UMS Talk: Symmetric Functions

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From high school algebra or calculus classes, you might have a sense of a real valued function which is 'symmetric'. That is, if have a function $f(x)$ then $f(-x) = f(x)$.

But this is *not* what we will be talking about today but it is kind of similar. The functions we will look at are what are called 'formal power series.' What this really means is that we won't have to worry about the specific numerical values since we won't be evaluating the function. Rather, we are interested in other aspects of the function.

Let me define some terms.

1. $[n] = \{1, 2, 3, ..., n\}$
2. $[[n]] = \{1^{\infty}, 2^{\infty}, ..., n^{\infty}\}$
3. $X_{[n]} = \{x_1, x_2, ..., x_n\}$
4. $f(X_{[n]})$ is a polynomial over the variables $x_1, x_2, ..., x_n$.

What do we mean when I refer to a symmetric function then? I will first give some intuition before formally defining it.

Suppose we have a polynomial $f(X_{[2]}) = x_1^2x_2 + x_1x_2^2$. This polynomial here seems to have a very nice property that the monomial terms in the polynomial have a similar 'structure.' This structure seems to be that permuting the indices of the variables of the polynomial actually don't change the polynomial itself.

Let $\pi : [n] \to [n]$ be a permutation. We note that $\pi \in S_n$. We can apply the permutation group action on these polynomials in a very intuitive way. That is, $\pi f(X_{[n]}) = \pi f(x_1, x_2, ..., x_n) = f(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$. A polynomial $f(X_{[n]})$ is symmetric if $\pi f(X_{[n]}) = f(X_{[n]})$ for all $\pi \in S_n$.

Let's reexamine our previous, very simple example.

$f(X_{[2]}) = x_1^2x_2 + x_1x_2^2$. What are the permutations in S_2? We have 12 and 21. We will write in one line notation.

Let's apply $\pi_1 = 12$. $\pi_1 f(X_{[2]}) = f(X_{[2]})$ since π_1 is the identity element. Now let's apply $\pi_2 = 21$. $\pi_2 f(X_{[2]}) = \pi_2 f(x_1, x_2) = f(x_2, x_1)$.

Everywhere we see x_1 originally, we put x_2 and vice versa. So $\pi_2 f(X_{[2]}) = x_2^2x_1 + x_2x_1^2 = f(X_{[2]})$.

I will three examples of polynomials and let's check if they are symmetric.

$$\begin{aligned} f(X_{[3]}) &= x_1^2x_2 + x_1x_2^2 \\ f(X_{[4]}) &= x_1 + x_2 + x_3 + x_4 \\ f(X_{[1]}) &= 1 \end{aligned}$$

The first one is not, the second and third ones definitely are.

Symmetric functions are important in the study of representation theory, though I am not very well versed in that. But they can also prove many combinatorial formulas as well.

We will introduce a specialized type of symmetric polynomial called the 'complete homogeneous' symmetric polynomial.

$$h_k(X_{[n]}) = \sum_{\substack{J \subseteq [[n]] \\ |J|=k}} \prod_{j \in J} x_j$$

Work through an example, $n = 3, k = 2$.

$$h_2(X_{[3]}) = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

Def: A set partition of $[n]$ is a collection of nonempty blocks so that each element of $[n]$ appears in exactly one block.

The number of set partitions into k non empty blocks is denoted $S(n, k)$ Stirling number of the second kind.

Theorem 0.1. *$h_k(1, 2, ..., n) = S(n + k, n)$ where $S(a, b)$ are the Stirling numbers of the second kind.*

Lemma 0.2. $S(a, b) = S(a - 1, b - 1) + b \cdot S(a - 1, b)$

Proof. If n is alone, there are $S(n - 1, k - 1)$ ways to locate the rest

If n is not alone, there are $S(n - 1, k)$ ways to fill the boxes, then k ways to put n. □

Lemma 0.3. $h_k(X_{[n]}) = h_k(X_{[n-1]}) + x_n \cdot h_{k-1}(X_{[n]})$

Proof. There are two types of terms in $h_k(X_{[n]})$, terms with an x_n factor and terms without. Thus, $h_k(X_{[n-1]})$ gives us all the terms without x_n and then $h_{k-1}(X_{[n]})$ gives us all the terms missing exactly one x_n term, so once we multiply $h_{k-1}(X_{[n]})$ by x_n we see that

$h_k(X_{[n]}) = h_k(X_{[n-1]}) + x_n \cdot h_{k-1}(X_{[n]})$. □

Finally, before the proof, I will give but not prove the following identity.

$S(a, a-1) = \binom{a}{2} = \frac{a(a-1)}{2}$

Proof of theorem. We will prove the claim by induction.

For $n = 1$, $h_k(X_{[1]}) = x_1^k$, thus $h_k(1) = 1 = S(1+k, 1)$.

Next, we show that for $k = 1$, then $h_1(X_{[n]}) = x_1 + x_2 + ... + x_n$ and thus, $h_1(1, 2, ..., n) = 1 + 2 + ... + n = \frac{k(k+1)}{2} = S(n+1, n)$. Next, suppose the claim holds up to $(n-1)$ for some $n \in \mathbb{N}$.

First note that by Lemma 2,

$$\begin{aligned} h_k(X_{[n]}) &= h_k(X_{[n-1]}) + x_n \cdot h_{k-1}(X_{[n]}) \\ &= h_k(X_{[n-1]}) + x_n \cdot (h_{k-1}(X_{[n-1]}) + x_n h_{k-2}(X_{[n]})) \\ &\vdots \\ &= h_k(X_{[n-1]}) + x_n \cdot h_{k-1}(X_{[n-1]}) + x_n^2 \cdot h_{k-2}(X_{[n-1]}) + ... \\ &+ x_n^{k-2} h_2(X_{[n-1]}) + x_n^{k-1} h_1(X_{[n]}) \end{aligned}$$

By induction hypothesis, $h_i(1, 2, ..., n-1) = S(n+i-1, n-1)$ for any i. Thus,

$$\begin{aligned} h_k(1, 2, ..., n) = \left\{ n+k-1 \atop n-1 \right\} + n \left\{ n+k-2 \atop n-1 \right\} + n^2 \left\{ n+k-3 \atop n-1 \right\} + ... \\ + n^{k-2} \left\{ n+1 \atop n-1 \right\} + n^{k-1} \left\{ n+1 \atop n \right\} \end{aligned}$$

1						
0	1					
0	1	1				
0	1	3	1			
0	1	7	6	1		
0	1	15	25	10	1	
0	1	31	90	65	15	1

Table 1: y axis- n starting at 0, x axis - k

We can now group in reverse order,

$$\begin{aligned} n^{k-2}\left\{ {n+1 \atop n-1} \right\} + n^{k-1}\left\{ {n+1 \atop n} \right\} &= n^{k-2} \cdot \left(\left\{ {n+1 \atop n-1} \right\} + n\left\{ {n+1 \atop n} \right\} \right) \\ &= n^{k-2}\left\{ {n+2 \atop n} \right\} \end{aligned}$$

by Lemma 1. Our next pair to be simplified is $n^{k-3}\left\{ {n+2 \atop n-1} \right\} + n^{k-2}\left\{ {n+2 \atop n} \right\}$. In general, we will be simplifying pairs that are of the form $n^{i-1}\left\{ {n+k-i \atop n-1} \right\} + n^{i}\left\{ {n+k-i \atop n} \right\} = n^{i-1}\left\{ {n+k-i+1 \atop n} \right\}$. Continuing to simplify via Lemma 2, we will eventually get

$$h_k(1, 2, ..., n) = \left\{ {n+k-1 \atop n-1} \right\} + n\left\{ {n+k-1 \atop n} \right\} = \left\{ {n+k \atop n} \right\}$$

$\square$