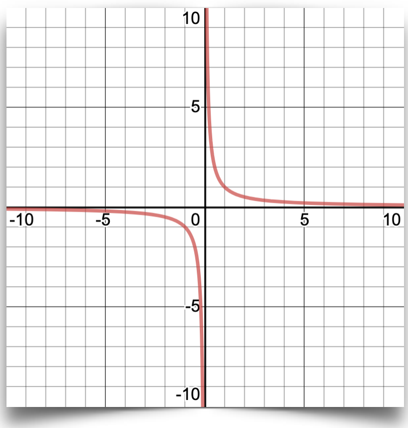
Asymptotes

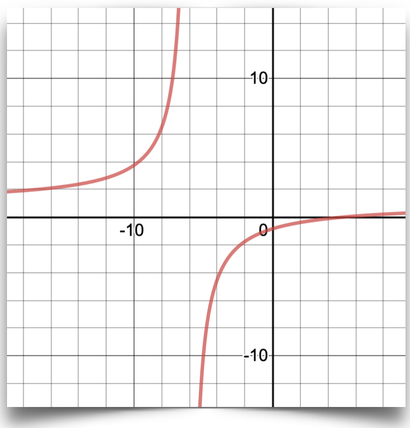
Vertical Asymptotes
End Behavior
Horizontal Asymptotes

Introduction

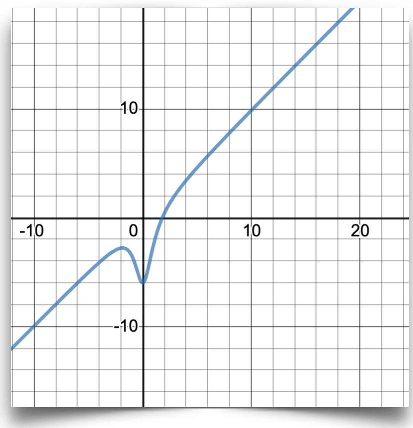
If you have been graphing things for a while (and you're encouraged to plot lots of different graphs on www.desmos.com), you will notice that if you plot graphs of the form $\dfrac{function}{function}$, they tend to become smoothly horizontal or smoothly vertical, or smoothly slant. Here are three examples.



$y = \dfrac{1}{x}$



$\dfrac{x-5}{x+6}$



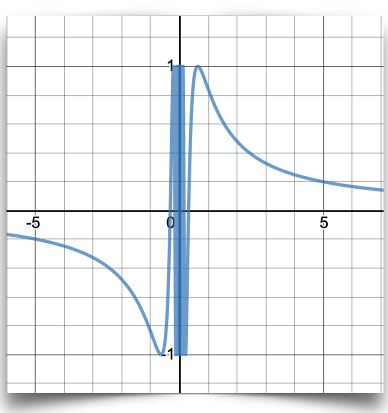
$\dfrac{x^3-7}{x^2-1}$

These are called *asymptotes.*

Definition: An asymptote for a graph is a straight line that the graph approaches *as x and y approach $\infty$and $-\infty$* but never touch.

The part in italics is actually very important. Consider the graph of $\sin(\dfrac{1}{x})$:

This graph actually has a horizontal asymptote of $y = x$, but the graph crosses this horizontal asymptote *an infinite number of times, but that's irrelevant, as this graph approaches but never touches* $y = 0$.



$\sin(\dfrac{1}{x})$

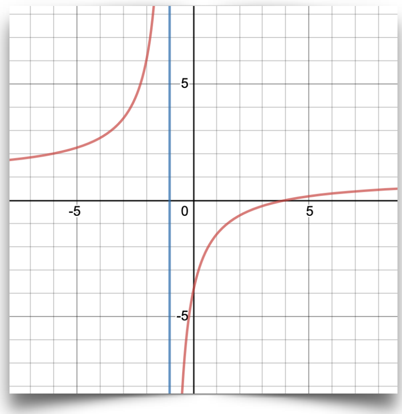
There are three types of asymptotes: horizontal, vertical, and slant asymptotes. Of these, horizontal and slant asymptotes can be calculated using the same technique, but vertical asymptotes require a different approach.

Vertical Asymptotes

Vertical asymptotes are usually a symptom that the domain is no longer working. In other words, the easiest way to find vertical asymptote is to find out where the domain fails.

Example 1: Find the vertical asymptote of $\dfrac{x-4}{x+1}$.

A couple of observations, notice that this function with any value of x, except if we have $x=-1$, because that will produce: $\dfrac{-1-4}{-1+1}=\dfrac{-5}{0}$, which is undefined. In other words, the domain is $\mathbb{R}\backslash\{-1\}$ or $(-\infty,-1)\cup(-1,\infty)$. The vertical asymptote is therefore $x=-1$. Check against the graph on the right. The blue line is the vertical asymptote of this graph.

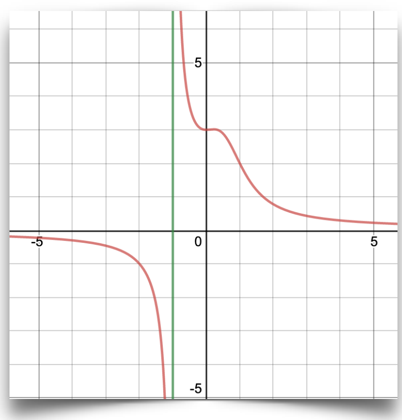


Example 2: Find the vertical asymptote of $\dfrac{x^2+3}{x^3+1}$.

Again, consider that this function should work for all values of x except those values of x that would cause the denominator to become zero. We can solve a quick equation to find out what those values would be:

$$x^3+1=0$$
$$x^3=-1$$
$$x=\sqrt[3]{-1}=-1$$

The vertical asymptote is therefore $x=-1$, that is the green line on the graph.

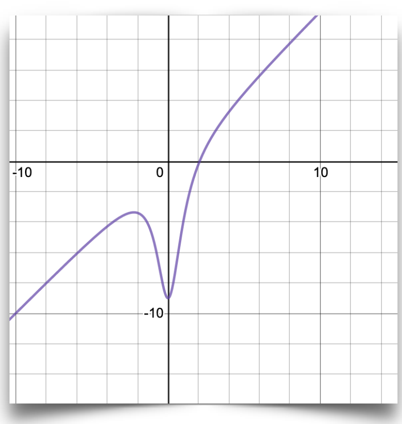


Example 3: Find the vertical asymptote of $\dfrac{x^3-9}{x^2+1}$.

Now consider the denominator. Is there any value of x that would cause the denominator to become zero? Let's check.

$$x^2+1=0$$
$$x^2=-1$$
$$x=\pm\sqrt{-1}$$

Which, for the purpose of this class, will be considered as undefined. Notice also that this graph has no vertical asymptotes.



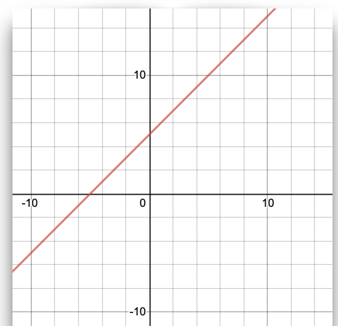
This graph, however, has what is called a *slant asymptote.*

End Behavior

Before we learn how to find horizontal and slant asymptotes, we need to understand something else, called *end behavior.* End behavior refers to the behavior of functions as x and y start to approach infinity. Really, the only way to think about this is to reason your way through an equation.

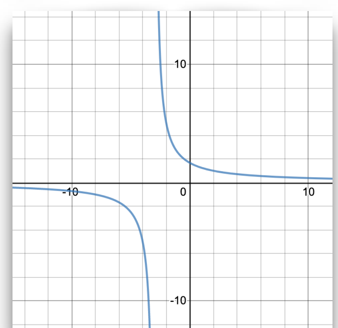
Note: We will make a shorthand, in that we will use arrows $\to$ to represent when a value starts to approach ∞ or some other number. For example we will say $x \to \infty$ to mean "as x approaches ∞", or as "x starts to look more and more like ∞".

Example 1: Find the end behavior of $x + 5$.



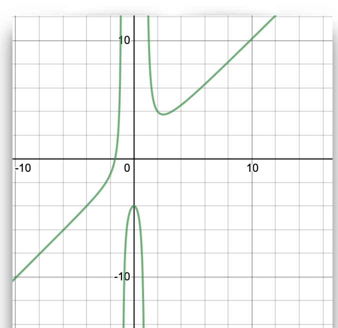
Notice that as $x \to \infty$, larger and larger values of x, cause the value of $x + 5$ to become even larger. In other words as $x \to \infty$, $x + 5 \to \infty$. On the other hand as $x \to -\infty$, we have $x + 5 \to -\infty$. Thus the end behaviors are, $x + 5$ approaches ∞ as x approaches ∞ and $x + 5$ approaches $-\infty$ as x approaches $-\infty$.

Example 2: Find the end behavior of $\dfrac{5}{x+3}$.



Notice that as $x \to \infty$, $x + 3 \to \infty$. This means the as we approach infinity $\dfrac{5}{x+3}$ starts to look more and more like $\dfrac{5}{\infty} = 0$. This is the end behavior as x approaches ∞. On the other side, $x \to -\infty$, $x + 3 \to -\infty$, and so $\dfrac{5}{x+3} \to \dfrac{5}{-\infty} \to 0$. So the end behavior is $\dfrac{5}{x+3}$ approaches 0 as x approaches both infinity and negative infinity.

Example 3: Find the end behavior of $\dfrac{x^3+4}{x^2-1}$.



Notice something interesting as $x \to \infty$, the constant terms on the numerator and denominator start to matter less and less. In other words $x^3 + 4 \to x^3$ and $x^2 - 1 \to x^2$. Together that means $\dfrac{x^3+4}{x^2-1} \to \dfrac{x^3}{x^2} = \dfrac{x^3}{x^2} = x$. But our starting assumption was that $x \to \infty$. Thus as $x \to \infty$, $\dfrac{x^3+4}{x^2-1} \to \infty$. So what happens as $x \to -\infty$. From our arguments above, we see that $x \to -\infty$, $\dfrac{x^3+4}{x^2-1} \to -\infty$.

Horizontal and Slant Asymptote

Now that we know how to calculate end behavior, we can very easily calculate horizontal and slant asymptote. The steps to calculate horizontal and slant asymptote is as follows:

1. Set $x \to \infty$
2. Find out how that transforms the function
3. Set $x \to -\infty$
4. Find out how that transforms the function
5. If the two transformations are the same then the function has one horizontal asymptote
6. If not, the function has two (or more) horizontal asymptotes.

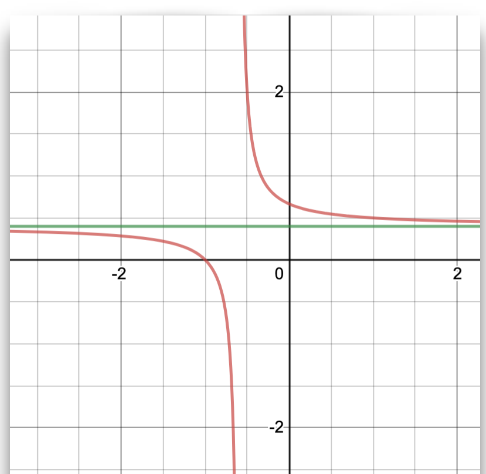
We will look at a few examples.

Example 1: Find the horizontal (or slant) asymptotes of $\dfrac{5}{x+8}$.

Consider that $x \to \infty$, $x+8 \to \infty$, and so $\dfrac{5}{x+8} \to \dfrac{5}{\infty} \to 0$. On the other hand, $x \to -\infty$, $x+8 \to -\infty$, and so $\dfrac{5}{x+8} \to \dfrac{5}{-\infty} \to 0$. Both end behaviors agree, so the graph has one horizontal asymptote of $y = 0$.

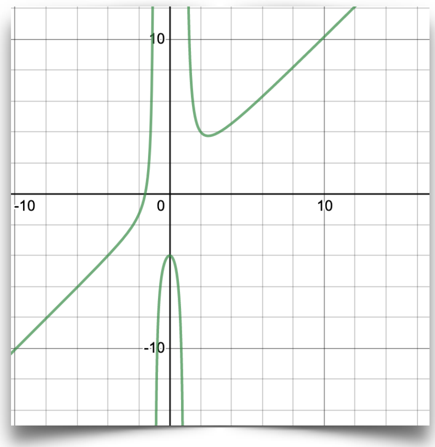
Example 2: Find the horizontal (or slant) asymptotes of $\dfrac{x+2}{x+3}$.

Consider that as as $x \to \infty$, the $+2$ on the numerator starts to matter less and less. In other words $x+2 \to x$ and same in the denominator $x+3 \to x$. Thus $\dfrac{x+2}{x+3} \to \dfrac{x}{x} = 1$. The same in the other end $\dfrac{x+2}{x+3} \to \dfrac{x}{x} = 1$. The horizontal asymptotes on both ends agree, and so there is one horizontal asymptote of $y = 1$.



Example 3: Find the horizontal (or slant) asymptotes of $\dfrac{2x+2}{5x+3}$.

Again, using the logic of the previous example, we notice that the $+2$ in the numerator, and the $+3$ in the denominator, matters less and less as $x \to \infty$. This means $\dfrac{2x+2}{5x+3} \to \dfrac{2x}{5x} = \dfrac{2}{5}$. On the other side, the same argument applies and we get $\frac{2}{5}$. Both sides agree, and we have the horizontal asymptote of $y = \frac{2}{5}$.



Example 4: Find the horizontal (or slant) asymptotes of $\frac{x^3+4}{x^2-1}$.

So, notice again that the constant terms start to matter less and less as $x \to \infty$. This means that $x^3 + 4 \to x^3$, and $x^2 - 1 \to x^2$, and thus we have that $\frac{x^3+4}{x^2-1} \to \frac{x^3}{x^2} = x$. Thus, this function starts to behave, or starts to look more and more like $y = x$, towards the end. The same thing applies as $y \to -\infty$. Thus the function has a slant asymptote of $y = x$.