1a]. $\|\theta - \theta^*\|_2 \leq \frac{C_1}{\sqrt{s}} \|\theta - \theta_s\|_1 + C_2 \varepsilon$

→ As s ~~decreases~~ increases, sparsity of θ decreases.

→ So, $\frac{\|\theta - \theta_s\|_1}{\sqrt{s}}$ ~~is increases~~ decreases

→ But on the other hand, due to θ becoming less sparse, A starts to deviate from obeying RIP. ⇒ The bound $(1-\delta_s)\|\theta\|^2 \leq \|A\theta\|^2 \leq (1+\delta_s)\|\theta\|^2$ becomes loose ⇒ ~~$\delta_s \uparrow$~~ as

⇒ δ_s increases as 's' increases

∵ C_1 & C_2 are increasing functions of δ_s, C_1 and C_2 increase ~~as~~ with s.

→ Hence, this appears to counter the decrease in the factor $\frac{\|\theta - \theta_s\|_1}{\sqrt{s}}$. So, the bound remains the same.

Q1].

(2). $y = \Phi x + \eta$ where $\|\eta\|_2 \leq \varepsilon$

Now, $\varepsilon \geq k\sigma^2 m$ where $k \in \mathbb{R}$; σ^2 is the variance of the random noise distribution for η.

$\therefore$ If m increases, ε increases. & $\therefore$, this leads to addition of more noise to the signal & $\therefore$ might degrade the quality of the reconstructed image.

Also, ~~in case~~ ~~when~~ for this case,

$$\|\theta - \theta^*\|_2 \leq \frac{C_0}{\sqrt{s}} \|\theta - \theta_s\|_1 + C_1 \varepsilon$$

This also shows that greater the m, greater is the value of ε & $\therefore$ error between reconstructed image & the original one increases.

(3). Apart from obeying RIP, A must additionally preserve the squared diff. between any two S-sparse ~~matri~~ vectors i.e.

$$(1-\delta_{2S})\|\theta^{(1)}-\theta^{(2)}\|^2 \leq \|A(\theta^{(1)}-\theta^{(2)})\| \leq \|A\theta^{(1)}-\theta^{(2)}\|^2(1+\delta_{2S})$$

Closer is δ_{2S} to 0, ~~Joll~~ better tighter is the bound ~~the~~ and better is the reconstruction

$\therefore$ Theorem 3A i.e. $\delta_{2S} \leq 0.1$ is a more useful theorem.

(1). ε is actually given by —

$\|\eta\|_2 \leq \varepsilon$

It is given that η has non-zero magnitude

$\Rightarrow \|\eta\|_2 > 0 \Rightarrow \varepsilon > 0$ always

Hence, ε cannot be set to 0 if the noise vector η has non-zero magnitude

[Q2]. Coherence between Φ & Ψ is given by —

$$\mu(\Phi,\Psi) = \sqrt{n} \max_{i\in\{0,1,\dots,m-1\},\, j\in\{0,1,\dots,n-1\}} |\phi^{iT}\psi_j| \quad \text{— (A)}$$

where $\Phi \in \mathbb{R}^{m\times n}$, $\Psi \in \mathbb{R}^{n\times n}$, $\phi^i \in \mathbb{R}^{n\times 1}$ (i^{th} row of Φ), $\psi_j \in \mathbb{R}^{n\times 1}$ (j^{th} col. of Ψ)

(1) Now, the product $v^T\cdot w = |v||w|\cos\theta$ where $v, w \in \mathbb{R}^{n\times 1}$

In eqn (A), $\because |\cos\theta| \le 1$ & ~~we~~ $|\phi^i| = |\psi_j| = 1$ (as rows of Φ are normalized & cols. of Ψ are ~~orthogonal~~ orthonormal), $|\phi^{iT}\psi_j| = |\phi^i|\cdot|\psi_j|\cdot\cos\theta \le 1$

$$\max(|\phi^{iT}\psi_j|) = 1$$

$$\therefore \mu(\Phi,\Psi) \le \sqrt{n}$$

(2) Any vector $v \in \mathbb{R}^n$ can be expressed as a ~~sum~~ weighted sum of columns of Ψ as Ψ is an orthonormal matrix.

$\therefore$ using ψ as an orthonormal basis,

$$\phi^i = \sum_{k=1}^{n} \alpha_k \psi_k \quad \text{such that} \quad \sum_{k=1}^{n} \alpha_k^2 = 1 \text{ as}$$

ϕ^i is a unit vector

$$\Rightarrow \phi^{i^T} \cdot \psi_j = (\alpha_0 \psi_1 + \dots + \alpha_j \psi_j + \dots + \alpha_{n-1}\psi_{n-1}) \cdot \psi_j = \alpha_j$$

$$\Rightarrow \cancel{\mu(g} \; \mu(\Phi;\Psi) = \sqrt{n} \max_{j \in \{0,1,\dots,n-1\}} |\alpha_j|$$

For finding the lower bound, we need to minimize ~~[illegible]~~ α_j with the constraint that

$$\sum_{k=1}^{n} \alpha_k^2 = 1$$

$$\text{i.e. } J_j = \alpha_j - \lambda\left(\sum_{k=1}^{n} \alpha_k^2\right)$$

We need to find minimum for all j $(j \in \{1, 2, \dots, n\})$

$$J_j = \alpha_j - \lambda_j \left(\sum_{k=1}^{n} \alpha_k^2 \right)$$

$$\frac{\partial J_j}{\partial \alpha_j} = 1 - 2\lambda \alpha_j = 0 \Rightarrow \alpha_j = \frac{1}{2\lambda}$$

Now $\sum_{k=1}^{n} \alpha_k^2 = 1$

$$\Rightarrow \frac{1}{(2d)^2} + \frac{1}{(2d)^2} + \cdots \text{(n times)} = 1$$

$$\Rightarrow \frac{n}{4d^2} = 1 \Rightarrow d = \frac{\sqrt{n}}{2}$$

$$\Rightarrow \cancel{\alpha_i = 1} \Rightarrow \alpha_j = \frac{1}{2d} = \frac{1}{\sqrt{n}}$$

This is true $\forall \alpha_j$ (where $j \in \{1, 2, \ldots, n\}$

$$\therefore \alpha_j = \frac{1}{\sqrt{n}} \quad \forall \{j = 1, 2, \ldots, n\}$$

$$\Rightarrow \max_j |\alpha_j| = \frac{1}{\sqrt{n}}$$

$$\Rightarrow \mu(\Phi, \Psi) = \sqrt{n} \cdot \frac{1}{\sqrt{n}} = 1$$

$$\therefore \mu(\Phi, \Psi) \geq 1$$

$$\therefore 1 \leq \mu(\Phi, \Psi) \leq \sqrt{n}$$

3]. $y = \Phi . x$

(a). • If $m = 1$, $y = \Phi x$ where $\Phi \in \mathbb{R}^{1 \times n}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$

i.e. $y = \Phi_i x_i$ where x_i is the only non-zero element of x. If index is not given, we ~~cannot~~ will not know which element of Φ, Φ_k ($k \in \{1, 2, \ldots, n\}$) to divide y with to ~~get~~ recover x_i.

~~will not~~

• If index is given, we will know 'i' & $\therefore \Phi_i$

$$\Rightarrow x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ y/\Phi_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ row}$$

(b). $m = 2$ and x has only one non-zero element x_I in the I^{th} row;

$$\Phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1I} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2I} & \cdots & \phi_{2n} \end{bmatrix}$$

$\phi_{jk} \in \mathbb{R} \;\forall\; j \in \{1, 2\}$ & $k \in \{1, \ldots, n\}$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \Phi x$$

So, we can perform the calculation $\frac{y_1}{\phi_{1j}} . \phi_{2j}$ for every $j \in \{1, 2, \ldots, n\}$ until $\frac{y_1}{\phi_{1j}} . \phi_{2j} = y_2$. This will happen at $j = I$. Then $x_I = \frac{y_1}{\phi_{1I}} = \frac{y_2}{\phi_{2I}}$.

But rows ϕ_1 & ϕ_2 must not be dependent. If $\phi_1 = \phi_2 . a$ for some scalar a, $\frac{y_1 \phi_{2j}}{\phi_{1j}} = \frac{y_1}{a} = y_2 \;\forall j$ and we won't be able to recover x.

(c). For reconstruction, the problem P0 : $\min \|\theta\|_0$ s.t. $y = \Phi x$ is guaranteed to have a unique solution if 2S columns of Φ are linearly independent (S is the no. of non-zero elements in x)

In this qn., $\because S = 2 \Rightarrow 2S = 4$

$\Rightarrow$ Φ needs to have 4 independent columns.

But, $\because$ rank $\leq \min(m, n)$, $m < n$ & $m = 3$,

$$\text{rank}(\Phi) \leq 3$$

This means ~~the~~ Φ cannot have 4 independent columns. Hence, x cannot be uniquely estimated.

d] $m = 4 \Rightarrow \text{rank}(\Phi) \leq 4$. So, it is guaranteed that x can be recovered uniquely if Φ has 4 independent columns.

The 4 columns can be found out iteratively by considering sets of 4 columns at once and solving the foll². eqⁿ :-

$$\alpha_1 \Phi_a + \alpha_2 \Phi_b + \alpha_3 \Phi_c + \alpha_4 \Phi_d = 0$$

$a, b, c, d \in \{1, 2, \ldots, n\}$ & $a \neq b \neq c \neq d$

the ~~pair~~ set of columns for which $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, ~~are~~ is selected. Then consider $\Phi_{new} = [\Phi_a\ \Phi_b\ \Phi_c\ \Phi_d]$

~~hie sel~~ So, $x = \Phi_{new}^{-1} y$

Unique x is guaranteed if Φ has $2S = 4$ independent columns.

4]. $J_1(v) = \|v\|_1 + d_1(\|y - Av\|_2 - e + a^2)$

$J_2(v) = \|Av - y\|_2 + d_2(\|v\|_1 - t + b^2)$

Consider $t' = \|x\|_1$

It is given that x is a unique minimizer of P_1 ($J_1(v)$). Consider a vector $z \in \mathbb{R}^n$

s.t. $\|z\|_1 \le t'$ i.e. $\|z\|_1 \le \|x\|_1$ — ①

$\because x$ is a unique ~~minimize~~ minimizer of P_1,

$J_1(x) \le J_1(z)$ — ②

From ① & ②

$\|x\|_1 + d_1(\|y - Ax\|_2 - e + a^2) \le \|z\|_1 + d_1(\|y - Az\|_2 - e + a^2) \le \|x\|_1 + d_1(\|y - Az\|_2 - e + a^2)$

$\cancel{\|x\|_1} + d_1(\|y - Ax\|_2 \cancel{- e} \cancel{+ a^2}) \le$

$\cancel{\|x\|_1} + d_1(\|y - Az\|_2 \cancel{- e} \cancel{+ a^2})$

$\Rightarrow \|y - Ax\|_2 \le \|y - Az\|_2$

$\Rightarrow x$ ~~als~~ is also a unique minimizer of Q_2.

6]

9. $E_u = \sum_{t=1}^{T} C_t \cdot F_t$ where $E_u, F_t, C_t \in \mathbb{R}^{H \times W}$ —— 1

elemental product

Now, E_u must be vectorized to form y.

Consider # pixels $= n$, $y \in \mathbb{R}^n$

C_t has 1's in those ~~pixel~~ positions where pixels are sensed by the array.

We can find $\mathrm{diag}(C_t)$ by placing these 1's at position's corresponding to r^{th} pixel i.e. if r^{th} pixel ~~must be~~ is sensed by the DMD array

$$C_{t_{r,r}} = 1 \; ; \; \mathrm{diag}(C_t) \in \mathbb{R}^{n \times n}$$

- Then $y_t = \mathrm{diag}(C_t) \cdot \Psi \cdot \theta_t \quad \forall \, t \in \{1, 2, \ldots, T\}$

- So $\Phi = [\mathrm{diag}(C_1) \,|\, \mathrm{diag}(C_2) \,|\, \cdots \,|\, \mathrm{diag}(C_T)]$

$\Phi \in \mathbb{R}^{n \times Tn}$

- $y = \sum_{t=1}^{T} y_t \Rightarrow y \in \mathbb{R}^{n \times 1}$ (from eqn 1, $y =$ vectorized E_u)

- $\theta = [\theta_1^T \,|\, \theta_2^T \,|\, \cdots \,|\, \theta_T^T]^T$

$\Rightarrow \theta \in \mathbb{R}^{Tn \times 1}$ (as $\theta_t \in \mathbb{R}^{n \times 1}$)

$\therefore \; y = \Phi \theta$

d). x := i^{th} patch from the video of size $p \times p \times T$

$A = [\mathrm{diag}(C_{1p}) \mid \mathrm{diag}(C_{2p}) \mid \ldots \mid \mathrm{diag}(C_{Tp})]$

where C_{tp} is the $p \times p$ patch corresponding to x.

x needs to be vectorized to form

$$x_v \in \mathbb{R}^{Tp^2 \times 1}$$

If $x_{v_t} = \Psi . \theta_t$ where x_{v_t} is the ~~$p \times p$~~ vectorized form of the $p \times p$ patch of the t^{th} frame. (t = 1, 2, ..., T)

$$\Rightarrow A = [\mathrm{diag}(C_{1p}).\Psi \mid \mathrm{diag}(C_{2p})\Psi \mid \ldots \mid \mathrm{diag}(C_{Tp}).\Psi]$$

$$\& \ \theta = [\theta_1^T \mid \theta_2^T \mid \cdots \mid \theta_T^T]^T, \ \theta \in \mathbb{R}^{Tp^2 \times 1}$$

$$E_{u_p} = \sum_{t=1}^{T} C_{tp} \, x(:,:,t)$$

$$y = \text{vectorize}(E_u) \Rightarrow y \in \mathbb{R}^{p^2}$$

So $y = A\theta$

- η is a Gaussian random vector

$\Rightarrow \eta_i \sim \mathcal{N}(0, \sigma^2)$

$\|\eta\|_2^2$ is a chi-square random variable

With very high probability, $\|\eta\|_2^2$ will lie within $m \cdot (3\sigma)^2 = 9\sigma^2 m$

$\therefore$ let $\varepsilon \geq 9\sigma^2 m$.

error bound

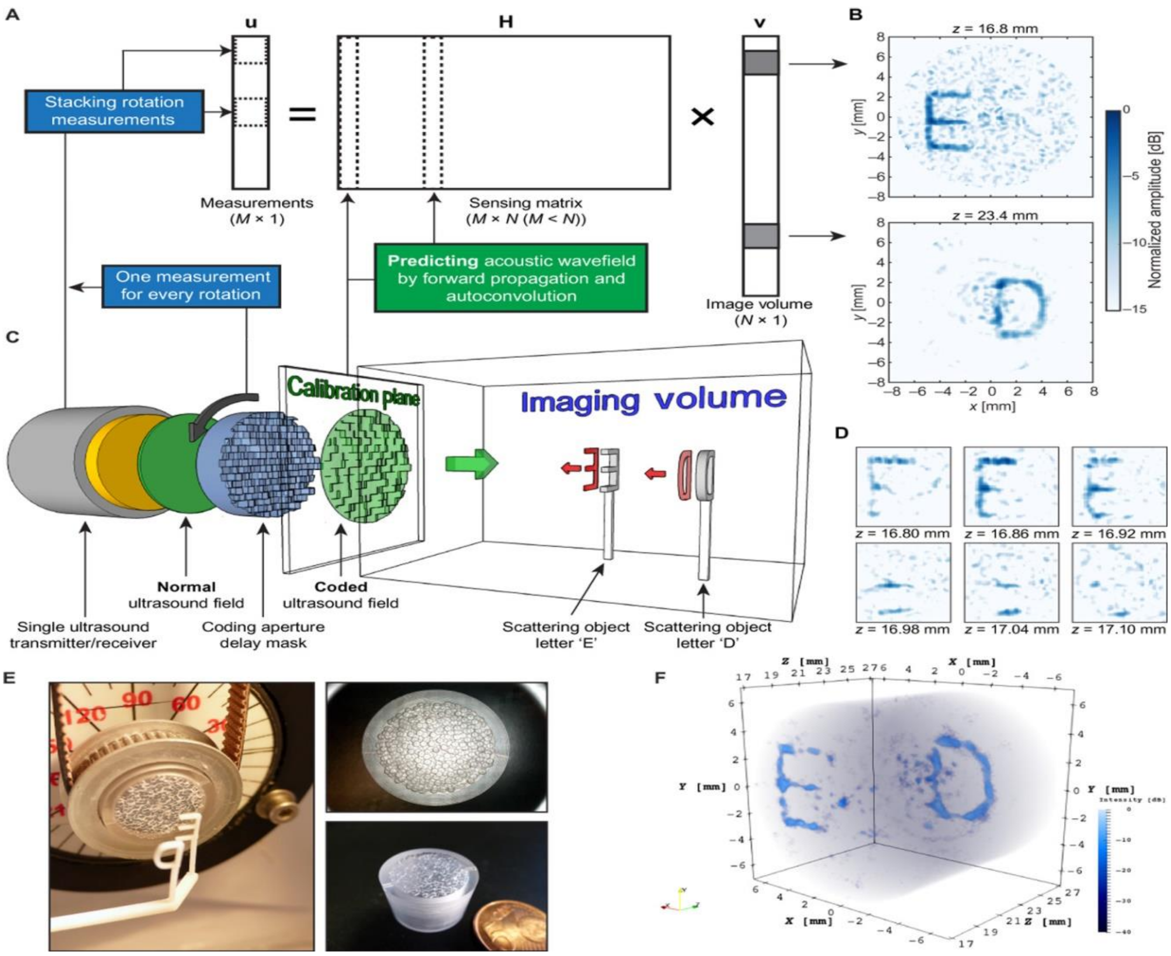
Picture Based Answers

Answer (5).

a). Mention the title of the paper, where and when it was published, which venue(name of journal or conference or workshop) and include a link to the paper.

Title: Compressive 3d Ultrasound imaging using a Single Sensor
Journal: Science Advances Vol.3, No. 12
Date: 8 December 2017
Link: https://www.science.org/doi/10.1126/sciadv.1701423

b). Very briefly describe the hardware architecture used in the paper. You may refer to the figures from the paper itself.



Compressive 3D ultrasound imaging using a single sensor

(**A**) Schematic sketch of the signal. Each column of the observation matrix **H** contains the ultrasound pulse-echo signal that is associated with a pixel in 3D space, which is contained in the image vector **v.** By rotating the coding mask in front of the sensor, we obtain new measurements that can be stacked as additional entries in the measurement vector **u** and additional rows in **H.**

(**C**) Schematic overview of the complete imaging setup. A single sensor transmits a phase uniform ultrasound wave through a coding mask that enables the object information (two

plastic letters “E” and “D”) to be compressed to a single measurement. Rotation of the mask enables additional measurements of the same object.
(**D**) Reconstruction of the letter “E” in six adjacent z slices. A small tilt of the letter (from top left corner to bottom right corner) can be observed, demonstrating the potential 3D imaging capabilities of the proposed device.
(**E**) Image showing the two 3D-printed letters and the plastic coding mask with a rubber band for rotating the mask over the sensor. The two right-hand panels show close-ups of the plastic coding mask.
(**F**) 3D rendering of the complete reconstructed image vector **v**, obtained by BPDN. The images shown in (B) and (D) were obtained using 72 evenly spaced mask rotations, and the full 3D image in (F) was obtained using only 50 evenly spaced rotations to reduce the total matrix size.

c). What reconstruction technique or cost function does the paper adopt for the sake of compressive reconstruction in this application?
They are solving the Basis Pursuit L1 norm cost function to do the compressive reconstruction.

$$\text{BP: } \min\left|\left|\boldsymbol{\theta}\right|\right|_1 \text{such that } \left|\left|\mathbf{y} - \boldsymbol{\phi}\boldsymbol{\psi}\boldsymbol{\theta}\right|\right|_2^2 \leq \epsilon$$

6.
Cars.avi

a. **T = 3**

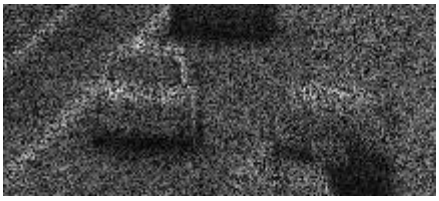
Coded Snapshot 1

t = 1

t = 2

t = 3

b. T = 5

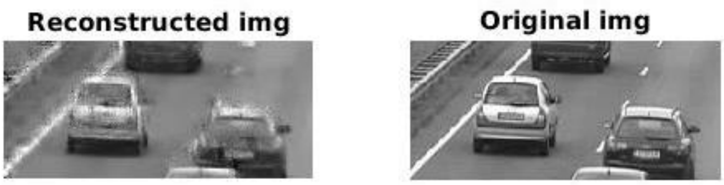


Coded Snapshot

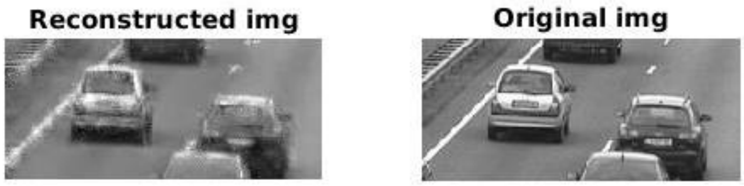
t = 1

t = 2

t = 3



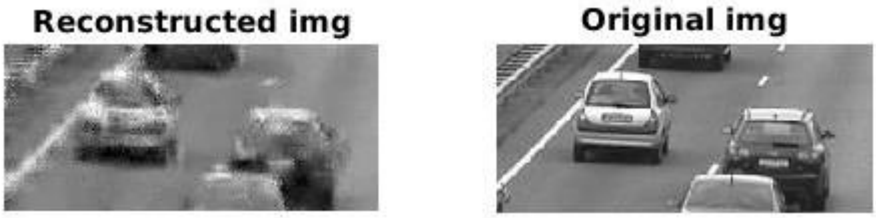
t = 4



t = 5

c. T = 7

Coded Snapshot



t = 1

t = 2

t = 3

t = 4

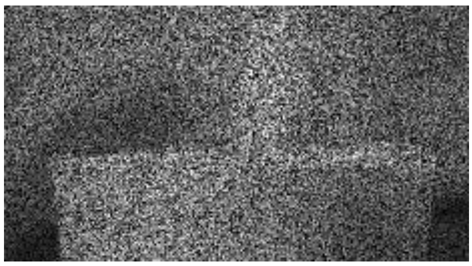
t = 5

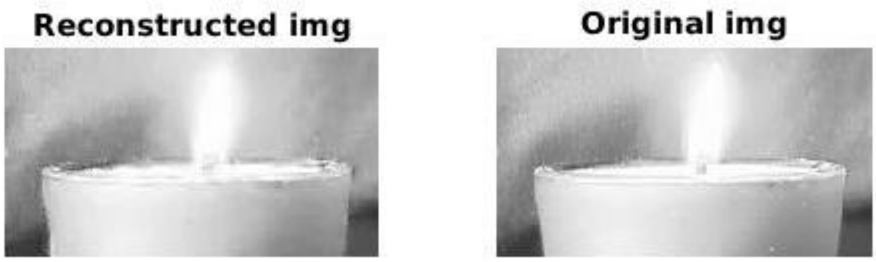
t = 6

t = 7

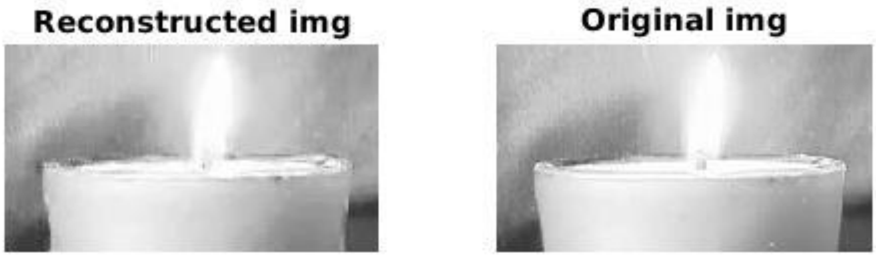
Flame:

a. T = 5

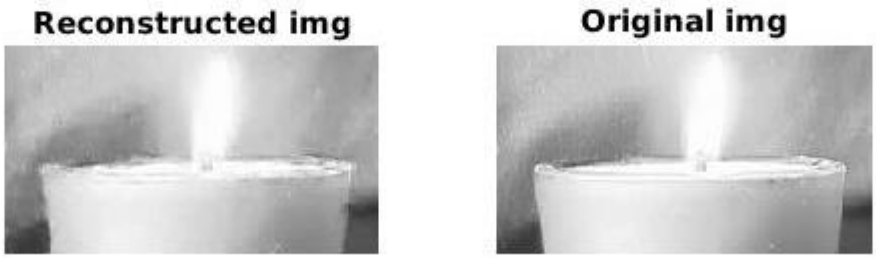

Coded Snapshot



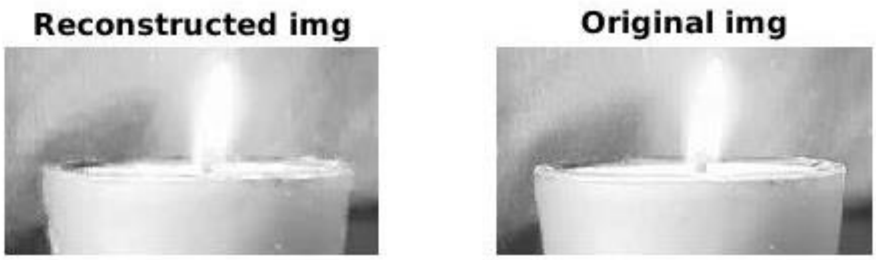
t = 1



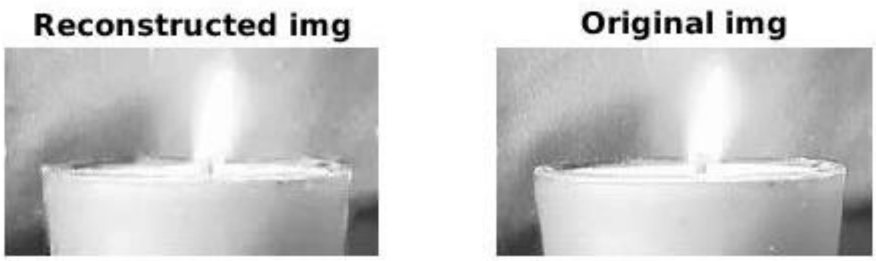
t = 2



t = 3



t = 4



t = 5

d).

RMSE:-
Cars video:
T = 3 0.012601
T = 5 0.021728
T = 7 0.034161
Flames video:
T = 5 0.031697