Polynomials

End Behavior
Zeros of Polynomials

The purpose of this class is to study algebraic equations, try and understand properties about them, and try to decipher what these functions can really do. There is a large system of trying to classify functions, and at the base of all of that classification are *polynomials*. They form the foundation of many different branches of mathematics, and form as the basis for trying to simulate other more complex functions.

The word polynomial comes from the Greek word *poly*-meaning many, and *nomial*-names, meaning *many names*. Polynomials are constructed from smaller parts called *monomials*. Here's the definition.

Definition: A monomial is an algebraic expression of the form: ax^n, where n is a natural number ($n \in \mathbb{N}$), and a is a real number ($a \in \mathbb{R}$).

Here are some examples of monomials: $x^2, 4x^3, -\frac{7}{8}x, 7, -3.5$ etc etc, not however that these are not monomials $\sqrt{2}, \frac{1}{x}$.

Monomials are the building blocks of polynomials. Their definition is:

Definition: A polynomial is a sum of monomials.

Here are some examples: $4x+9$, x^2+3x-9, $x+9$, and $\frac{4}{3}x^5+\pi+8x$.

Notice that some polynomials are more complex than others. One very simple way to capture this concept of complexity is called *degree*. This is the definition:

Definition: The degree of a polynomial is the highest power of x in the polynomial.

Usually we denote constant terms as *degree zero polynomials*. So for example the number 5 is a *degree zero polynomial.*

So this means that in the previous example, the degrees are, 1,2,1, and 5. In face we can go a little further and give polynomials of certain degree special names.

Degree one polynomials are called *linear equations*.

Degree two polynomials are called *quadratic equations.*

Degree three polynomials are called *cubic equations.*

Degree four polynomials are called *quartic equations,* and finally degree five are called *quintic.*

One distinction we will make is between *even* and *odd* degree functions. Linear, cubic and quintic functions are odd degree functions, while quadratic and quartic are even degree function. This distinction will come in handy when we try to find the end behavior of functions.

One more distinction we will make is the *sign of the leading term*. The *leading term*, is the co-efficient of the highest power of x in your polynomial no matter where it resides. Together with information about even and odd degrees we can predict the end behavior of the our polynomials.

One final thing to note is for a factored polynomial, the degree is the sum of the degrees of the individual factors. So for example, the degree of $(x-1)(x-9)$ is 2, and the degree of $x(x+1)^2(x-3)^2$, is $2+2+1=5$.

End Behavior

Recall that end behavior refers to the behavior of a function as $x \to \infty$, and as $x \to -\infty$. Notice that as x starts to approach infinity, the largest power in a function starts to take over. In other words, for example, as $x \to \infty$, $x^2+3x-9 \to x^2$.

This is true for any polynomial. Toward infinity and negative infinity, the function starts to look more and more like its leading term, which in tern starts to look more and more like x^n. This means that finding out the end behavior of polynomials is rather easy.

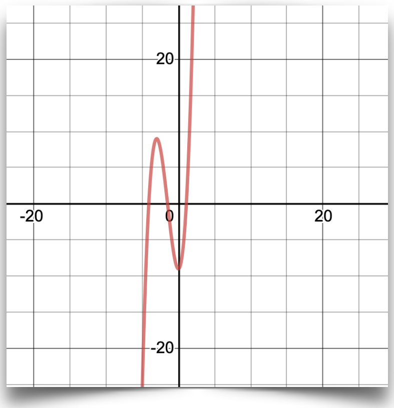
But we need one more ingredient before we can talk about the end behavior of polynomials. Notice that raising a positive number to an even power will always result in a positive number, and raising a negative number to an even power always results in a positive number. In other words:

$$\text{positive number}^{\text{even power}} = \text{positive number}$$
$$\text{negative number}^{\text{even number}} = \text{positive number}$$

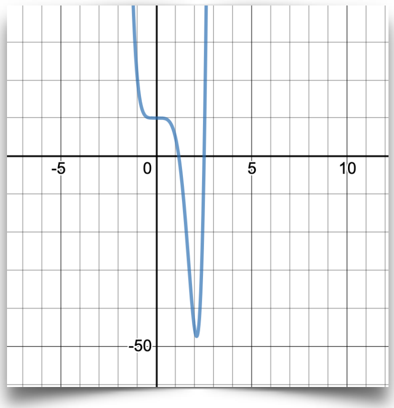
Notice something else:

$$\text{positive number}^{\text{odd power}} = \text{positive number}$$
$$\text{negative number}^{\text{odd number}} = \text{negative number}$$

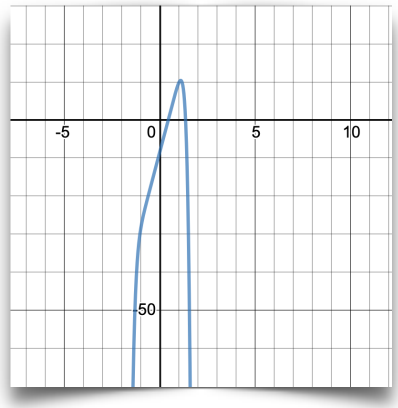
This means that with odd degree polynomials, the endpoints always point in different directions, while with even degree polynomials the endpoints point in the same direction. Here are several examples:



$\frac{4}{3}x^3 + 6x^2 - 9$

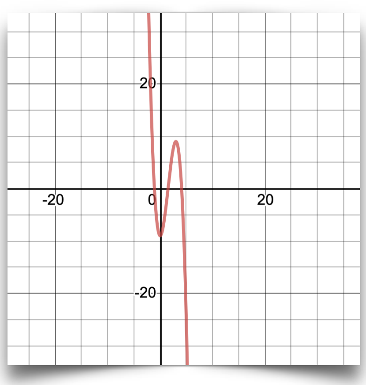


$\pi x^6 - 8x^5 + 10$



$-x^{10} + 19x - 8$

If you notice the last graph you'll see that even though both endpoints point to the same direction ($-\infty$), they point in the direction away from the second graph, where both endpoints point to $+\infty$. This is because the leading term in the third graph is -1, which is negative. This results in the endpoints pointing in the opposite direction. For example this is the first graph, but with the leading term changed to $-\frac{4}{3}$.



$-\frac{4}{3}x^3 + 6x^2 - 9$

Zeros of Polynomials

Where a function crosses the x-axis can tell you a lot about the shape of the function. A graph that crosses the x-axis five times is going to look a lot different than a function that crosses the x-axis four times.

We need to get a few terminology out of the way. The *zeros* of a polynomial correspond to the x-intercepts (where the graph crosses or touches) on the graph of this function. In order to be able to find the zeros of a polynomial we need to introduce a very fundamental concept. *Real numbers have no zero divisors.*

In layman's terms this means that if two numbers are multiplied to get zero, this means that one of the terms in the product is zero.

Principle: If $ab = 0$, this means that either $a = 0$, or $b = 0$.

This has important ramifications.. For example if $(x-1)(x+5) = 0$, we can confidently say:

$$
\begin{array}{lcl}
(x-1)(x+5) = 0 & & \\
x - 1 = 0 & \text{or} & x + 5 = 0 \\
x = 1 & \text{or} & x = -5
\end{array}
$$

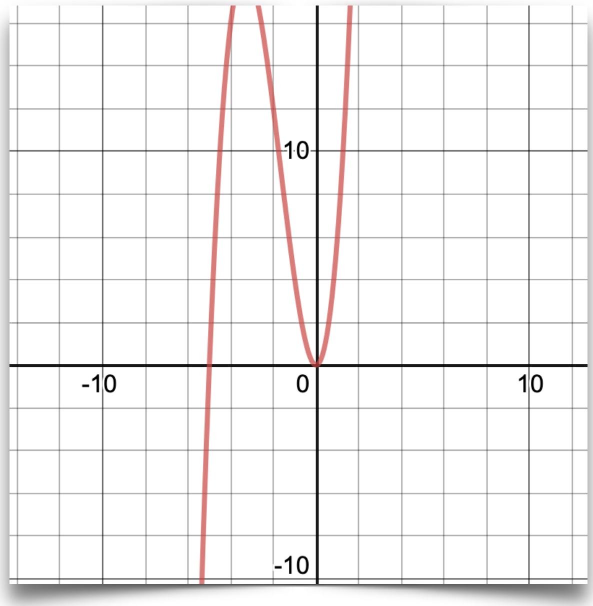
This gives us a powerful way to find the zeros of a polynomial. If we can factor a polynomial, then we have a neat way of finding the zero (or solutions) of a polynomials.

Example: Find the zeros of $x^2(x+5)$.

We do the usual calculations:

$$
\begin{array}{lcl}
x^2(x+5) = 0 & & \\
x^2 = 0 & \text{or} & x + 5 = 0 \\
x = 0 & \text{or} & x = -5
\end{array}
$$

Notice that as $x^2 = 0$, the zero is repeated. One last thing to notice is that because $x^2 = 0$, the zero at $x = 0$ is repeated. This means that the graph at $x = 0$, touches the point $x = 0$, and bounces. Notice also that $x + 5$ is actually $(x+5)^1$. This is an odd power, this means that the graph *crosses* the x-axis at $x = -5$. In fact, take a very close look at the graph below:



$y = x^2(x+5)$

This is a common theme in the graph of polynomials. In fact we can write this as:

- If a zero has odd power, then the graph *bounces*, at the x-axis at that point on the x-axis.
- If a zero has even power, then the graph *crosses*, at the x-axis at the point on the x-axis.

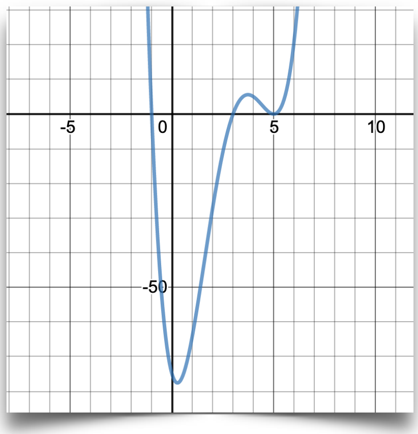
Example: Find the zeros of $(x-5)^2(x+1)(x-3)$.

The zeros of the function are as follows:

$$(x-5)^2(x+1)(x-3) = 0$$

$$\begin{aligned} (x-5)^2 &= 0 & x+1 &= 0 & x-3 &= 0 \\ x-5 &= 0 & x &= -1 & x &= 3 \\ x &= 5 & x &= -1 & x &= 3 \end{aligned}$$

The zeros of the function, occur at $5, -1$, and at 3. What's the behavior of the function at these points? Well, the graph *bounces* at $x = 5$, and crosses the x-axis at $x = -1$, and $x = 3$. The graph looks as follows:



$y = (x-5)^2(x+1)(x-3)$

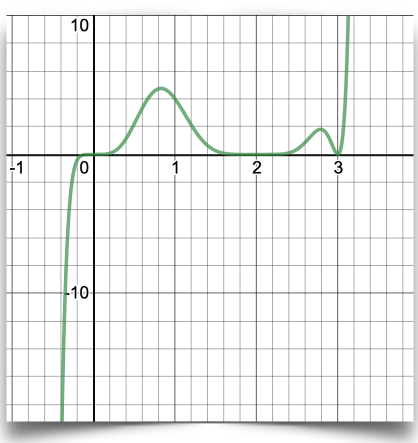
Example: Find the zeros of $x^5(x-3)^2(x-2)^6$.

The zeros of the function are as follows:

$$x^5(x-3)^2(x-2)^6 = 0$$

$$\begin{array}{rlll} x^5 = 0 & (x-3)^2 & x-2)^6 = 0 \\ x = 0 & x-3 = 0 & x-2 = 0 \\ x = 0 & x = 3 & x = 2 \end{array}$$

What about the zeros? The degree of the zero at $x = 0$ is odd, so we have a crossing at $x = 0$ and at $x = 3$, and at $x = 2$, we have the graph bouncing the x-axis. In fact, here's what the graph looks like:



$x^5(x-3)^2(x-2)^6$