NUMERICAL SIMULATION ON BLACK-SCHOLES MODEL PROJECT REPORT

Wenhao Yang
School of Mathematical Sciences
Peking University
yangwenhaosms@pku.edu.cn

ABSTRACT

In this report, I discover the theoritical results and simulation results of stochastic differential equation on Black-Scholes model. The two simulation algorithms are Euler-Maruyama and Milstein scheme. Besides, multilevel monte carlo method is also applied to this model to make comparison with single monte carlo method.

1 MODEL DESCRIPTION AND THEORITICAL RESULTS

In this section, I will give some theoritical results of Black-Scholes model. The stochastic differential equation of Black-Scholes model is:

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{1}$$

And the exact solution of S_t is: $S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}$. In fact, we have:

$$\begin{aligned} d\log S_t &= \frac{1}{S_t} dS_t - \frac{1}{2S_t^2}(dS_t)^2 \\ &= rdt + \sigma dW_t - \frac{1}{2}\sigma^2 dt \\ &= (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t \end{aligned} \tag{2}$$

$$\log S_t - \log S_0 = (r - \frac{1}{2}\sigma^2)t + \sigma W_t \tag{3}$$

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t} \tag{4}$$

And we care about the random variable $P = e^{-r}\max\{0, S_1 - 1\}$, which has following properties such as:

$$\mathbb{P}(P = 0) = \mathbb{P}(S_1 \leq 1) = \mathbb{P}(W_1 \leq \frac{1}{2}\sigma - \frac{r}{\sigma} - \frac{\log S_0}{\sigma}) \tag{5}$$

$$\mathbb{P}(P > x) = \mathbb{P}(W_1 > \frac{\log(xe^r + 1)}{\sigma} - \frac{\log S_0}{\sigma} + \frac{\sigma}{2} - \frac{r}{\sigma}) \tag{6}$$

$$\mathbb{E}[P] = \int_0^{+\infty} \mathbb{P}(P > x)dx \tag{7}$$

As it is difficult to get the exact value of the expectation of P, we just leave it alone. The next sections will compute its numerical simulation result.

2 NUMERICAL APPROXIMATION METHODS AND SIMULATION RESULTS

There are two mainly approximation methods to simulate the stochastic process.

2.1 Euler-Maruyama Approximation

The first approximation method is called Euler-Maruyama:

$$S_{t_{m+1}}^{(n)} = S_{t_m}^{(n)} + rS_{t_m}^{(n)} \Delta t_m + \sigma S_{t_m}^{(n)} \eta_m \tag{8}$$

where $\Delta t_m = T/n = \delta$ and $\eta_m = \Delta B_{t_m} \sim N(0, \delta)$

2.2 Milstein Approximation

The second approximation method is called Milstein:

$$S_{t_{m+1}}^{(n)} = S_{t_m}^{(n)} + rS_{t_m}^{(n)} \Delta t_m + \sigma S_{t_m}^{(n)} \eta_m + \frac{1}{2} \sigma^2 S_{t_m} (\eta_m^2 - \Delta t_m) \tag{9}$$

2.3 Simulation Results

All of the parameters are set as: $S_0 = 1$, $r = 0.05$ and $\sigma = 0.2$. In order to compare the efficiency of the two simulation methods, mean absolute error is taken as quanlification. As the orders of these two simulation methods are not the same, scaling factors $\sqrt{\delta}$ and δ are used for Euler-Maruyama and Milstein method, respectively. The mean absolute errors of two simulation methods are shown in Fig 1, from which the Milstein method has a more quick convergence rate.

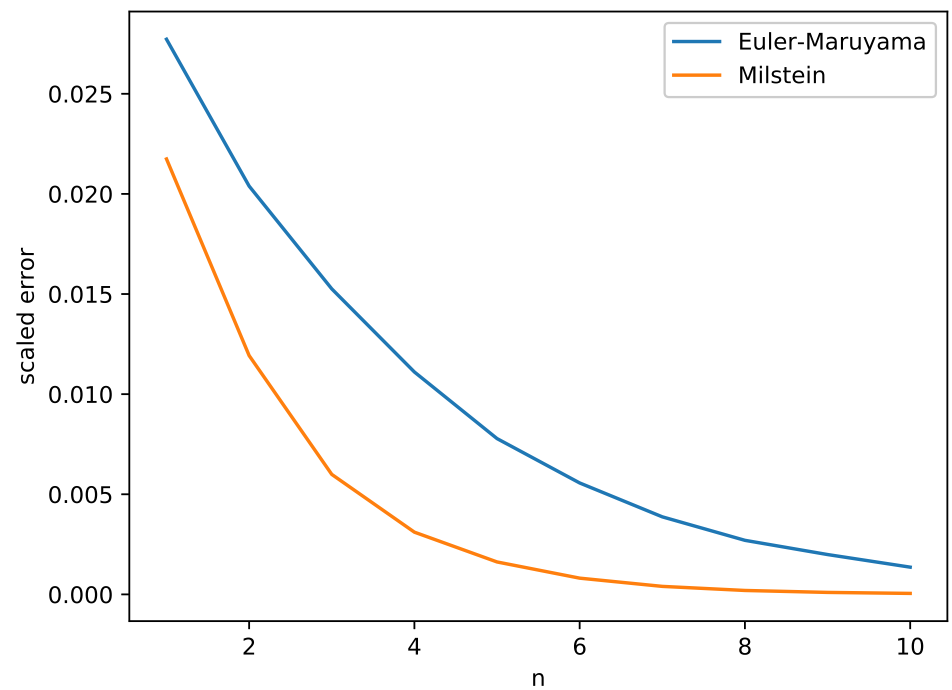

Figure 1: Scaled errors

3 Multilevel Monte Carlo Simulation method

To reduce estimation variance, multilevel monte carlo is promoted. In this section, multilevel monte carlo method is used to simulate the Black-Scholes model.

3.1 MULTILEVEL MONTE CARLO

As our aim is to estimate the expectation $\mathbb{E}[f(S_{n,T})]$, where n means discretize the time with M^{-n}. In this case, $M = 2$ is always supposed. In fact the expectation can be rewritten as:

$$\mathbb{E}[f(S_{n,T})] = \mathbb{E}[f(S_{1,T})] + \sum_{k=2}^{n} \mathbb{E}[f(S_{k,T}) - f(S_{k-1,T})] \tag{10}$$

So the estimation of $f(S_{n,T})$ could be written as:

$$\widehat{f}(S_{n,T}) = \frac{1}{N_1} \sum_{i=1}^{N_1} f(S_{1,T}^{(i)}) + \sum_{k=2}^{n} \frac{1}{N_k} \sum_{i=1}^{N_k} [f(S_{k,T}^{(i)}) - f(S_{k-1,T}^{(i)})] \tag{11}$$

To reduce variance, for each k, $f(S_{k,T}^{(i)}) - f(S_{k-1,T}^{(i)})$ is independent with $f(S_{k-1,T}^{(j)}) - f(S_{k-2,T}^{(j)})$, which means, the estimation of $f(S_{k,T}^{(i)}) - f(S_{k-1,T}^{(i)})$ is depended on the path simulated by $f(S_{k,T}^{(i)})$ instead of $f(S_{k-1,T}^{(j)})$. So the variance of $\widehat{f}(S_{n,T})$ is:

$$Var[\widehat{f}(S_{n,T})] = \sum_{k=1}^{n} \frac{V_k}{N_k} \tag{12}$$

where $V_k = Var[f(S_{k,T}) - f(S_{k-1,T})]$

4 GENERAL FORMATTING INSTRUCTIONS

The text must be confined within a rectangle 5.5 inches (33 picas) wide and 9 inches (54 picas) long. The left margin is 1.5 inch (9 picas). Use 10 point type with a vertical spacing of 11 points. Times New Roman is the preferred typeface throughout. Paragraphs are separated by 1/2 line space, with no indentation.

Paper title is 17 point, in small caps and left-aligned. All pages should start at 1 inch (6 picas) from the top of the page.

Authors' names are set in boldface, and each name is placed above its corresponding address. The lead author's name is to be listed first, and the co-authors' names are set to follow. Authors sharing the same address can be on the same line.

Please pay special attention to the instructions in section 6 regarding figures, tables, acknowledgments, and references.

5 HEADINGS: FIRST LEVEL

First level headings are in small caps, flush left and in point size 12. One line space before the first level heading and 1/2 line space after the first level heading.

5.1 HEADINGS: SECOND LEVEL

Second level headings are in small caps, flush left and in point size 10. One line space before the second level heading and 1/2 line space after the second level heading.

5.1.1 HEADINGS: THIRD LEVEL

Third level headings are in small caps, flush left and in point size 10. One line space before the third level heading and 1/2 line space after the third level heading.

6 CITATIONS, FIGURES, TABLES, REFERENCES

These instructions apply to everyone, regardless of the formatter being used.

6.1 Citations within the text

Citations within the text should be based on the `natbib` package and include the authors' last names and year (with the "et al." construct for more than two authors). When the authors or the publication are included in the sentence, the citation should not be in parenthesis (as in "See Hinton et al. (2006) for more information."). Otherwise, the citation should be in parenthesis (as in "Deep learning shows promise to make progress towards AI (Bengio & LeCun, 2007).").

The corresponding references are to be listed in alphabetical order of authors, in the References section. As to the format of the references themselves, any style is acceptable as long as it is used consistently.

6.2 Footnotes

Indicate footnotes with a number[1] in the text. Place the footnotes at the bottom of the page on which they appear. Precede the footnote with a horizontal rule of 2 inches (12 picas).[2]

6.3 Figures

All artwork must be neat, clean, and legible. Lines should be dark enough for purposes of reproduction; art work should not be hand-drawn. The figure number and caption always appear after the figure. Place one line space before the figure caption, and one line space after the figure. The figure caption is lower case (except for first word and proper nouns); figures are numbered consecutively.

Make sure the figure caption does not get separated from the figure. Leave sufficient space to avoid splitting the figure and figure caption.

You may use color figures. However, it is best for the figure captions and the paper body to make sense if the paper is printed either in black/white or in color.

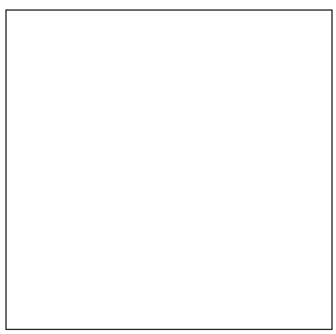

Figure 2: Sample figure caption.

6.4 Tables

All tables must be centered, neat, clean and legible. Do not use hand-drawn tables. The table number and title always appear before the table. See Table 1.

Place one line space before the table title, one line space after the table title, and one line space after the table. The table title must be lower case (except for first word and proper nouns); tables are numbered consecutively.

[1]Sample of the first footnote

[2]Sample of the second footnote

Table 1: Sample table title

PART	**DESCRIPTION**
Dendrite	Input terminal
Axon	Output terminal
Soma	Cell body (contains cell nucleus)

7 FINAL INSTRUCTIONS

Do not change any aspects of the formatting parameters in the style files. In particular, do not modify the width or length of the rectangle the text should fit into, and do not change font sizes (except perhaps in the REFERENCES section; see below). Please note that pages should be numbered.

8 PREPARING POSTSCRIPT OR PDF FILES

Please prepare PostScript or PDF files with paper size "US Letter", and not, for example, "A4". The -t letter option on dvips will produce US Letter files.

Consider directly generating PDF files using `pdflatex` (especially if you are a MiKTeX user). PDF figures must be substituted for EPS figures, however.

Otherwise, please generate your PostScript and PDF files with the following commands:

```
dvips mypaper.dvi -t letter -Ppdf -G0 -o mypaper.ps
ps2pdf mypaper.ps mypaper.pdf
```

8.1 MARGINS IN LATEX

Most of the margin problems come from figures positioned by hand using `\special` or other commands. We suggest using the command `\includegraphics` from the graphicx package. Always specify the figure width as a multiple of the line width as in the example below using .eps graphics

```
\usepackage[dvips]{graphicx} ...
\includegraphics[width=0.8\linewidth]{myfile.eps}
```

or

```
\usepackage[pdftex]{graphicx} ...
\includegraphics[width=0.8\linewidth]{myfile.pdf}
```

for .pdf graphics. See section 4.4 in the graphics bundle documentation (`http://www.ctan.org/tex-archive/macros/latex/required/graphics/grfguide.ps`)

A number of width problems arise when LaTeX cannot properly hyphenate a line. Please give LaTeX hyphenation hints using the `\-` command.

ACKNOWLEDGMENTS

Use unnumbered third level headings for the acknowledgments. All acknowledgments, including those to funding agencies, go at the end of the paper.

REFERENCES

Yoshua Bengio and Yann LeCun. Scaling learning algorithms towards AI. In *Large Scale Kernel Machines*. MIT Press, 2007.

Geoffrey E. Hinton, Simon Osindero, and Yee Whye Teh. A fast learning algorithm for deep belief nets. *Neural Computation*, 18:1527–1554, 2006.