Set Cut Out Classification

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Def: Let X a set. Let $\mathcal{C} \subset \mathcal{P}(X)$. Let $F \subset X$. Define $\mathcal{C} \cap F = \{ C \cap F : C \in \mathcal{C} \}$ as a "cut out."

Clearly $|\mathcal{C} \cap F| \leq 2^{|F|}$.

Def: $f_{\mathcal{C}}(n) = \max \{ \ | \mathcal{C} \cap F| : F \subset X, |F| = n \}$.

The following result is very interesting because it splits the whild behavior of sets into two very nice and manageable sets.

Theorem 0.1. *Either $f_{\mathcal{C}}(n) = 2^n$ or exists $d \in \mathbb{N}$ such that $f_{\mathcal{C}}(n) \leq n^d$.*

First we prove the following lemma.

Lemma 0.2. *Let $F \subset X$ of cardinality n. Let $\mathcal{D} \subset \mathcal{P}(F)$ such that $|D| > \sum_{i<d} \binom{n}{i}$ for $0 \leq d \leq n$. Then there exists $E \subset F$ of cardinality d such that $\mathcal{D} \cap E = \mathcal{P}(E)$.*

Proof. We prove by induction on d. Define $p_d(n) = \sum_{i<d} \binom{n}{i}$. For $d = 0$, the result is trivial by letting $E = \varnothing$. For $d = n$, the result is also easy by letting $E = F$ (another way to see this is that $p_n(n) = 2^n - 1$ and so if $\mathcal{D} \subset \mathcal{P}(F)$ but also $|\mathcal{D}| > 2^n - 1$, then it must be the case that $|\mathcal{D}| = 2^n$ so $\mathcal{D} = \mathcal{P}(F)$).

Next, let $0 < d < n$ given. Choose $x \in F$ and let $F' = F \setminus \{ x \}$. Define $\mathcal{D}' = \{ D - \{ x \} : D \in \mathcal{D} \}$. Note that for any $S \in \mathcal{D}'$, it is either the case that: only $S \cup \{ x \} \in \mathcal{D}$ OR both $S \cup \{ x \}$ and S are elements of $\mathcal{D}$.

We can partition $\mathcal{D}'$ based on this. Let $\mathcal{D}' = \mathcal{D}_1 \cup \mathcal{D}_2$ where $S \in \mathcal{D}_1$ if only $S \cup \{ x \} \in \mathcal{D}$ and let $S \in \mathcal{D}_2$ if both S and $S \cup \{ x \}$ belong in $\mathcal{D}$.

If $|\mathcal{D}'| > p_d(n-1)$ by induction hypothesis there exists $E \subset F' \subset F$ of cardinality d such that $\mathcal{D}' \cap E = \mathcal{P}(E)$, which implies $\mathcal{D} \cap E = \mathcal{P}(E)$.

Otherwise, suppose $|\mathcal{D}'| \leq p_d(n-1)$. Then

$$|\mathcal{D}| = |\mathcal{D}_1| + 2|\mathcal{D}_2| = |\mathcal{D}'| + |\mathcal{D}_2|.$$

We know by assumption that $|\mathcal{D}| > p_d(n)$ and so

$$|\mathcal{D}'| + |\mathcal{D}_2| > p_d(n) = p_d(n-1) + p_{d-1}(n-1)$$

by Pascal's identity. Since $|\mathcal{D}'| \leq p_d(n-1)$, then $|\mathcal{D}_2| > p_{d-1}(n-1)$. So by induction, there exists $E \subset F'$ of cardinality $d-1$ so that $\mathcal{D}_2 \cap E = \mathcal{P}(E)$. Then, $\mathcal{D} \cap (E \cup \{ x \}) = \mathcal{P}(E \cup \{ x \})$ since each set in $\mathcal{D}$ is able to cut out each subset of E not containing x but also each subset of E containing x. $\square$

Finally, we can utilize this result to prove our initial theorem, which we will reframe.

Theorem 0.3. *Suppose $d \geq 0$ with $f_{\mathcal{C}}(d) < 2^d$. Then we have $f_{\mathcal{C}}(n) \leq p_d(n)$.*

We note that $p_d(n)$ is exactly a polynomial on n of degree at most d.

Proof. If $n < d$ then $p_d(n) = 2^n$ and the result trivially holds.

Let $n \geq d$. Let $F \subset X$ and $|F| = n$. If $|\mathcal{C} \cap F| > p_d(n)$ then there exists, by our lemma, $E \subset F$ such that $|E| = d$ and $|\mathcal{C} \cap E| = 2^d$ contradicting $f_{\mathcal{C}}(d) < 2^d$. Hence, $|\mathcal{C} \cap F| \leq p_d(n)$. Since F was given, we are done. $\square$