q}} (f\overline{f}) \geq 1$. But there are only finitely many such x_0 and so only finitely many $y_0 = \pm\sqrt{C(x_0)}$.

If $f \in K(x,y)$, $f = \frac{r}{q}$, $r, q \in K[x,y]$ then $\operatorname{ord}_{x=x_0} f = \operatorname{ord}_{x=x_0} r - \operatorname{ord}_{x=x_0} q$, and again only finitely many x_0 exist for which this differs from zero.

For the second claim, give our sum the name
$$s(f) = \sum_{\mathcal{Q} \in H(\overline{K})} \operatorname{ord}_{\mathcal{Q}} f$$
and note that $s(f) = s(\overline{f})$ due to Lemma 5 and the fact that we take the sum over all $\mathcal{Q}$. Because $\operatorname{ord}_{\mathcal{Q}} (f\overline{f}) = \operatorname{ord}_{\mathcal{Q}} (f) + \operatorname{ord}_{\mathcal{Q}} (\overline{f})$ we can see that
$$s(f\overline{f}) = s(f) + s(\overline{f}) = 2s(f).$$
But because $f\overline{f} \in K(x)$ we can use Lemma 4 to write this out as
$$\begin{aligned}
2s(f) &= \sum_{\mathcal{Q} \in H(\overline{K})} \operatorname{ord}_{\mathcal{Q}} f\overline{f} \\
&= 2 \sum_{\substack{x_0 \neq \infty \\ C(x_0) \neq 0}} \operatorname{ord}_{x=x_0} f\overline{f} + 2 \sum_{C(x_0)=0} \operatorname{ord}_{x=x_0} f\overline{f} + 2 \sum_{x_0=\infty} \operatorname{ord}_{x=x_0} f\overline{f} \\
&= 2 \sum_{x_0 \in \overline{K} \cup \{\infty\}} \operatorname{ord}_{x=x_0} f\overline{f}.
\end{aligned}$$

Here α_i are the points on which $C(x)$ vanishes and since

$$\sum_{x_0 \in \overline{K} \cup \{\infty\}} \operatorname{ord}_{x=x_0} g = 0$$

for any $g \in K(x)$ we have $s(f) = 0$ in $\mathbb{Z}$. $\square$

2.2 Divisors and Lemmas

Definition 6: The divisor of a function $f \in \mathcal{F}^*$ is the formal sum

$$(f) = \sum_{\mathcal{Q} \in H(\overline{K})} \operatorname{ord}_{\mathcal{Q}} f \cdot \mathcal{Q}$$

Thanks to Lemma 6 the sum is finite and the sum of all coefficients is 0.

Points $\mathcal{Q} \in H(\overline{K})$ with a positive coefficient in (f) are called zeroes of f while those with a negative coefficient are called poles.

Lemma 7: If $f \in \mathcal{F}^*$ has no poles on $H(\overline{K})$, i.e. if $\operatorname{ord}_{\mathcal{Q}} f \geq 0$ for all $\mathcal{Q}$, then f is constant.

Proof. With $f = g + hy$, $\operatorname{ord}_{\mathcal{Q}} f \geq 0$ for every $\mathcal{Q} \in H(\overline{K})$ we take a look at $f + \overline{f} = 2g \in K(x)$ and $f\overline{f} = g^2 - h^2 C \in K(x)$ and observe that by well-known properties of orders

$$\begin{aligned} \operatorname{ord}_{\mathcal{Q}} (f + \overline{f}) &\geq \min\{\operatorname{ord}_{\mathcal{Q}} f, \operatorname{ord}_{\mathcal{Q}} \overline{f}\} \\ &= \min\{\operatorname{ord}_{\mathcal{Q}} f, \operatorname{ord}_{\overline{\mathcal{Q}}} f\} \geq 0. \end{aligned}$$

with Lemma 5 and similarly

$$\operatorname{ord}_{\mathcal{Q}} (f\overline{f}) = \operatorname{ord}_{\mathcal{Q}} f + \operatorname{ord}_{\overline{\mathcal{Q}}} f \geq 0.$$

With the help of Lemma 4 we conclude that $\operatorname{ord}_{x=x_0} (f + \overline{f}) \geq 0$ and $\operatorname{ord}_{x=x_0} (f\overline{f}) \geq 0$ respectively.

But in $\overline{K}(x)$, a function q with $\operatorname{ord}_{x=x_0} q \geq 0$ for every $x_0 \in \overline{K} \cup \{\infty\}$ must be constant, so both $f + \overline{f}$ and $f\overline{f}$ are constant functions. Since f is a root of $(T - f)(T - \overline{f}) = T^2 - (f + \overline{f})T + f\overline{f} \in \overline{K}[T]$, f lies in $\overline{K}$. $\square$

Theorem 1: Let K be a field and x, $y \in K(t)$ rational functions such that $y^2 = C(x)$. Then x and y are actually constants, i.e. x, $y \in K$.

Lemma 8: Suppose $f \in \mathcal{F}^*$ has a pole of order at most one at ∞ i.e. $\operatorname{ord}_{\infty} f \geq -1$ and no pole at any other $\mathcal{Q} \in H_0(\overline{K})$. Then f is a constant.

Proof. By Lemma 7 we can assume $\operatorname{ord}_\infty f = -1$ and $\operatorname{ord}_\mathcal{Q} f \geq 0$ for every $\mathcal{Q} \in H_0(\overline{K})$. Now $f = g + hy$ and $\lambda_\infty(f) = \kappa t^{-1} + \cdots$ with $\kappa \neq 0$ in $\overline{K}$. Since we are only interested in the order, replace f with $\kappa^{-1} f$ so $\lambda_\infty(f) = t^{-1} + \cdots$.

Now remember that $\lambda_\infty(x) = t^{-2}$ so $\lambda_\infty(x - f^2) = \alpha t^{-1} + \cdots$, $\alpha \in \overline{K}$ and finally we have a regular power series $\lambda_\infty(x - f^2 - \alpha f)$ so

$$\operatorname{ord}_\infty (x - f^2 - \alpha f) \geq 0.$$

Also for any other $\mathcal{Q} \in H_0(\overline{K})$ we have

$$\operatorname{ord}_\mathcal{Q} (x - f^2 - \alpha f) \geq \min\{\operatorname{ord}_\mathcal{Q} (x), \operatorname{ord}_\mathcal{Q} (f^2), \operatorname{ord}_\mathcal{Q} f\}$$

which is non-negative by virtue of the prerequisite on f and $\lambda_\mathcal{Q}(x) = x_0 + \cdots$. The previous Lemma now implies $x - f^2 - \alpha f \in \overline{K}$ so we see $x = f^2 + \alpha f + \beta$ as a polynomial $x = X(f)$ with $X(T) \in \overline{K}[T] \setminus \overline{K}$.

Do the same thing with $\quad \lambda_\infty(y) = t^{-5}\sqrt{\tau_3} = t^{-5} + \cdots$

so $\qquad \lambda_\infty(y - f^5) = \beta_4 t^{-4} + \cdots$

and $\qquad \lambda_\infty(y - f^5 - \beta_4 f^4) = \beta_3 t^{-3} + \cdots$

and so on, so $\lambda_\infty(y - f^5 - \beta_4 f^4 - \beta_3 f^3 - \beta_2 f^2 - \beta_1 f)$ is a power series as well. Again $y = f^5 + \beta_4 f^4 + \beta_3 f^3 + \beta_2 f^2 + \beta_1 f + \beta_0 = Y(f)$ with $Y(T) \in \overline{K}[T] \setminus \overline{K}$.

Combined we have the polynomial equation $Y(f)^2 - C(X(f)) = 0$ over $\overline{K}$ whose algebraic closure dictates either $f \in \overline{K}$ or $Y(T)^2 = C(X(T))$. The latter can't be true by Theorem 1 and the former is in contradiction to $\operatorname{ord}_\infty f = -1$. $\square$

Lemma 9: If $f \in \mathcal{F}^*$ has a pole of order at most two at ∞ and $\operatorname{ord}_\mathcal{Q} f \geq 0$ for every $\mathcal{Q} \in H_0(\overline{K})$ then f is of the form $f = \alpha + \beta x$ with $\alpha, \beta \in K$.

Proof. Because $\lambda_\infty(f) = \beta t^{-2} + \cdots$ where $\beta \in \overline{K}$ we have $\lambda_\infty(f - \beta x) = \gamma t^{-1} + \cdots$ and thanks to Lemma 8 we know this to imply $f - \beta x = \alpha$. $\square$

Lemma 10: If $f \in \mathcal{F}^*$ has a pole of order at most three at ∞ and $\operatorname{ord}_\mathcal{Q} f \geq 0$ for every $\mathcal{Q} \in H_0(\overline{K})$ then f is of the form $f = \alpha + \beta x$ with $\alpha, \beta \in K$.

Proof. We can see that for instance for α_1 one of the roots of $C(x)$ we have

$$\begin{aligned} \lambda_\infty\left(\frac{y}{x - \alpha_1}\right) &= (t^{-2} - \alpha_1)^{-1} t^{-5}\sqrt{\tau_3} \\ &= t^2(1 + \cdots)^{-1} t^{-5}(1 + \cdots) \\ &= t^{-3}(1 + \cdots). \end{aligned}$$

So if $\lambda_\infty(f) = \gamma t^{-3} + \cdots$ then per Lemma 9 we have

$$f - \gamma \frac{y}{x - \alpha_1} = \alpha + \beta x.$$

But now for $\mathcal{Q} = (\alpha_0, 0)$ we have

$$\begin{aligned} \lambda_{\mathcal{Q}}(f) &= \alpha + \beta(\alpha_1 + t^2) + \gamma \frac{\nu t \sqrt{\tau}_2}{t^2} \\ &= \gamma \nu t^{-1} + \cdots \end{aligned}$$

so $\gamma\nu$ must be 0. Since $\nu = C'(x_0) \neq 0$ and f does not have any poles in $H_0(\overline{K})$, γ must be zero and so $f = \alpha + \beta x$.

□

Lemma 11: If $f \in \mathcal{F}^*$ has a pole of order at most four at ∞ then f is of the form $f = \alpha + \beta x + \gamma x^2$ with $\alpha, \beta, \gamma \in K$.

Proof. With $\lambda_\infty(f) = \gamma t^{-4} + \cdots$ we know that $\lambda_\infty(f - \gamma x^2) = \delta t^{-3} + \cdots$ and we use the previous Lemma to see that $f - \gamma x^2 = \alpha + \beta x$. □

3 Associativity

Lemma 12: Suppose we have $\mathcal{Q}_i = (x_i, y_i)$, $\mathcal{Q}_i \in H(\overline{K})$, that fulfill

$$\{\mathcal{Q}_1, \mathcal{Q}_2\} + \{\mathcal{Q}_3, \mathcal{Q}_4\} = \{\overline{\mathcal{Q}}_5, \overline{\mathcal{Q}}_6\}$$

with $\mathcal{Q}_j = \infty$ for at most one $j \in \{1, .., 6\}$. This means that we are in one of the cases 1–5 of the addition law which involved passing a polynomial p through all $\mathcal{Q}_i \neq \infty$. Now remember that we defined $D(x) = C(x) - p^2(x)$.

If exactly e of the $\mathcal{Q}_i$ are equal, say to some $\mathcal{Q}_k$ for a $k \in \{1, .., 6\}$, then

$$D(x) = (x - x_k)^e g(x)$$

with $g(x_k) \neq 0$. This is of course equivalent to stating $\mathrm{ord}_{x_k} D(x) = e$.

Proof. By construction we have

$$D(x) = -p_3^2 \prod_{i=1}^{6} (x - x_i)$$

if $p_3 \neq 0$. If $p_3 = 0$ and therefore $\mathcal{Q}_j = \infty$ for exactly one $j \in \{1, .., 6\}$ then

$$D(x) = \prod_{\substack{i=1 \\ i \neq j}}^{6} (x - x_i).$$

So by construction, $\mathcal{Q}_i$ being equal to some $\mathcal{Q}_k$ contributes a factor $x - x_k$. $\square$

Theorem 2: Let $\mathbf{P}_i \in \mathbf{J}$, $\mathbf{P}_i = \{\mathcal{Q}_1, \mathcal{Q}_2\}$, $\mathbf{P}_2 = \{\mathcal{Q}_3, \mathcal{Q}_4\}$ and $\mathbf{P}_3 = \{\overline{\mathcal{Q}}_5, \overline{\mathcal{Q}}_6\}$,

$$\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_3.$$

Then there exists an $f \in \mathcal{F}^*$ with divisor

$$(f) = \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 + \mathcal{Q}_4 - \overline{\mathcal{Q}}_5 - \overline{\mathcal{Q}}_6 - 2\infty. \qquad (\star)$$

Remark 4: Looking back at the construction of $\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_3$ we used the equivalence $\mathcal{Q}_i = \mathcal{Q}_j \iff x_i = x_j$, $\mathcal{Q}_i, \mathcal{Q}_j \in H(\overline{K})$, $x_i, x_j \in \overline{K} \cup \{\infty\}$ for $i, j \leq 4$ whenever not in the zero case. We can now easily see that this extends to $i, j \in \{1, .., 6\}$. First note that at most one of the six $\mathcal{Q}_i$ may equal ∞. By construction, the polynomial $p(x)$ passes through all those $\mathcal{Q}_i$ that are in $H_0(\overline{K})$. This implies that if $\mathcal{Q}_i = \overline{\mathcal{Q}}_j$ we must have $-y_i = y_j = p(x_j) = p(x_i) = y_i$ so $y_i = 0$ in $\mathrm{char}(K) \neq 2$ i.e. $\overline{\mathcal{Q}}_i = \mathcal{Q}_j = \mathcal{Q}_i$.

Proof. First part

Note that if $\mathbf{P}_2 = 0$ then we want $(f) = \mathcal{Q}_1 + \mathcal{Q}_2 + 2\infty - \mathcal{Q}_1 - \mathcal{Q}_2 - 2\infty = 0$, a condition that is fulfilled by $f = 1$.

In all the other cases we have the polynomial $p(x)$ at our disposal. We can now assume that we are not in the zero case. Since $y_i = p(x_i)$ for all i except possibly $i = 6$, we define

$$q = y - p(x) \in K[x, y] \subset \mathcal{F}^*$$

with the intention of showing that $(q) = \sum_{i=1}^{6} \mathcal{Q}_i - 6\infty$.

By Lemma 3 (b) we know that $\operatorname{ord}_{\mathcal{Q}_i} q \geq 1$ for those $\mathcal{Q}_i$ that are not ∞.

Let's first assume that all $\mathcal{Q}_i \in H_0(\overline{K})$. This means that we are in the case where $p(x)$ is of degree 3, so as $\lambda_\infty(x) = t^{-2}$ we see that $\operatorname{ord}_\infty q = -6$. As $q \in K[x, y]$, the only negative order may be at ∞ which implies $\operatorname{ord}_{\mathcal{Q}} q = 0$ for all $\mathcal{Q} \in H_0(\overline{K})$ that are not one of the $\mathcal{Q}_i$.

With Lemma 12, if e_i of the $\mathcal{Q}_i$ are equal and $y_i \neq 0$ then with Lemma 4 (a)

$$\begin{aligned}
\operatorname{ord}_{\mathcal{Q}_i} q + \operatorname{ord}_{\mathcal{Q}_i} \overline{q} &= \operatorname{ord}_{\mathcal{Q}_i} q\overline{q} \\
&= \operatorname{ord}_{\mathcal{Q}_i} D(x) \\
&= e_i.
\end{aligned}$$

But by Remark 4 we know that $\operatorname{ord}_{\mathcal{Q}_i} \overline{q} = 0$ as $q(\overline{\mathcal{Q}}_i)$ can't be zero if $\overline{\mathcal{Q}}_i \neq \mathcal{Q}_i$, so $\operatorname{ord}_{\mathcal{Q}_i} q = e_i$. In the case where $y_i = 0$, the same lemmas imply

$$\operatorname{ord}_{\mathcal{Q}_i} q + \operatorname{ord}_{\mathcal{Q}_i} \overline{q} = 2e_i$$

and since $\overline{\mathcal{Q}}_i = \mathcal{Q}_i$, Lemma 5 gives $\operatorname{ord}_{\mathcal{Q}_i} \overline{q} = \operatorname{ord}_{\mathcal{Q}_i} q$ so again $\operatorname{ord}_{\mathcal{Q}_i} q = e_i$.

So in both cases, if $\mathcal{Q}_i$ appears e_i times in the sum $\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_3$ then it appears the same number of times in (q) and all in all

$$(q) = \sum_{\{\mathcal{Q}_i\}} e_i \mathcal{Q}_i - 6\infty = \sum_{i=1}^{6} \mathcal{Q}_i - 6\infty.$$

Now consider the case where $\mathcal{Q}_k = \infty$ for some k. Because $p(x)$ is now of degree two or less and $\lambda_\infty(y) = t^{-5}\sqrt{\tau}$ we have $\operatorname{ord}_\infty q = -5$. This leads to

$$(q) = \sum_{\substack{i=1 \\ i \neq k}}^{6} \mathcal{Q}_i - 5\infty = \sum_{i=1}^{6} \mathcal{Q}_i - 6\infty$$

completing the first part of our proof.

Second Part

The divisor of $x-x_i$ is $(x-x_i) = \mathcal{Q}_i + \overline{\mathcal{Q}}_i - 2\infty$ both for $\mathcal{Q}_i = \overline{\mathcal{Q}}_i$ and $\mathcal{Q}_i \neq \overline{\mathcal{Q}}_i$.

Without loss of generality let $\mathcal{Q}_5 \neq \infty$ and distinguish between $\mathcal{Q}_6 \neq \infty$ and $\mathcal{Q}_6 = \infty$. In the first case it is easy to check that for $f = q(x-x_5)^{-1}(x-x_6)^{-1}_{,}$

$$\begin{aligned}(f) &= \sum_{i=1}^{6} \mathcal{Q}_i - 6\infty - \mathcal{Q}_5 - \overline{\mathcal{Q}}_5 - \mathcal{Q}_6 - \overline{\mathcal{Q}}_6 + 4\infty \\ &= \sum_{i=1}^{4} \mathcal{Q}_i - \overline{\mathcal{Q}}_5 - \overline{\mathcal{Q}}_6 - 2\infty\end{aligned}$$

which corresponds precisely to the statement we aimed to prove. If $\mathcal{Q}_6 = \infty$, we can define $f = q(x-x_5)^{-1}$ and check that

$$\begin{aligned}(f) &= \sum_{i=1}^{5} \mathcal{Q}_i - 5\infty - \mathcal{Q}_5 - \overline{\mathcal{Q}}_5 + 2\infty \\ &= \sum_{i=1}^{4} \mathcal{Q}_i - \overline{\mathcal{Q}}_5 - \overline{\infty} - 2\infty.\end{aligned}$$

So the function f that we were looking for is

$$f = \frac{y - p(x)}{(x-x_5)(x-x_6)}$$

if $\mathcal{Q}_5$ and $\mathcal{Q}_6$ are both different from ∞ and otherwise if $\mathcal{Q}_6 = \infty$ then

$$f = \frac{y - p(x)}{x - x_5}.$$

□

Theorem 3: *Associativity.* The sums

$$\begin{aligned}\underbrace{(\mathbf{P}+\mathbf{Q})}_{=\mathbf{T}} + \mathbf{R} &= \mathbf{S} \\ \text{and}\quad \mathbf{P} + \underbrace{(\mathbf{Q}+\mathbf{R})}_{=\mathbf{W}} &= \mathbf{S}'\end{aligned}$$

give the same result, namely $\mathbf{S} = \mathbf{S}'$.

Proof. We will use the following notation:

$$\begin{gathered}\mathbf{P} = \{\mathcal{P}_1, \mathcal{P}_2\}, \mathbf{Q} = \{\mathcal{Q}_1, \mathcal{Q}_2\}, \mathbf{R} = \{\mathcal{R}_1, \mathcal{R}_2\}, \mathbf{T} = \{\mathcal{T}_1, \mathcal{T}_2\}, \mathbf{W} = \{\mathcal{W}_1, \mathcal{W}_2\}, \\ \mathbf{S} = \{\mathcal{A}, \mathcal{B}\}, \mathbf{S}' = \{\mathcal{U}, \mathcal{V}\}, \\ \mathcal{A} = (a, *), \mathcal{B} = (b, *).\end{gathered}$$

Thanks to the previous theorem we have functions $f_{\mathbf{PQ}}$, $f_{\mathbf{TR}}$, $f_{\mathbf{PW}}$ and $f_{\mathbf{QR}}$ in $\mathcal{F}^*$ such that the divisor of

$$f = \frac{f_{\mathbf{PQ}} f_{\mathbf{TR}}}{f_{\mathbf{PW}} f_{\mathbf{QR}}}$$

is

$$\begin{aligned}(f) = {} & \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{T}_1 - \mathcal{T}_2 - 2\infty \\ & + \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{R}_1 + \mathcal{R}_2 - \mathcal{A} - \mathcal{B} - 2\infty \\ & - \mathcal{P}_1 - \mathcal{P}_2 - \mathcal{W}_1 - \mathcal{W}_2 + \mathcal{U} + \mathcal{V} + 2\infty \\ & - \mathcal{Q}_1 - \mathcal{Q}_2 - \mathcal{R}_1 - \mathcal{R}_2 + \mathcal{W}_1 + \mathcal{W}_2 + 2\infty \\ = {} & \mathcal{U} + \mathcal{V} - \mathcal{A} - \mathcal{B}.\end{aligned}$$

This means that $\widetilde{f} = (x-a)(x-b)f$ has divisor $(\widetilde{f}) = \overline{\mathcal{A}} + \overline{\mathcal{B}} + \mathcal{U} + \mathcal{V} - 4\infty$. By Lemma 11, $\widetilde{f}$ must be of the form $\kappa(x-\varrho)(x-\varsigma) \in \overline{K}[x]$ and so

$$f = \frac{\kappa(x-\varrho)(x-\varsigma)}{(x-a)(x-b)}.$$

But this has divisor of the form

$$(f) = \mathcal{M} + \overline{\mathcal{M}} + \mathcal{N} + \overline{\mathcal{N}} - \mathcal{A} - \overline{\mathcal{A}} - \mathcal{B} - \overline{\mathcal{B}}.$$

So, without loss of generality, either $\mathcal{A} = \mathcal{M}$ and $\mathcal{B} = \overline{\mathcal{M}}$ or $\mathcal{A} = \mathcal{M}$ and $\mathcal{B} = \mathcal{N}$. First case implies $\mathbf{S}$ and $\mathbf{S}'$ equals 0, second case implies $(f) = \mathcal{A} + \mathcal{B} - \mathcal{A} - \mathcal{B}$ and in both cases we're done. $\square$