Inverses

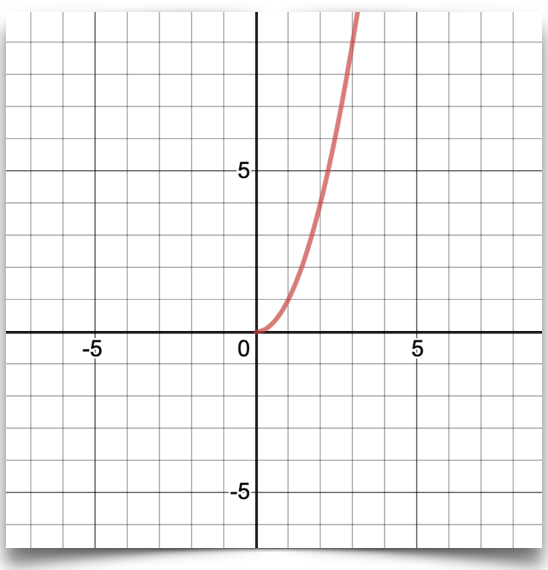
A lot of functions (but not all) can be undone. In other words we can come up with a function that undoes the action of another function. If f is a function that converts $4 \mapsto 5$ and $8 \mapsto 11$, we can come up with a function that does: $5 \mapsto 4$ and $11 \mapsto 8$. We call this the *inverse of* f, and represent it by f^{-1}.

What functions have inverses?

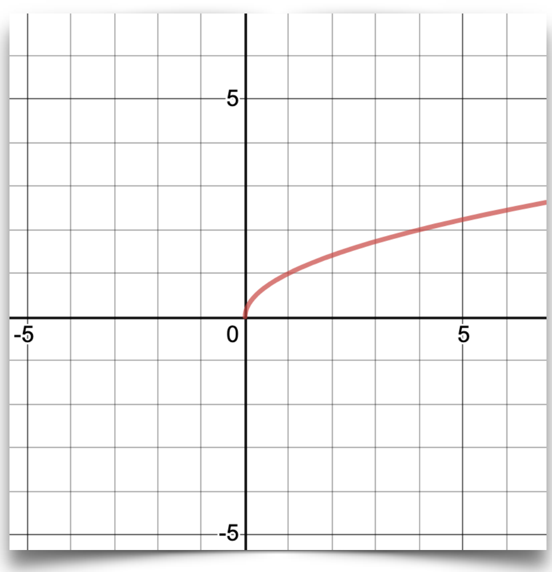
Consider the function $f(x) = x^2$. There are two different values of x which will produce the output 4. $f(2) = 4$ and $f(-2) = 4$. If there was an f^{-1} for this what would it produce on input 4? 2 or -2? That's a trick question, the answer is $f(x) = x^2$ *does not have an inverse*. If there was an inverse that inverse would not be a function. Is there anything we can do about it? Yes!

Domain Restriction

One of the easiest ways to get an inverse of a function that does not have an inverse is to *restrict the domain*. Take for example $f(x) = x^2$, under normal circumstances, it has a domain of $\mathbb{R}$ or $(-\infty, \infty)$. We already saw, that with this domain, $f(x)$ can't be inverted. We could however restrict the domain to $[0,\infty)$, which makes the graph look like the one on the left.



Notice that this function passes *both* the horizontal and vertical line test. This function has an inverse, it is $f^{-1}(x) = \sqrt{x}$, which looks like:



How do we take function inverses?

Notice at the beginning we said an inverse undoes a function. In other words, it flips x with y and flips input with output. To find an inverse of a function, our strategy is somehow replace x with y, to flip the arrow in a way, from $x \mapsto f(x)$ to $f(x) \mapsto x$.

This is done by re-arranging the function so that it goes from y being a function of x to x being a function of y. Let's look at a few examples:

Example: Find the inverse of $f(x) = 4x - 9$.

$$\begin{aligned} f(x) &= 4x - 9 \\ f(x) + 9 &= 4x \\ \frac{f(x)+9}{4} &= x \\ \frac{x+9}{4} &= f^{-1}(x) \qquad \text{re-writing} \end{aligned}$$

Example: Find the inverse of $f(x) = \dfrac{x+4}{x-1}$.

$$\begin{aligned} f(x) &= \frac{x+4}{x-1} \\ f(x) \cdot x - f(x) &= x + 4 \\ f(x) \cdot x - x &= f(x) + 4 \\ x(f(x) - 1) &= f(x) + 4 \\ x &= \frac{f(x)+4}{f(x)-1} \\ f^{-1}(x) &= \frac{x+4}{x-1} \end{aligned}$$

Notice that this function is its own inverse. We have a name for these, these are called *involutions.*

Example: Find the inverse of $f(x) = x^2 - 6$.

Before we begin, we need to note that this function is not *one-to-one as it stands*, we can't find the inverse of this function. We need to establish a few more concepts before we can find the inverse of this function. Let's begin.

What happens when a function is inverted?

Since input and output are flipped with an inverse, t*he domain of $f(x)$ is the range of $f^{-1}(x)$, and the range of $f(x)$ is the domain of $f^{-1}(x)$*. For example, the domain of $\log_e(x)$ is $(0,\infty)$, and thus the range of e^x is $(0,\infty)$. The range of $\log_e(x)$ is $(-\infty, \infty)$ and the domain of e^x is $(-\infty, \infty)$.

So domain and range are flipped when taking inverse and vice versa.

One thing you will notice throughout the course is that most concepts have a geometric and algebraic interpretation, both are equivalent. Inverses also have a geometric and algebraic interpretation. We will look at two ways of looking the inverse, first an algebraic way, and then a geometric way.

The Identity Function, and Function Inverse

There is an important function called the identity function, denoted id,1, or I. This function preserves everything in place. In other words, no matter the value, it takes x and keeps it as x. That is to say $1(x) = x$.

A function inverse has a special property that if you compose a function, with its inverse, both ways, you will always get back the identity function. In a more fancy way $f \circ f^{-1}(x) = x$ or $f^{-1} \circ f(x) = x$.

Consider our first example:

$$\begin{aligned} f \circ f^{-1}(x) &= 4(\frac{x+9}{4}) - 9 \\ &= x + 9 - 9 \\ &= x \end{aligned}$$

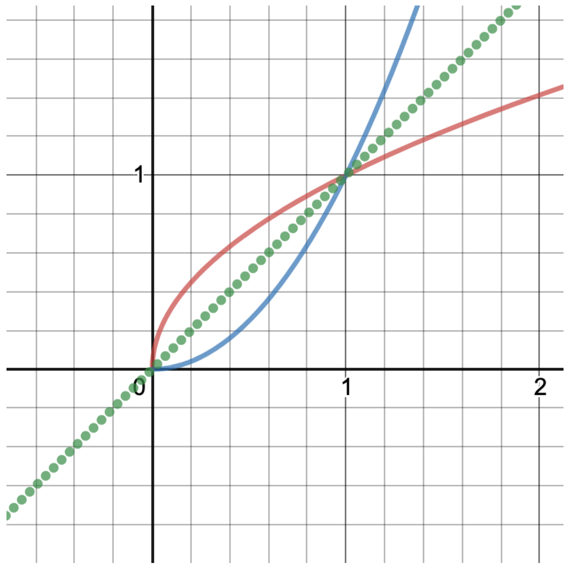
And vice versa:

$$\begin{aligned} f^{-1} \circ f(x) &= \frac{4x - 9 + 9}{4} \\ &= \frac{4x}{4} \\ &= x \end{aligned}$$

In fact, a good way to check if your function invert is correct is to compose it with f both ways and see if you get x.

The Geometric Interpretation of Inverse

Geometrically, the function inverse represents a reflection of the graph along the diagonal line $y = x$. For example, consider the graphs of $y = x^2$ and the graphs of $y = \sqrt{x}$. In order for this inverse to make sense, we need to restrict $y = x^2$ to $x \geq 0$. This gives us the following graph:



This gives us a small hint on how to solve example 3 prom the previous pages.

Example: Find the inverse of $f(x) = x^2 - 6$.

To find the inverse, we need to restrict the domain. This graph has a vertex at $x = 0$. We restrict the domain to $x \geq 0$. The inverse is therefore:

$$\begin{aligned} f(x) &= x^2 - 6 \\ f(x) + 6 &= x^2 \\ \sqrt{f(x) + 6} &= x \\ f^{-1}(x) &= \sqrt{x + 6} \qquad \text{re-writing} \end{aligned}$$

Thus the domain of $f(x)$ is $x \geq 0$ and the range. Is $x \geq -6$, and conversely the domain of $f^{-1}(x)$ is $f^{-1}(x) \geq -6$ and the range of $f^{-1}(x)$ is $f^{-1}(x) \geq 0$.