Basics

Number Systems
Identity and Inverse
Symmetry and Duality

In this page we will look at some of the basics and guiding principles of this course. We will look at number systems, the basic building blocks of algebra. Following that we will introduce four different concepts: identity and inverse, followed by symmetry and duality. These later four concepts, and especially the last two will be our guiding principles throughout the course.

We start with the first of our concepts, that of number systems

Number Systems

The basic point of algebra is to be able to solve equations and get numbers as a result. The problem is if we don't have a concept of the types of number there are out there we won't be able to solve these equations. Think of numbers as a playground. The types of numbers we know of, allows us to solve more and more equations. If this doesn't make sense, hold on, we will look at a few examples.

$\mathbb{N}$, the Naturals

The simplest set of number we will encounter are natural numbers. These are the numbers: $0,1,2,3,\ldots$ and so on. There are infinitely many of them, and they go on forever, and are represented by, what is called *blackboard N:* $\mathbb{N}$.

These numbers are sufficient for addition and multiplication. If you have a natural number and add another natural number you will always get a natural number. If you have a natural number and multiply with another natural number you will always get back a natural number. *Try it.*

If all you have is addition and multiplication, then the natural numbers are sufficient.

What happens when you introduce a new operation called subtraction? Well $5 - 3 = 2$, we're safe. But if we do $3 - 5$ we get -2, which is not a natural number. Our number system is not equipped to handle negative numbers.

We need to expand our set of numbers in order to be able to handle this new algebra.

$\mathbb{Z}$, the Integers

What happens when we take the natural number and add the negatives of all the numbers? We get what is called the *integers*, symbolized by *blackboard Z:* $\mathbb{Z}$.

$\ldots -3, -2, -1, 0, 1, 2, 3 \ldots$

Like the natural numbers, this is also an infinite set of numbers but it is infinite in *both directions*. Think of this as the natural numbers on steroids. This set is well equipped to handle both addition and multiplication. An integer times an integer is always an integer, and an integer plus an integer is also an integer. You can also safely subtract an integer from another integer to always get another integer.

Our number system can now safely handle addition, multiplication, and subtraction. What about division? For certain numbers, this is safe, for example: $\frac{4}{2} = 2$ an integer, but on the other hand $\frac{3}{5}$ is no longer an integer.

We need to expand our number system again.

$\mathbb{Q}$, the Rationals

We add an extra layer to the integers. If we add to the integers, all fractions of the form $\frac{a}{b}$ where a is an integer and b is an integer we get the rational numbers, symbolized by $\mathbb{Q}$. The fact of the matter is that even integers can be represented this way, for example $5 = \frac{5}{1} = \frac{10}{2} = \frac{20}{4}$.

With the rationals, our number system is now equipped to handle $+, -, \times, \div$. Check for yourself that dividing a rational by a rational is always a rational, e.g. $\frac{3}{2} \div \frac{5}{7} = \frac{21}{10}$.

Our number system is equipped to handle the four main algebraic operations: $+, -, \times, \div$. A set of numbers equipped to handle these four operations has a special name they are called a **field.**

But even the rationals are not "large" enough to handle most of the algebra we will be doing in this course. For example, how do you solve the equation $x^2 - 2 = 0$:

$$\begin{aligned} x^2 - 2 &= 0 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

But $\sqrt{2}$ is not a rational. In other words, our number system $\mathbb{Q}$ is not "large enough" to handle a solution to the equation $x^2 - 2$. We need to expand our number system **yet again. We need to include along with rationals (numbers expressible as fractions), those numbers that can't be expressed as fractions called *irrationals*.**

$\mathbb{R}$, the Reals

This is the largest set of numbers we will work with, in this class. This is the set of numbers that can be expressed as a fraction and those numbers that can't be expressed as a fraction. If this definition sounds a little cludgy, it's because it is. The proper definition of the reals requires quite a few years of mathematics.

For the vast majority of our class we will be dealing with the real numbers. They form the backbone of most algebraic operations and allow us the flexibility of using *most* algebraic operations safely.

There are larger sets of numbers. The complex numbers, $\mathbb{C}$, consist of the reals and real multiples of the imaginary constant i, but that's beyond the scope of this class. There are again the Hamiltonian Quaternions $\mathbb{H}$ and the octinions $\mathbb{O}$, but we don't use them.

Identity and Inverse

Every algebraic operation has what is called an *identity*. A special number that when you apply with the algebraic operation *keeps everything the same.*

For addition the identity is 0. Add zero, and no matter what number you add zero to, the number stay the same: $5 + 0 = 5$, $8 + 0 = 8$, and so on.

For multiplication, the identity is 1. Multiply anything by 1, and you get back the same number. So for example: $5 \times 1 = 5$, $8 \times 1 = 8$, and so on.

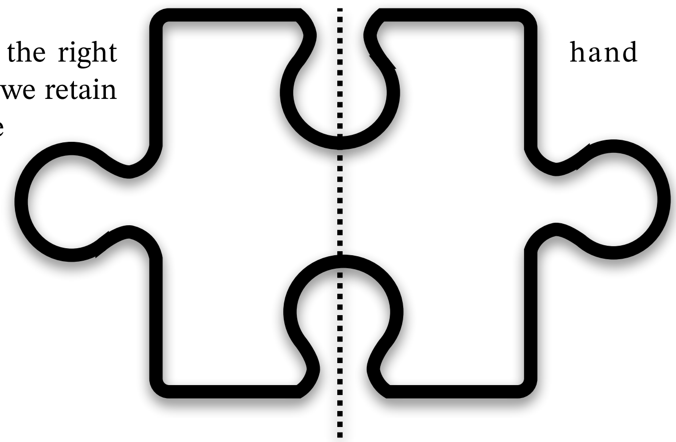
Identity is a number or operation that **keeps everything the same.** Hence the name *identity*. The identity depends on the operation you're applying. The additive identity, that is, the identity under addition is 0, but the multiplicative identity, that is, the identity under multiplication is 1.

The inverse is a number such that if you apply your operation you get back the identity for that operation. For examples, the inverse under addition of 5, is -5, because $5 + (-5) = 0$. The multiplicative inverse for 5 is $\frac{1}{5}$, because $5 \times \frac{1}{5} = 1$.

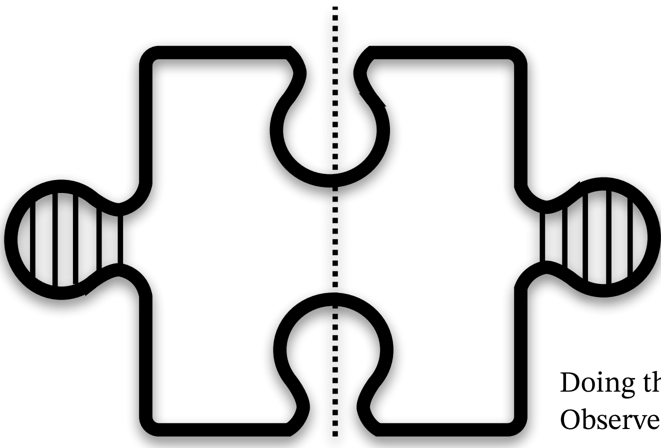
If you fix an operation (multiplication or addition), then you have a fixed and unique identity, and inverse.

Symmetry and Duality

Symmetry refers to how algebraic operations work. Imagine a symmetrical shape, as follows:



The left hand part of the shape is the same as the right hand part. If we do the same operation on both sides, we retain symmetry. For example, if we were to shade the folds on either side, in the same way, the symmetry is preserved:



Doing the same thing on both sides, preserves symmetry. Observe that an equation is also an expression of symmetry. An equations such as $2x + 1 = 5$, is an expression of symmetry. The left hand of the equation is the same as the right hand of the equation, just as in the diagram the left hand of the shape is the same as the right hand of the shape.

Just as in the diagram, doing the same thing on both sides should still retain the symmetry of the system:

$$\begin{aligned} 2x + 1 &= 5 \\ 2x &= 4 && \text{subtract 1 from both sides} \\ x &= 2 && \text{divide both sides by 2} \end{aligned}$$

This is the idea when it comes to solving equations. We want to preserve the symmetry of the statement, and the only way to retain symmetry while still trying to transform our equation is to do the same thing to both sides of the equation. In order retain the property of being symmetrical and balanced, any operation applied to one side of the equation must be applied to both sides of the equal sign.

The next big concept is by far the most abstract, and that is duality. In short, duality refers to the fact that **any operation doable with algebra has a geometric interpretation, and any geometric operation has an algebraic analogue.**

Here are a few examples:

1. When we say zeros of polynomials, that is an algebraic statement, geometrically they refer to the points on the graph where the graph crosses the x-axis.

2. Slope refers to how steep the graph of an equation is, but algebraically it is: $\frac{y_1 - y_2}{x_1 - x_2}$.

3. An asymptote is a line that the graph of an equation gets arbitrarily close to but never touches, but algebraically an asymptote is found by setting $x \to \infty$.

You will see many many examples throughout this course. A concept that appears geometric has an algebraic interpretation, and a concept that is algebraic has a geometric interpretation. **Algebra and geometry are two sides of the same coin.**

This concept is called **duality.** Duality is arguably one of the most important concepts in all of mathematics.