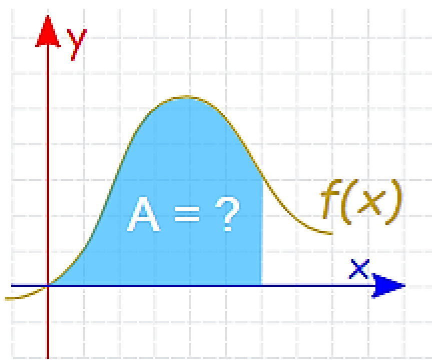
Module 4. INTEGRAL

At the end of the module the students should be able to:

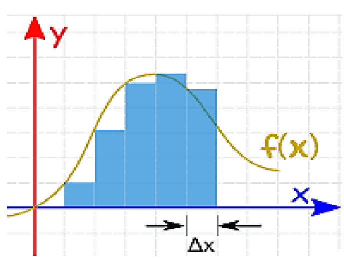
1. illustrate an antiderivative of a function;
2. compute the general antiderivatives of functions using the basic integration formulas;
3. determine the derivatives of functions using other integration techniques; and
4. solve real-life problems using integration.

Introduction to Integration

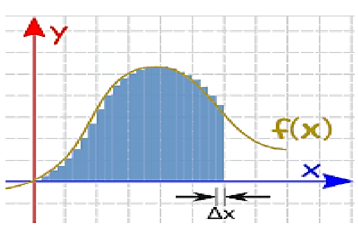
Integration can be used to find areas, volumes, central points and many useful things. But it is easiest to start with finding the area under the curve of a function like this:



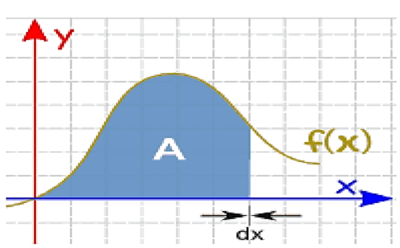
What is the area under **y** = **f(x)** ?



We could calculate the function at a few points and **add up slices of width Δx** like this (but the answer won't be very accurate).



We can make **Δx** a lot smaller and **add up many small slices** (answer is getting better).



And as the slices **approach zero in width**, the answer approaches the **true answer**.

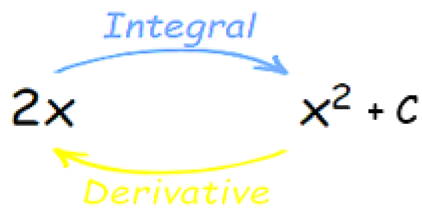
We now write **dx** to mean the **Δx** slices are approaching zero in width.

That is a lot of adding up, but we don't have to add them up, as there is a "shortcut". Because, finding an Integral is the reverse of finding a Derivative.

Example 1. What is an integral of ***2x***?

If integration is the reverse of differentiation then the derivative of $\boldsymbol{x^2}$ is ***2x***.

Can we say that the integral of ***2x*** is $\boldsymbol{x^2}$, not exactly? ***integrations are not unique*****.** A given function can have many integral.



We will discuss later why a **contant *(C)*** is neede in every integral.

Notation

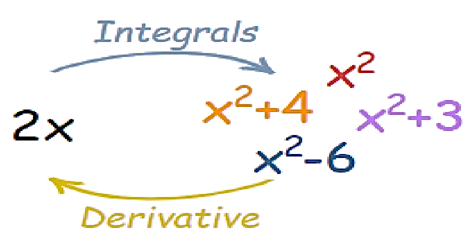
In example 1, you are asked to determine the integral of ***2x***. This can be written in symbol,

$$\int 2x \, dx = x^2 + C$$

The first symbol is the **Integral** ($\int$), the function we want to find is the **integrand (2x)**, and finish with **dx** to mean the slices go in the x direction (and approach zero in width).

We wrote the answer as x^2 but why + C ?

It is the "Constant of Integration". It is there because of **all the functions whose derivative is 2x**:



The derivative of $\boldsymbol{x^2 + 4}$ is **2x**, and the derivative of $\boldsymbol{x^2 + 99}$ is also **2x**, and so on! Because the derivative of a constant is zero. When we **reverse** the operation (to find the integral) we only know **2x**, but there could have been a constant of any value.

So, we wrap up the idea by just writing + C at the end.

Basic Integration Formulas

The following are the basic integration formulas.

1. $\int dx = x + C$

2. $\int k dx = kx + C$, where *k* is a constant

3. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, (*n* ≠ *0*)

4. $\int kf(x)dx = k \int f(x)\, dx$

5. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

6. $\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$

7. $\int a^{nx} = \frac{1}{n} \frac{a^{nx}}{\ln (a)} + C$

8. $\int e^{ax} = \frac{1}{a} e^{ax} + C$

Integration formulas for trigonometric functions:

1. $\int \sin(x)\, dx = -\cos(x) + C$

2. $\int \cos(x)\, dx = \sin (x) + C$

3. $\int \sec^2(x)\, dx = \tan (x) + C$

4. $\int \csc^2(x)\, dx = -\cot(x) + C$

5. $\int \csc(x)\cot (x)\, dx = -\csc (x) + C$

6. $\int \sec(x)\tan (x)\, dx = \sec (x) + C$

7. $\int \tan(x)\, dx = -\ln\, |\cos (x)| + C = \ln\, |\sec (x)| + C$

Example 2. Evaluate the following:

1. $\int 5dx = \mathbf{5x + C}$

2. $\int x^{12} dx = \frac{x^{12+1}}{12+1} + C = \frac{\boldsymbol{x^{13}}}{\mathbf{13}} + \boldsymbol{C}$

3. $\int 3x^5 dx = 3 \int x^5 dx = 3\left(\frac{x^{5+1}}{5+1}\right) + C = \frac{3x^6}{6} + C = \frac{\boldsymbol{x^6}}{\mathbf{2}} + \boldsymbol{C}$

4. $\int (3x + 5)dx = 3 \int xdx + \int 5dx = 3\left(\frac{x^{1+1}}{1+1}\right) + 5\,x + C = \frac{\mathbf{3}\boldsymbol{x^2}}{\mathbf{2}} + \mathbf{5}\boldsymbol{x} + \boldsymbol{C}$

5. $\int \frac{7}{x^4}\,dx = \int 7x^{-4}dx = 7\int x^{-4}dx = 7\left(\frac{x^{-4+1}}{-4+1}\right) + C = \frac{7x^{-3}}{-3} + C = -\boldsymbol{\frac{7}{3x^3}} + \boldsymbol{C}$

6. $\int \left(2x^{\frac{1}{3}} - 3x^{\frac{1}{4}}\right)dx = 2\int x^{\frac{1}{3}}\,dx - 3\int x^{\frac{1}{4}}\,dx = 2\left(\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right) - 3\left(\frac{x^{\frac{1}{4}+1}}{\frac{1}{4}+1}\right) + c$

$$= 2\left(\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right) - 3\left(\frac{x^{\frac{5}{4}}}{\frac{5}{4}}\right) + C = \frac{6x^{\frac{4}{3}}}{4} - \frac{12x^{\frac{5}{4}}}{5} + C$$

$$= \boldsymbol{\frac{3x^{\frac{4}{3}}}{2} - \frac{12x^{\frac{5}{4}}}{5} + C}$$

7. $\int(4x^8 - 9x^5 + 4x^2 - 6)dx = 4\int x^8dx - 9\int x^5dx + 4\int x^2dx - \int 6dx$

$$= 4\left(\frac{x^{8+1}}{8+1}\right) - 9\left(\frac{x^{5+1}}{5+1}\right) + 4\left(\frac{x^{2+1}}{2+1}\right) - 6x + C$$

$$= 4\left(\frac{x^9}{9}\right) - 9\left(\frac{x^6}{6}\right) + 4\left(\frac{x^3}{3}\right) - 6x + C$$

$$= \frac{4x^9}{9} - \frac{9x^6}{6} + \frac{4x^3}{3} - 6x + C$$

$$= \boldsymbol{\frac{4x^9}{9} - \frac{3x^6}{2} + \frac{4x^3}{3} - 6x + C}$$

8. $\int 5\cos(x)\,dx = 5\int \cos(x) = \mathbf{5sin}\,(\boldsymbol{x}) + \boldsymbol{C}$

9. $\int [5\csc(x)\cot(x) - 4\sec^2(x)]\;dx = 5\int \csc(x)\cot(x)\,dx - 4\int \sec^2(x)dx$

$$= -\,\mathbf{5csc}(\boldsymbol{x}) - \mathbf{4\,tan}(\boldsymbol{x}) + \boldsymbol{C}$$

10. $\int 9\sec^2(x)\;dx = 9\int \sec^2(x)dx = \mathbf{9\,tan}(\boldsymbol{x}) + \boldsymbol{C}$

11. $\int -\frac{5}{\csc(x)}dx = -5\int \frac{1}{\csc(x)}dx = -5\int sinxdx$

$$= -5(-\cos(x)) + C$$

$$= 5\,\mathbf{cos}(\boldsymbol{x}) + \boldsymbol{C}$$

*Using the trigonometric identities, we can simplify first the integrand before proceeding to integration. In the case of $\frac{1}{\csc x}$ it is equivalent to $\sin(x)$ in the reciprocal identities $\sin(x) = \frac{1}{\csc(x)}$.

Below are some of the trigonometries identities that can be helpful in evaluating trigonometric functions.

Reciprocal identities

$$\sin u = \frac{1}{\csc u} \quad \cos u = \frac{1}{\sec u} \quad \tan u = \frac{1}{\cot u}$$

$$\csc u = \frac{1}{\sin u} \quad \sec u = \frac{1}{\cos u} \quad \cot u = \frac{1}{\tan u}$$

Pythagorean Identities

$$\sin^2 u + \cos^2 u = 1 \quad 1 + \tan^2 u = \sec^2 u \quad 1 + \cot^2 u = \csc^2 u$$

Quotient Identities

$$\tan u = \frac{\sin u}{\cos u} \quad \cot u = \frac{\cos u}{\sin u}$$

12. $\int -6\tan(x)\,dx = -6\int \tan(x)dx$

$$= -6(-\ln|\cos(x)|) + C$$
$$= \mathbf{6ln\ |cos\ (\boldsymbol{x})| + \boldsymbol{C}}$$

13. $\int 3\sec^2(x)dx = 3\int \sec^2(x)dx$

$$= \mathbf{3\tan(\boldsymbol{x}) + \boldsymbol{C}}$$

14. $\int -3\csc(x)\cot(x)\,dx = -3\int \csc(x)\cot(x)\,dx$

$$= -3(-\csc(x) + C$$
$$= \mathbf{3\csc(\boldsymbol{x}) + \boldsymbol{C}}$$

15. $\int \frac{2}{\sec(x)}dx = 2\int \frac{1}{\sec(x)}dx = 2\int \cos(x)\,dx$

$$= \mathbf{2\sin(\boldsymbol{x}) + \boldsymbol{C}}$$

16. $\int e^{5x}dx = \frac{1}{5}e^{5x} + C$

$$= \frac{\boldsymbol{e^{5x}}}{\mathbf{5}} + \boldsymbol{C}$$

17. $\int (2 - 3e^x)dx = \int 2dx - 3\int e^x dx$

$$= \mathbf{2\boldsymbol{x} - 3\boldsymbol{e^x} + \boldsymbol{C}}$$

18. $\int 7^{2x+3}dx = \frac{\mathbf{7}}{\mathbf{2}}\frac{\mathbf{7^{2x+3}}}{\mathbf{ln\,(7)}} + \boldsymbol{C}$

19. $\int (e^{4x} - e^{-4x})dx = \int e^{4x}dx - \int e^{-4x}dx$

$$= \frac{1}{4}e^{4x} - \left(-\frac{1}{4}e^{-4x}\right) + C$$
$$= \frac{\mathbf{1}}{\mathbf{4}}\boldsymbol{e^{4x}} + \frac{\mathbf{1}}{\mathbf{4}}\boldsymbol{e^{-4x}} + \boldsymbol{C}$$
$$= \frac{\mathbf{1}}{\mathbf{4}}(\boldsymbol{e^{4x}} + \boldsymbol{e^{-4x}}) + \boldsymbol{C}$$

20. $\int (3^{2x} + e^{-3x})dx = \int 3^{2x}dx + \int e^{-3x}dx$

$$= \frac{1}{2}\frac{3^{2x}}{\ln(3)} + \frac{1}{-3}e^{-3x} + C$$

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Activity 1
Basic Integration

I. Evaluate the following integrals.

1. $\int -5dx$
2. $\int 7xdx$
3. $\int -32x^5dx$
4. $\int 49x^6dx$
5. $\int (-x^6 + x^5 + 7x^2)dx$
6. $\int (2x^{11} - 5x^7 - 10x^6 + 4)dx$
7. $\int (-x^6 + 5x^5 + 9x^2 - 10x - 12)dx$
8. $\int 8x^{-7}dx$
9. $\int (-4x^{-3} + 24x^{-6})dx$
10. $\int \left(x^{\frac{1}{3}} + 4x^{\frac{1}{6}}\right)dx$
11. $\int \left(\frac{4}{x^3} - \frac{8}{x^5}\right)dx$
12. $\int \left(\frac{16}{x^5} - \frac{3}{x^3}\right)dx$
13. $\int -\frac{14x^{\frac{5}{2}}}{3}dx$
14. $\int \left(5\sqrt{2x} - \sqrt[3]{x^2}\right)dx$
15. $\int \left(\sqrt[4]{3x^3} - 2\sqrt{5x}\right)dx$
16. $\int 5tan(x)dx$
17. $\int (2cos(x) - 3\sin(x))dx$
18. $\int \left(\frac{2}{\cos(x)} - \frac{3}{\sin(x)}\right)dx$
19. $\int (2cos(x) - 3\sin(x))dx$
20. $\int (x^2 + 3x - 3\sin(x))dx$
21. $\int (sec(x) - 4e^{-5x})dx$
22. $\int (2 + 3cos(x) - \sec^2(x))dx$
23. $\int (csc(x) - 7\cot(x))dx$
24. $\int (-7e^{3x})dx$
25. $\int (11 \cdot 12^x)dx$
26. $\int \left(\frac{cos(x)}{\sin(x)} - 5\sin(x)\right)dx$
27. $\int \frac{(\sin^2(x) + \cos^2(x))}{\csc^2(x)}dx$
28. $\int (2^{-7x} - 3^{2x} + \sin(x))dx$
29. $\int (2x^3 + 2cos(x) - \cot(x))dx$
30. $\int \left(3^{\frac{1}{3}x} - 5^{\frac{2}{5}x} + \frac{1}{3x^3}\right)dx$

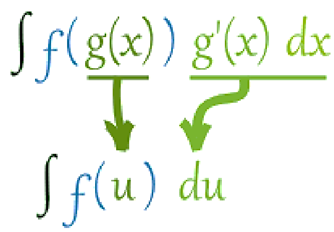
Other Integration Techniques

Integration by Substitution

"Integration by Substitution" (also called "u-Substitution" or "The Reverse Chain Rule") is a method to find an integral, but only when it can be set up in a special way.

The first and most vital step is to be able to write our integral in this form:

When our integral is set up like that, we can do **this substitution**:



Then we can **integrate f(u)**, and finish by **putting g(x) back as u**.

Example 3. Find the integral of the following algebraic functions:

1. $\int (x^2+5)^3 2x\, dx$

Let $u = x^2 + 5$ $\quad du = 2xdx$

$$\int (x^2+5)^3 2x\, dx = \int u^3 du = \frac{u^{3+1}}{3+1} + C = \frac{u^4}{4} + C$$
$$= \boldsymbol{\frac{(x^2+5)^4}{4} + C}$$

2. $\int 12x^2\sqrt{4x^3+7}\ dx$

Let $u = 4x^3 + 7$ $\quad du = 12x^2 dx$

$$\int 12x^2\sqrt{4x^3+7}\ dx = \int u^{\frac{1}{2}} du = \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$$
$$= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2u^{\frac{3}{2}}}{3} + C$$

$$= \boldsymbol{\frac{2(4x^3+7)^{\frac{3}{2}}}{3} + C}$$

3. $\int \sqrt{2x^3+7}\, x^2 dx$

Let $u = 2x^3 + 7$ $\quad du = 6x^2 dx$

$$\frac{du}{6} = x^2 dx$$

$$\int \sqrt{2x^3+7}\, x^2 dx = \int u^{\frac{1}{2}} \frac{du}{6} = \frac{1}{6}\int u^{\frac{1}{2}} du = \frac{1}{6}\left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right] + C = \frac{1}{6}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] + C$$

$$= \frac{1}{6}\left[\frac{2u^{\frac{3}{2}}}{3}\right] + C = \frac{u^{\frac{3}{2}}}{9} + C$$

$$= \boldsymbol{\frac{(2x^3+7)^{\frac{3}{2}}}{9} + C}$$

4. $\int \frac{5x^2+1}{\sqrt[3]{(5x^3+3x-2)^2}} dx$

Let $u = 5x^3 + 3x - 2$ $\quad du = (15x^2+3)dx = 3(5x^2+1)dx$

$$\frac{du}{3} = (5x^2+1)dx$$

$$\int \frac{5x^2+1}{\sqrt[3]{(5x^3+3x-2)^2}} dx = \int \frac{\frac{du}{3}}{u^{\frac{2}{3}}} = \frac{1}{3}\int \frac{1}{u^{\frac{2}{3}}} du = \frac{1}{3}\int u^{-\frac{2}{3}} du = \frac{1}{3}\left[\frac{u^{-\frac{2}{3}+1}}{-\frac{1}{3}+1}\right] + C$$

$$= \frac{1}{3}\left[\frac{u^{\frac{1}{3}}}{\frac{1}{3}}\right] + C = \frac{1}{3}\left[3u^{\frac{1}{3}}\right] + C$$

$$= u^{\frac{1}{3}} + C$$

$$= \boldsymbol{\left(5x^3+3x-2\right)^{\frac{1}{3}} + C}$$

or

$$= \boldsymbol{\sqrt[3]{5x^3+3x-2}}$$

5. $\int (5x+2)^9 dx$

Let $u = 5x + 2$ $\quad du = 5dx$

$$\frac{du}{5} = dx$$

$$\int (5x+2)^9 dx = \int u^9 \frac{du}{5} = \frac{1}{5}\int u^9 du$$

$$= \frac{1}{5}\left(\frac{u^{9+1}}{9+1}\right) + C$$

$$= \frac{1}{5}\left(\frac{u^{10}}{10}\right) + C$$

$$= \frac{u^{10}}{50} + C$$

$$= \boldsymbol{\frac{(5x+2)^{10}}{50} + C}$$

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Activity 2
Integration by Substitution (Algebraic Functions)

I. Evaluate the following integrals. Show all the necessary solution.

1. $\int (3x+2)^5 dx$

2. $\int (5x-6)^{13} dx$

3. $\int 2x(4x^2-6)^9 dx$

4. $\int \sqrt[3]{1-3x}\,dx.$

5. $\int (20x+30)(x^2+3x-5)^9 \, dx$

6. $\int \frac{dx}{\sqrt{1+4x}}$

7. $\int \frac{x}{(x^2+4)^5} dx$

8. $\int \frac{x^2}{x^3+1}$

9. $\int (2x^3)\sqrt[3]{3x^3+4}\,dx$

10. $\int \frac{x+1}{x^2+2x-5} dx$

Example 4. Find the integral of the following trigonometric functions:

1. $\int \sin(3x)\, dx$

Let $u = 3x$ $\quad$ $du = 3dx$

$dx = \frac{du}{3}$

$\int \sin(3x)\, dx = \int \sin(u) \frac{du}{3} = \frac{1}{3} \int \sin(u)\, du = \frac{1}{3}(- \cos(u) + C$

$= -\frac{1}{3} \cos\,(u) + C$, but $u = 3x$

$= -\frac{\cos(3x)}{3} + C$

2. $\int \cos(5x)\, dx$

Let $u = 5x$ $\quad$ $du = 5dx$

$dx = \frac{du}{5}$

$\int \cos(5x)\, dx = \int \cos(u) \frac{du}{5} = \frac{1}{5} \int \cos(u)\, du = \frac{1}{5} \sin\,(u) + C$

$= \frac{1}{5}(\sin\,(5x) + C$

3. $\int \mathrm{x} \cos(x^2 + 5)\, dx$

Let $u = x^2 + 5$ $\quad$ $du = 2xdx$

$xdx = \frac{du}{2}$

$\int \mathrm{x} \cos(x^2 + 5)\, dx = \int \cos(u) \frac{du}{2} = \frac{1}{2} \int \cos(u)\, du$

$= \frac{1}{2} \sin(u) + C$

$= \frac{1}{2} \sin(x^2 + 5) + C$

4. $\int \sin^3(x) \cos\,(x)\, dx$

Let $u = \sin\,(x)$ $\quad$ $du = \cos\,(x) dx$

$\int \sin^3(x) \cos\,(x)\, dx = \int u^3 du = \frac{u^{3+1}}{3+1} + C$

$= \frac{u^4}{4} + C$

$= \frac{sin^4(x)}{4} + C$

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Activity 3
Integration by Substitution (Trigonometric Functions)

I. Evaluate the following integrals. Show all the necessary solution.

1. $\int \sin(x^2)\, 2x\, dx$

2. $\int \cos(x^3)\, 3x^2\, dx$

3. $\displaystyle\int 5\sin^4(x)\cos(x)dx$

4. $\displaystyle\int 2\sec^2(x)\tan(x)dx$

5. $\int \sin(x^2)\, 2x\, dx$

6. $\int \cos(x^3)\, 3x^2\, dx$

7. $\int \cot(3x+5)dx$

8. $\displaystyle\int \cos(2x+1)\, dx$

9. $\displaystyle\int \sin^{10}(x)\cos(x)\, dx$

10. $\displaystyle\int \frac{\sin(x)}{(\cos(x))^5}\, dx$

Example 5. Find the integral of the following exponential functions:

1. $\int x^2 e^{x^3} dx$

Let $u = x^3$ $\qquad du = 3x^2 dx$

$$x^2 dx = \frac{du}{3}$$

$$\int x^2 e^{x^3} dx = \int e^u \frac{du}{3} = \frac{1}{3}\int e^u du$$

$$= \frac{1}{3}e^u + C$$

$$= \boldsymbol{\frac{1}{3}e^{x^3} + C}$$

2. $\int e^x \sqrt{1 + e^x} dx$

Let $u = 1 + e^x$ $\qquad du = e^x dx$

$$\int e^x \sqrt{1 + e^x} dx = \int u^{\frac{1}{2}} du = \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2u^{\frac{3}{2}}}{3} + C$$

$$= \boldsymbol{\frac{2(1+e^x)^{\frac{3}{2}}}{3} + C}$$

3. $\int e^x (4e^x - 2)^2 dx$

Let $u = 4e^x - 2$ $\qquad du = 4e^x dx$

$$e^x dx = \frac{du}{4}$$

$$\int e^x (4e^x - 2)^2 dx = \int u^2 \frac{du}{4} = \frac{1}{4}\int u^2 du = \frac{1}{4}\left(\frac{u^{2+1}}{2+1}\right) + C$$

$$= \frac{1}{4}\left(\frac{u^3}{3}\right) + C = \frac{u^3}{12} + C$$

$$= \boldsymbol{\frac{(4e^x - 2)^3}{12} + C}$$

4. $\int 3x^2 e^{2x^3}$

Let $u = 2x^3$ $\qquad du = 6x^2 dx$

$$3x^2 dx = \frac{du}{2}$$

$$\int 3x^2 e^{2x^3} = \int e^u \frac{du}{2} = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C$$

$$= \boldsymbol{\frac{1}{2}e^{2x^3} + C}$$

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Activity 4
Integration by Substitution (Exponential Functions)

I. Evaluate the following integrals. Show all the necessary solution.

1. $\int \boldsymbol{2x^3e^{x^4}\,dx}$

2. $\int \boldsymbol{(25x^4+4x)e^{5x^5+2x^2}dx}$

3. $\int \boldsymbol{80x^3\cdot 3^{5x^4-2}dx}$

4. $\int \frac{20e^{5x}}{e^{5x}+3}dx$

5. $\int xe^{x^2}dx$

6. $\int \boldsymbol{(-6x-4)e^{3x^2+4x}dx}$

7. $\int \boldsymbol{(315x^4+35)e^{9x^5+5x}\,dx}$

8. $\int 36x^2e^{4x^3+3}dx$

9. $\int \boldsymbol{10\,sin(-2x)\cdot e^{cos(-2x)}dx}$

10. $\int \cos(5x)\,e^{\sin(5x)}dx$

Integration: The Basic Logarithmic Form

$$\int \frac{du}{u} = \ln|u| + C$$

Example 6. Evaluate the following:

1. $\int \frac{2x^3}{x^4+1} dx$

Let $u = x^4 + 1$ $\quad du = 4x^3 dx$

$2x^3 dx = \frac{du}{2}$

$$\int \frac{2x^3}{x^4+1} dx = \int \frac{du}{2u} = \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}\ln|u| + C$$

$$= \boldsymbol{\frac{1}{2}\ln\left|x^4 + 1\right| + C}$$

2. $\int \frac{3}{x} dx = 3\int \frac{1}{x} dx = \boldsymbol{3\ln|x| + C}$

3. $\int \frac{2}{2x+5} dx$

Let $u = 2x + 5$ $\quad du = 2dx$

$$\int \frac{2}{2x+5} dx = \int \frac{du}{u} = \ln|u| + C$$

$$= \boldsymbol{\ln|2x + 5| + C}$$

4. $\int \frac{x^2-2x+3}{x^3-3x^2+9x+27} dx$

Let $u = x^3 - 3x^2 + 9x + 27$ $\quad du = (3x^2 - 6x + 9)dx = 3(x^2 - 2x + 3)dx$

$\frac{du}{3} = (x^2 - 2x + 3)dx$

$$\int \frac{x^2-2x+3}{x^3-3x^2+9x+27} dx = \int \frac{du}{3u} = \frac{1}{3}\int \frac{1}{u} du$$

$$= \frac{1}{3}\ln|u| + C$$

$$= \boldsymbol{\frac{1}{3}\ln\left|x^3 - 3x^2 + 9x + 27\right| + C}$$

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Activity 5
Integration by Substitution (Logarithmic Functions)

I. Evaluate the following integrals. Show all the necessary solution.

1. $\int \frac{2}{2x+5}dx$

2. $\int \frac{3-4x}{6+3x-2x^2}dx$

3. $\int \frac{1-e^{-x}}{x+e^{-x}}dx$

4. $\int \frac{1}{x-2}dx$

5. $\int \frac{-2x}{3-x^2}dx$

6. $\int \frac{2x-4}{(x-2)^2}dx$

7. $\int \frac{x^2}{1-x^3}dx$

8. $\int \frac{3e^{2x}+3}{e^{2x}+2x}dx$

9. $\int \frac{-2x}{3-x^2}dx$

10. $\int \frac{x^2+1}{x^3+3x+7}dx$

Integration of Inverse Trigonometric Functions

The following integration formulas yield inverse trigonometric functions:

1. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C; a > 0$

2. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C\,; a > 0$

3. $\int \frac{du}{u\sqrt{a^2-u^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C\,, u > a > 0$

Example 7. Evaluate the following:

1. $\int \frac{dx}{\sqrt{25-x^2}} = \int \frac{dx}{\sqrt{5^2-x^2}}$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$$
$$\int \frac{dx}{\sqrt{5^2-x^2}} = \boldsymbol{\arcsin\left(\frac{x}{5}\right) + C}$$

2. $\int \frac{dx}{\sqrt{4-9x^2}} = \int \frac{dx}{\sqrt{2^2-(3x)^2}}$

Let $u = 3x$ $\qquad du = 3dx$

$$\frac{du}{3} = dx$$

$$\int \frac{\frac{du}{3}}{\sqrt{2^2-u^2}} = \frac{1}{3}\int \frac{dx}{\sqrt{2^2-u^2}}$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$$

$$\int \frac{dx}{\sqrt{2^2-(3x)^2}} = \frac{1}{3}\arcsin\left(\frac{u}{2}\right) + C = \boldsymbol{\frac{1}{3}\arcsin\left(\frac{3x}{2}\right) + C}$$

3. $\int \frac{1}{9+x^2}dx$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$$

$$\int \frac{dx}{3^2+x^2} = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

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Activity 6
Integration of Inverse Trigonometric Functions

I. Evaluate the following integrals. Show all necessary solution.

1. $\int \frac{dx}{16 + x^2}$

2. $\int \frac{dx}{\sqrt{1 - 16x^2}}$

3. $\int \frac{dx}{25 + 4x^2}$

4. $\int \frac{dx}{4 + x^2}$

5. $\int \frac{dx}{1 + 4x^2}$

6. $\int \frac{5dx}{3x\sqrt{4x^2 - 1}}$

7. $\int \frac{dx}{\sqrt{9 - x^2}}$

8. $\int \frac{x + 2}{\sqrt{4 - x^2}} dx$

9. $\int \frac{2dx}{x\sqrt{x^2 - 1}}$

10. $\int \frac{5x + 4}{9x^2 + 1} dx$

Definite Integral

A **definite integral** is a formal calculation of area beneath a function, using **infinitesimal slivers or stripes** of the region. Integrals may represent the (signed) area of a region, the accumulated value of a function changing over time, or the quantity of an item given its density. They were first studied by 17th-century mathematicians Isaac Newton and Gottfried Liebniz, who independently developed their own systems of integration. The modern notation follows from Liebniz's notes, and given a real-valued function f*f* and real numbers ***a* < *b***, the definite integral is written

$$\int_{a}^{b} f(x)\,dx$$

This value represents the signed area between the function *f*, the *x* – axis, and the lines *x* = *a* and *x* = *b*; regions above the x – axis have positive area, while regions below the *x* – axis have negative area.

By the fundamental theorem of calculus, given a function *f* defined on the interval [*a*, *b*] with antiderivative *F*,

$$\int_{a}^{b} f(x)\,dx = F(b) - F(a)$$

Integration Rules for Definite Integral

1. $\int_{b}^{a} f(x)dx = -\int_{a}^{b} f(x)dx$

2. $\int_{a}^{a} f(x)dx = 0$

3. $\int_{a}^{b} c\ dx = c(b-a)$ where *c* is a constant

4. $\int_{a}^{b}[f(x) \pm g(x)]dx = \int_{a}^{b} f(x)dx \pm \int_{a}^{b} g(x)dx$

5. $\int_{a}^{b} cf(x)\ dx = c\int_{a}^{b} f(x)\ dx$ where *c* is a constant

6. $\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx$

Definite integral of an even function

7. $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

Definite integral of an odd functions

8. $\int_{-a}^{a} f(x)dx = 0$

Example 1. Evaluate the following definite integral:

a. $\int_1^2 4xdx = \ 4\int_1^2 xdx$

$$= \frac{4x^2}{2}\Big|_1^2$$

$$= 2x^2|_1^2$$

$$= [2(2)^2] - [2(1)^2]$$

$$= 8 - 2$$

$$= \mathbf{6}$$

b. $\int_{-1}^2 (4x^2 + 3x)dx \ = 4\int_{-1}^2 x^2 dx + 3\int_{-1}^2 xdx$

$$= \left(\frac{4x^3}{3} + \frac{3x^2}{2}\right)\Big|_{-1}^2$$

$$= \left[\frac{4(2)^3}{3} + \frac{3(2)^2}{2}\right] - \left[\frac{4(-1)^3}{3} + \frac{3(-1)^2}{2}\right]$$

$$= \left[\frac{32}{3} + \frac{12}{2}\right] - \left[-\frac{4}{3} + \frac{3}{2}\right]$$

$$= \frac{\mathbf{50}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{6}}$$

$$= \frac{\mathbf{33}}{\mathbf{2}} = \mathbf{16.5}$$

c. $\int_0^2 (4x+1)^3 4dx$

Let $u = 4x + 1$ $\qquad du = 4dx$

$$\int_0^2 (4x+1)^3 4dx = \int_0^2 u^3 du$$

$$= \frac{u^4}{4}\Big|_0^2$$

$$= \frac{(4x+1)^4}{4}\Big|_0^2$$

$$= \left[\frac{(4(2)+1)^4}{4}\right] - \left[\frac{(4(0)+1)^4}{4}\right]$$

$$= \left[\frac{(9)^4}{4}\right] - \left[\frac{(1)^4}{4}\right]$$

$$= \left[\frac{6561}{4}\right] - \left[\frac{1}{4}\right]$$

$$= \mathbf{1640}$$

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Activity 7
Definite Integral

I. Evaluate the following definite integrals. Show all the necessary solution.

1. $\int_{1}^{2} x^3 dx$

6. $\int_{2}^{5} (2x^4 - 3x^2 + x - 1) dx$

2. $\int_{2}^{5} (5x^2 - 5x) dx$

7. $\int_{1}^{2} (5x^2 - 4) 5x dx$

3. $\int_{1}^{3} (6x^2 + 2x + 1) dx$

8. $\int_{1}^{3} \sqrt{2x + 1} \cdot 2 dx$

4. $\int_{1}^{2} (2x - 1)^4 dx$

9. $\int_{2}^{4} (x - 3)^6 dx$

5. $\int_{-2}^{1} (4x^2 - 3x + 2) dx$

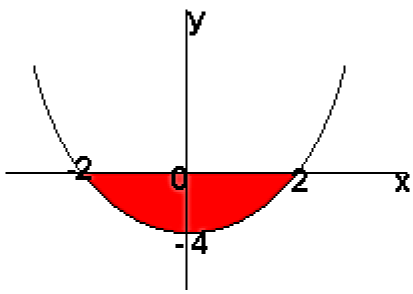
10. $\int_{2}^{3} (4x^4 - 3x^3 + 1) dx$

Area Under the Curve and Area Between Curves

The Area Under a Curve

The area under a curve between two points can be found by doing a definite integral between the two points. To find the area under the curve y = f(x) between x = a and x = b, integrate y = f(x) between the limits of a and b.

Example 1. What is the area between the curve $y = x^2 - 4$ and the x axis?



The shaded area is the area that we want.

We can easily work out that the curve crosses the x axis when x = -2 and x = 2. To find the area, therefore, we integrate the function between -2 and 2.

$$\boldsymbol{Area_{below} = -\int_a^b f(x)dx}$$

$$-\int_{-2}^{2}(x^2-4)dx = -\left[\int_{-2}^{2}x^2dx - \int_{-2}^{2}4dx\right]$$

$$= -\left(\frac{x^3}{3} - 4x\right)\Big|_{-2}^{2}$$

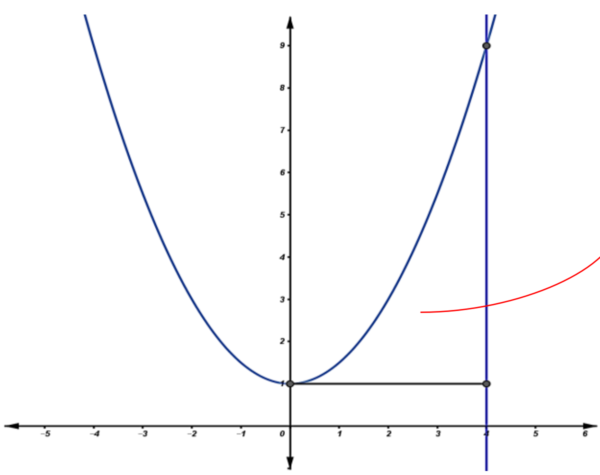
$$= -\left[\left(\frac{(2)^3}{3} - 4(2)\right) - \left(\frac{(-2)^3}{3} - 4(-2)\right)\right]$$

$$= -\left[\left(\frac{8}{3} - 8\right) - \left(\frac{-8}{3} + 8\right)\right]$$

$$= -\left[\left(-\frac{16}{3}\right) - \left(\frac{16}{3}\right)\right]$$

$$= \frac{32}{3} = \boldsymbol{10.67\ square\ unit}$$

Example 2. Find the area beneath the curve $y = \frac{1}{2}x^2 + 1$ and above the x-axis from o to 4.



If the area is above the curve, we use

$$Area = \int_a^b f(x)dx$$

$$Area = \int_0^4\left(\frac{1}{2}x^2 + 1\right)dx = \frac{1}{2}\int_0^4 x^2\,dx + \int_0^4 1dx$$

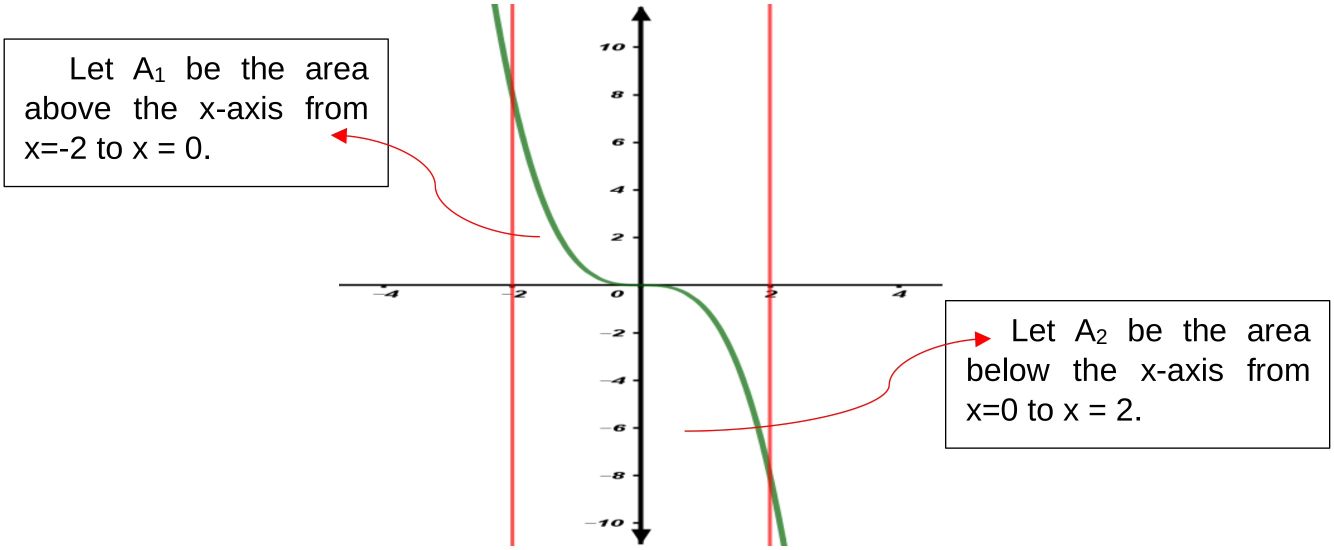
$$= \left(\frac{x^3}{6} + x\right)\Big|_0^4$$

$$= \left(\frac{4^3}{6} + 4\right) - \left(\frac{0^3}{6} + 0\right)$$

$$= \left(\frac{64}{6} + 4\right) + 0$$

$$= \frac{44}{3} = \boldsymbol{14.67\ square\ units}$$

Example 3. Find the area between the graph of $y = -x^3$ and the x – axis from x = –2 to x = 2.



$$A_1 = \int_{-2}^{0} -x^3 dx$$

$$= \frac{-x^4}{4}\Big|_{-2}^{0}$$

$$= \left(\frac{-(0)^4}{4}\right) - \left(\frac{-(-2)^4}{4}\right)$$

$$= 0 - \left(\frac{-16}{4}\right)$$

$$= \mathbf{4\ \mathit{square\ units}}$$

$$A_2 = -\int_{0}^{2} -x^3 dx$$

$$= -\left[\frac{-x^4}{4}\right]\Big|_{-2}^{0}$$

$$= -\left[\left(\frac{-(2)^4}{4}\right) - \left(\frac{-(0)^4}{4}\right)\right]$$

$$= -\left[\left(\frac{-16}{4}\right) - \left(\frac{0}{4}\right)\right]$$

$$= -\left(\frac{-16}{4}\right) = -(-4)$$

$$= \mathbf{4\ \mathit{square\ units}}$$

$$Area = A_1 + A_2$$

$$Area = 4 + 4$$

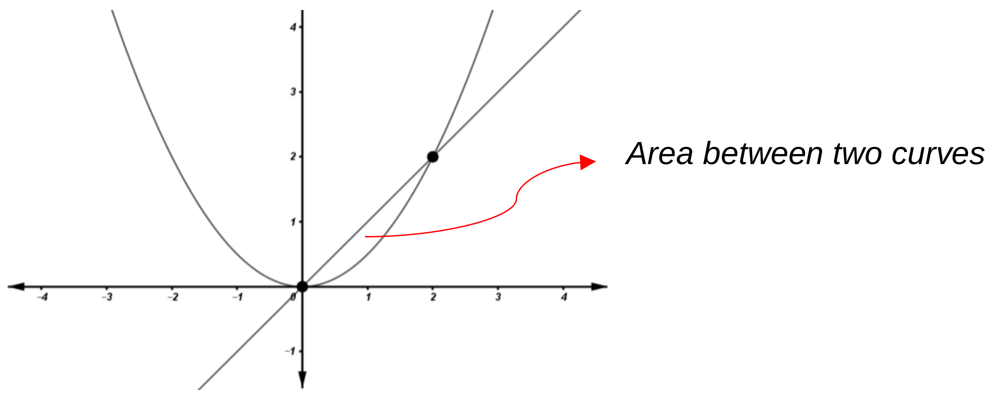
$$\boldsymbol{Area = 8\ square\ units}$$

Area Between Curves

The area between a positive-valued curve and the horizontal axis, measured between two values aa and bb (bb is defined as the larger of the two values) on the horizontal axis, is given by the integral from ***a*** to ***b*** of the function that represents the curve. The area between the graphs of two functions is equal to the integral of one function, ***f(x)***, minus the integral of the other function, ***g(x)***:

$A = \int_a^b \big(f(x) - g(x)\big)dx$ where f(x) is the curve with the greater y – value.

Example 1. Find the area between the two curves $f(x) = x$ and $f(x) = \frac{x^2}{2}$ over the interval x = 0 and x = 2.



Without the graphs of the functions, the points of intersection can be calculated using substitution.

$f(x) = x$ equation 1
$f(x) = \frac{x^2}{2}$ equation 2

Replace f(x) in the first equation by $\frac{x^2}{2}$ and solve for x.

$$\frac{x^2}{2} = x$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$\boldsymbol{x = 0 \text{ and } x = 2}$$

In $f(x) = x$, when x = 0, y = 0.
In $f(x) = \frac{x^2}{2}$, when x = 0, y = 0 too.
In $f(x) = x$, when x = 2, y = 2.
In $f(x) = \frac{x^2}{2}$, when x = 2, y = 2 too.

Based on the evaluation done, the points of intersections are (0, 0) and (2, 2).

$$\boldsymbol{Area = \int_0^2 \left(x - \frac{x^2}{2}\right) dx = \int_0^2 x dx - \frac{1}{2}\int_0^2 x^2 dx}$$

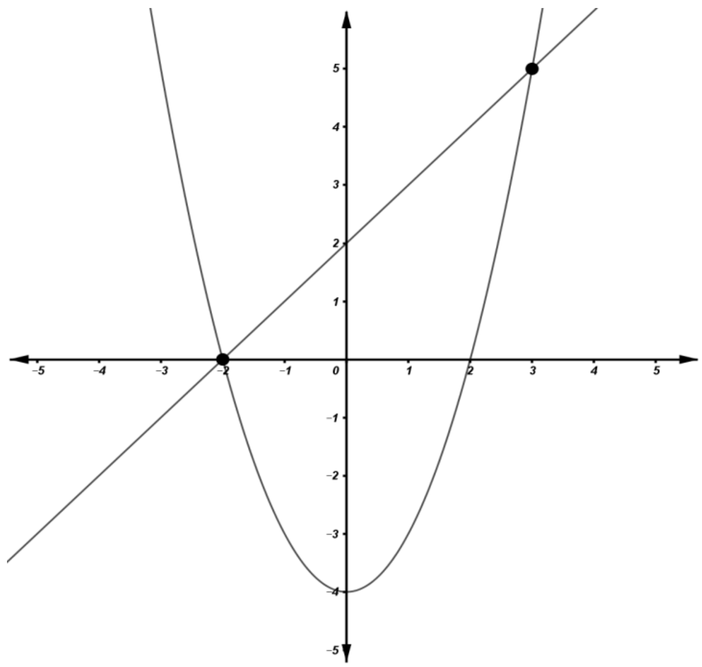
$$\boldsymbol{= \left(\frac{x^2}{2} - \frac{x^3}{6}\right)\Big|_0^2}$$

$$\boldsymbol{= \left(\frac{2^2}{2} - \frac{2^3}{6}\right) - \left(\frac{0^2}{2} - \frac{0^3}{6}\right)}$$

$$\boldsymbol{= \left(\frac{4}{2} - \frac{8}{6}\right) - 0}$$

$$\boldsymbol{= \frac{2}{3} = 0.67}\ \textbf{square unit}$$

Example 2. Find the area between $y = x^2 - 4$ and $y = x + 2$ from x = −2 and x = 3.



The graphs show that the point of intersection between the two curves are (-2, 0) and (3, 5).

$$Area = \int_{-2}^{3} [(x+2) - (x^2 - 4)]dx$$

$$= \int_{-2}^{3} (-x^2 + x + 6)dx$$

$$= -\int_{-2}^{3} x^2 dx + \int_{-2}^{3} x\, dx + \int_{-2}^{3} 6\, dx$$

$$= \left(\frac{-x^3}{3} + \frac{x^2}{2} + 6x\right)\Big|_{-2}^{3}$$

$$= \left(\frac{-(3)^3}{3} + \frac{(3)^2}{2} + 6(3)\right) - \left(-\frac{(-2)^3}{3} + \frac{(-2)^2}{2} + 6(-2)\right)$$

$$= \left(\frac{-27}{3} + \frac{9}{2} + 18)\right) - \left(\frac{8}{3} + \frac{4}{2} - 12\right)$$

$$= \left(\frac{27}{2}\right) - \left(-\frac{22}{3}\right) = \frac{27}{2} + \frac{22}{3} = \frac{81+44}{6}$$

$$= \frac{\mathbf{125}}{\mathbf{6}} = \boldsymbol{20.83\ square\ units}$$

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Activity 8
Areas Under the Curve and Between Curves

I. Answer the following and show all the necessary solution.

A. Find the area enclosed by the given curve, the x-axis, and the given lines.

1. $y = x^3 - 8,$ $\quad x = -1 \; and \; x = 2$

2. $y = x^2 - 6,$ $\quad x = -2 \; and \; x = 2$

B. Find the area between the given curves.

3. $y = x^2 \; and \; y = \sqrt{x}$

4. $y = x^2 - 4 \;\; and \; y = -2x$

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