classmate

Date

Page

DISCRETE STRUCTURE

Introduction

Discrete Mathematics deals with descrete objects. Discrete object are those objects that can be counted. For example:- Trees, integeres, houses, etc.

Logic

Logic is a language for resoning. Since logic can help us to reason the mathematical models. It need some rules associated with logic. So that we can apply those rules for mathematical resoning. Application maybe for designing circuits, programing, program verification, etc.

Propositions and Proportional Calculus

→ Proposition is a fundamental concept of logic.

→ Proposition is a declarative sencentence that is either true or false, but noth both.

e.g. $2+2=5$ (false) ; $7-1=6$ (true) ; Kathmandu is capital city of Nepal (true) are some of the examples of proposition.

$x>5$; come here ; $3+4$ are some examples which are not propositions.

→ Propositions are denoted by using small letters like p, q, r, s... The truth value of proposition is denoted by T for true proposition and F for false proposition.

→ The logic that deals with proposition is called propositional logic or propositional calculus.

Logical Operators / Connectives

→ Logical operators are used to connect mathematical statements having one or more propositions by combining the propositions.

→ The combinal propositional is called compound proposition.

→ The truth table is used to get the relationship between truth values of propositions.

1. Negation (NOT)

Gives a proposition p, negation operator (¬) is used to get negation of p, denoted by ¬p called "not p".

Eg:- p = I love birds.
¬p = I don't love birds.

p	¬p
T	F
F	T

2. Conjuction (AND)

Gives two propositions p and q, the proposition "p ∧ q" denoted by p^q is the proposition that is true whenevor p and q are true, false otherwise.

Eg:- p = "Ram is intelligent"
q = "Ram is diligent"
p^q = Ram is intelligent and diligent.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

3. Disjunction (OR):

Given two propositions p and q, the propositions "p OR q" denoted by $p \vee q$ is the proposition that is false whenever both the proposition p and q are false, true otherwise.

Eg:- p = "Ram is intelligent"
q = "Ram is deligent"
$p \vee q$ = "Ram is intelligent or deligent".

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Construct a truth table

1. $\neg p \wedge (p \vee \neg q)$

p	q	$\neg p$	$\neg q$	$(p \vee \neg q)$	$\neg p \wedge (p \vee \neg q)$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	F
F	F	T	T	T	T

11. $(p \wedge q) \vee (\neg q \wedge r)$

p	q	r	$\neg q$	$p \wedge q$	$\neg q \wedge r$	$(p \wedge q) \vee (\neg q \wedge r)$
T	T	T	F	T	F	T
T	T	F	F	T	F	T
T	F	T	T	F	T	T
T	F	F	T	F	F	F
F	T	T	F	F	F	F
F	T	F	F	F	F	F
F	F	T	T	F	T	T
F	F	F	T	F	F	F

4. Exclusive Or (XOR)

Given two propositions p and q, the proposition exclusive or of p and q denoted by $p \oplus q$ is the propositions that is true whenever only one of the propositions p and q is true, false otherwise.

Eg:- p: "Ram drinks coffee in the morning"

q = "Ram drinks coffee in the morning".

$p \oplus q$ = Ram drinks tea or coffee in the morning

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

5. Implication

Given two propositional p and q, the proposition implication $p \to q$, is the proposition that is false when p is true and q is false, true otherwise. Here p is called "Hypothesis" or "antecedent" or "premise" and q is called "conclusive" or "consequence".

Different terminologies to express $p \to q$ are like:-

1. "if p then q"
2. "q is consequence of p"
3. "p is sufficient of q"
4. "q if p"
5. "q is necessary for p"
6. "q follows from p"
7. "if p, q"
8. "p implies q"
9. "p only if q"
10. "q whenever p"
11. "q provides p"

P	q	$P \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Contrapositive, Inverse and Converse

Some of related implication formed from $p \to q$ are:-

$$\boxed{\begin{array}{l} \text{Converse}: q \to p \\ \text{Inverse}: \neg p \to \neg q \\ \text{Contrapositive}: \neg q \to \neg p \end{array}}$$

P = "today is sunday"
q = "it is not today"

implication = If today is sunday, it is hot.
Converse :- It is hot today only if today is sunday.
Inverse :- If today is not sunday, it is not hot.
Contrapositive :- If it is not hot today, it is not sunday.

Is contrapositive same as $p \to q$? Verify.

p	q	$p \to q$	$\neg p$	$\neg q$	$\neg q \to \neg p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

$$p \to q = \neg q \to \neg p$$

Q. Let p, q and r be the positions.

p = "You have the flu"
q = "You ~~have~~ miss the final examination"
r = "You pass the course".

Express each of the proportion as an English Sentence and construct the truth table

① $p \to q$
If You have flue, you miss final examination.

(ii) $q \to \neg r$

If you miss the final exam, you will not pass the course.

(iii) $(p \to \neg r) \vee (q \to \neg r)$

If you have the flu, you will not pass the course or if you miss final exam, you will not pass the course.

p	q	r	$\neg r$	$p \to \neg r$	$q \to \neg r$	$p \to q$	$(p \to \neg r) \vee (q \to \neg r)$
T	T	T	F	F	F	T	F
T	T	F	T	T	T	T	T
T	F	T	F	F	T	F	T
T	F	F	T	T	T	F	T
F	T	T	F	T	F	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	T	T	T	T	T

Note:-

To translate english sentence to the proposition symbolic form, follows these steps:-

1. Restart the given sentence into building block sentence.
2. Give the symbol to each sentence and substitute the symbol using connectives:-

Q. If it is snowing then I will go to the beach.

$p \to$ "It is snowing" $q =$ "I will go to beach"

$\Rightarrow p \to q$

q if p $p \to q$

p p if q

Q. You can access the internet from campus, if you are a computer science major or you are not the freshman.

$\Rightarrow$

p = You can access the internet from campus.
q = You are a computer science major
r = You are the freshman

$\Rightarrow (q \vee \neg r) \to p$

Construct the truth table:

$((p \leftrightarrow q) \oplus (\neg p \to q)) \vee (q - \neg r)$ (*)

p	q	r	$\neg p$	$\neg r$	$p \leftrightarrow q$	$\neg p \to q$	$q \to \neg r$	(*)
T	T	T	F	F	T	T	F	T
T	T	F	F	T	T	T	F	T
T	F	T	F	F	F	T	T	T
T	F	F	F	T	F	T	T	T
F	T	T	T	F	F	T	T	T
F	T	F	T	T	F	T	F	T
F	F	T	T	F	T	F	T	T
F	F	F	T	T	T	F	T	T

6. Biconditional

Given propositions p and q, the biconditional $p \longleftrightarrow q$ is a proposition that is true when p & q have same truth values. Alternatively $p \longleftrightarrow q$ is true whenever both $p \rightarrow q$ and $q \rightarrow p$ are true. Some of the technologies used for biconditional are:-

1. "p if and only if q"
2. "if p then q and coversly"
3. "p is necessary and sufficient for q".

Eg:- p = "today is sunday"
q = "it is hot today"

$p \longleftrightarrow q$: today is sunday if and only if it is hot day.

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \longleftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

let p, q and r be the proposition with truth value T, F, T respectively. Evaluate the following:

i) $\neg r \vee \neg(p \vee q)$

ii) $\neg(p \vee q) \wedge (\neg r) \vee q$

i) $\neg r \vee \neg(p \vee q)$

p	q	r	$\neg r$	$p \vee q$	$\neg(p \vee q)$	$\neg r \vee \neg(p \vee q)$
T	F	T	F	F	T	F

ii) $\neg(p \vee q) \wedge (\neg r) \vee q$

p	q	r	$\neg r$	$p \vee q$	$\neg(p \vee q)$	$(\neg r) \vee q$	$\neg(p \vee q) \wedge (\neg r) \vee q$
T	F	T	F	F	T	F	F

Tautology and Contradiction

A compound proposition that is always true, no matter what the truth values of the atomic propositions that contain it is called tautology.

For example:-

$p \vee \neg p$ is always true verify

p	$\neg p$	$p \vee \neg p$
T	F	T
T	F	T
F	T	T
F	T	T

A compound proposition that is always false is called contradiction. For eg:- $p \wedge \neg p$.

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

A compound proposition that is neither a tautology nor a contradiction is called contingency.

Logical Equivalences:

The compound propositions p and q are logically equivalent denoted by $p \Leftrightarrow q$ or $p \equiv q$, if proposition $p \leftrightarrow q$ is tautology.

Some important logical equivalences:-

1. $p \wedge T \Leftrightarrow p$ — Identity Law
2. $p \vee F \Leftrightarrow p$ — "
3. $p \wedge F \Leftrightarrow F$ — Domination Law
4. $p \vee T \Leftrightarrow T$ — "
5. $p \wedge p \Leftrightarrow p$ — Idempotent law
6. $p \vee p \Leftrightarrow p$ — "
7. $\neg(\neg p) \Leftrightarrow p$ — Double negation law
8. $p \wedge q \Leftrightarrow q \wedge p$ — Commutative law
9. $p \vee q \Leftrightarrow q \vee p$ — "
10. $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ — Associative law
11. $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ — "
12. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ — Distributive law
13. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ — "
14. $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ — De-Morgan's Law
15. $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ — "

16. $p \wedge \neg p \Leftrightarrow F$ Trival Tautology

17. $p \vee \neg p \Leftrightarrow T$ "

18. $p \to q \Leftrightarrow \neg p \vee q$ Implication

19. $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$ Equivalence

20. $(p \wedge q) \to r \Leftrightarrow p \to (q \to r)$ Exportation

21. $(p \to q) \wedge (p \to \neg q) \Leftrightarrow \neg p$ Absurdity

22. $p \to q \Leftrightarrow \neg q \Leftrightarrow \neg p$ Contrapositive

23. $p \wedge (p \vee q) \Leftrightarrow p$ Absorption

24. $p \vee (p \wedge q) \Leftrightarrow p$ "

① Truth Table

② Symbolic Derivation

12. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

$p \wedge (q \vee r)$

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

$(p \wedge q) \vee (p \wedge r)$

p	q	r	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

Hence, $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

Prove that:- $(p \to q) \wedge (p \to \neg q) \equiv \neg p$ by the help of truth table.

p	q	$\neg q$	$(p \to q)$	$(p \to \neg q)$	$(p \to q) \wedge (p \to \neg q)$
T	T	F	T	F	F
T	F	T	F	T	F
F	T	F	T	T	T
F	F	T	T	T	T

$\neg p$
F
F
T
T

Hence, $(p \to q) \wedge (p \to \neg q) \equiv \neg p$

Show that $\neg(p \to q)$ and $(p \wedge \neg q)$ are logically equivalent by the help of symbolic derivation.

Solution:-

LHS: $\neg(p \to q)$

$\equiv \neg(\neg p \vee q)$ [∵ Implication law]

$\equiv \neg(\neg p) \wedge \neg(q)$ [∵ Demorgan's law]

$\equiv p \wedge \neg q$ [∵ double negation]

$\equiv$ RHS

proved!

$(p \to q) \wedge (p \to \neg q) \equiv \neg p$

LHS $\equiv (p \to q) \wedge (p \to \neg q)$

$\equiv (\neg p \vee q) \wedge (\neg p \vee \neg(\neg q))$ [Implication law]

$\equiv [\neg p \vee q] \wedge [\neg p \vee q]$ ~~[Double negation]~~

$\equiv \neg p$ [Absurdity law]

Show that $\neg(p \vee (\neg p \wedge q))$ and $(\neg p \wedge \neg q)$ are logically equivalent

LHS $\equiv \neg(p \vee (\neg p \wedge q))$

$\equiv \neg p \wedge \neg(\neg p \wedge q)$ De Morgan's Law

$\equiv \neg p \wedge (p \vee \neg q)$ double negation

$\cancel{\equiv \neg p \wedge p \vee \neg q}$

$\cancel{\equiv F \vee \neg q}$

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ Distributive. ~~(Double Negation)~~

$\equiv F \vee (\neg p \wedge \neg q)$ [Tautology]

$\equiv \neg p \wedge \neg q$ [Identity

$\equiv (\neg p \wedge \neg q) \vee F$ (commutative)

Lhs:- $\neg(p \vee (\neg p \wedge q))$

$\equiv \neg[(p \vee \neg p) \wedge (p \vee q)]$ (distributive law)

$\equiv \neg(T \wedge (p \vee q))$ Tri. tautology.

$\equiv \neg((T \wedge p) \vee (T \wedge q))$ Distributive

$\equiv \neg((p \wedge T) \vee (q \wedge T))$ Commutative

$\equiv \neg(p \vee q)$ Identity

$\equiv \neg p \wedge \neg q$ De Morgan's

proved!

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

$\equiv (p \wedge q) \rightarrow (p \vee q)$

~~$\equiv (\neg p \vee q)$~~

~~$\equiv \neg(p \wedge q) \vee (p \vee q)$~~ [Implication Law]

~~$\equiv (\neg p \vee \neg q) \vee (p \vee q)$~~ [De Morgans Law]

~~$\equiv (\neg p \vee p)$~~

$(p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$ [Implication Law

$\equiv (\neg p \vee \neg q) \vee (p \vee q)$ [De Morgan Law

$\equiv (\neg p \vee p) \vee (\neg q \vee q)$ [Associative.

$\equiv (p \vee \neg p) \vee (q \vee \neg q)$ [commutative

$\equiv T \vee T$ [Trival. Taulogy]

$\equiv T$

Dual of compound proposition

The dual of compound proposition that contains only the logical operators $\wedge$, $\vee$ and $\neg$ is the compound proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each T by F and each F by T. The dual of s is denoted by s^*.

Find the dual of each of these compound proposition.

(a) $p \vee \neg q$

$\Rightarrow s = p \vee \neg q$

$s^* = p \wedge \neg q$

(b) $p \wedge (q \vee (r \wedge T))$

$s = p \wedge (q \vee (r \wedge T))$

$s^* = p \vee (q \wedge (r \vee F))$

(c) $(p \wedge \neg q) \vee (q \wedge F)$

$s = (p \wedge \neg q) \vee (q \wedge F)$

$s^* = (p \vee \neg q) \wedge (q \vee T)$

(d) $(p \vee F) \wedge (q \vee T)$

$s = (p \vee F) \wedge (q \vee T)$

$s^* = (p \wedge T) \vee (q \wedge F)$

When does $S^* = S$, where S is a compound proposition?

Solution:-

Since S is a compound proposition, Let p be proposition then

$$S = p \vee p$$

The dual of $S \Rightarrow S^* = p \wedge p$

$$p \vee p = p \wedge p = p \text{ (Idempotent law)}$$

$$\therefore S = S^*$$

Show that $(S^*)^* = S$ where S is a compound proposition.

Solution:-

S is a compound proposition then,

Let $S = p \wedge \neg q$

Now,

$$S^* = p \vee \neg q$$

$$(S^*)^* = p \wedge \neg q$$

$$\therefore (S^*)^* = S$$

proved!

Predicate.

lets take a statement $x > 3$, there are two parts one is the variable part called "subject" and another is relation part ">3" called predicate. We can ~~den~~ denote the statement "$x > 3$" by $P(x)$ where P is a predicate ">3" and x is the variable. Once the value is assigned to the propositional function then we can tell whether it is true or false i.e. proposition.

→ The logic involving predicates is called predicate logic or predicate calculus.

Let $P(x)$ denotes the statements "$x > 10$". What are the truth value of $P(12)$ and $P(5)$?

Solution:-

$P(x) = x > 10$

$P(12) = 12 > 10$ (Truth)

$P(5) = 5 > 10$ (false)

$P(x)$ denotes $x > 10$

$P(12)$ denotes $12 > 10$ which is "true"

$P(5)$ denotes $5 > 10$ which is "false"

Let $Q(x,y)$ denote the statement "$x = y+3$". What are the truth value for the propositions $Q(1,2)$ and $Q(3,0)$?

Solution:-

$Q(x,y)$ denotes $x = y+3$

$Q(1,2)$ denotes $1 = 2+3$
which is "False"

$Q(3,0)$ denotes $3 = 0+3$
which is "True"

Let $R(x,y,z)$ denote the statement "$x+y=z$". What are the truth value of $R(1,2,3)$ & $R(0,0,1)$?

Solution:-

$R(x,y,z)$ denotes $x+y=z$

$R(1,2,3)$ denotes $1+2=3$
which is "True"

$R(0,0,1)$ denotes $0+0=1$
which is "false"

Quantifiers

Quantifiers are the tools to make the propositional function a proposition. Construction of propositions from the predicates using quantifiers is called quantification. The variables that appear in the statement can take different possible values and all the possible values that the variables can take forms a domain called "Universe of Discourse" or "Universal set".

Types of Quantifiers:

1. Universal Quantifier
2. Existential Quantifier

Universal Quantifier

It is denoted by ($\forall$) symbol "for all". The universal Quantification of $P(x)$ denoted by for all x $P(x)$:

$$\forall x P(x)$$

is a proposition. "$P(x)$ is true for all the values of x in the Universal Discourse".

We can represent the universal quantification by using the english sentences like (i) "for all x $P(x)$ holds".

(ii) "for every x $P(x)$ holds."

(iii) "for each x $P(x)$ holds".

Example:-

Express the statement into quantified statement. "all students of CSIT takes discrete mathematics class", where UoD is set of all CSIT students".

$\Rightarrow$ $D(x)$ denotes x takes discrete mathematics class.

$\forall x \; D(x)$

$D(x)$ = predicate

Existential Quantifiers

It is denoted by "$\exists$". The existential Quantification of $p(x)$ denoted by there exists x $P(x)$ "$\exists x P(x)$".

"$P(x)$ is true for some values of x in UoD".

It can be represented like

i) "there exists x such that $P(x)$ is true".

ii) "$P(x)$ is true for at least one x."

Examples:-

Let $Q(x)$ be the statement "$x < 2$". What is truth value of the quantification $\forall x Q(x)$ where domain consists of all real numbers?

$\Rightarrow$ $Q(x)$ denotes "$x < 2$"

UoD = $\{\ldots -2, -1, 0, 1, 2 \ldots\}$

$x = 3$

$\forall x Q(x) \Rightarrow Q(3) \Rightarrow 3 > 2$

False $\therefore$ The truth value...

What is the truth value of $\exists x\, P(x)$ where $P(x)$ is the statement "$x^2 > 10$" and the UoD consists of the integer not exceeding 4?

$\Rightarrow$ $P(x)$ denote "$x^2 > 10$"

UoD = $\{0, 1, 2, 3, 4\}$

$\exists x\, P(x)$ (True)

Because: $P(2)$.

$4^2 > 10$

$16 > 10$

(True)

The true value of $\exists x\, P(x)$ is true.

Free and Bound Variables

When the variable is assigned a value or it is quantified. It is called Bound variable. If the variable is not bounded then it is called free variables

Examples:-

Identify the free and bound variables:

1. $P(x, y)$, both are free variables.
2. $P(2, y)$, $\Rightarrow$ y is free.
3. $P(2, y)$ where $y = 4$ $\Rightarrow$ both bounded.
4. $\forall x\, P(x)$, x is bounded variable
5. $\forall x\, P(x, y)$, x bounded, y free

Expression with no free variable is <u>proposition</u>?

Expression with at least one free variables is <u>predicate</u>?

Order of Quantification :-

Example :-

Let $L(x, y)$ denotes x loves y where UoD for x, y is set of all people in the world.
Translate the given quantified statement in English.

i) $\forall x \exists y \, L(x, y)$

→ for all x there exist some y such that x loves y i.e. everyone loves someone.

ii) $\exists y \forall x \, L(x, y)$

→ There exist some y for all x such that x loves y i.e. someone is loved by everyone.

iii) $\forall x \forall y \, L(x, y)$

⇒ for all x and y such that x loves y i.e. everyone loves everybody.

iv) $\exists x \exists y \, L(x, y)$

⇒ there exist some x and some y such that x loves y

someone loves somebody

Negation of Quantified Expression.

$$\forall x\, P(x) \Rightarrow \neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$

$$\exists x\, P(x) \Rightarrow \neg \exists x\, P(x) \equiv \forall x\, \neg P(x).$$

Example:-

Let $P(x)$ denotes x is lovely, UoD for x is girls in Kathmandu.

$\forall x\, P(x)$

$\Rightarrow$ Every girl in Kathmandu are lovely

$\exists x\, P(x)$

$\Rightarrow$ Some girls in Kathmandu are lovely

$\neg \forall x\, P(x)$

$\Rightarrow$ Not all girls in Kathmandu are lovely

$\neg \exists x\, P(x)$

$\Rightarrow$ All girls in Kathmandu are not lovely

Translate the sentence into logical expression.

1. "not every integer is even.

let $P(x)$ denotes x is even integer, UoD for integers

$$\neg \forall x\, P(x)$$

Translate "If a person is female and is a parent then this person is someone's mother". into logical expression where UoD is set of all peoples.

$\Rightarrow$ Let $F(x)$ denotes female

$P(x)$ denotes parent

UoD = { set of all people }

$M(x, y)$ denotes x is mother of y.

$$\forall x\, \exists y\, ((F(x) \wedge P(x)) \rightarrow M(x, y))$$

Mathematical Reasoning.

Rules of Reasoning

To draw conclusion from the given premise we must be able to apply some well defined steps that helps ~~reaching~~ reaching the conclusion. These steps of reaching the conclusion are provided by rule of inference

Rule 1: Modus Ponens (or Law of Detachment)

Whenever two propositions p & $p \to q$ are both true then we confirm q is true i.e.

$$\frac{\begin{array}{l} p \\ p \to q \end{array}}{q}$$

, this rule is value rule of inference because the implication $[p \wedge (p \to q)] \to q$ is a tautology.

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Rule 2: Hypothetical Syllogism (Transitive Rule)

Whenever two proposition $p \to q$ and $q \to r$ are both true then we confirm that implication $p \to r$ is true.

i.e.

$$\frac{\begin{array}{l} p \to q \\ q \to r \end{array}}{\therefore p \to r}$$

, this rule is valid rule of inference because the implication

$[(p \to q) \wedge (q \to r)] \to (p \to r)$ is tautology.

Similarly,

$$\begin{array}{l} p \to q \\ q \to r \\ \underline{r \to s} \\ \therefore p \to s \end{array}$$

p	q	r	$p \to q$	$p \to r$	$q \to r$	$[(p \to q) \wedge (q \to r)]$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	F	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Rule 3:- Addition

Due to tautology $p \to (p \vee q)$, rule $\dfrac{p}{\therefore p \vee q}$ is valid rule of inference.

$$\begin{array}{ll} p \to (p \vee q) & \\ \equiv \neg p \vee (p \vee q) & \text{(implication)} \\ \equiv \neg p \vee p \vee q & \\ \equiv T \vee q & \text{(Trivial tautology)} \\ \equiv T & \text{[Domination]} \end{array}$$

$[(p \to q) \wedge (q \to r)] \to (p \to r)$

$\equiv \neg[(p \to q) \wedge (q \to r)] \vee (p \to r)$ [∵ Implication law]

$\equiv \neg[(p \to q) \wedge (q \to r)] \vee (\neg p \vee r)$ [∵ Implication law]

$\equiv \neg(p \to q) \vee \neg(q \to r) \vee (\neg p \vee r)$ [∵ De Morgan's law]

$\equiv \neg(\neg p \vee q) \vee \neg(\neg q \vee r) \vee (\neg p \vee r)$ [Implication law]

$\equiv (\neg(\neg p) \wedge \neg q) \vee (\neg(\neg q) \wedge \neg r) \vee (\neg p \vee r)$ [De Morgan's law]

$\equiv (p \wedge \neg q) \vee (q \wedge \neg r) \vee (\neg p \vee r)$ [Double Negation]

$\equiv$

$[(p \to q) \wedge (q \to r)] \to (p \to r)$

$\equiv (p \wedge \neg q) \vee (q \wedge \neg r) \vee (\neg p \vee r)$ [Implication]

$\equiv (p \wedge \neg q) \vee \{(q \wedge \neg r) \vee r\} \vee \neg p$ [Associative]

$\equiv (p \wedge \neg q) \vee \{(q \vee r) \wedge (\neg r \vee r)\} \vee \neg p$ [Distributive]

$\equiv (p \wedge \neg q) \vee \{(q \vee r) \wedge T\} \vee \neg p$ [Trivial Tautology]

$\equiv \{(p \wedge \neg q) \vee \neg p\} \vee (q \vee r) \wedge T$ [Associative]

$\equiv (p \vee \neg p) \wedge (\neg q \vee \neg p) \vee (q \vee r) \wedge T$ (Distributive)

$\equiv T \wedge (\neg q \vee \neg p) \vee (q \vee r) \wedge T$ (Trivial tautology)

$\equiv T \wedge (\neg q \vee q) \vee \neg p \vee r \wedge T$ (Associative)

$\equiv T \wedge \{T \vee (\neg p \vee r)\} \wedge T$ (Dominative)

$\equiv T \wedge T \wedge T$ (Trivial tautology)

$\equiv T$

Rule 4:- Simplification:-

Due to the tautology $(p \wedge q) \rightarrow p$, rule

$$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$$

is a valid rule of inference.

$$\begin{aligned}
& (p \wedge q) \rightarrow p \\
\equiv\ & \neg(p \wedge q) \vee p && \text{[Implification]} \\
\equiv\ & \neg p \vee \neg q \vee p && \text{(De-Morgan's Law)} \\
\equiv\ & p \vee \neg p \vee \neg q \\
\equiv\ & T \vee \neg q && \text{(Trival tautology)} \\
\equiv\ & T && \text{(Trival tautology)}
\end{aligned}$$

Rule 5: Conjuction

Due to the tautology $[(p) \wedge (q)] \rightarrow (p \wedge q)$, rule

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

is valid rule of inference.

$$\begin{aligned}
& [(p) \wedge (q)] \rightarrow (p \wedge q) \\
\equiv\ & (p \wedge q) \rightarrow (p \wedge q) \\
=\ & \neg(p \wedge q) \vee (p \wedge q) && \text{[Implication]} \\
\equiv\ & \neg p \vee \neg q \vee (p \wedge q) && \text{[De-Morgan's law]}
\end{aligned}$$

Rule 6: Modes Tollens

Due to tautology $[\neg q \wedge (p \to q)] \to \neg p$; rule $\begin{array}{l} \neg q \\ p \to q \\ \hline \therefore \neg p \end{array}$

is valid rule of inference.

p	q	$\neg p$	$\neg q$	$(p \to q)$	$\neg q \wedge (p \to q)$	$\neg q \wedge (p \to q) \to \neg p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

$\neg q \wedge (p \to q) \to \neg p$

Rule 7: Disjunctive Syllogism

Due to tautology $[(p \vee q) \wedge \neg p] \rightarrow q$, rule $\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$ is a valid rule of inference.

p	q	$\neg p$	$p \vee q$	$(p \vee q) \wedge \neg p$	$((p \vee q) \wedge \neg p) \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

$[(p \vee q) \wedge \neg p] \rightarrow q$

Rule 8: Resolution

Due to tautology $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$, rule is valid rule of inference

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$$

$$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$$

Example:-

For the set of premises "If I play football, then I am sore the next day", "I will take rest if I am sore", "I did not take rest". What relevant conclusion can be drawn? Explain the rule of inference to draw the conclusion.

Solution:-

p = I play football

q = I am sore

r = I ~~tak~~ will take rest

Hypothesis:-

(i) $p \to q$

(ii) $q \to r$

(iii) $\neg r$

Steps	Reason.
(i) $p \to q$	Hypothesis.
(ii) $q \to r$	Hypothesis
(iii) $p \to r$	Transitive from (i) & (ii)
(iv) $\neg r$	Hypothesis
(v) $\neg p$	Modes Tollen

Conclusion:

Therefore we can conclude that, "I did not play football.

Example:-

Show that the hypothesis "It is not sunny this afternoon and it is ~~called~~ cooler than yesterday". "We will go swimming only if it is sunny", "If we do not go swimming, then we will take a trip" and "If we take a trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset".

Solution:-

p = It is sunny this afternoon.
q = It is cooler
r = We will go swimming
s = We will take a trip
t = We will be home by sunset.

Hypothesis

(I) $\neg p \wedge q$
(II) $r \rightarrow p$
(III) $\neg r \rightarrow s$
(IV) $s \rightarrow t$

Steps	Reason
(I) $\neg p \wedge q$	Hypothesis
(II) $r \rightarrow p$	Hypothesis
(III) $\neg r \rightarrow s$	Hypothesis
(IV) $s \rightarrow t$	Hypothesis
(V) $\neg r \rightarrow t$	Transitive from (III) & (IV)
(VI) $\neg p$	Simplification (I).
(VII) $\neg r$	(Modus Tollens) (VI) & (II)
(VIII) s	Modus Ponen (VII) & (III)
(IX) t	Modus Ponen (VIII) & (IV)

Fallacies:-

The fallacies are arguement that are convincing but not correct. So fallacies produce faulty inference. Fallacies are contingency rather than tautology.

① Fallacy of affirming the conclusion:

The kind of fallacy has the form

$$\begin{array}{l} q \\ p \to q \\ \hline \therefore p \end{array}$$

i.e. $q \wedge (p \to q) \to p$. This is not a tautology, hence it is fallacy.

p	q	$p \to q$	$q \wedge (p \to q)$	$q \wedge (p \to q) \to p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Example:-

"If the economy of Nepal is poor, then the education system in Nepal will be poor."
"The education system in Nepal is poor. Therefore, "Economy of Nepal is poor".

Solution:-

p = economy of Nepal is poor
q = education system in Nepal is poor.

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$

(which is fallacy (fallacy of affirming the conclusion)) Hence we can conclude economy of Nepal is not poor.

(iii) Fallacy of denying the hypothesis

This kind of fallacy has the form
$$\begin{array}{l} \neg p \\ p \to q \\ \hline \therefore \neg q \end{array}$$
i.e. $[\neg p \wedge (p \to q)] \to \neg q$. This is not a tautology hence it is a fallacy.

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \wedge (p \to q)$	$[(\neg p \wedge (p \to q)] \to \neg q$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

Example:-
"If today is sunday, then it rains today". "Today is not sunday". Therefore, "it doesnot rain today".

$\Rightarrow$ p : today is sunday
q : it rains today.

$$\begin{array}{l} p \to q \\ \neg p \\ \hline \therefore \neg q \end{array}$$

(iii) Begging the question (Circular Reasoning)

If the statement that is used for loop is equivalent to the statement that is being proved then it is called circular reasoning

Example:-
Ram is black because he is black.

Rules of inference for Quantified Statement

Universal Instantiation:-
If the proposition of the form $\forall x\, P(x)$ is supposed to be true then the universal quantifier can be dropped out to get $P(c)$ is true for arbitary c in the universe of discourse

i.e. $\dfrac{\forall x\, P(x)}{\therefore P(c), \text{ for all } c}$

Universal Generalization:-
If all the instances of c makes $P(c)$ true, then $\forall x\, P(x)$ is true. This can be written as.

$\dfrac{P(c) \text{ for all } c}{\therefore \forall x\, P(x)}$,

Here the choosen c must be arbitrary not a specific element from the UoD.

Existential Instantiation:

If the proposition of the form $\exists x P(x)$ is supposed to be true then there is an element c in the UoD such that $P(c)$ is true.

i.e. $\dfrac{\exists x P(x)}{\therefore P(c) \text{ for some } c}$

Here c is not an arbitary, it must be scientific such that $P(x)$ is true.

Extential Generalization

If at least an element c from the UoD make $P(c)$ true then $\exists x P(x)$ is true.

i.e. $\dfrac{P(c), \text{ for some } c}{\therefore \exists x P(x)}$

Show that the premises, "Everyone in this discrete mathematics class taken a course in computes Science" and "Marla is a student in this class" imply the conclusion "Marla has take a course in Computes Science."

Solution:-

$D(x)$ denotes x is in discrete mathematics class

$C(x)$ denotes x has taken a course in CS.

Quantified statement (Hypothesis)

1. $\forall x (D(x) \rightarrow C(x))$
2. $D(\text{Marla})$

③ C (Marla)

Steps	Result
① $\forall x(D(x) \rightarrow C(x))$	Hypothesis.
② $D(Marla) \rightarrow C(Marla)$	Universal Instantiation from ①.
③ $D(Marla)$	Hypothesis
④ $C(Marla)$	Modes ponens ~~fs~~ on ② & ③.

Prove or disprove the validity of the arguement, "every living thing thing is a plant or animal.", "Nari's dog is alive and it is not a plant.", "All animals have heart", hence "Nari's dog has a heart".

Solution:- UoD denote "every living thing is a plant or animal".
$H(x)$ denotes All animals have heart.
$P(x) \rightarrow x$ is plant, $A(x) \rightarrow x$ is animal, $J(x) \rightarrow x$ is alive.

1. $\forall x(P(x) \vee A(x))$	Hypothesis.
2. $J(\text{Nari's dog}) \wedge \neg P(\text{Nari's dog})$	~~from~~ hypothesis.
3. $J(\text{Nari's dog}) \wedge A(\text{Nari's dog})$	from ① & ②.
4. $\forall x\, H(x)$	Hypothesis
5. $H(\text{Nari's Dog})$	Universal instation from (5)

Hypothesis:
(i) $\forall x(P(x) \vee A(x))$
(ii) $J(\text{Nari's Dog}) \wedge \neg P(\text{Nari's Dog})$
(iii) $\forall x\, H(x)$
(iv) $H(\text{Nari's dog})$.