Progress Report

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Table of Contents

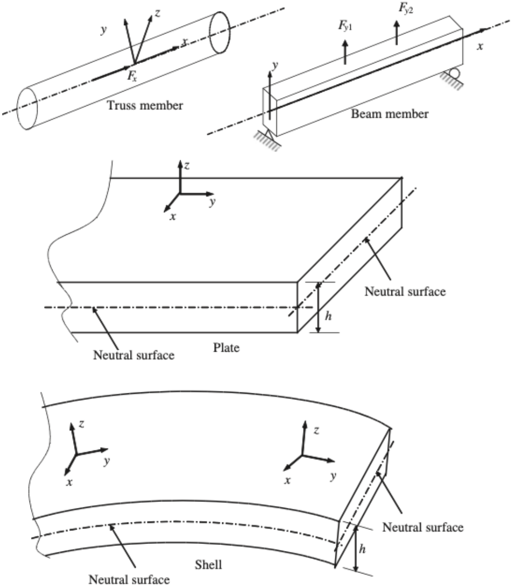
What to Solve

We seek to solve dynamic equilibrium equations. The formulation from Liu and Quek assumes among other things:

1. *Linear Elastic* Deformation grows proportionally with external force.
2. *Isotropic* Material property is direction independent.

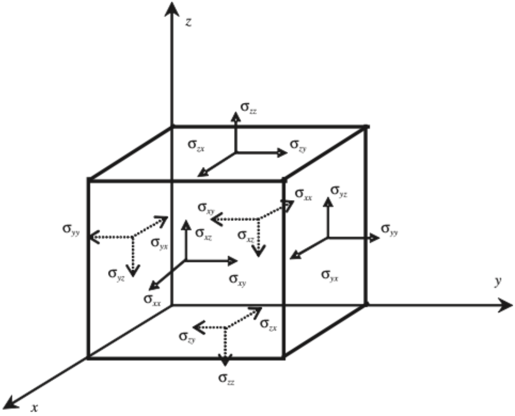
We distinguish between four kinds of objects: beams, trusses, plates and shells.

What to Solve



What to Solve

Taking a queue from Liu and Quek, we start off by defining DEE for an idealized infinitesimal, linearly elastic, isotropic material.



Observe that because of equilibrium $\sigma_{xy} = \sigma_{yx}; \sigma_{xz} = \sigma_{zx}; \sigma_{yz} = \sigma_{yz}$
Which give sus the stress tensors: $\sigma^{T} = \{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma xz, \sigma xy\}$

What to Solve

We also get six strain components $\epsilon^T = \{\epsilon_{xx}, \epsilon_{yy}, \epsilon zz, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}$ Where:

$$\epsilon_{xx} = \frac{\partial u}{\partial x}; \quad \epsilon_{yy} = \frac{\partial v}{\partial y}; \quad \epsilon_{zz} = \frac{\partial w}{\partial z};$$

$$\gamma_{xy} = 2\epsilon_{xy} = \frac{\partial u}{\partial_y} + \frac{\partial v}{\partial x};$$

$$\gamma_{xy} = 2\epsilon_{xz} = \frac{\partial u}{\partial_z} + \frac{\partial w}{\partial x};$$

$$\gamma_{yz} = 2\epsilon_{yz} = \frac{\partial v}{\partial_z} + \frac{\partial w}{\partial y};$$

What to Solve

Provided $\mathbf{U} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ the displacement matrix we get the system $\epsilon = \mathbf{LU}$

Where the differential operator is: $\mathbf{L} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix}$

What to solve

For an external force in the y direction the equilibrium forces are:

$$(\sigma_{xx} + d\sigma_{xx})dydz - \sigma_{xx}dydz + (\sigma_{yx} + d\sigma_{yx})dxdz - \sigma_{yx}dxdz \\ + (\sigma_{zx} + d\sigma_{zx})dxdy - \sigma_{zx}dxdy + f_x dxdydz = \rho\ddot{u}dxdydz \tag{1}$$

But since:

$$d\sigma_{xx} = \frac{\partial\sigma_{xx}}{\partial x}dx, d\sigma_{yx} = \frac{\partial\sigma_{yx}}{\partial y}dy, d\sigma_{zx} = \frac{\partial\sigma_{zx}}{\partial z}dz \tag{2}$$

Substituting (2) into (1) gives us:

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z} + f_x = \rho\ddot{u} \tag{3}$$

What to solve

Using analogous equations for different directions, y and z, we get:

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + f_y = \rho \ddot{v}$$
$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = \rho \ddot{w}$$

Yielding the linear system:

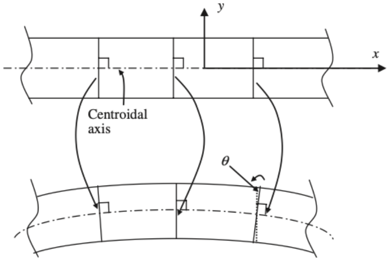
$$L^T \sigma + f_b = \rho \ddot{U} \tag{4}$$

And since Hooke's Law states $\sigma = c\epsilon$ we can further simplify to:

$$L^T cLU + f_b = \rho \ddot{U}$$

What to Solve

For beams, the ruling paradibm is *Euler-Bernoulli* beam theory. This assumes that transverse sections of the beam remain normal to the centroidal axis.



Leads to assumption: $\gamma_{xy} = 0, u = -y\theta, \theta = \frac{\partial v}{\partial x}$

What to Solve

Leading us to conclude

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial(-y\theta)}{\partial x} = -y\frac{\partial \theta}{\partial x} = -y\frac{\partial^2 v}{\partial x^2} = -yLv \quad (5)$$

With differential operator: $L = \frac{\partial^2}{\partial x^2}$
Recall that Hooke's Law then says $\sigma_{xx} = E\epsilon_{xx}$
Substituting (5) into Hooke's Law then gives:

$$\sigma_{xx} = -yELv \quad (6)$$

Following further moment derivations from Liu and Quek, the dynamic equilibrium equations for beam is then:

$$EI\frac{\partial^4 v}{\partial x^4} = F_y \quad (7)$$

How to Solve It

General structure of a deal.ii step.

1. include deal.ii headers
2. Create `class StepN` with public methods `StepN()`, `run()` and private methods `make_grid()`,`setup_system()`,`assemble_system()`,`solve()`, `output_results()`, and attributes often `Triangulation<dim>`, `FE_Q <dim>`, `DoFHandler<dim>`
3. Define the methods outside, sometimes all encaspulated in a template for dimension independent programming.

How to Solve It

```cpp
#include <deal.II/grid/tria.h> ⋯

using namespace dealii;



class Step3 ⋯

Step3::Step3() ⋯

void Step3::make_grid() ⋯

void Step3::setup_system() ⋯

void Step3::assemble_system() ⋯

void Step3::solve() ⋯

void Step3::output_results() const ⋯

void Step3::run() ⋯

int main() ⋯
```

How to Solve It

Results of solving Laplace Equation on Different Domains:

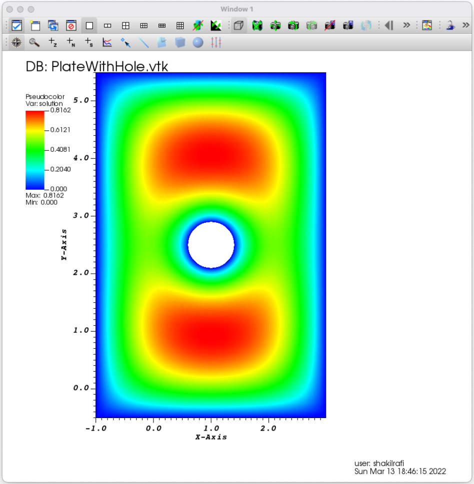


Figure: Plate with one hole

How to Solve It

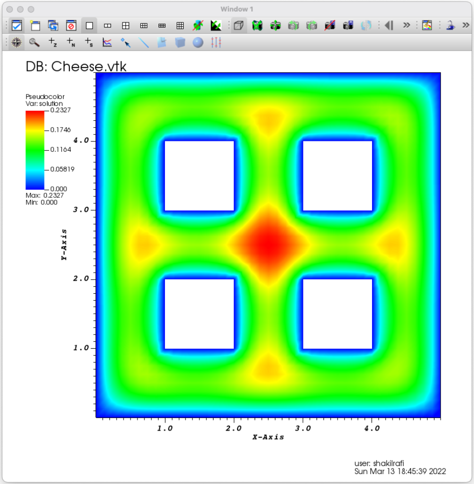


Figure: Cheese domain