Domain & Range

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Not every function can handle every kind of number. If you have the function $f(x) = \frac{1}{x}$, then this function can't handle 0, that is $f(0) = \frac{1}{0} =$ undefined. Again, if you have $g(x) = \log(x)$, then this function is undefined if you plug in any number ≤ 0 (try this on your calculator).

We want to be able to talk about this, so we have a definition:

Definition: The *domain* of a function is the set of acceptable (or permissible) values of input.

Using this definition, we can go back to our examples. We see clearly that the domain of $f(x)$ is every real number except 0. But how do we represent that?

Interval Notation

We want to start with a brief reminder of interval notation. Remember that, the number-line is represented by what is called *blackboard R*: $\mathbb{R}$. $\mathbb{R}$ includes everything from $-\infty$ to $+\infty$. Sometimes, we need a smaller part of that line. We use [and] to represent $\leq$ and $\geq$, and (and) to represent $<$ and $>$.

In other words, $5 \leq x \leq 10$ is represented by $x \in [5,10]$. Similarly $5 < x \leq 10$ is represented by $(5,10]$. Infinity is traditionally represented by), so for example, it's always $[5.\infty)$ but never $[5,100]$, and same with $-\infty$.

One last thing, we represent "broken" domains using the cup symbol: $\cup$. We "glue" together domains $[7,8]$ and $[11,14)$ to give us $[7,8] \cup [11,14)$. Notice that it is one object, but with two parts.

Domain

There are two philosophical approaches to representing domains. Imagine all real numbers. Most functions will accept input from most parts of the real line, why not just accept all the real numbers and throw away the parts of the real number that do not work?

Using this, from the start of the page the domain of $f(x)$ is $\mathbb{R}\backslash\{0\}$ or $(-\infty,0) \cup (0,\infty)$. The backslash is called "toss". We're "tossing away" 0 from the real numbers. Similarly the domain $(-\infty,2] \cup [3,\infty)$ would be represented as $\mathbb{R}\backslash(2,3)$.

What do you think the domain for $g(x)$ would be?

If you play around with your calculator, you will notice three hard and fast rules:

- You can't divide by zero.
- You can't take the square root of a negative number. You can if you consider imaginary numbers, but that is for a higher level class.
- You can't take the $\log(x)$ of a negative number or zero. You can take the log of a negative number if you use imaginary numbers, but that is for a higher level class.

Using this you can try and find out the domain of more complicated functions than the one looked at.

Example 1: What is the domain of $\dfrac{1}{x^2+3x}$?

Notice that this is a fractions. Fractions are fine, and really no number will choke this function, unless it results in a division by zero. How do we know what kind of numbers will cause the denominator to become zero? Well...algebra! We solve the denominator for zero.

$$\begin{aligned} x^2 + 3x &= 0 \\ x(x+3) &= 0 \\ x &= 0 \qquad\qquad x + 3 = 0 \\ x &= 0 \qquad\qquad\quad x = -3 \end{aligned}$$

So the domain of this function is all real numbers except 0 and 3. We can write this as $\mathbb{R}\backslash\{0,-3\}$or $(-\infty,-3) \cup (-3,0) \cup (0,\infty)$.

Example 2: What is the domain of $\frac{x^2-5}{x^2-8}$?

Again, we make a few initial observations. Despite the initial complicated appearance. This is a fractions. Fractions are fine, except if the denominator is zero. So as long as we throw away the values of x that cause the denominator to become zero, we can plug in any value for x and still get a valid output.

To find out the values of x that cause the denominator to become zero, we solve for zero:

$$\begin{aligned} x^2 - 8 &= 0 \\ x^2 &= 8 \\ x &= \pm\sqrt{8} = \pm 2\sqrt{2} \end{aligned}$$

So, the domain is$\mathbb{R}\backslash\{\pm\sqrt{2}\}$, or $(-\infty, -\sqrt{2}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{2}, \infty)$.

Example 3: What is the domain of $\sqrt{x^2 - 12}$?

Pause for a moment and consider that $\sqrt{\ }$ works for all numbers except negative numbers. $\sqrt{\ }$ is well defined as long as what we're taking the square root of, is ≥ 0. So we need the find the values of x that keep the operand (the stuff inside the square root) positive. We use algebra:

$$\begin{aligned} x^2 - 12 &\geq 0 \\ x^2 &\geq 12 \\ x \leq -\sqrt{12} &\quad \text{or} \quad x \geq \sqrt{12} \\ x \leq -2\sqrt{3} &\quad \text{or} \quad x \geq 2\sqrt{3} \end{aligned}$$

So the domain is $\mathbb{R}\backslash(-2\sqrt{3}, 2\sqrt{3})$ or $(-\infty, -2\sqrt{3}) \cup (2\sqrt{3}, \infty)$.

Example 4: What is the domain of $\log(x^3 - 10)$?

Thinking back, we realize that log (no matter the base) is defined when the operand (the thing inside the log) is > 0. Clearly we have lots of stuff inside the log, so to find the domain, all we need to do is find out the values of x that result in the operand being > 0. We start to solve:

$$\begin{aligned} x^3 - 10 &> 0 \\ x^3 &> 10 \\ x &> \sqrt[3]{10} \approx 2.1544... \end{aligned}$$

The domain is $\mathbb{R}\backslash(-\infty, \sqrt[3]{10}]$, or $(\sqrt[3]{10}, \infty)$.

Range

We now look at a closely related concept called *range*. The definition of range is thus:

Definition: The range is the set of output values for a given domain.

Range is highly dependent on domain. If you restrict the domain, chances are your range will be restricted too. Take for example $f(x)=x+2$. Its domain is $\mathbb{R}$ and its output is $\mathbb{R}$. But if I restrict its domain to, say $x > 0$, notice that plugging in $x = 0$, gives us 2, its range is $y > 2$. This means that finding the range is *much* harder than finding the domain. It involves first finding the domain, and then finding the set of y values that this domain could possibly give rise to.

We will look at a few examples:

Example: What is the range of $x^2 + 3$?

Stepping back we realize that$x^2 \geq 0$. If that's the case, then $x^2 + 2 \geq 2$. The range of $x^2 + 2$ is$f(x) \geq 2$ or $(2,\infty)$.

Example: What is the range of $\log(x^2 + 4)$?

Well let's look at the operand $x^2 + 4$. Notice that $x^2 \geq 0$, and as such $x^2 + 4 \geq 4$. Since the operand (the stuff inside the log) is ≥ 4, the entire function has a possible lowest value of$\log(4)$, and a possible highest value of ∞ since the operand has a possible highest value of ∞. The range of our function is $\mathbb{R}\backslash(-\infty, \log(4)]$ or $(\log(4), \infty)$.

Example: What is the range of $\frac{1}{x}$?

We saw earlier that the domain of $\frac{1}{x}$ is $(-\infty,0) \cup (0,\infty)$. This is the key to finding its range. Is it ever possible for $\frac{1}{x}$ to be 0? Yes, if x is ∞ or$-\infty$, but the domain (remember this is set of acceptable or permissible values of x) doesn't allow x to be ∞ or $-\infty$ for that matter. Thus the range is all values of x except 0, in other words $\mathbb{R}\backslash\{0\}$ or $(-\infty,0) \cup (0,\infty)$.

Example: What is the range of $\frac{3x-5}{8x+6}$?

For rational functions such as this, it is useful to determine end behavior. Notice as $x \to \infty$ and $x \to -\infty$ the 5 and 6 make less and less sense. In this regard we can say $\frac{3x-5}{8x+6} \to \frac{3x}{8x} = \frac{3}{8}$. But notice that the graph never touches $\frac{3}{8}$. Thus the horizontal asymptote is $y = \frac{3}{8}$. And thus the range is $\mathbb{R}\backslash\{\frac{3}{8}\}$ or $(-\infty, \frac{3}{8}) \cup (\frac{3}{8}, \infty)$.