Finite Element Methods: HW 3

Answer 1 From definitions we know that $-\Delta u = -\nabla^2 u = -(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}) = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$, and so we get:

$$
\begin{aligned}
u &= x_1x_2 - x_1x_2^2 - x_1^2x_2 + x_1^2x_2^2 \\
\frac{\partial u}{\partial x_1} &= x_2 - x_2^2 - 2x_1x_2 + 2x_1x_2^2 \\
\frac{\partial^2 u}{\partial x_1^2} &= -2x_2 + 2x_2^2
\end{aligned}
$$

And by symmetry $\frac{\partial^2}{\partial x_2^2} = -2x_1 + 2x_1^2$

Thus $-\Delta u = f = -(-2x_2 + 2x_2^2) - (-2x_1 + 2x_1^2)$

Again starting from definitions, we see that ∇u is:

$$x_2 - x_2^2 - 2x_1x_2 + 2x_1x_2^2 + x_1 - x_1^2 - 2x_1x_2 + 2x_2x_1^2$$

Similarly following from definitions we have that $D^2 f$ is:

In [ ]:
```
from sympy import *
import math

init_printing(use_unicode=False, wrap_line=False)
x = Symbol('x')
y = Symbol('y')

u = x*y - x*y**2-x**2*y+x**2*y**2

D_squared_f = sqrt(abs(diff(u,x,2))**2+2*abs(diff(u,x,y))**2+abs(diff(u,y,y))**2
D_squared_f
```

Out[ ]:
$$\sqrt{4|x\,(x-1)|^2 + 4|y\,(y-1)|^2 + 2|4xy - 2x - 2y + 1|^2}$$

For this we use SymPy, but we simplify a bit further:

$$
\begin{aligned}
||u||_{L^2(\Omega)} &= \left(\int_\Omega u^2 d\Omega \right)^{\frac{1}{2}} \\
&= \left(\int_0^1 \int_2^1 u^2 dx_1 dx_2 \right)^{\frac{1}{2}} \\
&= \left(\int_0^1 \int_2^1 (x_1 - x_1^2)^2 (x_2 - x_2^2)^2 dx_1 dx_2 \right)^{\frac{1}{2}}
\end{aligned}
$$

In [ ]:
```
sqrt(integrate(u**2,(x,0,1),(y,0,1)))
```

For this we use SymPy, but we simplify a bit further:

$$
\begin{aligned}
||u||_{L^2(\Omega)} &= \left(\int_\Omega u^2 d\Omega \right)^{\frac{1}{2}} \\
&= \left(\int_0^1 \int_2^1 u^2 dx_1 dx_2 \right)^{\frac{1}{2}} \\
&= \left(\int_0^1 \int_2^1 (x_1 - x_1^2)^2 (x_2 - x_2^2)^2 dx_1 dx_2 \right)^{\frac{1}{2}}
\end{aligned}
$$

In [ ]:
```
sqrt(integrate(u**2,(x,0,1),(y,0,1)))
```

Out[ ]: $\frac{1}{30}$

Finally for $||\nabla u||_{L^2(\Omega)}$ we have:

In [ ]:
```
nabla_u = diff(u,x) + diff(u,y)
nabla_u
sqrt(integrate(nabla_u,(x,0,1),(y,0,1)))
```

Out[ ]: 0

Answer 2

We will use techniques from Lecture-4. We realize that in our case $|\overline{K}| = \frac{1}{2}$, and using the identity $\int_K \phi_1^m \phi_2^n \phi_3^p = \frac{2m!n!p!}{(m+n+p+2)!}|\overline{K}|$ Since our domain is only a triangle we expect a 3×3 matrix, which then gives us: $\begin{pmatrix} \phi_1\phi_1 & \phi_1\phi_2 & \phi_1\phi_3 \\ \phi_2\phi_1 & \phi_2\phi_2 & \phi_2\phi_3 \\ \phi_3\phi_1 & \phi_3\phi_2 & \phi_3\phi_3 \end{pmatrix}$

$$
M^{\overline{K}} = \frac{1}{12} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \frac{1}{2}
$$

First we notice that our triangle has nodes $N_1 = (0,0), N_2 = (1,0), N_3 = (0,1)$, however $a = 1$. We have already observed that $|\overline{K}| = \frac{1}{2}$. Thus by plugging in we see that $b_1 = -1, c_1 = -1, b_2 = 1, c_2 = 0, b_3 = 0, c_3 = 1$. This means that the stiffnmess matrix is:

$$A^{\overline{K}} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \times \frac{1}{2}$$

Answer 3

We use the pderect tool in matlab to give us (with code given at the end of the document), a square with a square cutout. We use Mesh > Initialize Mesh to create a mesh and use Mest > Export mesh to give us the (p, e, t) matrix, all of which are included at the end of the document.

Answer 4

Observe that in our particular case we have that $a = n = 1$ and $f = 0$, and so our first equation is the one below:

$$-\nabla \cdot (\nabla u) = 0 \qquad \in \text{int}(\Omega)$$

For the boundary equation we need to do some simplification

In [ ]:
```
from sympy import *


x, y = symbols('x y')
init_printing(use_unicode=True)

u = exp(x) * atan(y)
del_u = diff(u,x) + diff(u,y)
del_u
```

Out[ ]:
$$e^x \operatorname{atan}(y) + \frac{e^x}{y^2 + 1}$$

this means that on the boundary we have (for simplification let $x_1 = x, x_2 = y$):

$$-n \cdot (a \nabla u) = \kappa(u - g_D) - g_N \qquad \in \partial\Omega$$

$$-ne^x \arctan(y) - \frac{ne^x}{y^2+1} = \kappa e^x \arctan(y) - \kappa g_D - g_N$$

```
u = exp(x) * atan(y)
del_u = diff(u,x) + diff(u,y)
del_u
```

Out[ ]:

$$e^x \operatorname{atan}(y) + \frac{e^x}{y^2+1}$$

this means that on the boundary we have (for simplification let $x_1 = x, x_2 = y$):

$$-n \cdot (a\nabla u) = \kappa(u - g_D) - g_N \qquad \in \partial\Omega$$

$$-ne^x \arctan(y) - \frac{ne^x}{y^2+1} = \kappa e^x \arctan(y) - \kappa g_D - g_N$$

Comparing coefficients then yields that: $\kappa = 1$, $n = -1$, $g_N = 0$ and $g_D = \frac{-e^x}{y^2+1}$. Please see the end of the document for the matlab implementation. We then modify the code from Larson to indicate that κ is a constant 1 and a is a constant 1. Our p, e, t, ξ matrices are at the end of the document.

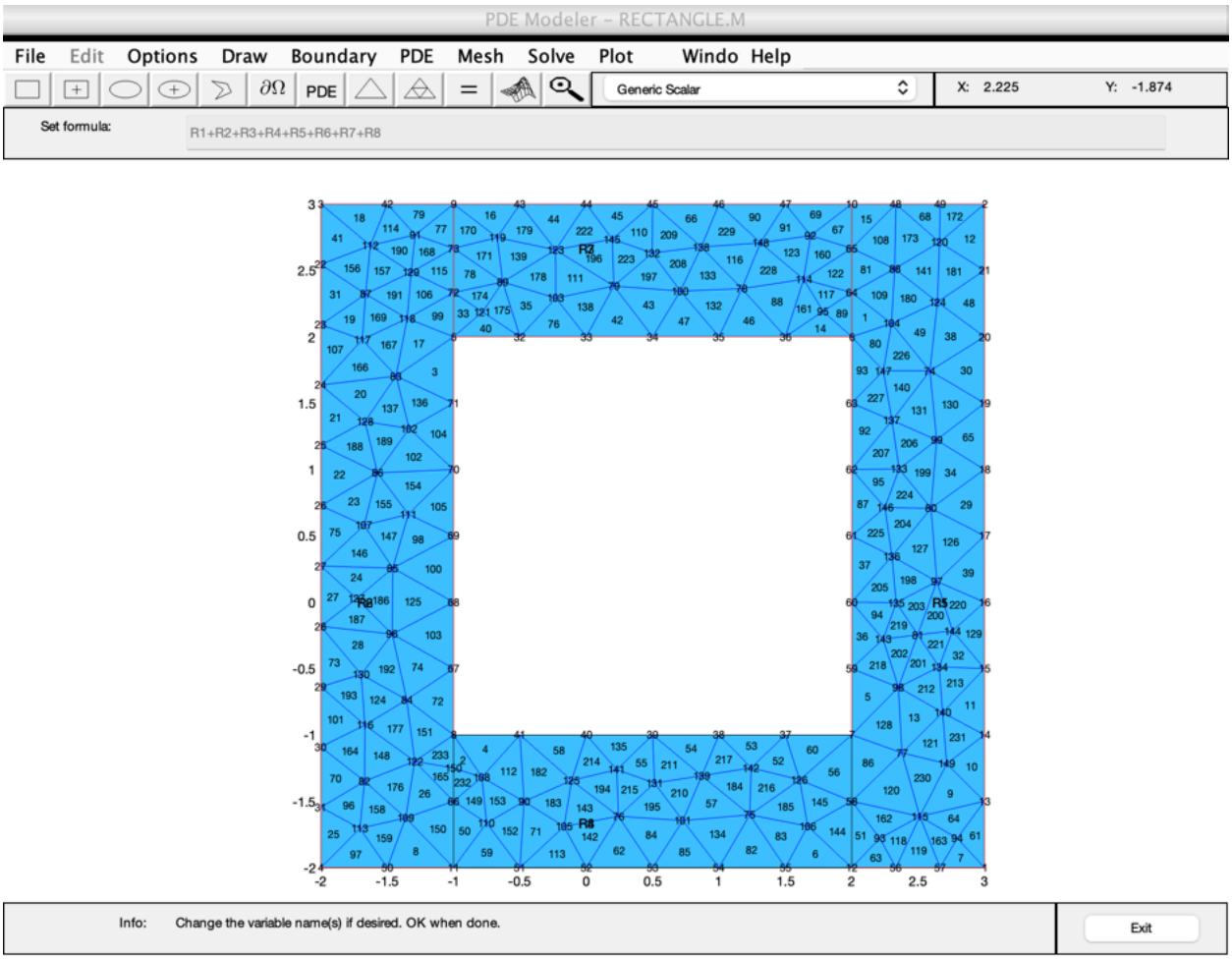
Answer 5

We will adapt Step-3 for our purpose. We replace `GridGenerator::hyper\_cube(triangulation, -1,1)` on line 98 (stripped comments) with `GridGenerator::hyper\_rectangle(triangulation,{-2,-6.28},{2,6.28})`;

We then replace Line 180 with `Functions::SymbolicFunction<2> fun("exp(x)*atan(y)")` however this will require the following headers: #include <deal.II/base/function_lib.h>}$ #include <deal.II/base/symbolic_function.h> #include <deal.II/base/function_spherical.h>

Unfortunately I could not get symbolic functions to work, but I did get cosine boundaries to work and exponential boundaries to work.

```
pderect([2,3,-2,3])
pderect([-2 -1 -2 3])
pderect([-1 2 2 3])
pderect([-1 2 -2 -1])
```



Published with MATLAB® R2022a

```
A = StiffnessAssembler2D(p,t,1);
R = RobinMassMatrix2D(p,e,1);
r = RobinLoadVector2D(p,e,1,@gD,@gN);

xi = r\(A+R);
```

Published with MATLAB® R2022a

```
function A = StiffnessAssembler2D(p,t,a)
np = size(p,2);
nt = size(t,2);

A = sparse(np,np);
for K = 1:nt
    loc2glb = t(1:3,K);
    x = p(1,loc2glb);
    y = p(2,loc2glb);
    [area, b, c] = HatGradients(x,y);
    xc = mean(x); yc = mean(y);
    abar = a;   %since a is a constant, we replace with constant a
    AK = abar*(b*b'+ c*c')*area;
    A(loc2glb,loc2glb) = A(loc2glb,loc2glb) +AK;
end
```

```
Not enough input arguments.

Error in StiffnessAssembler2D (line 2)
np = size(p,2);
```

Published with MATLAB® R2022a

```
function R = RobinMassMatrix2D(p,e,kappa)

np = size(p,2);
ne = size(e,2);
R = sparse(np,np);
for E = 1:ne
    loc2glb = e(1:2,E);
    x = p(1,loc2glb);
    y = p(2,loc2glb);
    len = sqrt((x(1)-x(2))^2+(y(1)-y(2))^2);
    xc = mean(x); yc = mean(y);
    k = kappa;  %since kappa is 1 we simply replace the matrix with the
 constant kappa
    RE = k/6*[2 1; 1 2]*len;
    R(loc2glb,loc2glb) = R(loc2glb,loc2glb)+RE;
end
```

```
Not enough input arguments.

Error in RobinMassMatrix2D (line 3)
np = size(p,2);
```

Published with MATLAB® R2022a

```
function r = RobinLoadVector2D(p,e,kappa,gD,gN)
np = size(p,2);
ne = size(e,2);
r = zeros(np,1);
for E = 1:ne
    loc2glb = e(1:2,E);
    x = p(1,loc2glb);
    y = p(2,loc2glb);
    len = sqrt((x(1)-x(2))^2+(y(1)-y(2))^2);
    xc = mean(x); yc = mean(y);
    tmp = kappa*gD(xc,yc)+gN(xc,yc); rE = tmp*[1; 1]*len/2; %since kappa is a
 constant we replace with constant variable
r(loc2glb) = r(loc2glb) + rE;
end
```

```
Not enough input arguments.

Error in RobinLoadVector2D (line 2)
np = size(p,2);
```

Published with MATLAB® R2022a

```
function[R,r] = RobinAssembler2D(p,e,kappa, gD, gN)
R = RobinMassMatrix2D(p,e,kappa);
r = RobinLoadVector2D(p,e,kappa,gD,gN);
```

```
Not enough input arguments.

Error in RobinAssembler2D (line 2)
R = RobinMassMatrix2D(p,e,kappa);
```

Published with MATLAB® R2022a

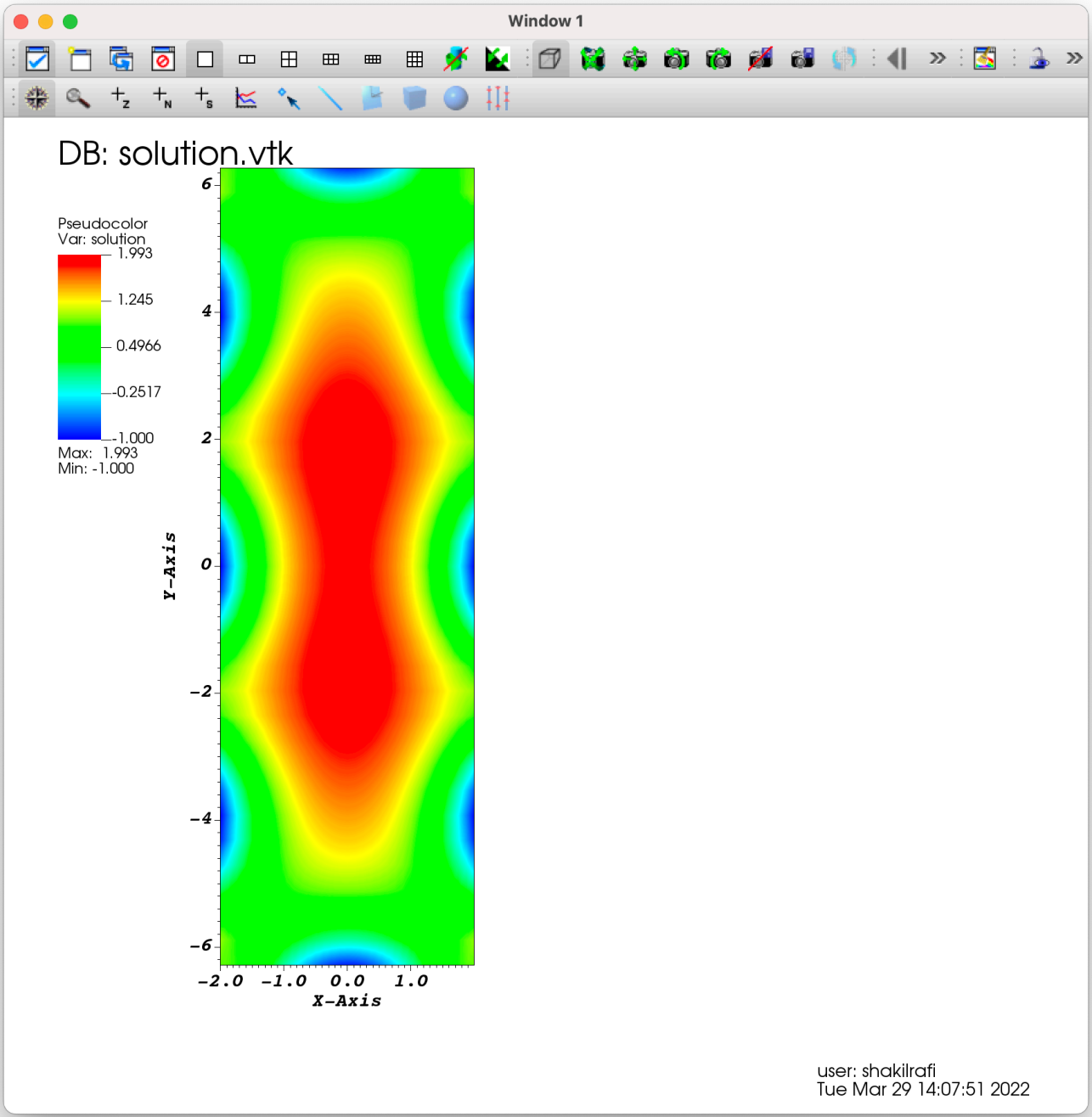
```
function out = gD(x,y)

out = (-exp(x))/(y^2+1);    %we use the function u as gD, the Dirichlet
 boundary condition

Not enough input arguments.

Error in gD (line 3)
out = (-exp(x))/(y^2+1);    %we use the function u as gD, the Dirichlet
 boundary condition
```

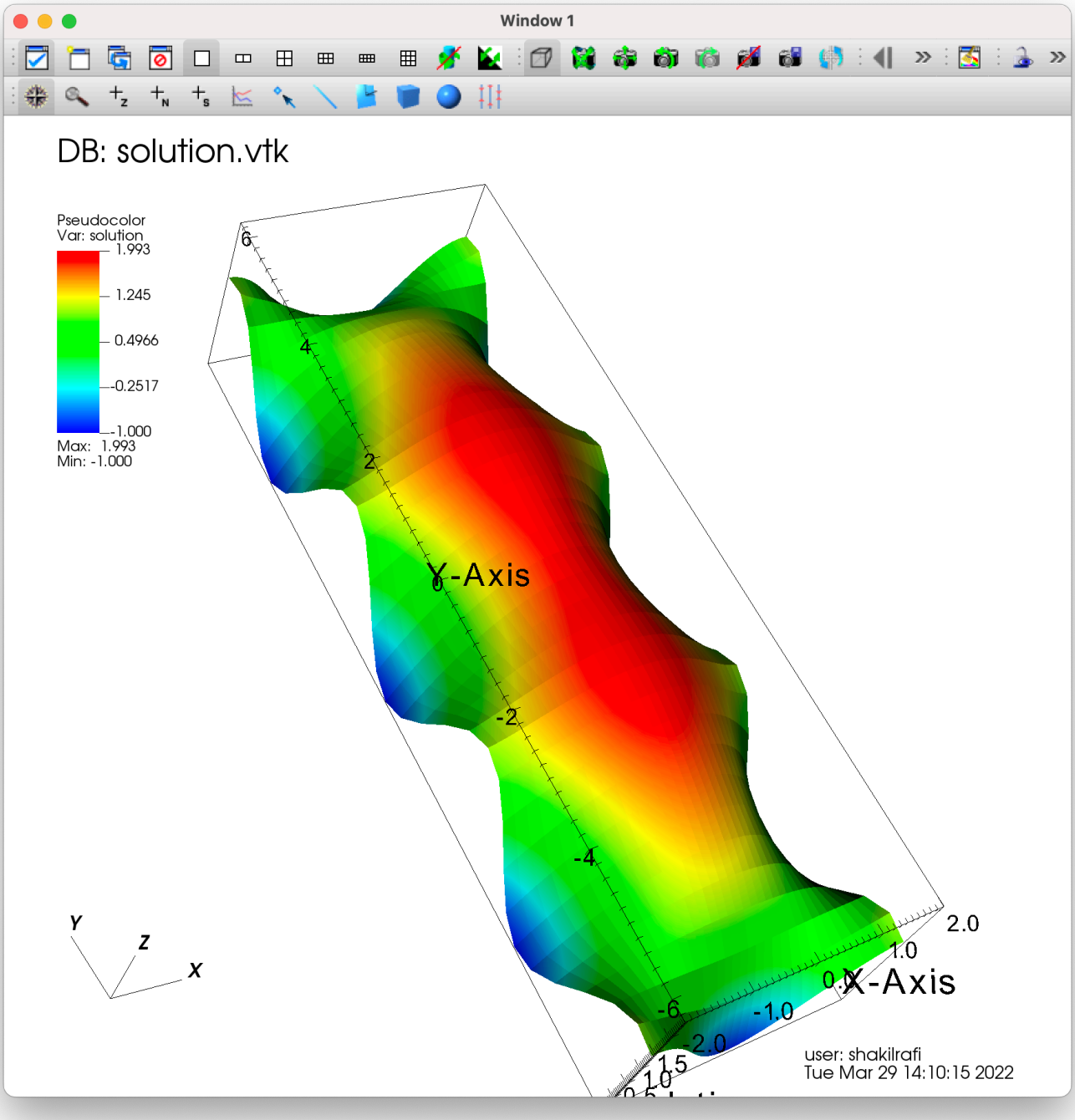
Published with MATLAB® R2022a

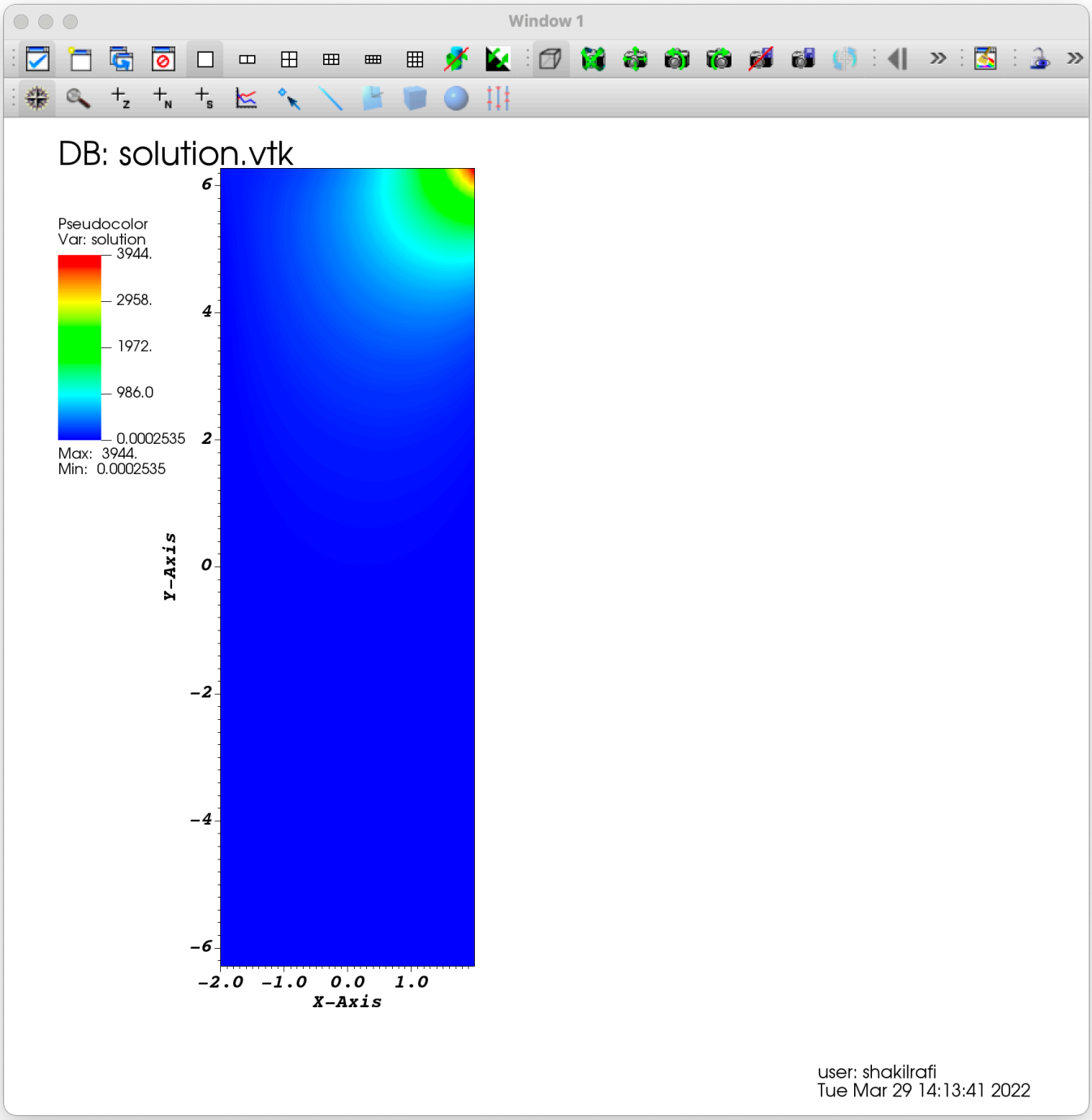
```
function output = gN(~,~)
    output = 0; %we replace gN with zero
```

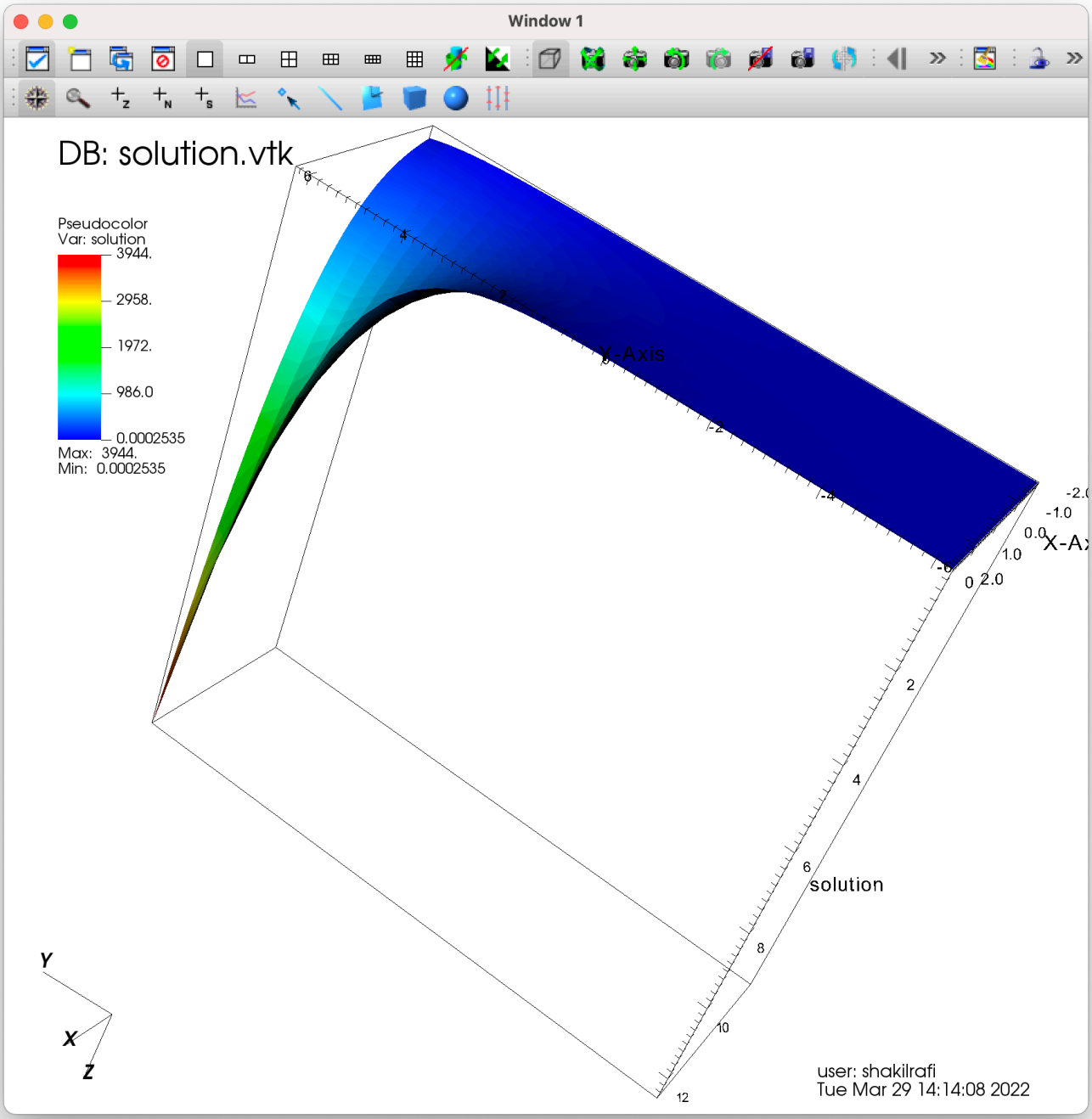
```
ans =

     0
```

Published with MATLAB® R2022a


Window 1
DB: solution.vtk
Pseudocolor
Var: solution
1.993
1.245
0.4966
-0.2517
-1.000
Max: 1.993
Min: -1.000
6
4
2
0
-2
-4
-6
Y-Axis
-2.0
-1.0
0.0
1.0
X-Axis
user: shakilrafi
Tue Mar 29 14:07:51 2022


Window 1
DB: solution.vtk
Pseudocolor
Var: solution
1.993
1.245
0.4966
-0.2517
-1.000
Max: 1.993
Min: -1.000
6
4
2
0
-2
-4
-6
Y-Axis
2.0
1.0
0.5
-1.0
-2.0
1.5
1.0
X-Axis
Y
Z
X
user: shakilrafi
Tue Mar 29 14:10:15 2022


Window 1
DB: solution.vtk
Pseudocolor
Var: solution
3944.
2958.
1972.
986.0
0.0002535
Max: 3944.
Min: 0.0002535
6
4
2
0
-2
-4
-6
Y-Axis
-2.0
-1.0
0.0
1.0
X-Axis
user: shakilrafi
Tue Mar 29 14:13:41 2022


Window 1
DB: solution.vtk
Pseudocolor
Var: solution
3944.
2958.
1972.
986.0
0.0002535
Max: 3944.
Min: 0.0002535
X-Axis
solution
Y
X
Z
user: shakilrafi
Tue Mar 29 14:14:08 2022

```
disp(p)
disp(e)
disp(t)
disp(xi)
```

```
  Columns 1 through 7

    3.0000    3.0000   -2.0000   -2.0000   -1.0000    2.0000    2.0000
   -2.0000    3.0000    3.0000   -2.0000    2.0000    2.0000   -1.0000

  Columns 8 through 14

   -1.0000   -1.0000    2.0000   -1.0000    2.0000    3.0000    3.0000
   -1.0000    3.0000    3.0000   -2.0000   -2.0000   -1.5000   -1.0000

  Columns 15 through 21

    3.0000    3.0000    3.0000    3.0000    3.0000    3.0000    3.0000
   -0.5000         0    0.5000    1.0000    1.5000    2.0000    2.5000

  Columns 22 through 28

   -2.0000   -2.0000   -2.0000   -2.0000   -2.0000   -2.0000   -2.0000
    2.5455    2.0909    1.6364    1.1818    0.7273    0.2727   -0.1818

  Columns 29 through 35

   -2.0000   -2.0000   -2.0000   -0.5000         0    0.5000    1.0000
   -0.6364   -1.0909   -1.5455    2.0000    2.0000    2.0000    2.0000

  Columns 36 through 42

    1.5000    1.5000    1.0000    0.5000         0   -0.5000   -1.5000
    2.0000   -1.0000   -1.0000   -1.0000   -1.0000   -1.0000    3.0000

  Columns 43 through 49

   -0.5000         0    0.5000    1.0000    1.5000    2.3333    2.6667
    3.0000    3.0000    3.0000    3.0000    3.0000    3.0000    3.0000

  Columns 50 through 56

   -1.5000   -0.5000         0    0.5000    1.0000    1.5000    2.3333
   -2.0000   -2.0000   -2.0000   -2.0000   -2.0000   -2.0000   -2.0000

  Columns 57 through 63

    2.6667    2.0000    2.0000    2.0000    2.0000    2.0000    2.0000
   -2.0000   -1.5000   -0.5000         0    0.5000    1.0000    1.5000

  Columns 64 through 70
```

```
  2.0000    2.0000   -1.0000   -1.0000   -1.0000   -1.0000   -1.0000
  2.3333    2.6667   -1.5000   -0.5000         0    0.5000    1.0000

Columns 71 through 77

 -1.0000   -1.0000   -1.0000    2.5896    1.2316    0.2422    2.3784
  1.5000    2.3333    2.6667    1.7433   -1.6022   -1.6127   -1.1350

Columns 78 through 84

  1.1720    0.2074    2.5979    2.4977   -1.6713   -1.4339   -1.3533
  2.3624    2.3823    0.7074   -0.2455   -1.3490    1.7037   -0.7352

Columns 85 through 91

 -1.4583   -1.5739   -1.6634    2.3238   -0.6301   -0.4694   -1.2890
  0.2575    0.9747    2.3215    2.5125    2.4128   -1.5016    2.7683

Columns 92 through 98

  1.6905    2.2118    2.7952    1.7847   -1.4626    2.6414    2.3518
  2.7624   -1.7788   -1.7788    2.1921   -0.2398    0.1562   -0.6408

Columns 99 through 105

  2.6404    0.7087    0.7271   -1.3349   -0.2299    2.2999   -0.1676
  1.2222    2.3446   -1.6455    1.3085    2.2917    2.0996   -1.6900

Columns 106 through 112

  1.6683   -1.6745   -0.7869   -1.3615   -0.7513   -1.3412   -1.6280
 -1.6873    0.5793   -1.3189   -1.6257   -1.6637    0.6626    2.6872

Columns 113 through 119

 -1.7063    1.6377    2.5130   -1.6689   -1.6886   -1.3500   -0.6718
 -1.7044    2.4332   -1.6144   -0.9263    1.9785    2.1377    2.7478

Columns 120 through 126

  2.6619   -0.7823   -1.2932   -0.2316    2.6461   -0.1109    1.6078
  2.7125    2.1870   -1.1985    2.6549    2.2617   -1.3428   -1.3386

Columns 127 through 133

 -1.7301   -1.6682   -1.3207   -1.6967    0.5143    0.4973    2.3572
  0.0272    1.3615    2.4862   -0.5443   -1.3631    2.6309    1.0042

Columns 134 through 140

  2.6594    2.3356    2.3033    2.3036    0.8666    0.8702    2.6850
 -0.4875   -0.0040    0.3465    1.3701    2.6766   -1.3078   -0.8314

Columns 141 through 147
```

```
  0.2296    1.2425    2.2371    2.7599    0.1961    2.2505    2.2384
 -1.2618   -1.2482   -0.2783   -0.2158    2.7359    0.7130    1.7433

Columns 148 through 150

  1.3120    2.7161   -1.0000
  2.7075   -1.2172   -1.2500

Columns 1 through 7

  1.0000   13.0000   14.0000   15.0000   16.0000   17.0000   18.0000
 13.0000   14.0000   15.0000   16.0000   17.0000   18.0000   19.0000
       0    0.1000    0.2000    0.3000    0.4000    0.5000    0.6000
  0.1000    0.2000    0.3000    0.4000    0.5000    0.6000    0.7000
  1.0000    1.0000    1.0000    1.0000    1.0000    1.0000    1.0000
  3.0000    3.0000    3.0000    3.0000    3.0000    3.0000    3.0000
       0         0         0         0         0         0         0

Columns 8 through 14

 19.0000   20.0000   21.0000    3.0000   22.0000   23.0000   24.0000
 20.0000   21.0000    2.0000   22.0000   23.0000   24.0000   25.0000
  0.7000    0.8000    0.9000         0    0.0909    0.1818    0.2727
  0.8000    0.9000    1.0000    0.0909    0.1818    0.2727    0.3636
  1.0000    1.0000    1.0000    2.0000    2.0000    2.0000    2.0000
  3.0000    3.0000    3.0000    1.0000    1.0000    1.0000    1.0000
       0         0         0         0         0         0         0

Columns 15 through 21

 25.0000   26.0000   27.0000   28.0000   29.0000   30.0000   31.0000
 26.0000   27.0000   28.0000   29.0000   30.0000   31.0000    4.0000
  0.3636    0.4545    0.5455    0.6364    0.7273    0.8182    0.9091
  0.4545    0.5455    0.6364    0.7273    0.8182    0.9091    1.0000
  2.0000    2.0000    2.0000    2.0000    2.0000    2.0000    2.0000
  1.0000    1.0000    1.0000    1.0000    1.0000    1.0000    1.0000
       0         0         0         0         0         0         0

Columns 22 through 28

  5.0000   32.0000   33.0000   34.0000   35.0000   36.0000    7.0000
 32.0000   33.0000   34.0000   35.0000   36.0000    6.0000   37.0000
       0    0.1667    0.3333    0.5000    0.6667    0.8333         0
  0.1667    0.3333    0.5000    0.6667    0.8333    1.0000    0.1667
  3.0000    3.0000    3.0000    3.0000    3.0000    3.0000    4.0000
  2.0000    2.0000    2.0000    2.0000    2.0000    2.0000    4.0000
       0         0         0         0         0         0         0

Columns 29 through 35

 37.0000   38.0000   39.0000   40.0000   41.0000    3.0000   42.0000
 38.0000   39.0000   40.0000   41.0000    8.0000   42.0000    9.0000
  0.1667    0.3333    0.5000    0.6667    0.8333         0    0.5000
```

```
    0.3333    0.5000    0.6667    0.8333    1.0000    0.5000    1.0000
    4.0000    4.0000    4.0000    4.0000    4.0000    5.0000    5.0000
    4.0000    4.0000    4.0000    4.0000    4.0000         0         0
         0         0         0         0         0    1.0000    1.0000

Columns 36 through 42

    9.0000   43.0000   44.0000   45.0000   46.0000   47.0000   10.0000
   43.0000   44.0000   45.0000   46.0000   47.0000   10.0000   48.0000
         0    0.1667    0.3333    0.5000    0.6667    0.8333         0
    0.1667    0.3333    0.5000    0.6667    0.8333    1.0000    0.3333
    6.0000    6.0000    6.0000    6.0000    6.0000    6.0000    7.0000
         0         0         0         0         0         0         0
    2.0000    2.0000    2.0000    2.0000    2.0000    2.0000    3.0000

Columns 43 through 49

   48.0000   49.0000    4.0000   50.0000   11.0000   51.0000   52.0000
   49.0000    2.0000   50.0000   11.0000   51.0000   52.0000   53.0000
    0.3333    0.6667         0    0.5000         0    0.1667    0.3333
    0.6667    1.0000    0.5000    1.0000    0.1667    0.3333    0.5000
    7.0000    7.0000    8.0000    8.0000    9.0000    9.0000    9.0000
         0         0    1.0000    1.0000    4.0000    4.0000    4.0000
    3.0000    3.0000         0         0         0         0         0

Columns 50 through 56

   53.0000   54.0000   55.0000   12.0000   56.0000   57.0000   12.0000
   54.0000   55.0000   12.0000   56.0000   57.0000    1.0000   58.0000
    0.5000    0.6667    0.8333         0    0.3333    0.6667         0
    0.6667    0.8333    1.0000    0.3333    0.6667    1.0000    0.5000
    9.0000    9.0000    9.0000   10.0000   10.0000   10.0000   11.0000
    4.0000    4.0000    4.0000    3.0000    3.0000    3.0000    4.0000
         0         0         0         0         0         0    3.0000

Columns 57 through 63

   58.0000    7.0000   59.0000   60.0000   61.0000   62.0000   63.0000
    7.0000   59.0000   60.0000   61.0000   62.0000   63.0000    6.0000
    0.5000         0    0.1667    0.3333    0.5000    0.6667    0.8333
    1.0000    0.1667    0.3333    0.5000    0.6667    0.8333    1.0000
   11.0000   12.0000   12.0000   12.0000   12.0000   12.0000   12.0000
    4.0000         0         0         0         0         0         0
    3.0000    3.0000    3.0000    3.0000    3.0000    3.0000    3.0000

Columns 64 through 70

    6.0000   64.0000   65.0000   11.0000   66.0000    8.0000   67.0000
   64.0000   65.0000   10.0000   66.0000  150.0000   67.0000   68.0000
         0    0.3333    0.6667         0    0.5000         0    0.1667
    0.3333    0.6667    1.0000    0.5000    0.7500    0.1667    0.3333
   13.0000   13.0000   13.0000   14.0000   14.0000   15.0000   15.0000
    2.0000    2.0000    2.0000    1.0000    1.0000    1.0000    1.0000
    3.0000    3.0000    3.0000    4.0000    4.0000         0         0
```

```
Columns 71 through 77

 68.0000   69.0000   70.0000   71.0000    5.0000   72.0000   73.0000
 69.0000   70.0000   71.0000    5.0000   72.0000   73.0000    9.0000
  0.3333    0.5000    0.6667    0.8333         0    0.3333    0.6667
  0.5000    0.6667    0.8333    1.0000    0.3333    0.6667    1.0000
 15.0000   15.0000   15.0000   15.0000   16.0000   16.0000   16.0000
  1.0000    1.0000    1.0000    1.0000    1.0000    1.0000    1.0000
       0         0         0         0    2.0000    2.0000    2.0000

Column 78

150.0000
  8.0000
  0.7500
  1.0000
 14.0000
  1.0000
  4.0000

Columns 1 through 13

  64   108    71    41    59    55    57    50   115    13    14    21    98
   6     8     5     8     7    12     1    11    13    14    15     2    77
 104   150    83   108    98   106    94   109   149   149   140   120   140
   3     4     1     4     3     4     3     1     3     3     3     3     3

Columns 14 through 26

  36    48    43    83    42    87    83    24    25    86    85    31   109
   6    10     9     5     3    23    24    25    26    26    27     4    66
  95    65   119   118   112   117   128   128    86   107   127   113   122
   2     3     2     1     1     1     1     1     1     1     1     1     1

Columns 27 through 39

  27    96    17    19    22   134    72    80    89    60    61    74    16
  28    28    18    20    23    15     5    18    32    59    60    20    17
 127   130    80    74    87   144   121    99   103   143   136   124    97
   1     1     3     3     1     3     2     3     2     3     3     3     3

Columns 40 through 52

   5     3    33    79    44    45    35    34    20   104    66    58   126
  32    22    34    34    43    44    36    35    21    74    11    12    37
 121   112    79   100   123   145    78   100   124   124   110    93   142
   2     1     2     2     2     2     2     2     3     3     4     3     4

Columns 53 through 65

  37    38   131    58   101    40    11     7     1    52    12    94    18
  38    39    39     7    75    41    51    37    13    53    56    13    19
 142   139   141   126   139   125   110   126    94    76    93   115    99
```

```
    4     4     4     4     4     4     4     4     3     4     3     3     3

Columns 66 through 78

   46    65    49    10    30    90     8    28    84    26    32    73    73
   45    10    48    47    31    51    67    29    67    27    33     9    72
  138    92   120    92    82   105    84   130    96   107   103    91    89
    2     2     3     2     1     4     1     1     1     1     2     1     2

Columns 79 through 91

    9   104    65    54    75    76    53     7    62    78     6    47    92
   42     6    64    55    55    53    54    58    61    36    64    46    47
   91   147    88    75   106   101   101    77   146   114    95   148   148
    1     3     3     4     4     4     4     3     3     2     2     2     2

Columns 92 through 104

   63     6   135   133    82     4    85     5    68    29    86    67    70
   62    63    60    62    31    50    69    72    69    30    70    68    71
  137   147   143   146   113   113   111   118    85   116   102    96   102
    3     3     3     3     1     1     1     1     1     1     1     1     1

Columns 105 through 117

   69   118    23    48    88   132   103    90    51    91    72   138    95
   70    72    24    65    64    45    79    41    52    42    73    78    64
  111   129   117    88   104   145   123   108   105   112   129   148   114
    1     1     1     3     3     2     2     4     4     1     1     2     2

Columns 118 through 130

   93    56    77   140    64   114   116    68    17    97     7    15    19
   56    57    58    77    65    92    84    85    80    80    77    16    74
  115   115   115   149   114   148   130    96    97   136    98   144    99
    3     3     3     3     2     2     1     1     3     3     3     3     3

Columns 131 through 143

   99    35   100    54    39    71   102    33   119   137   120    52   105
   74    78    78    75    40    83    83    79    89    74    88    76    76
  137   100   138   101   141   102   128   103   123   147   124   105   125
    3     2     2     4     4     1     1     2     2     3     3     4     4

Columns 144 through 156

   12   106    27   107   116   108    11     8    51    90    70    86    22
   58    58    85    85    82    66    66    84    90   108    86   107    87
  106   126   107   111   122   110   109   122   110   110   111   111   112
    4     4     1     1     1     4     1     1     4     4     1     1     1

Columns 157 through 169

  112   109    50    65    36    58    57    30   122    24   117    73    87
```

```
  87    82   109    92    95    93    94    82    66    83    83    91   117
 129   113   113   114   114   115   115   116   150   117   118   129   118
   1     1     1     2     2     3     3     1     1     1     1     1     1

Columns 170 through 182

   9    73     2    48    89    32    82    84    89    43    88    21    41
  73    89    49    88    72    89   109   116   103   119   104   120    90
 119   119   120   120   121   121   122   122   123   123   124   124   125
   2     2     3     3     2     2     1     1     2     2     3     3     4

Columns 183 through 195

  90   139    75    96    28    25    86    91    87    84    29   125    76
 105    75   106    85    96    86   102   112   118    96   116    76   101
 125   142   126   127   127   128   128   129   129   130   130   141   131
   4     4     4     1     1     1     1     1     1     1     1     4     4

Columns 196 through 208

 123    79   135    80    97    81    98    81   136    60   133    62   132
  79   100    97    99    81    98    81    97    80   135    99   133   100
 145   132   136   133   144   134   143   135   146   136   137   137   138
   2     2     3     3     3     3     3     3     3     3     3     3     2

Columns 209 through 221

  45   131    39   134    15    40    76    75    38    59    81    16    81
 132   101   131    98   134   125   131   126   139    98   135    97   134
 138   139   139   140   140   141   141   142   142   143   143   144   144
   2     4     4     3     3     4     4     4     4     3     3     3     3

Columns 222 through 233

  44    79    80    61    74    63    78    46    77    14    66     8
 123   132   133   136   104   137   114   138   115   140   108   122
 145   145   146   146   147   147   148   148   149   149   150   150
   2     2     3     3     3     3     2     2     3     3     4     1

Columns 1 through 7

 -0.0081   -0.0021   -0.0000   -0.0000   -0.0005   -0.0095   -0.0207

Columns 8 through 14

 -0.0011   -0.0001   -0.0017   -0.0003   -0.0065   -0.0221   -0.0316

Columns 15 through 21

 -0.0503   -0.0606   -0.0417   -0.0264   -0.0165   -0.0104   -0.0078

Columns 22 through 28

 -0.0000   -0.0001   -0.0001   -0.0001   -0.0002   -0.0003   -0.0003
```

```
Columns 29 through 35

 -0.0002   -0.0001   -0.0001   -0.0004   -0.0006   -0.0009   -0.0015

Columns 36 through 42

 -0.0034   -0.0066   -0.0048   -0.0025   -0.0017   -0.0008   -0.0001

Columns 43 through 49

 -0.0002   -0.0003   -0.0005   -0.0009   -0.0017   -0.0008   -0.0018

Columns 50 through 56

 -0.0001   -0.0003   -0.0006   -0.0009   -0.0014   -0.0026   -0.0029

Columns 57 through 63

 -0.0047   -0.0105   -0.0191   -0.0219   -0.0190   -0.0119   -0.0077

Columns 64 through 70

 -0.0025   -0.0014   -0.0003   -0.0007   -0.0008   -0.0008   -0.0005

Columns 71 through 77

 -0.0003   -0.0001   -0.0001    0.0092    0.0011    0.0004    0.0086

Columns 78 through 84

  0.0013    0.0004    0.0252         0    0.0001    0.0001    0.0006

Columns 85 through 91

  0.0004    0.0001    0.0000    0.0017    0.0001    0.0002    0.0001

Columns 92 through 98

  0.0021    0.0088    0.0179    0.0053    0.0005    0.0311    0.0093

Columns 99 through 105

  0.0183    0.0009    0.0009    0.0003    0.0004    0.0026    0.0004

Columns 106 through 112

  0.0056    0.0002    0.0006    0.0002    0.0004    0.0006    0.0001

Columns 113 through 119

  0.0001    0.0010    0.0027    0.0001    0.0001    0.0001    0.0002

Columns 120 through 126
```

```
    0.0059    0.0004    0.0003    0.0001    0.0070    0.0008    0.0077

  Columns 127 through 133

    0.0003    0.0001    0.0000    0.0002    0.0008    0.0002    0.0040

  Columns 134 through 140

    0.0167    0.0076    0.0104    0.0048    0.0005    0.0022    0.0199

  Columns 141 through 147

    0.0014    0.0043    0.0147    0.0418    0.0003    0.0111    0.0050

  Columns 148 through 150

    0.0008    0.0195    0.0001
```

Published with MATLAB® R2022a