Computation and Topology

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Computation: Turing Machines

A k-tape Turing Machine M is a triple (Γ, Q, δ) where:

- Γ is a finite set of symbols that M's tapes can contain including a designated start symbol $\blacktriangleright$and designated blank symbol $\blacksquare$, and the numbers $0, 1$. Γ is called the *alphabet* for M.
- Q is a finite set of states for M including designated Q_{start} and Q_{halt} states.
- $\delta : Q \times \Gamma^k \to Q \times \Gamma^{k-1} \times \{L, S, R\}^k$ and $(k \geq 2)$, the *transition function*.

Computing a function & running time

Let $f : \{0,1\}^* \to \{0,1\}^*$ and let $T : \mathbb{N} \to \mathbb{N}$. M computes f if whenever M is configured with input x it halts with output $f(x)$ written on its output tape. M computes f in $T(n)$ time if every input x takes at most $T(|x|)$ steps.

Computation: Robustness of Turing Machines

A k tape Turing machine M can be simulated by $\tilde{M}$ a single tape TM in polynomial time.

A bidirectional Turing M machine can can be simulated in polynomial time by a unidirectional $\tilde{M}$.

Church-Turing (CT) Thesis

Any physically realizable computation device can be simulated by a Turing Machine.

Computation: Complexity Class P

Let $T : \mathbb{N} \to \mathbb{N}$ be a function. A language L is in DTIME($T(n)$) iff there is a Turing Machine running in time $c \cdot T(n)$ for $c > 0$ deciding L.

The Class P

$\mathrm{P} = \bigcup_{k \geq 1} \mathrm{DTIME}(n^k)$.

This class is desirable because it is closed under composition and (nearly) independent of the computation model.

Computation: Complexity Class NP

The Class NP

A language $L \subseteq \{0,1\}^*$ is in NP if $\exists p : \mathbb{N} \to \mathbb{N}$ a polynomial and a polynomial-time TM M s.t. $(\forall x \in \{0,1\}^*)(x \in L \iff \exists u \in \{0,1\}^{p(|x|)}$ s.t. $M(x,u) = 1)$.

Example: Factoring Given numbers N, L, U, does N have a prime factor $p \in [L, U]$. The certificate is the factor p.
Example: Connectivity Given a graph G and given $\{s, t\} \in V(G)$, is s connected to t in G? The certificate is a path from s to t.
Example: VC Given G, and $n \in \mathbb{N}$, does G have a vertex cover of size $\leq n$. The certificate is a vertex cover V.

Remark: $\mathrm{P} \subseteq \mathrm{NP}$ as we can set $p(|x|) = 0$, however it is believed that $\mathrm{P} \neq \mathrm{NP}$.
Remark: M is called the *verifier* for L and u is the *certificate* or *witness* for x w.r.t. language L and machine M.

Computation: Karp Reductions & NP-Hardness

Karp Reductions

$L \subseteq \{0,1\}^*$ is *polynomial time Karp-reducible* to to $L' \subseteq \{0,1\}^*$, denoted $L \leq_p L'$ if $\exists$ a polynomial time computable $f : \{0,1\}^* \to \{0,1\}^*$ s.t. $(\forall x \in \{0,1\}^*)(x \in L \iff f(x) \in L')$,

Definition: L' is *NP-hard* if $(\forall L \in \text{NP})(L \leq_p L')$.
Definition: L' is *NP-complete* if L' is NP-hard and $L' \in \text{NP}$.

Transitivity of reductions

- If $L \leq_p L'$ and $L' \leq_p L''$ then $L \leq_p L''$.
- If $L \in$ NP-hard and $L \in \text{P}$ then $\text{P} = \text{NP}$.
- If $L \in$ NP-complete then $L \in \text{P}$ iff $\text{P} = \text{NP}$.

Remark: To show a problem is NP-hard we polynomially reduce a known NP-hard problem to it.

Computation: The Cook-Levin Theorem

A CNF formula is a Boolean formula of form $\bigwedge_i \left(\bigvee_j v_{i_j} \right)$, where v_{i_j} is a literal (variable u_j or its negation $\neg u_j$).
A k-CNF formula is a CNF formula where each clause has exactly k variables.

The Problem SAT or $k-$SAT, $k \geq 3$

Given a CNF (or k-CNF instance), $k \geq 3$, is there a satisfying assignment of variables?

Remark: 2-SAT is in P.

Cook-Levin Theorem (Cook '71, Levin '73)

- SAT is NP-complete.
- $3-$SAT is NP-complete.

Remark: More precisely SAT $\leq_p$ 3-SAT.

A 3-manifold is a second countable Hausdorff space M s.t. every point $x \in M$ has a neighborhood homeomorphic to an open subset of $\mathbb{R}^3$.

Examples: $\mathbb{R}^3, \mathbb{B}^3, \mathbb{S}^3, \mathbb{T}^3, V = \mathbb{S}^1 \times \mathbb{D}^2, \mathbb{S}^1 \times \mathbb{S}^2, \widetilde{SL(2,\mathbb{R})}$.

3-Manifolds: Connect Sums

Definition: Oriented Connect Sum of (oriented) 3-Manifolds

Let M_1 and M_2 be two (oriented) 3-manifolds. Let $\mathbb{B}_i \subsetneq int(M_i)$,$i = 1, 2$ be 3-balls embedded in M_i, then $M_1 \# M_2 := (M_1 \setminus int(\mathbb{B}_1)) \bigcup_{\partial \mathbb{B}_1 \sim \partial \mathbb{B}_2} (M_2 \setminus int(\mathbb{B}_2))$. The identification is orientation reversing.

Given maps $h_1, h_2 : \mathbb{S}^2 \to \mathbb{S}^2$, they are both isotopic if they both preserve or both reverse orientation.

Given two $\mathbb{B}_1^3$ and $\mathbb{B}_2^3$ whose closure is embedded in the interior of a closed connected 3-manifold M, there exists an isotopy taking B_1 to B_2.

Oriented 3-manifolds un-
der oriented connect sums exhibit a monoid structure with $\mathbb{S}^3$ as the identity.

3-Manifolds: Primes & Irreducibility

Prime Manifolds

A 3-manifold M is prime if whenever $M = M_1 \# M_2$, M_1 or M_2 is $\mathbb{S}^3$.

Equivalently, a 3-manifold is prime iff it contains no separating essential $\mathbb{S}^2$.

Irreducible Manifolds

A 3-manifold M is *irreducible* if every $\mathbb{S}^2$ in M bounds $\mathbb{B}^3$.

$\mathbb{S}^1 \times \mathbb{S}^2$ is reducible.
(Alexander Theorem). $\mathbb{R}^3$ is irreducible. By extension so is $\mathbb{B}^3$ and $\mathbb{S}^3$.

Prime Manifolds that are Reducible

An irreducible closed connected 3-manifold is prime. An orient-able closed connected prime 3-manifold is either irreducible or $\mathbb{S}^1 \times \mathbb{S}^2$.

3-Manifolds: Prime Decomposition

Existence (Kneser '29)

Every compact orient-able 3-manifold can be expressed as a connected sum of a finite number of prime factors.

Uniqueness (Milnor '62)

Let M be a compact orient-able 3-manifold. If $M = M_1 \# ... \# M_j = N_1 \# ... \# N_k$ are prime decompositions then $j = k$ and after reordering $M_i \cong N_i$ for $i = 1, ..., j = k$.

Up to connect summing $\mathbb{S}^3$.

The decomposing spheres are not unique upto diffeomoerphism of M.

(Canonical) Torus Decomposition

Let M be a compact orientable irreducible 3-manifold.

An embedded surface with $g \geq 1$ is *incompressible* if it is π_1-injective.

A surface $S \subset M$ is *essential* if it is incompressible, ∂-incompressible, and not ∂-parallel.

An embedded torus in M is *canonical* if it is essential and can be isotoped off any incompressible embedded torus.

A manifold M is *atoroidal* if it has no essential embedded torus and is itself not $\mathbb{T}^2 \times [-1,1]$ or $\mathbb{K}\tilde{\times}[-1,1]$.

3-Manifolds: JSJ Decomposition & Thurston Geometrization Conjecture

JSJ Decomposition Theorem (Jaco-Shalen '79, Johansson '79)

A maximal (possibly empty) collection of disjoint, non-parallel, canonical tori is finite and unique upto isotopy. It cuts M into submanifolds that are atoroidal (based on $\mathbb{H}^3$ or Sol) or Seifert-fibered.

The Geometrization Theorem (Thurston '82, Perelman '02)

The interior of any compact, orientable 3-manifold can be split along a finite collection of essential, pari-wise disjoint, embedded 2-spheres and 2-tori into a canonical collection of geometric 3-manifolds after capping off all boundary spheres by 3-balls.

Remark: This implies the Poincaré conjecture: Every simply connected closed 3-manifold M is diffeomorphic to $\mathbb{S}^3$.

3-Manifolds: The Eight Geometries

A *geometry* is a simply connected, homogenous, unimodular Riemannian manifold $\mathbb{X}$.

A manifold M is *geometric* if M is diffeomorphic to $\mathbb{X}/\Gamma$, where $\Gamma \leq Isom(\mathbb{X})$, is discrete and acting freely on $\mathbb{X}$. We also say M has a *geometric structure modeled on* $\mathbb{X}$

There are exactly eight maximal, three dimensional geometries: $\mathbb{E}^3, \mathbb{H}^3, \mathbb{S}^3, \mathbb{S}^2 \times \mathbb{E}, \mathbb{H}^2 \times \mathbb{E}, \widetilde{SL(2,\mathbb{R})}, Nil, Sol$.

Six are Seifert fibered, and classified according to e (Euler number of fibration) and χ (the Euler number of the base orbifold):

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$\mathbb{S}^2 \times \mathbb{E}$	$\mathbb{E}^3$	$\mathbb{H}^2 \times \mathbb{E}$
$e \neq 0$	$\mathbb{S}^3$	Nil	$\widetilde{SL(2,\mathbb{R})}$

$\pi_1(M)$ and classification

Let M be a closed manifold modeled on one $\mathbb{X}$ of the eight geometries, then:

- If $\pi_1(M)$ is finite, then $\mathbb{X} = \mathbb{S}^3$, else
- If $\pi_1(M)$ is virtually cyclic, then $\mathbb{X} = \mathbb{S}^2 \times \mathbb{E}$, else
- If $\pi_1(M)$ is virtually abelian, then $\mathbb{X} = \mathbb{E}^3$, else
- If $\pi_1(M)$ is virtually nilpotent, then $\mathbb{X} =$Nil, else
- If $\pi_1(M)$ is virtually solvable, then $\mathbb{X} =$Sol, else
- If $\pi_1(M)$ has a normal cyclic group K, then:
    - If a finite index subgroup of the quotient lifts, then $\mathbb{X} = \mathbb{H}^2 \times \mathbb{E}$.
    - Otherwise $\mathbb{X} = \widetilde{SL(2, \mathbb{R})}$, else
- $\mathbb{X} = \mathbb{H}^3$.

Preliminaries 1

The unknot recognition problem UR asks, given a knot representation (e.g. a knot diagram), is the knot the unknot?

UR is decidable (Haken '61).
UR $\in$ NP (Haas, Lagarias, Pippenger '99).
UR $\in$ co-NP assuming GRH (Kuperberg '14).
UR $\in$ co-NP (Lackenby '16)

Since UR $\in \mathrm{co} - \mathrm{NP} \cap \mathrm{NP}$ this suggests that UR is not NP-hard.

Preliminaries 2

We may ask a slightly related question: Given an unknot diagram D and an integer k, can D be untangled with at most k Rademeister moves?

Given an unknot diagram with c crossings we need at most $(236c)^{11}$ moves to untangle said diagram. (Lackenby '15)

Previously the bound was 2^{nc} where $n = 10^{11}$. (Haas-Lagarias '01)

There are examples of unknot diagrams where untangling requires $O(c^2)$ Rademeister moves to untangle. (Haas-Nowik '10)

Hardness of The Trivial Sub-link Problem

The trivial sublink problem asks, given a link L and an integer n, does L admit an n-component trivial sub-link?

Theorem (deMesmay, Rieck, Sedgewick, Tancer '18)

The trivial sublink problem is NP-complete.

Related: Showing that a link is a sublink of another is NP-Hard. (Lackenby '17)

Deciding a link is trivial is in NP. (Haas, Lagarias, Pippenger '99). Adding a collection of n components of L to the certificate, yields certificate for the trivial sublink problem, i.e. the trivial sublink problem is NP.

We need to prove that the trivial sublink problem is NP-hard, we do this via reducing 3-SAT to the trivial sublink problem.

The Trivial Sub-link Problem

Motivation: Given a 3-SAT instance Φ on n variables we create a link diagram L_Φ s.t. L_Φ admits an n-component trivial sublink iff Φ is satisfiable.

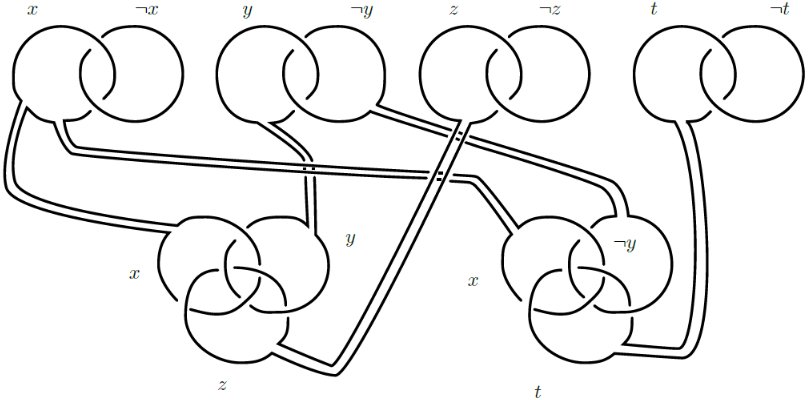

Figure: D_Φ for $\Phi = (x \vee y \vee \vee z) \wedge (x \vee \neg y \vee t)$.

Trivial Sub-link Problem: Proof

Setup: For every literal l_i there is a Hopf link associated to it labeled with x_i and $\neg x_i$. For every clause $c_j = l_1 \vee l_2 \vee l_3$ there is a Borromean ring associated with the clause, with links labeled l_1, l_2, l_3.
The sets of Hopf links and Borromean rings are band summed according to the way the literals appear in the clause, making sure not to weave.

Forward implication

If Φ is satisfiable then L_Φ admits an n-component trivial sublink.

Given an assignment, we remove κ_{x_i} if $x_i = TRUE$ and remove $\kappa_{\neg x_i}$ if $x_i = FALSE$.

Claim: The resulting diagram is an n-component trivial sublink.

Exactly one component is removed from each Hopf link and by extension at-least one from each Borromean ring. The resulting rings can be isotoped back to a set of n rings that are unlinked.

Trivial Sub-link Problem: Proof

Backward Implication

If L_Φ admits an n-component trivial sublink, then Φ is satisfiable.

Assume L_Φ admits an n-component trivial sublink $\mathcal{U}$. $\mathcal{U}$ does not admit the Hopf link and has n components, so exactly one of κ_{x_i} or $\kappa_{\neg x_i}$ is in $\mathcal{U}$.

If κ_{x_i} is in $\mathcal{U}$ set $x_i = FALSE$ and if $\kappa_{\neg x_i}$ is in $\mathcal{U}$ set $x_i = TRUE$.

Claim: This results in a satisfying assignment for Φ.

For contradiction assume that a clause c_i is not satisfied, for some i. This is equivalent to saying that all variables x_j in c_i are assigned false for $j \in \{1, 2, 3\}$. This means κ_{x_j} for $j \in \{1, 2, 3\}$ is present as a Borromean ring in $\mathcal{U}$, a contradiction.

Homeomorphism Problem

The Homeomorphism Problem asks given finite simplicial complexes M and N representing orientable smooth manifolds, do they represent homeomorphic manifolds?
The problem is undecidable for $d \geq 4$. (Markov '58)

It is decidable in dimension 2 due to classification theorems. Indeed χ and the number of boundary components and punctures is enough to define an orientable 2-manifold.

The
problem is decidable for dimension 3 due to the existence of Geometrization.

Homeomorphism Problem

The oriented homeomorphism problem for closed, oriented 3-manifolds is elementary recursive. (Kuperberg '17)

The class ELEMENTARY

$$\mathrm{ELEMENTARY} = \mathrm{DTIME}(2^n) \cup \mathrm{DTIME}(2^{2^n}) \cup \mathrm{DTIME}(2^{2^{2^n}}) \cup \ldots$$

Sphere Recognition Problem

A restricted version of the problem is the Sphere Recognition Problem.
Given a finite simplicial complex N representing a smooth manifold, is N homeomorphic to $\mathbb{S}^d$.
Undecidable for $d \geq 5$. (Novikov)
Unknown for $d = 4$.
Decidable for $d = 3$. (Rubinstein '94, Thompson '94)
Sphere recognition is NP. (Ivanov '01, Schleimer '14)
Sphere recognition is co-NP mod GRH. (Zentner '16)

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