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Answer 1

$\overline{(A \setminus B)} \cap \overline{(B \setminus A)} = \{x \mid x \in \overline{(A \setminus B)} \land x \in \overline{(B \setminus A)}\}$ by definition of intersection
$= \{x \mid x \notin (A \setminus B) \land x \notin (B \setminus A)\}$ by definition of complement
$= \{x \mid \neg(x \in (A \setminus B)) \land \neg(x \in (B \setminus A))\}$ by definition of d.n.b symbol
$= \{x \mid \neg\big(x \in (A \setminus B) \lor x \in (B \setminus A)\big)\}$ by De Morgan's equivalences
$= \{x \mid \neg\big((x \in A \land x \notin B) \lor (x \in B \land x \notin A)\big)\}$ by definition of difference
$= \{x \mid \neg\Big(\big(x \in A \lor (x \in B \land x \notin A)\big) \land \big(x \notin B \lor (x \in B \land x \notin A)\big)\Big)\}$ by distributive law
$= \{x \mid \neg\Big(\big((x \in A \lor x \in B) \land (x \in A \lor x \notin A)\big) \land \big((x \notin B \lor x \in B) \land (x \notin B \lor x \notin A)\big)\Big)\}$ by dist. law
$= \{x \mid \neg\Big(\big((x \in A \lor x \in B) \land (x \in U)\big) \land \big((x \in U) \land (x \notin B \lor x \notin A)\big)\Big)\}$ by negation law
$= \{x \mid \neg\big((x \in A \lor x \in B) \land (x \notin B \lor x \notin A)\big)\}$ by identity law
$= \{x \mid \neg\Big(\big(x \in A \lor x \in B\big) \land \big(\neg(x \in B) \lor \neg(x \in A)\big)\Big)\}$ by definition of d.n.b symbol
$= \{x \mid \neg\big(x \in A \lor x \in B\big) \lor \neg\big(\neg(x \in B) \lor \neg(x \in A)\big)\}$ by De Morgan's equivalences
$= \{x \mid \neg\big(x \in A \lor x \in B\big) \lor \big(\neg\neg(x \in B) \land \neg\neg(x \in A)\big)\}$ by De Morgan's equivalences
$= \{x \mid \neg(x \in A \lor x \in B) \lor (x \in B \land x \in A)\}$ by double negation law
$= \{x \mid (x \in B \land x \in A) \lor \neg(x \in A \lor x \in B)\}$ by commutative law
$= \{x \mid (x \in A \land x \in B) \lor \neg(x \in A \lor x \in B)\}$ by commutative law
$= \{x \mid (x \in A \cap B) \lor \neg(x \in A \cup B)\}$ by definition of intersection and union
$= \{x \mid (x \in A \cap B) \lor (x \notin A \cup B)\}$ by definition of d.n.b symbol
$= \{x \mid (x \in A \cap B) \lor (x \in \overline{A \cup B})\}$ by definition of complement
$= \{x \mid x \in \big((A \cap B) \cup \overline{(A \cup B)}\big)\}$ by definition of union
$= (A \cap B) \cup \overline{(A \cup B)}$

Answer 2

a) We want to show that if $x \in f((A \cap B) \times (C \cap D))$ then $x \in f(A \times C) \cap f(B \times D)$.
Suppose $x \in f((A \cap B) \times (C \cap D))$. Then there exists an ordered pair $(a, b) \in ((A \cap B) \times (C \cap D))$ such that $f(a, b) = x$.
Let's use some definitions and properties on this expression.

$$\begin{aligned}
(a,b) \in ((A \cap B) \times (C \cap D)) &\equiv (a \in (A \cap B)) \wedge (b \in (C \cap D)) \text{ by definition of cartesian product} \\
&\equiv (a \in A \wedge a \in B) \wedge (b \in C \wedge b \in D) \text{ by definition of intersection} \\
&\equiv a \in A \wedge a \in B \wedge b \in C \wedge b \in D \text{ since it's well-defined} \\
&\equiv a \in A \wedge b \in C \wedge a \in B \wedge b \in D \text{ by commutative law} \\
&\equiv (a \in A \wedge b \in C) \wedge (a \in B \wedge b \in D) \text{ by associative law} \\
&\equiv ((a,b) \in (A \times C)) \wedge ((a,b) \in (B \times D)) \text{ by definition of cartesian product}
\end{aligned}$$

Then from the last expression, we have the following:

$$(f(a,b) \in f(A \times C)) \wedge (f(a,b) \in f(B \times D))$$

which is equivalent to

$$f(a,b) \in (f(A \times C) \cap f(B \times D))$$

Recall that $f(a,b) = x$, hence $x \in f(A \times C) \cap f(B \times D)$ and our proof is complete.

b) Suppose gof is onto. Then for all $z \in Z$, there exist an $x \in X$ such that $gof(x) = z$.
Let $y = f(x)$. Then $y \in Y$ since $f\colon X \to Y$.
By definition of composite functions $z = gof(x) = g(f(x)) = g(y)$.
This shows g is onto since we took an arbitrary $z \in Z$ and found some $y \in Y$ such that $g(y) = z$.

Answer 3

First, suppose $x \in f^{-1}(A \cup B)$. Then $f(x) \in A \cup B$, which means $f(x) \in A \vee f(x) \in B$.
From this we have $x \in f^{-1}(A) \vee x \in f^{-1}(B)$, and therefore, $x \in f^{-1}(A) \cup f^{-1}(B)$.
Hence, $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$

Second, assume $x \in f^{-1}(A) \cup f^{-1}(B)$, which means $x \in f^{-1}(A) \vee x \in f^{-1}(B)$. From this we have $f(x) \in A \vee f(x) \in B$, which is equivalent to $f(x) \in A \cup B$. Therefore $x \in f^{-1}(A \cup B)$.
Hence, $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$

Since we have shown that both the following,

$$f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$$
$$f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$$

it follows that,

$$f^{-1}(A \cup B) \equiv f^{-1}(A) \cup f^{-1}(B)$$

Answer 4

a) Let $n = 2k + 1$ by hypothesis, where $k \in N$. Then we have $n^2 - 1 = 4k^2 + 4k = 4k(k + 1)$. Without loss of generality, let's assume k is even. This means $k = 2p$ for $p \in N$. Now we have $n^2 - 1 = 4k(k + 1) = 8p(2p + 1)$. Since $p(2p + 1)$ is an integer, it follows that $n^2 - 1$ is divisible by 8.

b) $(a - b)/2n = k$ for some integer k by hypothesis.
Then $a = 2kn + b$ and therefore $a^2 = b^2 + 4k^2n^2 + 4knb$.
From this we have $a^2 - b^2 = 4kn(kn + b)$. Since $(a^2 - b^2)/4n = k(kn + b)$ is an integer, it turns out that $a^2 - b^2$ is divisible by $4n$, hence $a^2 \equiv b^2 (\text{mod } 4n)$.

Answer 5

Let $h_1(n) = n^2$ and $h_2(n) = n$. Since big oh notation is about upper bounds;
$g_1(n) = g_2(n) = n^2$ such that $h_1(n)$ is $O(g_1(n))$ and $h_2(n)$ is $O(g_2(n))$.
Now we have $h_1(n)/h_2(n) = n$, and $O(g_1(n)/g_2(n)) = 1$.
This provides a counterexample, hence the statement in the question is false.

Answer 6

Let x, y, and z be defined as the following:

$$\begin{aligned} x &= p_1^{a_1} p_2^{a_2} ... p_n^{a_n} \\ y &= p_1^{b_1} p_2^{b_2} ... p_n^{b_n} \\ z &= p_1^{c_1} p_2^{c_2} ... p_n^{c_n} \end{aligned}$$

where $p_1, p_2, ..., p_n$ are all the primes occuring in the prime factorization of x, y, and z. Then we have,

$$\begin{aligned} xy &= p_1^{a_1+b_1} p_2^{a_2+b_2} ... p_n^{a_n+b_n} \\ xz &= p_1^{a_1+c_1} p_2^{a_2+c_2} ... p_n^{a_n+c_n} \end{aligned}$$

Least common multiple of these two is the following

$$\begin{aligned} lcm(xy, xz) &= p_1^{max(a_1+b_1, a_1+c_1)} p_2^{max(a_2+b_2, a_2+c_2)} ... p_n^{max(a_n+b_n, a_n+c_n)} \\ &= p_1^{a_1+max(b_1,c_1)} p_2^{a_2+max(b_2,c_2)} ... p_n^{a_n+max(b_n,c_n)} \\ &= (p_1^{a_1} p_2^{a_2} ... p_n^{a_n}) p_1^{max(b_1,c_1)} p_2^{max(b_2,c_2)} ... p_n^{max(b_n,c_n)} \end{aligned}$$

Hence it is equal to $x.lcm(y, z)$.