Student Information

Full Name : Murat TOPAK
Id Number : 2036218

Answer 1

Let $P(k)$ denote the statement $C^k - 1$ is divisible by $C - 1$.
We will use mathematical induction to show that $P(k)$ is true for $k \geq 1$.

BASIS STEP:
When $k = 1$, $P(k)$ is true since $C - 1$ is divisible by $C - 1$. This completes the basis step.

INDUCTIVE STEP:
We have to show that $P(k) \implies P(k+1)$. That means we should assume $C - 1 \mid C^k - 1$.
Under this assumption we must show that $C - 1 \mid C^{k+1} - 1$.
Assume $P(k)$ is true whenever $k \geq 1$, that is, assume $C - 1 \mid C^k - 1$ for $k \geq 1$. Then,

$C^k - 1 \equiv 0 \ (mod\ C - 1)$ //by definition of mod on inductive hypothesis
$C^{k+1} - C \equiv 0 \ (mod\ C - 1)$ //both sides multiplied with C
$C^{k+1} - 1 \equiv C - 1 \ (mod\ C - 1)$ //$C - 1$ added to both sides
$C^{k+1} - 1 \equiv 0 \ (mod\ C - 1)$ //rhs is 0 because $(C - 1)\ mod\ (C - 1) = 0$

Note that multiplication and addition in congruences are legal.
The last congruence above means $C - 1 \mid C^{k+1}$, which is $P(k+1)$. This completes the inductive step.

Since both basis and inductive steps have been proven, by mathematical induction, $C^k - 1$ is divisible by $C - 1$ whenever $k \geq 1$.

Answer 2

Let $P(n)$ be the proposition

$$(1 - \frac{1}{1+2}) \times (1 - \frac{1}{1+2+3}) \times ... \times (1 - \frac{1}{1+2+...+n}) = \frac{n+2}{3n}$$

We will prove that $P(n)$ is true for all $n \geq 2$ by using mathematical induction.

BASIS STEP:
When $n = 2$, $P(n)$ is true because $1 - \frac{1}{1+2} = \frac{2+2}{3 \times 2}$. This completes the basis step.

INDUCTIVE STEP:
Now we need to show that $P(n) \implies P(n+1)$.
Let's assume $P(n)$ is true whenever $n \geq 2$, that is, assume

$$(1 - \frac{1}{1+2}) \times (1 - \frac{1}{1+2+3}) \times ... \times (1 - \frac{1}{1+2+...+n}) = \frac{n+2}{3n}$$

Under this assumption we must show that

$$(1 - \frac{1}{1+2}) \times (1 - \frac{1}{1+2+3}) \times ... \times (1 - \frac{1}{1+2+...+n+n+1}) = \frac{(n+1)+2}{3(n+1)} = \frac{n+3}{3n+3}$$

Let's form the following product to accomplish this

$$(1 - \frac{1}{1+2}) \times (1 - \frac{1}{1+2+3}) \times ... \times (1 - \frac{1}{1+2+...+n}) \times (1 - \frac{1}{1+2+...+n+n+1})$$

Firstly, by inductive hypothesis, all the terms up to the last can be changed with $\frac{n+2}{3n}$.
Secondly, the denominator of the last term can be written as $\frac{(n+1)(n+2)}{2}$.
Thus we equivalently have the following

$$\begin{aligned}(1 - \frac{1}{1+2}) \times (1 - \frac{1}{1+2+3}) \times ... \times (1 - \frac{1}{1+2+...+n+n+1}) &= \frac{n+2}{3n} \times (1 - \frac{2}{(n+1)(n+2)}) \\ &= \frac{n+2}{3n} \times \frac{n(n+3)}{(n+1)(n+2)} \\ &= \frac{n+3}{3n+3}\end{aligned}$$

This completes the inductive step.
Since we proved both the basis and the inductive step, it follows that

$$(1 - \frac{1}{1+2}) \times (1 - \frac{1}{1+2+3}) \times ... \times (1 - \frac{1}{1+2+...+n}) = \frac{n+2}{3n}$$

is true whenever $n \geq 2$.

Answer 3

We have to select 4 points out of total 12. However, there are two cases to consider. Firstly, 4 points we have selected may be lying on the same edge, forming a line. Secondly, 3 points we have selected may be lying on the same edge, connected with another point not on that edge, forming a triangle. We have to substract both of these. Hence, the answer is

$$\binom{12}{4} - 3\binom{5}{4} - 3\binom{5}{3}\binom{7}{1} = 270$$

Answer 4

Let x_1, x_2, x_3, x_4, and x_5 be defined as the following

$x_1 = 2k + 1, \qquad x_2 = 2p + 2, \qquad x_3 = 2l + 1, \qquad x_4 = 2q + 2, \qquad x_5 = 2m + 1$

where $k, l, m, p, q \in N$. Then,

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= (2k+1) + (2p+2) + (2l+1) + (2q+2) + (2m+1) \\ &= 2(k+l+m+p+q) + 7 = 67 \end{aligned}$$

Now we have $k + l + m + p + q = 30$ and $k, l, m, p, q \in N$.

This is a simple stars and bars problem. Consider each plus sign as a bar, so we have 4 bars. 30 stars are to be distributed amongst these bars. There are no constraints. For example, one possible distribution is the following, which has equally six stars everywhere.

$$* * * * * * \,|\, * * * * * * | * * * * * * | * * * * * * | * * * * * *$$

It turns out this is actually a selection problem of 30 star positions out of 34 available -or 4 bar positions out of 34 available. In how many ways can we do this selection? The answer is,

$$\binom{34}{30} = \binom{34}{4} = 46376$$

Answer 5

Let a_n denote the valid arrangements of n hours. Let's also use the letters M,F,T to denote a must course, a free elective course, and a technical elective course, respectively.
An arrangement of n hours may end with either a must course, or an elective course(technical/free).

When it ends with an M, the next-to-last hour is also an M because a must course lasts for 2 hours(...MM). Thus we have a_{n-2} number of ways to end with a must course.

When it ends with an elective course, things get messy. There are two cases to consider:

Firstly, the two hours before an elective course may be filled with a must course(...MMT),(...MMF). We get $2a_{n-3}$ ways here.

Secondly, the next-to-last hour may be an elective course, and in this case, a must course has to precede these(...MMTT),(...MMTF),(...MMFT),(...MMFF). There are $4a_{n-4}$ valid arrangements generated from this case.

Consequently, $a_n = a_{n-2} + 2a_{n-3} + 4a_{n-4}$ for $n \geq 4$ and $a_0 = 1, a_1 = 2, a_2 = 5, a_3 = 4$

Answer 6

a)Let's first write $G(x)$ in terms of partial fractions.

$$G(x) = \frac{2x^4}{2x^3 - x^2 - 2x + 1} = \frac{2x^4}{(1-2x)(1-x^2)} = 2x^4 \frac{1}{(1-2x)(1-x)(1+x)}$$
$$= 2x^4 \left(\frac{A}{(1-2x)} + \frac{B}{(1-x)} + \frac{C}{(1+x)} \right)$$

When the corresponding system of linear equations is solved, it turns out $A = 4/3, B = -1/2, C = 1/6$, and thus

$$\begin{aligned} G(x) &= 2x^4 \left(\frac{4/3}{(1-2x)} + \frac{-1/2}{(1-x)} + \frac{1/6}{(1+x)} \right) \\ &= 2x^4 \left(\frac{4}{3} \sum_{k=0}^{\infty} (2x)^k - \frac{1}{2} \sum_{k=0}^{\infty} (x)^k + \frac{1}{6} \sum_{k=0}^{\infty} (-x)^k \right) \\ &= 2x^4 \left(\sum_{k=0}^{\infty} \left(\frac{2^{k+2}}{3} - \frac{1}{2} + \frac{(-1)^k}{6} \right) x^k \right) \\ &= \sum_{k=0}^{\infty} \left(\frac{2^{k+3}}{3} - 1 + \frac{(-1)^k}{3} \right) x^{k+4} \end{aligned}$$

Consequently, $a_k = \left(\frac{2^{k+3}}{3} - 1 + \frac{(-1)^k}{3} \right)$ for k=0,1,2...

b)

$$a_n = \frac{(6^n + 1)^2}{2^n} = \frac{(36^n + 2.6^n + 1)}{2^n} = (18^n + 2.3^n + \frac{1}{2^n})$$

$$\begin{aligned} G(x) = \sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} (18x)^n + 2 \sum_{n=0}^{\infty} (3x)^n + \sum_{n=0}^{\infty} \left(\frac{x}{2} \right)^n \\ &= \frac{1}{1-18x} + \frac{2}{1-3x} + \frac{1}{1-x/2} \end{aligned}$$