Function Composition

Sometimes we want to be able to *combine functions*, more precisely we want to *compose* functions. This is done by taking the output of one function and using it as the input for a second function.

For example, assume that we have $f(x) = 5x + 9$ and $g(x) = \frac{1}{x}$. Then composing them, using our arrow notation, would look something like:

$4 \xrightarrow{f} 29 \xrightarrow{g} \frac{1}{29}$ and similarly $-1 \xrightarrow{f} 4 \xrightarrow{g} \frac{1}{4}$.

More algebraically we would say $g \circ f(4) = \frac{1}{29}$ or $g \circ f(-1) = \frac{1}{4}$. Notice that we read this composition inside out: f acts on 4 first, converting 4 to 29 after which g is applied to 29 to give $\frac{1}{29}$. The same thing happens in $g \circ f(-1)$ to give $\frac{1}{4}$.

In fact, conceptually, we could also say: $g \circ f(4) = g(29) = \frac{1}{29}$, and equally $g \circ f(-1) = g(4) = \frac{1}{4}$.

Note that, we often drop $\circ$ to just write $gf(4) = \frac{1}{29}$ or $gf(-1) = \frac{1}{4}$, but this is somewhat discouraged as this results in confusion.

Warning: The sequence in which functions take place really *does matter*. In our case $f \circ g(4) = f(\frac{1}{4}) = \frac{5}{4} + 9 = 10\frac{1}{4}$ and $f \circ g(-1) = f(-1) = 4$. In baking a cake, you can't add the eggs after you're done baking.

Domain and Range of Composed Function

It's very hard to deduce the domain and range of a composed function from the constituent functions. Usually the best thing to do in this case is to consider the function as a whole. For example, if $f(x) = x^2 + 4$ and $g(x) = \log(x)$. What is the domain and range of $g \circ f(x)$?

Consider that $g \circ f(x) = \log(x^2 + 4)$. The domain is $\mathbb{R}$ or $(-\infty, \infty)$, as no matter what the value of x is, $x^2 + 4 > 0$, and this is safe as an operand of log. The range is $[\log(4), \infty)$, as the lowest the operand can be is 4, thus in consequence the lowest value of y is $\log(4)$, and thus the range is $[\log(4), \infty)$.