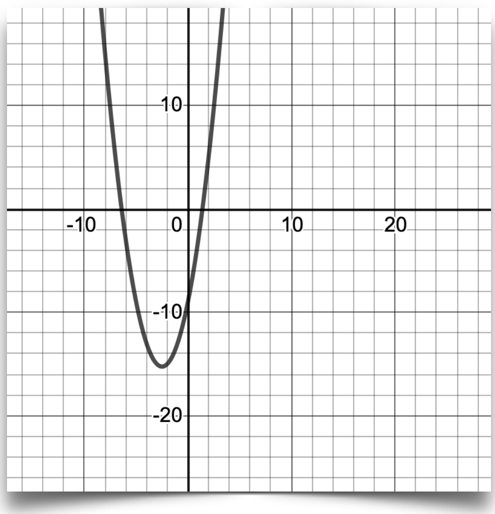
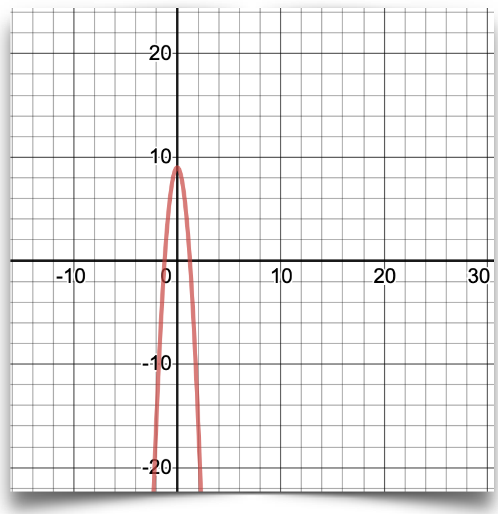
Quadratic Equations

Quadratic Formula
Discriminant

Degree two equations are called quadratic equations. They are usually of the form $ax^2 + bx + c$. Now the signage of a gives us an indication as to whether the graph is "U" shaped or "n" shaped. $a > 0$



$y = x^2 + 5x - 9$



$y = -6x^2 + 9$

indicates the graph is "U" shaped and $a < 0$ indicates the graph is "n" shaped, as these examples demonstrate:

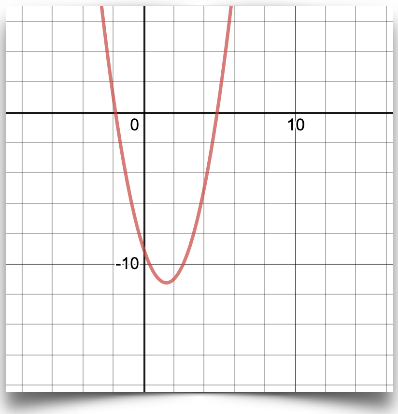
Once again, you are encouraged to play around with this **graph**. Roughly speaking, what you will realize is that a determines how thick or narrow the graph is, and whether it's U or n shaped. b determines, where the tip (technically called the vertex) lies, and finally c determines where the graph crosses the y-axis.

In fact, if you play around with the graph more, you will realize that both a and b affect where the tip of the quadratic equation lies. In fact, if you look carefully, **the effect of changing a is twice as pronounced as the effect of changing b**.

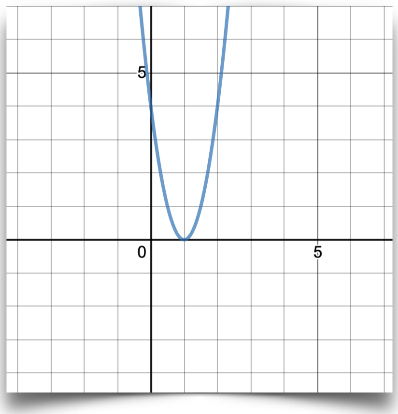
This gives us an insight to finding where the tip (called *vertex* from now on). The x *co-ordinate*, of the vertex of a quadratic equation is always at $\frac{-b}{2a}$. In fact, for emphasis:

$$\text{x co-ordinate of vertex} = \frac{-b}{2a}$$

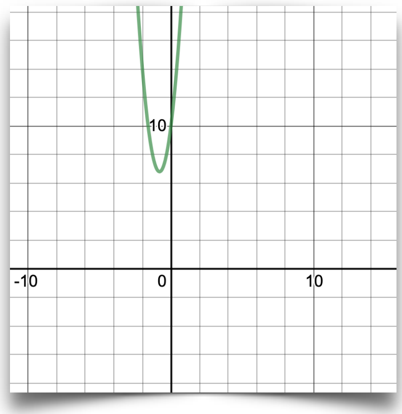
One thing to notice is that a graph of a quadratic could possible cross the x-axis, once, twice, or never. The following examples demonstrate:



$y = x^2 - 3x - 9$



$y = 4x^2 - 8x + 4$



$y = 5x^2 + 8x + 10$

We already established that the x co-ordinate of the vertex lies at $x = \dfrac{-b}{2a}$. That said, notice that where the graph crosses the x-axis twice the points of crossing are on either side of the vertex and the same distance away, this is fascinating.

The Quadratic Formula & Discriminant

You have probably seen this in high-school. We want to derive this. Given a generic quadratic equation: $y = ax^2 + bx + c$, what is the value of x? Well, we set this equal to zero and solve for x, which gives us:

$$
\begin{aligned}
ax^2 + bx + c &= 0 \\
a(x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\
x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 + c &= 0 \\
(x + \frac{b}{2a})^2 &= (\frac{b}{2a})^2 - c \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

This is the quadratic equation, which you might have learned in high-school as:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Which is the one showed above, but simplified. We have three scenarios here:

Repeated Roots: One crucial fact about the above equation is that if $b^2 - 4ac = 0$ we get $\frac{-b+\sqrt{0}}{2a} = \frac{-b-\sqrt{0}}{2a} = \frac{-b}{2a}$. In other words two solutions to the quadratic equation are the same. We call this situation as having *repeated roots*. Graphically this means that graph does not cross the x-axis but simply touches the x-axis.

Distinct Roots: If we have that $b^2 - 4ac > 0$ then we get two separate answers when we compute $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This result is known as having *distinct roots*.

Imaginary Roots: If we have that $b^2 - 4ac < 0$, then we end up having to take the square root of a negative number, which in this class we will assume is not possible. This scenario is called, *having imaginary roots*.

In fact, because $b^2 - 4ac$ is so important to a quadratic equation we have a special name for it:

Definition: Given a quadratic equation, $y = ax^2 + bx^2 + c$, the *discriminant* is: $b^2 - 4ac$. Think of this as a way to *discriminate*, between different types of quadratic equations.