RANDOM MATRIX PROJECT REPORT

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ABSTRACT

In this report, GOE and GUE will be discussed.

1 MODEL DESCRIPTION AND THEORITICAL RESULTS

In this part, I will give the model description and some theoritical results.

1.1 GOE

The GOE(N) is ensembled by generating $N \times N$ matrices H with $H = H^T$, $H_{ij} \sim \mathbb{N}(0,1)$ and $H_{ii} \sim \sqrt{2}\mathbb{N}(0,1)$. So the density of matrix H is:

$$\begin{aligned}\rho(H) &= (\frac{1}{\sqrt{4\pi}})^N(\frac{1}{\sqrt{2\pi}})^{N(N-1)/2}e^{-\frac{1}{4}\sum_{i=1}^N H_{ii}^2}e^{-\frac{1}{2}\sum_{i<j}^N H_{ij}^2} \\ &= (\frac{1}{\sqrt{4\pi}})^N(\frac{1}{\sqrt{2\pi}})^{N(N-1)/2}e^{-\frac{1}{4}\|H\|_F^2} \end{aligned} \tag{1}$$

Under orthogonal transformation, the density is invariant because $\|H\|_F^2 = tr(HH) = tr(P^THPP^THP) = \|P^THP\|_F^2$.

Suppose $\Lambda = diag\{\lambda_1, ..., \lambda n\}$, where λ_i are eigenvalues of matrix H. H could be written as $H = P\Lambda P^T$. By measurement transformation, we have:

$$\rho(H)dH = \rho(\Lambda)|J|dPd\Lambda \tag{2}$$

where J is Jacobian matrix. According to Liu (2000), J has the following formation:

$$J = \prod_{i<j}(\lambda_i - \lambda_j)h(p_1, ..., p_{N(N-1)/2}) \tag{3}$$

where p_i decide the matrix P. Integrate all the p_i, we have

$$\mathbb{P}(\Lambda) = \rho(\Lambda)\left|\prod_{i<j}(\lambda_i - \lambda_j)\right|d\Lambda \tag{4}$$

Take $N = 2$ as example, we have the density of spaceing:

$$\rho_{|\lambda_1-\lambda_2|}(x) = C\int_{-\infty}^{+\infty} xe^{-\frac{1}{4}(t^2+(t+x)^2)}dt = C'xe^{-\frac{1}{8}x^2} \tag{5}$$

As $\int_0^{+\infty}\rho_{|\lambda_1-\lambda_2|}(x)dx = 1$, we have $C' = \frac{1}{4}$. So the mean of $|\lambda_1 - \lambda_2|$ is:

$$\mathbb{E}|\lambda_1 - \lambda_2| = \int_0^{+\infty}\frac{1}{4}x^2e^{-\frac{1}{8}x^2}dx = \sqrt{2\pi} \tag{6}$$

As the normalized spacing is $|\lambda_1 - \lambda_2|/\mathbb{E}|\lambda_1 - \lambda_2|$, its density is:

$$\frac{\pi}{2}xe^{-\frac{\pi}{4}x^2} \tag{7}$$

1.2 GUE

The GUE(N) is ensembled by generating $N \times N$ complex matrices $H = H^*$ with $H_{ij} \sim \mathbb{N}(0, \frac{1}{2}) + i\mathbb{N}(0, \frac{1}{2})$ and $H_{ii} \sim \mathbb{N}(0, 1)$. The density of matrix H is:

$$\begin{aligned}\rho(H) &= (\frac{1}{\sqrt{\pi}})^{N(N-1)/2} e^{-\sum_{i<j}^{N} Re(h_{ij})^2} (\frac{1}{\sqrt{\pi}})^{N(N-1)/2} e^{-\sum_{i<j}^{N} Im(h_{ij})^2} (\frac{1}{\sqrt{2\pi}})^{N} e^{-\frac{1}{2}\sum_{i}^{N} h_{ii}^2} \\ &= (\frac{1}{\sqrt{\pi}})^{N(N-1)} (\frac{1}{\sqrt{2\pi}})^{N} e^{-\frac{1}{2}\|H\|_F^2} \end{aligned} \tag{8}$$

The similar as GOE(N), the density of eigenvalues is:

$$\mathbb{P}(\Lambda) = \rho(\Lambda) \left| \prod_{i<j} (\lambda_i - \lambda_j)^2 \right| d\Lambda \tag{9}$$

For $N = 2$, we have the density of normalized spacing:

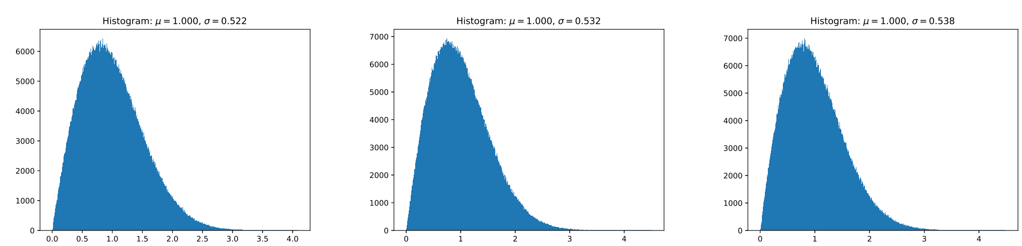
$$\frac{32}{\pi} x^2 e^{-\frac{4}{\pi} x^2} \tag{10}$$

2 Numerical Results

The number of samples is set to be 10^6.

2.1 GOE spacings

The matrix H is generated by $\frac{A+A^T}{\sqrt{2}}$, where $A_{ij} \sim \mathbb{N}(0, 1)$. As the density of spacings is invariant, we only consider the distribution of $\lambda_N - \lambda_{N-1}$, where $\lambda_N \geq \lambda_{N-1} \geq ... \geq \lambda_1$, like Fig 1 shows.



(a) N=2 (b) N=4 (c) N=10

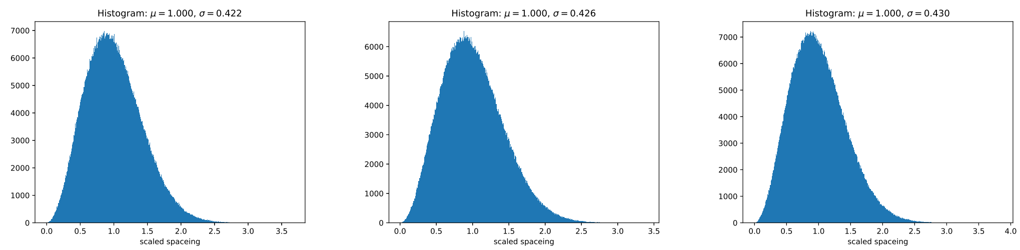
Figure 1: GOE

2.2 GUE spacings

The matrix H is generated by $\frac{A+A^*}{2}$, where $A_{ij} \sim \mathbb{N}(0, 1) + i\mathbb{N}(0, 1)$. The same as before, only $\lambda_N - \lambda_{N-1}$ is considered, like Fig 2 shows.

2.3 Eigenvalues

In this part, the size of matrix is 5000, GOE and GUE are simulated because they are special cases of Real Wigner and Complex Wigner, respectively. Besides, $H_{ij} \sim \mathbb{N}(0, 1)$ with $H_{ij} = H_{ji}$ is also simulated. Complex Wigner Matrix is ignored as GUE is a special case.



(a) N=2 (b) N=4 (c) N=10

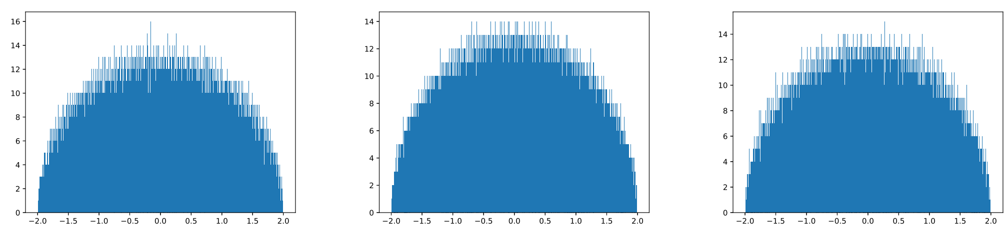
Figure 2: GUE

In fact, we have:

$$\frac{1}{N}\sum_{i=1}^{N}\lambda_i = \frac{1}{N}\sum_{i=1}^{N}H_{ii} \tag{11}$$

$$\frac{1}{N}\sum_{i=1}^{N}\lambda_i^2 = \frac{1}{N}\sum_{i=1}^{N}H_{ij}^2 \tag{12}$$

By large law, the first equation is about $o(1)$ but the second equation is about $O(N)$, which means we have to scale the eigenvalues by $\frac{1}{\sqrt{N}}$. The Fig 3 shows the empirical distribution of GOE, GUE and RW:



(a) GOE (b) GUE (c) RW

Figure 3: Eigenvalue distribution

In fact, by Wigner's semicircle law, the empirical distribution of scaled eigenvalues converges to ρ almost surely, where ρ satisifies:

$$\frac{d\rho}{dx} = \frac{1}{2\pi}\sqrt{4-x^2}1_{|x|\leq 2} \tag{13}$$

Experiment Details

All of my code and results can be found on website `https://github.com/yangwenh/stochastic-simulation`.

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References

Yi Kai Liu. Statistical behavior of the eigenvalues of random matrices. 2000.