AA203: Optimal and Learning-based Control Course Notes

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4 Indirect Methods

4.1 Calculus of Variations

We will begin by restating the optimal control problem. We will to find an admissible control sequence $\boldsymbol{u}^*$ which causes the system

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}(t), \boldsymbol{u}(t), t) \tag{1}$$

to follow an *admissible* trajectory $\boldsymbol{x}^*$ that minimizes the functional

$$J = c_f(\boldsymbol{x}(t_f), t_f) + \int_{t_0}^{t_f} c(\boldsymbol{x}(t), \boldsymbol{u}(t), t)dt. \tag{2}$$

To find the minima of functions of a finite number of real numbers, we rely on the first order optimality conditions to find candidate minima, and use higher order derivatives to determine whether a point is a local minimum. Because we are minimizing a function that maps from some n dimensional space to a scalar, candidate points have zero gradient in each of these dimensions. However, in the optimal control problem, we have a cost *functional*, which maps functions to scalars. This is immediately problematic for our first order conditions — we are required to check the necessary condition at infinite points. The necessary notion of optimality conditions for functionals is provided by calculus of variations.

Concretely, we define a functional J as a rule of correspondence assining each function $\boldsymbol{x}$ in a class Ω (the domain) to a unique real number. The functional J is linear if and only if

$$J(\alpha_1\boldsymbol{x}_1 + \alpha_2\boldsymbol{x}_2) = \alpha_1 J(\boldsymbol{x}_1) + \alpha_2 J(\boldsymbol{x}_2) \tag{3}$$

for all $\boldsymbol{x}_1, \boldsymbol{x}_2, \alpha_1\boldsymbol{x}_1 + \alpha_2\boldsymbol{x}_2$ in Ω. We must now define a notion of "closeness" for functions. Intuitively, two points being close together has an immediate geometric interpretation. We

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first define the norm of a function. The norm of a function is a rule of correspondence that assigns each $\boldsymbol{x} \in \Omega$, defined over $t \in [t_0, t_f]$, a real number. The norm of $\boldsymbol{x}$, which we denote $\|\boldsymbol{x}\|$, satisfies:

1. $\|\boldsymbol{x}\| \geq 0$, and $\|\boldsymbol{x}\| = 0$ iff $\boldsymbol{x}(t) = 0$ for all $t \in [t_0, t_f]$
2. $\|\alpha \boldsymbol{x}\| = |\alpha| \|\boldsymbol{x}\|$ for all real numbers α
3. $\|\boldsymbol{x}_1 + \boldsymbol{x}_2\| \leq \|\boldsymbol{x}_1\| + \|\boldsymbol{x}_2\|$.

To compare the closeness of two functions $\boldsymbol{y}, \boldsymbol{z}$, we let $\boldsymbol{x}(t) = \boldsymbol{y}(t) - \boldsymbol{z}(t)$. Thus, for two identical functions, $\|\boldsymbol{x}\|$ is zero. Generally, a norm will be small for "close" functions, and large for "far apart" functions. However, there exist many possible definitions of norms that satisfy the above conditions.

4.1.1 Extrema for Functionals

A functional J with domain Ω has a local minimum at $\boldsymbol{x}^*(t) \in \Omega$ if there exists an $\epsilon > 0$ such that $J(\boldsymbol{x}(t)) \geq J(\boldsymbol{x}^*(t))$ for all $\boldsymbol{x}(t) \in \Omega$ such that $\|\boldsymbol{x}(t) - \boldsymbol{x}^*(t)\| < \epsilon$. Maxima are defined similarly, just with $J(\boldsymbol{x}(t)) \leq J(\boldsymbol{x}^*(t))$.

Analogously to optimization of functions, we define the variation of the functional as

$$\Delta J(\boldsymbol{x}(t), \delta \boldsymbol{x}(t)) := J(\boldsymbol{x}(t) + \delta \boldsymbol{x}(t)) - J(\boldsymbol{x}(t)) \tag{4}$$

where $\delta \boldsymbol{x}(t)$ is the *variation* of $\boldsymbol{x}(t)$. The increment of a functional can be written as

$$\Delta J(\boldsymbol{x}, \delta \boldsymbol{x}) = \delta J(\boldsymbol{x}, \delta \boldsymbol{x}) + g(\boldsymbol{x}, \delta \boldsymbol{x}) \|\delta \boldsymbol{x}\| \tag{5}$$

where δJ is linear in $\delta \boldsymbol{x}$. If

$$\lim_{\|\delta \boldsymbol{x}\| \to 0} \{g(\boldsymbol{x}, \delta \boldsymbol{x})\} = 0 \tag{6}$$

then J is said to be differentiable on $\boldsymbol{x}$ and δJ is the variation of J at $\boldsymbol{x}$. We can now state the *fundamental theorem of the calculus of variations.*

Theorem 4.1 (Fundamental Theorem of CoV)**.** *Let $\boldsymbol{x}(t)$ be a vector function of t in the class Ω, and $J(\boldsymbol{x})$ be a differentiable functional of $\boldsymbol{x}$. Assume that the functions in Ω are not constrained by any boundaries. If $\boldsymbol{x}^*$ is an extremal, the variation of J must vanish at $\boldsymbol{x}^*$, that is $\delta J(\boldsymbol{x}^*, \delta \boldsymbol{x}) = 0$ for all admissible $\delta \boldsymbol{x}$ (i.e. such that $\boldsymbol{x} + \delta \boldsymbol{x} \in \Omega$).*

Proof. [Kir12], Section 4.1. □

We will now look at how calculus of variations may be leveraged to approach practical problems. Let $\boldsymbol{x}$ be a continuous function in C^1. We would like to find a function $\boldsymbol{x}^*$ for which the functional

$$J(\boldsymbol{x}) = \int_{t_0}^{t_f} g(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), t) dt \tag{7}$$

has a relative extremum. We will assume $g \in C^2$, that t_0, t_f are fixed, and x_0, x_f are fixed. Let $\boldsymbol{x}$ be any curve in Ω, and we will write the variation δJ from the increment

$$\begin{aligned}\Delta J(\boldsymbol{x}, \delta\boldsymbol{x}) &= J(\boldsymbol{x} + \delta\boldsymbol{x}) - J(\boldsymbol{x}) &(8)\\ &= \int_{t_0}^{t_f} g(\boldsymbol{x} + \delta\boldsymbol{x}, \dot{\boldsymbol{x}} + \delta\dot{\boldsymbol{x}}, t)dt - \int_{t_0}^{t_f} g(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)dt &(9)\\ &= \int_{t_0}^{t_f} g(\boldsymbol{x} + \delta\boldsymbol{x}, \dot{\boldsymbol{x}} + \delta\dot{\boldsymbol{x}}, t) - g(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)dt. &(10)\end{aligned}$$

Expanding via Taylor series, we get

$$\Delta J(\boldsymbol{x}, \delta\boldsymbol{x}) = \int_{t_0}^{t_f} g(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) + \underbrace{\frac{\partial g}{\partial \boldsymbol{x}}}_{g_{\boldsymbol{x}}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)\delta\boldsymbol{x} + \underbrace{\frac{\partial g}{\partial \dot{\boldsymbol{x}}}}_{g_{\dot{\boldsymbol{x}}}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)\delta\dot{\boldsymbol{x}} + o(\delta\boldsymbol{x}, \delta\dot{\boldsymbol{x}}) - g(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)dt \quad (11)$$

which yields the variation

$$\delta J = \int_{t_0}^{t_f} g_{\boldsymbol{x}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)\delta\boldsymbol{x} + g_{\dot{\boldsymbol{x}}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)\delta\dot{\boldsymbol{x}}\ dt. \quad (12)$$

Integrating by parts, we have

$$\delta J = \int_{t_0}^{t_f} \left[g_{\boldsymbol{x}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) - \frac{d}{dt} g_{\dot{\boldsymbol{x}}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)\right] \delta\boldsymbol{x}\delta t + [g_{\dot{\boldsymbol{x}}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)\delta\boldsymbol{x}(t)]_{t_0}^{t_f}. \quad (13)$$

We have assumed $\boldsymbol{x}(t_0), \boldsymbol{x}(t_f)$ given, and thus $\delta\boldsymbol{x}(t_0) = 0$, $\delta\boldsymbol{x}(t_f) = 0$. Considering an extramal curve, applying the CoV theorem yields

$$\int_{t_0}^{t_f} \left[g_{\boldsymbol{x}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) - \frac{d}{dt} g_{\dot{\boldsymbol{x}}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)\right] \delta\boldsymbol{x}\delta t. \quad (14)$$

We can now state the fundamental lemma of CoV.

Lemma 4.2 (Fundamental Lemma of CoV)**.** *If a function h is continuous and*

$$\int_{t_0}^{t_f} h(t)\delta\boldsymbol{x}(t)dt = 0 \quad (15)$$

for every function $\delta\boldsymbol{x}$ that is continuous in the interval $[t_0, t_f]$, then h must be zero everywhere in the interval $[t_0, t_f]$.

Proof. [Kir12], Section 4.2. □

Applying the fundamental lemma, we find that a necessary condition for $\boldsymbol{x}^*$ being an extremal is

$$g_{\boldsymbol{x}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) - \frac{d}{dt} g_{\dot{\boldsymbol{x}}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) = 0 \quad (16)$$

for all $t \in [t_0, t_f]$, which is the *Euler equation.* This is a nonlinear, time-varying second-order ordinary differential equation with split boundary conditions (at $\boldsymbol{x}(t_0)$ and $\boldsymbol{x}(t_f)$).

4.1.2 Generalized Boundary Conditions

In the previous subsection, we assumed that $t_0, t_f, \boldsymbol{x}(t_0), \boldsymbol{x}(t_f)$ were all given. We will now relax that assumption. In particular, t_f may be fixed or free, and each component of $\boldsymbol{x}(t_f)$ may be fixed or free.

We begin by writing the variation around $\boldsymbol{x}^*$

$$\begin{aligned}\delta J = &\left[g_{\dot{\boldsymbol{x}}}(\boldsymbol{x}^*(t_f), \dot{\boldsymbol{x}}^*(t_f), t_f)\right] \delta \boldsymbol{x}(t_f) + \left[g(\boldsymbol{x}^*(t_f), \dot{\boldsymbol{x}}^*(t_f), t_f)\right] \delta t_f \\ &+ \int_{t_0}^{t_f} \left[g_{\boldsymbol{x}}(\boldsymbol{x}^*, \dot{\boldsymbol{x}}^*, t) - \frac{d}{dt} g_{\dot{\boldsymbol{x}}}(\boldsymbol{x}^*, \dot{\boldsymbol{x}}^*, t)\right] \delta \boldsymbol{x} \delta t\end{aligned} \tag{17}$$

by using the same integration by parts approach as before. Note that for fixed t_f and $\boldsymbol{x}(t_f)$, the variations δt_f and $\delta \boldsymbol{x}(t_f)$ vanish, and so we are left with (14). Because δt_f and $\delta \boldsymbol{x}(t_f)$ do not vanish in this case, we are left with additional boundary conditions that must be satisfied. Note that

$$\delta \boldsymbol{x}_f = \delta \boldsymbol{x}(t_f) + \dot{\boldsymbol{x}}^*(t_f) \delta t_f \tag{18}$$

and substituting this, we have

$$\begin{aligned}\delta J = &\left[g_{\dot{\boldsymbol{x}}}(\boldsymbol{x}^*(t_f), \dot{\boldsymbol{x}}^*(t_f), t_f)\right] \delta \boldsymbol{x}_f + \left[g(\boldsymbol{x}^*(t_f), \dot{\boldsymbol{x}}^*(t_f), t_f) - g_{\dot{\boldsymbol{x}}}(\boldsymbol{x}^*(t_f), \dot{\boldsymbol{x}}^*(t_f), t_f) \dot{\boldsymbol{x}}^*(t_f)\right] \delta t_f \\ &+ \int_{t_0}^{t_f} \left[g_{\boldsymbol{x}}(\boldsymbol{x}^*, \dot{\boldsymbol{x}}^*, t) - \frac{d}{dt} g_{\dot{\boldsymbol{x}}}(\boldsymbol{x}^*, \dot{\boldsymbol{x}}^*, t)\right] \delta \boldsymbol{x} \delta t.\end{aligned} \tag{19}$$

Stationarity of this variation thus requires

$$g_{\dot{\boldsymbol{x}}}(\boldsymbol{x}^*(t_f), \dot{\boldsymbol{x}}^*(t_f), t_f) = 0 \tag{20}$$

if $\boldsymbol{x}_f$ is free, and

$$g(\boldsymbol{x}^*(t_f), \dot{\boldsymbol{x}}^*(t_f), t_f) - g_{\dot{\boldsymbol{x}}}(\boldsymbol{x}^*(t_f), \dot{\boldsymbol{x}}^*(t_f), t_f) \dot{\boldsymbol{x}}^*(t_f) = 0 \tag{21}$$

if t_f is free, in addition to the Euler equation being satisfied. For a complete reference on the boundary conditions associated with a variety of problem specifications, we refer the reader to Section 4.3 of [Kir12].

4.1.3 Constrained Extrema

Previously, we have not considered constraints in the variational problem. However, constraints (and in particular, dynamics constraints) are central to most optimal control problems. Let $\boldsymbol{w} \in \mathbb{R}^{n+m}$ be a vector function in C^1. As previously, we would like to find a function $\boldsymbol{w}^*$ for which the functional

$$J(\boldsymbol{w}) = \int_{t_0}^{t_f} g(\boldsymbol{w}(t), \dot{\boldsymbol{w}}(t), t) dt \tag{22}$$

has a relative extremum, although we additionally introduce the constraints

$$f_i(\boldsymbol{w}(t), \dot{\boldsymbol{w}}(t), t) = 0, \quad i = 1, \dots, n. \tag{23}$$

We will again assume $g \in C^2$ and that $t_0, \boldsymbol{w}(t_0)$ are fixed. Note that as a result of these n constraints, only m of the $n + m$ components of $\boldsymbol{w}$ are independent.

One approach to solving this constrained problem is re-writing the n dependent components of $\boldsymbol{w}$ in terms of the m independent components. However, the nonlinearity of the constraints typically makes this infeasible. Instead, we will turn to Lagrange multipliers. We will write our *augmented functional* as

$$\hat{g}(\boldsymbol{w}(t), \dot{\boldsymbol{w}}(t), \boldsymbol{p}(t), t) := g(\boldsymbol{w}(t), \dot{\boldsymbol{w}}(t), t) + \boldsymbol{p}^T(t)\boldsymbol{f}(\boldsymbol{w}(t), \dot{\boldsymbol{w}}(t), t) \tag{24}$$

where $\boldsymbol{p}(t)$ are Lagrange multipliers that are functions of time. Based on this, a necessary condition for optimality is

$$\hat{g}_{\boldsymbol{w}}(\boldsymbol{w}^*(t), \dot{\boldsymbol{w}}^*(t), \boldsymbol{p}^*(t), t) - \frac{d}{dt}\hat{g}_{\dot{\boldsymbol{w}}}(\boldsymbol{w}^*(t), \dot{\boldsymbol{w}}^*(t), \boldsymbol{p}^*(t), t) = 0 \tag{25}$$

with

$$\boldsymbol{f}(\boldsymbol{w}^*(t), \dot{\boldsymbol{w}}^*(t), t) = 0. \tag{26}$$

4.2 Indirect Methods for Optimal Control

4.3 Pontryagin's Maximum Principle

4.4 Numerical Aspects of Indirect Optimal Control

4.5 Further Reading

References

[Kir12] Donald E Kirk. *Optimal control theory: an introduction.* Courier Corporation, 2012.