Functions

Function Representation
Many-to-one Functions

Introduction

A function (sometimes called a *map*) is a tool for converting an object to a **one** other object. In this class that object happens to be numbers.

Functions are arguably the most important concept in all of mathematics. They underly every single branch of mathematics that exist and are actively being studied: category theory, abstract algebra, geometric topology, algebraic geometry etc.

One fun thing about the definition of functions is that *this is it*. Most concepts in math, tend to get redefined as you go higher in math, but the definition of function is still the same. It must convert one object into *one* other object, not two, not three, one object to *one* object. This is the defining characteristic of functions, no matter what branch of math you study, or even outside of math, in say theoretical physics.

Function Representation

There are many different ways to represent function, and in this class you are expected to show mastery over all forms.

Arrows

The simplest and most straightforward to represent functions is with arrows. Imagine a function that converts 2 to 4, 3 to 10, 4 to 17 and so on. Essentially we have $2 \mapsto 4, 3 \mapsto 10, 4 \mapsto 17$. The number on the left hand of the arrow is the *input* or *source* and the numbers on the right are the *output* or *target.*

Notice however that this is not a function: $2 \mapsto 4, 6 \mapsto 12, 2 \mapsto 3, 9 \mapsto 11$. Why?

Full function representation

Imagine a function again, this time, it goes something like: $1 \mapsto 5, 2 \mapsto 10, 3 \mapsto 15 \ldots$. There has to be a quicker way to represent this? We write this function out explicitly:

Function name

Input

$$f(x) = 5x$$

Explicit function description

Notice a few things, the input of the function is *usually* represented with x, but not always. The output of the function is represented by $f(x)$, or y. Now, if we plug in 5 to this function, we should get $5 \times 5 = 25$. This can be represented as $f(5) = 25$. Again, also $f(6) = 30, f(7) = 35$. The way to read $f(5) = 25$ is *if you input 5 into the function f, the output will be 25*, and so on.

Side-note: Functions can be named g as in $g(x)$, h as in $h(x)$, and so on.

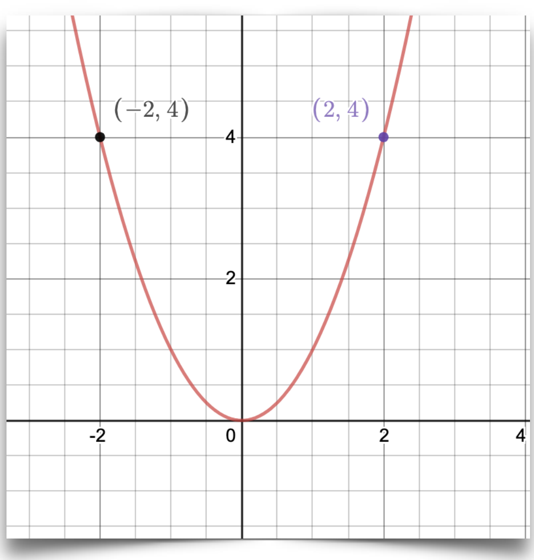
Graphs

Graphs are the most sophisticated ways to represent functions, and where we will spend the most time. On every graph we have two lines perpendicular, called x and y. We draw a few points on the graph, and connect them with curved lines.

The co-ordinates of these points are determined by a pair $(x, f(x))$. We will look at graphs throughout the semester.

Many-to-one functions

One curious fact is the existence of *many-to-one* functions. These are functions where multiple input values of x can give rise to the same output value y. One example is $y = x^2$, whose graph looks like this:



Notice that two very different inputs result in the same output, since $2^2 = (-2)^2 = 4$. When a function is many-to-one, we say that the function *fails the horizontal line test*, meaning that somewhere along the graph you can draw a horizontal line that crosses the function more than once.

Vertical line test: Notice as well that a function must always pass the *vertical line test*. That is, at nowhere along the graph should it be possible to draw a vertical line and have the function cross it twice. Because, to cross any vertical line twice means that an x value gives rise to y values, and those are simply not functions.

Passing both the horizontal and vertical line test: Functions that pass both the horizontal and vertical line tests are called *one-to-one* functions. These are the nicest kinds of functions, in the sense that these are the types of functions that have inverses. See the pages on function inverse to find out.