Unit 3 :-

Recurrence relation

→ A recurrence relation for a sequence {a_n} is an equation that express a_n in terms of one or more of the previous terms of the sequence namely $a_0, a_1, \ldots a_{n-1}$ for all integer n with $n \geq n_0$, where n_0 is a non-negative integer.

→ A sequence is called a solution of recurrence relation if its terms satisfies the recurrence relation.

Example:-

For $a_n = 3a_{n-1}$ and $a_0 = 1$ find a_1, a_2, a_3, a_4 & a_5.

Solution:-

$a_0 = 1$

$a_n = 3a_{n-1}$

then

$a_1 = 3 \cdot a_0 = 3$

$a_2 = 3 \cdot a_1 = 9$

$a_3 = 3 \cdot a_2 = 27$

$a_4 = 3 \cdot a_3 = 81$

$a_5 = 3 a_4 = 243$

Find the recurrence relation to find the total amount after 30 yrs of a person deposits Rs. 10,000 in a saving account at a bank yielding 11% per year with interest compounded annually.

Soln:-

Basis step:

$n = 1$.

$a_1 = 10,000 \times 11\% + 10,000$

$= 10,000 + 1100$

$= \cancel{1100}\ \cancel{10,}\ 11,100$

$= a_0 + 0.11 a_0$

Let $a_0 = 10,000$

Recursive step:

$\cancel{a_n = (a_{n-1}) - a_0 + 0.11 a_0 \cdot n}$

$a_n = a_{n-1} + 0.11 \times a_{n-1}$

or, $a_n = (1.11)^n P_0$.

$a_1 = 11,100$

$a_2 = 11\%$ of $a_1 + a_1$

$= (1.11)^2 a_0$

$a_3 = (1.11)^3 a_0$

Solving Recurrence Relations.

1. Linear Homogenous Recurrence Relation of Degree k with constant Coefficients.

Linear Homogenous Recurrence Relation of Degree k with constant coefficient with recurrence relation of the form:-

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots C_K a_{n-K}$$ where, $C_1, C_2 \dots C_K$ are real numbers and $C_K \neq 0$.

The above relation is linear since, right hand side is the sum of the multiples of previous terms of the sequence. It is homogenous because no terms occurs without being multiple of some a_j. All the coefficients of the terms are constant because they do not depend on n. And the degree of the relation is k because a_n is expressed in terms of previous k terms of the sequence.

(i) $P_n = (1.11) P_{n-1}$

(ii) $a_n = a_{n-5}$

(iii) $a_n = a_{n+1} + a^2_{n-2}$

(iv) $H_n = 2H_{n-1} + 1$ ⊗ (because its not homogenous)

(v) $B_n = n B_{n-1}$ ⊗ (because there is no constant.

Solving linear Homogeneous Recurrence Relation of Degree K with constant coefficients.

In solving the recurrence relation of this type, the approach is to look for the solution of the form $a_n = r^n$, where r is a constant. $a_n = r^n$ is a solution of a recurrence relation $a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_n a_{n-k}$ If and only if $r^n = C_1 r^{n-1} + C_2 r^{n-2} + \dots + C_k r^{n-k}$. When we divide both sides by r^{n-k} and transpose the right hand side we have,

$$r^k - C_1 r^{k-1} - C_2 r^{k-2} \dots C_k = 0.$$

Here, we can say $a_n = r^n$ is a solution iff r is the solution of eq^n $r^k - C_1 r^{k-1} \dots C_k = 0$. which is called characteristics of eq^n and the sol^n to this eq^n is called characteristics root.

Theorem 1:-

Let C_1 and C_2 be real numbers. Suppose $r^2 - C_1 r - C_2 = 0$ has two distinct roots r_1 and r_2. Thus the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = C_1 a_{n-1} + C_2 a_{n-2}$ iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2 \dots$ where α_1 and α_2 are constants.

Example:-

Solve the recurrence relation $a_n = a_{n+1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$.

Solution:-

Characteristics eq^n:-

$$r^2 - C_1 r - C_2 = 0$$
$$\Rightarrow r^2 - r - 6 = 0$$
$$\text{or, } r_1 = 3 \; ; \; r_2 = -2$$

The solution of this recourrence relⁿ is:

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

$\therefore a_n = \alpha_1 3^n + r_2 (-2)^n$

We know that,

$a_0 = 3$

$\alpha_1 (3)^0 + \alpha_2 (-2)^0 = 3$ — (i)

$a_1 = 6$

$\alpha_1 (3)^1 + \alpha_2 (-2)^1 = 6$ — (ii)

from (i) & (ii)

$\alpha_1 + \alpha_2 = 3$

$3\alpha_1 - 2\alpha_2 = 6$

$\therefore \alpha_1 = 12/5$

$\alpha_2 = 3/5$

$\{a_n\}$ is $a_n = \frac{12}{5} (3)^n + (\frac{3}{5})(-2)^n$

Exercise

What is the solution of the recurrence relation $a_n = a_{n+1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution:-

Characteristic eqⁿ:-

$r^2 - c_1 r - c_2 = 0$

$r^2 - r - 2 = 0$

or, $r^2 - 2r + r - 2 = 0$

or, $r(r-2) + 1(r-2) = 0$

or, $(r-2)(r+1) = 0$

$\therefore r_1 = 2,$

$r_2 = -1$

The solution of the recurrence eq^n is.

$$a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n}$$

$$\therefore a_n = \alpha_1 (2)^n + \alpha_2 (-1)^n \quad \text{—} (*)$$

We know that,

$$a_0 = 2$$

$$\alpha_1 (2)^0 + \alpha_2 (-1)^0 = 2 \quad \text{—} (i)$$

$$a_1 = 7$$

$$\alpha_1 (2)^1 + \alpha_2 (-1)^1 = 7 \quad \text{—} (ii)$$

From (i) & (ii),

$$\alpha_1 + \alpha_2 = 2 \qquad \text{or, } \alpha_1 = 2 - \alpha_2$$

$$2\alpha_1 - \alpha_2 = 7$$

$$\text{or, } 2(2-\alpha_2) - \alpha_2 = 7$$

$$\text{or, } 4 - 2\alpha_2 - \alpha_2 = 7$$

$$\text{or, } 4 - 3\alpha_2 = 7$$

$$\text{or, } 3\alpha_2 = -3$$

$$\text{or, } \alpha_2 = -1.$$

$$\therefore \alpha_1 = 2 - (-1) = 3$$

Putting value of α_1 and α_2, we get.

$$a_n = 3(2)^n + (-1)(-1)^n$$

When $n = 0$

$$a_0 = 3(2)^0 + (-1)(-1)^0 = 3 - 1 = 2$$

$$a_1 = 3(2)^1 + (-1)(-1)^1 = 6 + 1 = 7.$$

$\therefore$ The solution of recourrence relation $\{a_n\}$ is $a_n = 3(2)^n + (-1)(-1)^n$ //

Exercise:-

Find the explict formula for the fibonacci numbers. [Use $f_n = f_{n-1} + f_{n-2}$ as recursive defⁿ and $f_0 = 0$ and $f_1 = 1$ as initial condition].

Solution:-

Recursive Relation:-

$$f_n = f_{n-1} + f_{n-2}$$

Characteristic eqⁿ:-

$$r^2 - r - 1 = 0$$

$$\text{or, } r = \frac{-1 \pm \sqrt{1 - 4 \times (-1)}}{2}$$

$$= \frac{-1 \pm \sqrt{5}}{2}$$

$$\therefore r_1 = \frac{-1 + \sqrt{5}}{2} \qquad r_2 = \frac{-1 - \sqrt{5}}{2}$$

The solⁿ of recursive relation is.

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$a_n = \alpha_1 \left(\frac{-1 + \sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{-1 - \sqrt{5}}{2}\right)^n$$

We know,

$a_0 = 0$. $\quad$ $a_1 = 1$.

$$\therefore \alpha_1 = -\alpha_2 \quad \text{---} (1)$$

$$a_1 = 1$$

$$\alpha_1\left(\frac{-1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{-1-\sqrt{5}}{2}\right) = 1$$

$$\text{or, } \frac{-\alpha_1}{2} + \frac{\alpha_1\sqrt{5}}{2} - \frac{\alpha_2}{2} - \frac{\alpha_2\sqrt{5}}{2} = 1$$

$$\text{or, } \cancel{\frac{\alpha_2}{2}} - \frac{\alpha_2\sqrt{5}}{2} - \cancel{\frac{\alpha_2}{2}} - \frac{\alpha_2\sqrt{5}}{2} = 1$$

$$\text{or, } -\alpha_2 = \frac{2}{\sqrt{5}\cdot 2}$$

$$\text{or, } \alpha_2 = \frac{-1}{\sqrt{5}}$$

$$\therefore \alpha_1 = \frac{1}{\sqrt{5}}.$$

Putting value of α_1 & α_2 we get.

$$\{a_n\} \text{ is } a_n = \frac{1}{\sqrt{5}}\left(\frac{-1+\sqrt{5}}{2}\right)^n + \left(\frac{-1}{\sqrt{5}}\right)\left(\frac{-1-\sqrt{5}}{2}\right)^n$$

Theorem 2:

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. Then the sequence $\{a_n\}$ is a sol^n of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0, 1, 2 \dots$ where α_1 and α_2 are constant.

Example:-

Solve the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$

Solution:-

Characteristic eq^n:-

$r^2 - 2r + 1 = 0$

or, $r^2 - r - r + 1 = 0$

or, $r(r-1) - 1(r-1) = 0$

or, $(r-1)(r-1) = 0$

$\therefore r_1 = 1; r_2 = 1$

The sol^n of recurrence relation is

$a_n = \alpha_1 r_1^n + \alpha_2 n r_2^n$

we have, $a_0 = 3$

$\therefore \alpha_1 (1)^0 + \alpha_2 0 (1)^0 = 3$

or, $\alpha_1 = 3$ — (i).

$a_1 = 6$

$\alpha_1 (1)^1 + \alpha_2 \cdot 1 (1)^1 = 6$

$\alpha_1 + \alpha_2 = 6$

$\alpha_2 = 3$

$\therefore \{a_n\}$ is $a_n = 3(1)^n + 3 \cdot n (1)^n$

What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial condition $a_0 = 1$ & $a_1 = 6$?

Sol^n:-

Characteristic eq^n is

$r^2 - c_1 r - c_2 = 0$

or, $r^2 - 6r + 9 = 0$

or, $r^2 - 3r - 3r + 9 = 0$

or, $r(r-3) - 3(r-3) = 0$

$\therefore (r-3)(r-3) = 0$

$\therefore r_1 = 3$

$r_2 = 3$

The sol^n of recourrence relation is

$a_n = \alpha_1 (r_1)^n + \alpha_2 n (r_2)^n$

When,

$a_0 = 1$

$\alpha_1 (3)^0 + \alpha_2 0 (3)^0 = 1$

$\alpha_1 = 1$

$a_1 = 6$

$\alpha_1 (3)^1 + \alpha_2 \cdot 1 \cdot (3)^1 = 6$

$3\alpha_1 + 3\alpha_2 = 6$

$\alpha_2 = 1$

$\therefore \{a_n\}$ is $a_n = 1(3)^n + 1n(3)^n$.

Theorem 3:-

let $c_1, c_2 \dots c_k$ be real numbers suppose $r^k - c_1 r^{k-1} \dots c_k = 0$ has k distinct roots $r_1, r_2 \dots r_k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation ~~n=0, 1, 2 ... who~~

$$c_n = c_1 a_{n-1} + c_2 a_{n-2} + c_k a_{n-k} \text{ iff}$$

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots \alpha_k r_k^n$ for $n = 0, 1, 2 \dots$

where $\alpha_1 \dots \alpha_k$ are constant.

Example:-

Solve the ~~reco recur~~ recurrence relation $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n \geq 3$, $a_0 = 3$, $a_1 = 6$ & $a_2 = 9$

Characteristics Eq^n:-

$$r^3 - 2r^2 - r + 2 = 0$$

$$\text{or, } r^2(r-2) - 1(r-2) = 0$$

$$\text{or, } (r^2 - 1)(r-2) = 0$$

$$\text{or, } (r+1)(r-1)(r-2) = 0$$

$$\therefore r_1 = -1$$

$$r_2 = 1$$

$$r_3 = 2$$

The sol^n of recurrence relation is:-

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n$$

$$\text{or, } a_n = \alpha_1 (-1)^n + \alpha_2 1^n + \alpha_3 (2)^n$$

Where,

$$a_0 = 3.$$

$$\alpha_1 (-1)^0 + \alpha_2 (1)^0 + \alpha_3 (2)^0 = 3$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 3 \quad \text{— (i)}$$

When $a_1 = 6$

$$\alpha_1 (-1)^1 + \alpha_2 (1)^1 + \alpha_3 (2)^1 = 6$$

$$-\alpha_1 + \alpha_2 + 2\alpha_3 = 6 \quad \text{— (ii)}$$

When $a_2 = 9$

$$\alpha_1(-1)^2 + \alpha_2(1)^2 + \alpha_3(2)^2 = 9$$

$$\alpha_1 + \alpha_2 + 4\alpha_3 = 9 \quad \text{--- (iii)}$$

From (i) (ii) & (iii)

$$\alpha_1 + \alpha_2 + \alpha_3 = 3 \qquad \text{or, } \alpha_1 + \alpha_2 = 3 - \alpha_3 \quad \text{--- (x)}$$

$$-\alpha_1 + \alpha_2 + 2\alpha_3 = 6$$

$$\alpha_1 + \alpha_2 + 4\alpha_3 = 9$$

from (x)

$$(3 - \alpha_3) + 4(\alpha_3) = 9$$

$$3 - \alpha_3 + 4\alpha_3 = 9$$

$$3 + 3\alpha_3 = 9$$

$$-3 + 9 = 3\alpha_3$$

$$\alpha_3 = 2$$

$\therefore$ from (i) & (ii)

$$\cancel{\alpha_1} + \alpha_2 = 1$$

$$\cancel{-\alpha_1} + \alpha_2 = 2$$

$$2\alpha_2 = 3$$

$$\therefore \alpha_2 = \frac{3}{2}$$

$$\therefore \alpha_1 = 3 - 2 - \frac{3}{2}$$

$$= 1 - \frac{3}{2} = \frac{-1}{2}$$

$\therefore \{a_n\}$ is $a_n = \frac{-1}{2}(-1)^n + \frac{3}{2}(1)^n + 2(2)^n$,

Theorem 4:-

Let $C_1, C_2, \ldots C_k$ be real numbers. Suppose that $r^k - C_1 r^{k-1} \ldots C_k = 0$ has t distinct roots $r_1, r_2 \ldots r_t$ with multiplicity $m_1, m_2 \ldots m_t$ respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots t$ and $m_1, m_2 + \ldots m_t = k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = C_1 a_{n-1} + C_2 a_{n-2} + \ldots C_k a_{n-k}$

iff

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \ldots + \alpha_{1, m_1 - 1} n^{m_1 - 1}) r_1^n$$

$$+ (\alpha_{2,0} + \alpha_{2,1} n + \ldots + \alpha_{2, m_2 - 1} n^{m_2 - 1}) r_2^n.$$

$$+ \ldots + (\alpha_{t,0} + t_{t,1} n + \ldots + \alpha_{t, m_t - 1} n^{m_t - 1}) r_t^n$$

for $n = 0, 1, 2 \ldots$ where α_{ij} are constant for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Example:

Solve the recurrence relation $a_n = 5a_{n-1} - 7a_{n-2} + 3a_{n-3}$ for $n \geq 3$ $a_0 = 1$, $a_1 = 9$ and $a_2 = 15$.

Soln:-

Characteristics eqn:-

$r^3 - 5r^2 + 7r - 3 = 0$

or, $r^3 - r^2 - 4r^2 + 4r + 3r - 3 = 0$

or, $r^2(r-1) - 4r(r-1) + 3(r-1) = 0$

or, $(r-1)(r^2 - 4r + 3) = 0$

or, $(r-1)(r^2 - r - 3r + 3) = 0$

or, $(r-1)\{r(r-1) - 3(r-1)\} = 0$

or, $(r-1)(r-1)(r-3) = 0$

$\therefore r_1 = 1 \quad r_2 = 1 \quad r_3 = 3$

$\therefore r_1 = 1$ and $m_1 = 2$

$r_2 = 3$ and $m_2 = 1$

$$\therefore a_n = (\alpha_{1,0} + \alpha_{1,1} n + \dots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n + (\alpha_{2,0} + \alpha_{2,1} n + \dots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n$$

$$a_n = \left[\left[(\alpha_{1,0}) + (\alpha_{1,1}) n\right] 1^n + (\alpha_{2,0}) 3^n\right]$$

We have

$a_0 = 1$ — (i)

$a_1 = 9$ — (ii)

$a_2 = 15$ — (iii)

$$\cancel{a_0 = [\alpha_{1,0} + \alpha_{1,1} \cdot 0]\, 1^0 = 1}$$

$$\cancel{\alpha_{1,0} = 1}$$

$$\cancel{[\alpha_{1,0} + \alpha_{1,1} \cdot 1]\, 1^1 = 9}$$

$$\cancel{\alpha_{1,0} + \alpha_{1,1} = 9}$$

$$\cancel{[\alpha_{1,0} + \alpha_{1,1} \cdot 2]\, 1^2 + \alpha_{2,0} \cdot 3^2 = 15}$$

$$(\alpha_{1,0} + \alpha_{1,1} \cdot 0)\, 1^0 + \alpha_{2,0}\, 3^0 = 1$$

$$\alpha_{1,0} + \alpha_{2,0} = 1 \quad \text{— (i)}$$

$$(\alpha_{1,0} + \alpha_{1,1} \cdot 1)\, 1^1 + \alpha_{2,0}\, 3^1 = 9$$

$$\alpha_{1,0} + \alpha_{1,1} + 3\alpha_{2,0} = 9 \quad \text{— (ii)}$$

$$(\alpha_{1,0} + \alpha_{1,1} 2)\, 1^2 + \alpha_{2,0}\, 3^2 = 15$$

$$\alpha_{1,0} + 2\alpha_{1,1} + 9\alpha_{2,0} = 15 \quad \text{— (iii)}$$

from (ii) & (iii)

$$2\alpha_{1,0} + \cancel{2\alpha_{1,1}} + 6\alpha_{2,0} = 18$$
$$\alpha_{1,0} + \cancel{2\alpha_{1,1}} + 9\alpha_{2,0} = 15$$
$$- \quad - \quad - \quad -$$

$$\alpha_{1,0} - 3\alpha_{2,0} = 3 \quad \text{— (iv)}$$

from (i) & (iv)

$$\cancel{\alpha_{1,0}} - 3\alpha_{2,0} = 3$$
$$\cancel{\alpha_{1,0}} + \alpha_{2,0} = 1$$
$$- \quad - \quad -$$

$$-4\alpha_{2,0} = 2$$
$$\alpha_{2,0} = \frac{-1}{2}$$

$$\therefore \alpha_{1,0} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$\alpha_{1,1} = 9 - 3\alpha_{2,0} - \alpha_{1,0}$$
$$= 9 + \frac{3}{2} - \frac{3}{2}$$

$$= 9$$

The solution of sequence is :-

$\{a_n\}$ is $a_n = \left(\frac{3}{2} + 9 \cdot n\right) 9^n + \left(\frac{-1}{2}\right) 3^n$

For $n = 0$,

$$a_0 = \frac{3}{2} - \frac{1}{2} = 1$$

Exercise:-

Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

Solution:-

Characteristic eqⁿ is

$$r^3 + 3r^2 + 3r + 1 = 0$$

$$\text{or, } \cancel{r^2(3r+1)} \quad \text{or, } \cancel{(r+1)^3 = 0}$$

$$\text{or, } (r+1)(r+1)(r+1) = 0$$

$$\therefore r = -1 \text{ and } m = 3$$

$$\cancel{\therefore a_n = (\alpha_{1,0} + \alpha_{1,1} n + \alpha_{2,1} n^2) 1^n}$$

$$\therefore a_n = (\alpha_{1,0} + \alpha_{1,1} n + \dots + \alpha_{1,m-1} n^{m-1}) r^n$$

$$\text{or, } a_n = (\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2)(-1)^n$$

We have,

$a_0 = 1$

$a_1 = -2$

$a_2 = -1$

$$(\alpha_{1,0} + \alpha_{1,1} \cdot 0 + \alpha_{1,2} \cdot 0^2)(-1)^0 = 1$$

$$\alpha_{1,0} = 1 \quad \text{— (i)}$$

$$(\alpha_{1,0} + \alpha_{1,1} \cdot 1 + \alpha_{1,2} \cdot 1^2)(-1)^1 = -2$$

$$-(\alpha_{1,0} + \alpha_{1,1} + \alpha_{1,2}) = -2 \quad \text{— (ii)}$$

$$\cancel{\alpha_{1,0} + \alpha_{1,1} + \alpha_{1,2} = 2 \quad \text{— (ii)}}$$

~~From (i),~~ $\cancel{\alpha_{1,1} + \alpha_{1,2} = -1 \quad \text{— (ii)}}$

$$(\alpha_{1,0} + \alpha_{1,1}\, 2 + \alpha_{1,2}\, 2^2)(-1)^2 = -1$$

$$\alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2} = -1. \quad \text{--- (iii)}$$

from (ii) & (iii).

$$-\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2} = -2$$

$$\alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2} = -1$$

$$\alpha_{1,1} + 3\alpha_{1,2} = -3$$

from (i) & (ii)

$$-1 - \alpha_{1,1} - \alpha_{1,2} = -2$$

$$-\alpha_{1,1} - \alpha_{1,2} = -1$$

$$\alpha_{1,1} + \alpha_{1,2} = 1$$

$$\text{or, } \alpha_{1,1} = 1 - \alpha_{1,2}.$$

Now, in (iii)

$$1 + 2 - 2\alpha_{1,2} + 4\alpha_{1,2} = -1$$

$$1 + 2 + 2\alpha_{1,2} = -1$$

$$3 + 2\alpha_{1,2} = -1.$$

$$2\alpha_{1,2} = -4$$

$$\alpha_{1,2} = -2$$

$$\text{Or, } \alpha_{1,1} = 1 - (-2) = 3.$$

$\therefore$ The sol^n of sequence $\{a_n\}$ is

$$a_n = \{1 + 3n + (-2)n^2\}(-1)^n$$

2. Solving linear Non-homogenous Recurrence Relation of degree K with constant coefficients.

The recurrence relation of the form
$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} + F(n)$ where $C_1, C_2 \dots C_n$ are real numbers and $F(n)$ is a function depending upon n. The recurrence relation preceding $F(n)$ is called associated homogenous recurrence relation.

Theorem 5:-
If $\{a_n^{(p)}\}$ is a particular solution of the non homogenous linear recurrence relation with constant coefficient $a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} + F_n$ then every solution of the form $\{a_n^{(p)} + a_n^{(h)}\}$ where $a_n^{(h)}$ is a solution of a associated homogenous recurrence relation,
$a_n = C_1 a_{n-1} + C_2 a_{n-k} + \dots a_k a_{n-k}$.

Example:-
Find all the solution of the recurrence relation $a_n = 4a_{n-1} + n^2$. Also find the solution of relation with initial condition $a_1 = 1$

Solution,
here the associated recurrence relation is
$a_n = 4a_{n-1}$.

Characteristic eq^n:

$r^2 - 4r + 1 = 0$ $\quad$ $r - 4 = 0$

$r^2 - 2r - 2r + 1 = 0$ $\quad$ $r = 4$.

$\therefore a_n = \alpha(4)^n$

$\{a_n^{(n)}\} = \alpha 4^n$.

Now,

$F(n) = n^2$ is a polynomial of degree 2, a trial solution is a quadratic function in n, say $p_n = an^2+bn+c$ where a, b & c are constants.

To determine whether there are any solutions of the form, suppose that $p_n = an^2+bn+c$ is such solution. Then the eq^n is $a_n = 4a_{n-1} + n^2$ becomes.

$$an^2+bn+c = 4(a(n-1)^2+b(n-1)+c)+n^2$$
$$\{4a(n-1)^2+4b(n-1)+4c)\}+n^2$$
$$= \{4a(n^2-2n+1)+4b(n-1)+4c\}+n^2$$
$$\cancel{= (4an^2-4a+4bn-4b+4c)+n^2}$$
$$\cancel{= (4a+1)n^2+(4b-1)n} =$$
$$= (4a+1)n^2+(-8a+4b)n+(4a-4b+4c)$$

an^2+bn+c is a sol^n :

$$\cancel{a = 4a+1 \Rightarrow 3a = -1/4}$$
$$\cancel{b = -8a+4b \Rightarrow 2+4b \Rightarrow b = 1/2}$$
$$\cancel{c = 4a-4b+4c \Rightarrow -1-2+4c \Rightarrow c = 3/4}$$
$$a = 4a+1 \therefore a = -1/3$$
$$b = -8a+4b \therefore b = -8/9$$
$$c = 4a-4b+4c \therefore c = -20/27$$

$$\therefore \{a_n^{(p)}\} = an^2+bn+c = -\frac{1}{3}n^2 - \frac{8}{9}n - \frac{20}{27}$$

Now,

the solution of $\{a_n\}$ is

$$\{a_n^{(p)} + a_n^{(n)}\} =$$
$$\therefore a_n = \alpha 4^n - \frac{1}{3}n^2 - \frac{8}{9}n - \frac{20}{27}$$

Now, $a_1 = 1$.

$$a_n = \alpha(4)^n + \left(-\frac{1}{3}n^2 - \frac{8}{9}n - \frac{20}{27}\right)$$

$$\alpha(4)^1 + \left(-\frac{1}{3}\cdot 1^2 - \frac{8}{9}\cdot 1 - \frac{20}{27}\right) = 1$$

$$4\alpha + \left(\frac{-53}{27}\right) = 1.$$

$$\cancel{4\alpha} = \frac{53}{27}$$

$$\alpha = \frac{20}{27}.$$

$\therefore$ The final solution of $\{a_n\}$

$$a_n = \frac{20}{27}(4)^n + (an^2 + bn + c)$$

Exercise :-

Find all the solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$.

Solution:-

here the associated recurrence relation is

$$a_n = 3a_{n-1}$$

Characteristic eqⁿ :-

$$r - 3 = 0$$
$$r = 3$$
$$\therefore a_n = \alpha(3)^n$$
$$\{a_n^{(h)}\} = \alpha 3^n.$$

Now, $f(n) = 2n$ is a polynomial of degree 1, a trial solution of a linear function is n, say $p_n = an + c$ where a & c are constants.

To determine whether there are solution of the form, suppose $p_n = an + c$ is such solⁿ. Then eqⁿ is $a_n = 3a_{n-1} + 2n$ becomes.

$$an + c = 3[a(n-1) + c] + 2n$$
$$= 3[an - a + c] + 2n$$
$$= 3an + 3c + 2n - 3a.$$
$$= \cancel{3a_{n-1} + 3n + 3c}\ n(3a+2) + (-3a+3c)$$
$$= (3a+2)n + (-3a+3c)$$

$\therefore$ $an + c$ is solⁿ if

$$a = 3a + 2 \quad \text{or, } 2a = -2 \Rightarrow a = -1$$
$$c = 3c - 3a \quad \text{or, } c = 3c + 3$$
$$\text{or, } 3c - c = -3 \quad \text{or, } c = -3/2$$

$\therefore \{a_n^{(p_n)}\} = a_n + c = -n - \frac{3}{2}$

Now,

The solution of $\{a_n\}$ is

$$\{a_n^{(p)} + a_n^{(n)}\}$$

$$\therefore a_n = \alpha 3^n - n - \frac{3}{2}$$

Now,

$$a_1 = 3$$

$$a_n = \alpha 3^n - n - \frac{3}{2}$$

$$\text{or, } 3 = \alpha 3^1 - 1 - \frac{3}{2}$$

$$\text{or, } 3 = \alpha 3 - \frac{2-3}{2}$$

$$\text{or, } 3 = 3\alpha - \frac{5}{2}$$

$$\text{or, } 3\alpha = 3 - \frac{5}{2}$$

$$\text{or, } 3\alpha = \frac{11}{2}$$

$$\therefore \alpha = \frac{11}{6}$$

$\therefore$ Final solution of $\{a_n\}$ is

$$a_n = \frac{11}{6}(3)^n - n - \frac{3}{2}$$

Theorem 6:-

Suppose that $\{a_n\}$ satisfies the linear ~~la~~ non-homogenous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$, where $c_1, c_2 \dots c_k$ are real numbers and $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \dots b_1 n + b_0) s^n$, where $b_0, b_1 \dots b_t$ and s are real numbers.

When s is not a root of the characteristic eq^n of the associated linear homogenous recurrence relation, there is a particular solution of the form.

$$(P_t n^t + P_{t-1} n^{t-1} + \dots + P_1 n + P_0) s^n.$$

When s is a root of the characteristic eq^n and its multiplicity is m, there is a particular solution of the form $n^m (P_t n^t + P_{t-1} n^{t-1} + \dots + P_1 n + P_0) s^n$.

Example:-

Find a solution of the recurrence relation $a_n = 2a_{n-1} + n 2^n$.

Sol^n:-

here the associative linear homogenous recurrean relation is

$$a_n = 2 a_{n-1}$$

$\{a_n\} = n^n (P_t n^t + P_t n^{t-1} \dots P_0) s^n$

$= n^1 (p_1 n^1 + P_0 n^0) 2^n$

$= n (p_1 n + p_0) 2^n$

Characteristic eq^n is

$$r - 2 = 0$$

$$r = 2.$$

$\therefore$ Sol^n of $\{a_n\}$ is $\alpha 2^n$

$F(n) = n \cdot 2^n$

$\therefore$ The function is $p = (b_1 n + b_0) S^n$