Expansion of power sum symmetric functions as Schur's P functions

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Define $\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z}$ be the positive integers, naturals containing 0, and all integers respectively. Let $[n] = \{1, 2, ..., n\}$.

1 Rudimentary and minimally marked shifted tableaux

Definition 1.1. Let $\prec$ be the total order on $\mathbb{Z} - \{0\}$ with $-1 \prec 1 \prec -2 \prec 2 \prec ....$ This forms the alphabet $\{\mathbb{Z} - 0\}_\prec$ which we use to fill in the entries of tableaux. We say a tableau T is a *minimally marked shifted tableau* if the following conditions hold:

- The entries of T are weakly increasing with respect to $\prec$ along each row and column
- No two entries in the same column of T contain the same positive number.
- No two entries in the same row contain the same negative number.
- Every entry of T on the main diagonal $\{(i, i) : i \in \mathbb{P}\}$ is positive.
- Changing the mark of any entry from $-i$ to i results in an invalid shifted tableau.

We note that this definition is similar to the definition of a *semi-standard shifted marked tableaux* (SSMT), except for the inclusion of the last condition. We denote the set of minimally marked shifted tableaux of shape λ as $\text{MMST}(\lambda)$. We note that $\text{MMST}(\lambda) \subset \text{SSMT}(\lambda)$.

For $T \in \text{MMST}(\lambda)$, we can can flip $\gamma \geq 0$ of the entries in T in order to enumerate all valid semi-standard marked shifted tableaux with T as a 'base'. Define an entry as 'flippable' if flipping sign of a positive entry to negative still yields a valid semi-standard shifted marked tableau. Then, define the function $\text{flip} : \text{SSMT}(\lambda) \to \mathbb{N}$ which maps T to the number of flippable entries. The following theorem shows that the order in which an entry is flipped does not matter.

Theorem 1.1. *Let $T \in \text{SSMT}(\lambda)$. Let T' be the valid semi-standard shifted marked tableaux achieved by flipping the sign of any flippable entry. Then, $flip(T') = flip(T) - 1$. $flip(T') = flip(T)$ otherwise.*

Proof. Let $T \in \text{SSMT}(\lambda)$. Suppose there exists an entry, (i, j) such that it is flippable. Suppose the value of entry (i, j) is n.

Let us examine row i of T. In any given row there can only be one entry with value n that can be of a negative number. By our assumption, (i, j) must be the only one in the row. Flipping the sign of (i, j) will decrease the number of possible flippable entries in the row by exactly one.

Let us now examine column j of T. It is possible there are many entries whose value is n in the column. These entries must be in a constant sequence. Let (i', j) for $p \leq i' \leq q$ be the set of entries in column j such that (i', j) has value n. By definition, if (i, j) is positive, then it must be the lowest entry in the constant sequence (i.e. $i = q$) and all previous entries for $p \leq i \leq q - 1$ must be negative. Thus, flipping the sign of (i, j) will only decrease the number of flippable entries by one.

If there does not exist any (i, j) that is flippable, then clearly the number of flippable entries remains constant at 0. This proves the theorem. $\square$

We can define a partial order on elements of SSMT(λ) where
$T \leq_\alpha T'$ if flipping any number of flippable entries in T (including not flipping any entry at all) results in T'. It can be that this partial order forms a chain decomposition of SSMT(λ) where each tableau in MMST(λ) form the minimal elements in each chain, thus we can associate each chain with an element of MMST(λ). If we fix $T \in$ MMST(λ), it can be observed that each chain in the partial order will have flip(T) many elements.

Let us analyze an individual minimally marked tableau. Let the following tableau be a member of MMST(λ) for $\lambda = (4, 1)$.

$$X = \begin{array}{|c|c|c|c|} \hline 1 & -2 & 5 & 5 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{2}{c}{} \\ \cline{2-2} \end{array}$$

Although there are 5 entries in T, some entries make use of the same value in our alphabet. Notably, we only have $1 \prec 2 \prec 5$ from our alphabet. Structurally speaking, this tableau is no different than one of the following form:

$$X' = \begin{array}{|c|c|c|c|} \hline 1 & -2 & 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{2}{c}{} \\ \cline{2-2} \end{array}$$

since the value of 5 in T readily takes the structural property of 3 in T', in that it serves as the 'greatest' value in the tableau. We make the following definition.

Definition 1.2. A tableau is **rudimentary** if it uses the alphabet $\{n\}_\prec = \{-1 \prec 1 \prec ... \prec -n \prec n\}$ for minimal n over the structural property of row and column weak order.

With this definition, we see that X' is a rudimentary tableau, specifically over the alphabet $\{3\}_\prec$, whereas X is not. We can say X' is the **rudiment** of X, since both tableaux share the same structural properties and X' is rudimentary. It should be noted that rudiments are unique.

We define the set of rudimentary minimally marked shifted tableaux, RMMT(λ) to be the set of all rudimentary tableaux in MMST(λ). In fact, we can define an equivalence relation on elements of MMST(λ) where $T \sim_\beta T'$ if T and T' have the same rudiment. Thus, every tableau $T \in$ RMMT(λ) forms an equivalence class of tableaux in MMST(λ) that also partitions MMST(λ).

Under the equivalence relation $\sim_\beta$, the flip function remains invariant. So for $T, T' \in$ MMST$(\lambda), T' \neq T$, if $T \sim_\beta T'$ then flip$(T) =$ flip(T'). Therefore, we can enumerate the set SSMT(λ) from RMMT(λ).

Theorem 1.2. *Let* $\mathrm{SSMT}_k(\lambda)$ *be the set of all semi-standard marked shifted tableau using the alphabet* $\{k\}_{\prec}$*. A tableau* $T_j \in \mathrm{RMMT}(\lambda)$ *is over the alphabet* $\{j\}_{\prec}$ *for* $j \leq k$*. Then,*

$$|\mathrm{SSMT}_k(\lambda)| = \sum_{T_j \in \mathrm{RMMT}(\lambda)} \binom{k}{j} 2^{\mathrm{flip}(T_j)}. \tag{1}$$

Proof. Choose $T_j \in \mathrm{RMMT}(\lambda)$. Let $[T_j]$ be the equivalence class on the $\sim_\beta$ relation in $\mathrm{MMST}(\lambda)$. Since $\mathrm{MMST}(\lambda)$ is over the same alphabet as $\mathrm{SSMT}_k(\lambda)$, we can choose j many letters from the alphabet $\{k\}_{\prec}$ to replace each of the entries in T_j to form all the elements in $[T_j]$. For example,

$$T_3 = \begin{array}{|c|c|c|c|}\hline 1 & -2 & 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{2}{c}{} \\ \cline{2-2} \end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline x & -y & z & z \\ \hline \multicolumn{1}{c|}{} & y & \multicolumn{2}{c}{} \\ \cline{2-2} \end{array}$$

for $x \prec ... \prec -y \prec y \prec ... \prec z \in \{k\}_{\prec}$. There are $\binom{k}{j}$ ways to choose j many elements from the alphabet in $[T_j]$. From there, each element in the equivalence class $[T_j]$ enumerates chain on the $\leq_\alpha$ relation. For any given $T \in [T_j]$, there are $\mathrm{flip}(T)$ many choices of entries to flip the sign of, thus there are $2^{\mathrm{flip}(T)}$ elements that it enumerates. Taking the sum over all $T_j \in \mathrm{RMMT}(\lambda)$ thus enumerates every tableau in $\mathrm{SSMT}(\lambda)$. $\square$

Remark. As we take $\lim_{k\to\infty} |\mathrm{SSMT}_k(\lambda)|$, we begin to enumerate the number of terms in P_λ, Schur's P function.

Corollary 1.2.1. *Let* $\mathrm{SSMT}_k(\lambda)$ *be the set of all semi-standard marked shifted tableau using the alphabet* $\{k\}_{\prec}$*. A tableau* $T_j \in \mathrm{RMMT}(\lambda)$ *is over the alphabet* $\{j\}_{\prec}$ *for* $j \leq k$*. Then,*

$$|\mathrm{SSMT}_k(\lambda)| = \sum_{T_j \in \mathrm{MMST}(\lambda)} 2^{\mathrm{flip}(T_j)}. \tag{2}$$

Proof. The proof follows naturally from Theorem 1.2. $\square$

The nature of Equation 1 remains relatively elusive due to the nature of the flip function. However, we can find a rich characterization of flip once we begin work on the path towards our final bijection.

2 Ping ponging shifted tableau

We will focus our efforts on tableaux with only two rows or less. Further, we assume that all partitions λ are partitions of an odd integer k and contains only distinct parts unless otherwise stated.

Let $T \in \mathrm{RMMT}(\lambda)$ for $l(\lambda) \leq 2$. We define the 'pong' action on T, $\mathrm{pong}(T)$, by sliding the last entry in the first row of T down and to the left, stopping at a suitable spot. If the entry is below another entry with the same value, the top entry flips its sign to be negative. The sliding entry does not bump any entries already existing in the second row, thus certain tableaux cannot be ponged.

The pong actions takes a tableau in RMMT(λ) and maps them to another tableau in RMMT(λ), since we flip signs of entries only when necessary. Further, the pong action does not change the value of entries themselves, meaning a rudimentary tableau remains rudimentary over the same alphabet afterwards.

We give the following examples.

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&3\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&2&3\\\hline \multicolumn{1}{c|}{}&3&\multicolumn{1}{c}{}\\\cline{2-2}\end{array}$$

$$\begin{array}{|c|c|c|c|}\hline 1&2&2&2\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&-2&2\\\hline \multicolumn{1}{c|}{}&2&\multicolumn{1}{c}{}\\\cline{2-2}\end{array}$$

$$\begin{array}{|c|c|c|c|}\hline 1&-2&2&2\\\hline \multicolumn{1}{c|}{}&2&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{2-2}\end{array} \not\to \begin{array}{|c|c|c|}\hline 1&-2&-2\\\hline \multicolumn{1}{c|}{}&2&2\\\cline{2-3}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&-2&3\\\hline \multicolumn{1}{c|}{}&2&\multicolumn{1}{c}{}\\\cline{2-2}\end{array} \not\to \begin{array}{|c|c|c}\cline{1-2} 1&-2&\multicolumn{1}{c}{}\\\hline \multicolumn{1}{c|}{}&2&\multicolumn{1}{c|}{3}\\\cline{2-3}\end{array}$$

We note the last two examples are not a valid pong action since there cannot exist two positions in the same row containing the same negative number nor can the pong action form an invalid shifted tableau shape.

A tableau T is **k-pongable** if we can perform a valid pong action on the tableau k times. We write a tableau that has been ponged k times as $\text{pong}^k(T)$ and we write $\text{pong}^{-j}(T)$ for the inverse pong action j many times.

Theorem 2.1. *Let $w = w_1w_2...w_n$ be the reading word of the first row of T. Then T is k-pongable for $k \leq \#\{\text{Entries equal to } w_n\}$.*

Proof. For w a reading word of T, since the rows are weakly increasing, w_n is the greatest entry in w. Suppose for the sake of contradiction,
$k > \#$ entries equal to w_n, then one there exists an element w_j for $j \leq n-k$ such that $w_j < w_n$. Since pong cannot bump any elements, we cannot pong a tableau where w_j is the last element of its first row. Thus, $k \leq \#\{\text{Entries equal to } w_n\}$. □

Theorem 2.2. *Let $T \in \text{RMMT}(\lambda)$ where T is at most 1-pongable and $\text{pong}^{-1}(T)$ is not valid. Then $\text{flip}(T) - 1 = \text{flip}(\text{pong}(T))$ if $l(\lambda) = 1$. Otherwise, if $l(\lambda) = 2$, then $\text{flip}(T) = \text{flip}(\text{pong}(T)$.*

Proof. Let $T \in \text{RMMT}(\lambda)$ such that T is at most 1-pongable and $\text{pong}^{-1}(T)$ is not valid. There are four cases to consider.

Case 1: There are no entries in the second row, i.e. $l(\lambda) = 1$.

Case 1.1: There is exactly 1 entry equal to $(1, \lambda_1)$ in the first row. We note that before we pong T, the entry $(1, \lambda_1)$ contributes an addend to $\text{flip}(T)$ since that element would be flippable. After we pong T, that entry would be on $(2, 2)$, the main diagonal, meaning it is no longer flippable. Thus, $\text{flip}(\text{pong}(T)) = \text{flip}(T) - 1$.

Case 1.2: There are more than 1 entries equal to $(1, \lambda_1)$ in the first row. Let $(1, m)$ be the leftmost entry in the first row with equal value to $(1, \lambda_1)$. The entry $(1, m)$ contributes one addend to $\text{flip}(T)$. Since T is only 1-pongable, if we pong twice then that results in an invalid tableau. The only way that can happen is if $m = 2$, and ponging twice would result in the entry $(1, 3)$ to also be negative, violating our rule that no row can have two negative numbers.

Thus, ponging once will force the entry at $(1, 2)$ to be negative, removing an addend from $\text{flip}(\text{pong}(T))$.

Case 2: There exists at least one entry in row 2, i.e. $l(\lambda) = 2$.

Case 2.1: There is exactly 1 entry equal to $(1, \lambda_1)$ in the first row. Since $\text{pong}^{-1}(T)$ is invalid, every entry in the second row must be of lesser value than the entry $(1, \lambda_1)$. After ponging T, since there exists an entry in the second row, the ponged entry retains the capability to be positive or negative, meaning our total number of flip choices remains constant.

Case 2.2: There are more than 1 entries equal to $(1, \lambda_1)$ in the first row. Let $(1, m)$ be the leftmost entry in the first row with equal value to $(1, \lambda_1)$. Consider $\text{pong}(T)$. Since T is at most 1-pongable, by a similar argument as *Case 1.2*, $(1, m)$ becomes negative. The ponged entry will be the leftmost entry in row 2 of its value. Then where $(1, m)$ loses its flippable property, our ponged entry gains it by a similar logic as *Case 2.1*. □

Theorem 2.3. *Let* $T \in \text{RMMT}(\lambda)$ *be* k*-pongable for* $k > 1$*. Then* $\text{flip}(T) \leq \text{flip}(\text{pong}(T))$.

Proof. Let $T \in \text{RMMT}(\lambda)$ such that T is k-pongable for $k > 1$. By Theorem 2.1, there are at least $k > 1$ many entries in the first row of T equal to the value of the entry $(1, \lambda_1)$. Thus the entry we pong will not be flippable in T. There are two cases to consider.

Case 1: After $\text{pong}(T)$, then ponged entry becomes flippable in the second row. Then $\text{flip}(\text{pong}(T)) = \text{flip}(T) + 1 \geq \text{flip}(T)$

Case 2: After $\text{pong}(T)$, the ponged entry remains unflippable in the second row. Then $\text{flip}(T) = \text{flip}(\text{pong}(T))$. □

Corollary 2.3.1. *Let* $T \in \text{RMMT}(\lambda)$ *be* k*-pongable for* $k > 1$*. Then* $\text{flip}(T) = \text{flip}(\text{pong}(T))$ *if* $l(\lambda) = 1$ *or* $\text{pong}^{-1}(T)$ *is valid. Otherwise,* $\text{flip}(\text{pong}(T)) = \text{flip}(T) + 1$.

Proof. The proof follows naturally from the proof of Theorem 2.3 □

Remark. For Theorem 2.2, we may wonder why we insist on the condition that for $T \in \text{RMMT}(\lambda)$, $\text{pong}^{-1}(T)$ is invalid. This is because if T was at most 1-pongable, but $\text{pong}^{-1}(T)$ was valid, then we would instead take the tableau $T' = \text{pong}^{-1}(T)$ and apply Theorem 2.3. Thus, we can gain a notion of a tableau that is 'minimally ponged' if $\text{pong}^{-1}(T)$ is not valid. We will note later that tableau which are *not* minimally ponged can be accounted for in a specific way.

Theorem 2.4. *Let* $T \in \text{RMMT}(\lambda)$ *where* T *is at most* 1*-pongable and* $\text{pong}^{-1}(T)$ *is valid. Then* $\text{flip}(T) - 1 = \text{flip}(\text{pong}(T))$

Proof. Let $T \in \text{RMMT}(\lambda)$ where T is at most 1-pongable and $\text{pong}^{-1}(T)$ is valid. Therefore, there exists at least one entry in the second row, and further, that entry must have the same value as the entry in the first row. Since T is at most 1-pongable, $(1, \lambda_1)$ is flippable and contributes one addend to $\text{flip}(T)$. However, after $\text{pong}(T)$, our entry is no longer flippable and thus we lose an addend in $\text{flip}(\text{pong}(T))$. □

3 Constructing bijections on $\mathrm{SSMT}(\lambda)$

Define $\mathrm{CYT}_q(n)$ to be the set of tableaux of a single row of length n where all entries are equal and all entries are over the positive letters in the alphabet $\{q\}_{\prec}$. For an k odd positive integer, and $q \geq k$ a positive integer, let

$$I_{\lambda,q} = \left\{ (T_j, \binom{q}{j} 2^{\mathrm{flip}(T)+[l(\lambda)=2]}) : T_j \in \mathrm{RMMT}(\lambda) \right\}$$

where we use Iverson bracket notation for $[l(\lambda) = 2]$ and where T_j is the rudimentary minimally marked shifted tableau with entries over the alphabet $\{j\}_{\prec}$. We will write (T, c) for an element of the set $I_{\lambda,q}$ for brevity, but note that $c = \binom{q}{j} 2^{\mathrm{flip}(T)+[l(\lambda)=2]}$.

Then we define the following two sets,

$$G_q = \bigcup_{\substack{\lambda \vdash k \\ \lambda \text{ distinct} \\ l(\lambda) \leq 2 \\ \lambda_1 \text{ odd}}} I_{\lambda,q}, \quad H'_q = \bigcup_{\substack{\lambda \vdash k \\ \lambda \text{ distinct} \\ l(\lambda) = 2 \\ \lambda_1 \text{ odd}}} I_{\lambda,q}$$

and

$$H_q = H'_q \cup \{(T, 1) : T \in \mathrm{CYT}_q(k)\}$$

We note here that elements of G, H come as an ordered pair of a tableau T and a constant, however we wish to make use of the constant for our enumeration. Let $(T, c), (T', c')$ be elements of either G or H. Define $(T, c) \oplus (T', c') = c + c'$. We define the operation $\ominus$ analogously. Then for a chosen $\lambda \vdash k$, λ distinct and $l(\lambda) \leq 2$,

$$\bigoplus_{(T,c) \in I_{\lambda,q}} (T, c) = 2^{[l(\lambda)=2]} * |\mathrm{SSMT}_q(\lambda)|$$

which is a result due to Theorem 1.2. Thus,

$$\bigoplus_{(T,c) \in G_q \cup H'_q} (T, c) = \sum_{\substack{\lambda \vdash k \\ \lambda \text{ distinct} \\ l(\lambda) \leq 2}} 2^{[l(\lambda)=2]} * |\mathrm{SSMT}_q(\lambda)|$$

and from this, it is not hard to see that

$$\bigoplus_{(T,c) \in G_q \cup H_q} (T, c) = |\mathrm{CYT}_q(k)| + \sum_{\substack{\lambda \vdash k \\ \lambda \text{ distinct} \\ l(\lambda) \leq 2}} 2^{[l(\lambda)=2]} * |\mathrm{SSMT}_q(\lambda)|.$$

Next, we note what the pong function does to an element $(T, c) \in G$ or H. For $(T, c) \in G$ or H,

$$\mathrm{pong}((T, c)) = (\mathrm{pong}(T), \mathrm{pong}(c)) = (\mathrm{pong}(T), \binom{q}{j} 2^{\mathrm{flip}(\mathrm{pong}(T))+[l(\lambda)=2]}).$$

Let us take a moment to step back and look at the broader picture and examine what an element $(T, c) \in I_{\lambda,q}$ tells us. We know that $\binom{q}{j} 2^{\mathrm{flip}(T)}$ gives us the number of semi-standard shifted marked tableaux enumerated by a given $T_j \in \mathrm{RMMT}(\lambda)$, but we also give a 'tableau weight' of 2 for those semi-standard tableaux for $l(\lambda) = 2$. Let us consider the mapping from semi-standard shifted marked tableaux to monomials of a symmetric function. Then $2^{\text{flip}(T)+[l(\lambda)=2]}$ will be the coefficient of a monomial term x^T for some $T \in \text{SSMT}(\lambda)$. We note that there will be exactly $\binom{q}{j}$ many monomials with these exact coefficients which are quasisymmetric to x^{T_j} for $T_j \in \text{RMMT}(\lambda)$.

Thus, simply restating, an element $(T, c) \in I_{\lambda,q}$ enumerates for us every $\binom{q}{j}$ monomial terms x^T for $T \in \text{SSMT}(\lambda)$ where each x^T term has coefficient $2^{\text{flip}(T)+[l(\lambda)=2]}$.

Then what $\text{pong}(T)$ gives us is a way of translating between the sets G_q and H_q, as the pong action changes the number of entries in the first row. However, since the values of each entry in T and $\text{pong}(T)$ remain invariant under the pong action, $x^T = x^{\text{pong}(T)}$. Thus, it comes down to showing that for T, at most k-pongable, we will be 'ping ponging' (hence the namesake) between the two sets G_q and H_q to match the coefficients. It may be helpful to give an example.

Example. Consider the following tableaux.

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & 3 \\ \hline\end{array} \quad \text{pong}(T) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array} \quad \text{pong}^2(T) = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 3 & 3 \\ \cline{2-3}\end{array}$$

By definition, $x^T = x^{\text{pong}(T)} = x^{\text{pong}^2(T)}$. Let us include the coefficients of each term. x^T has coefficient $2^2 = 4$, $x^{\text{pong}(T)}$ has coefficient $2^{2+1} = 8$ and $x^{\text{pong}^2(T)}$ has coefficient $2^{1+1} = 4$. It's clear to see that $4x^T + 4x^{\text{pong}^2(T)} = 8x^{\text{pong}(T)}$. Further, T and $\text{pong}^2(T)$ are 'in' the set G_q whilst $\text{pong}(T)$ is 'in' the set H_q. This lends to the following observation.

Lemma 3.1. *Let $T \in \text{RMMT}(\lambda)$ be at most a j-pongable tableau such that $\text{pong}^{-1}(T)$ is not valid for $\lambda \vdash k$, λ distinct, $l(\lambda) = 2$ and for k an odd positive integer. Then*

$$\bigoplus_{\substack{n\in[j]\cup\{0\},\\ n \text{ even}}} (\text{pong}^n(T), \text{pong}^n(c)) = \bigoplus_{\substack{n\in[j],\\ n \text{ odd}}} (\text{pong}^n(T), \text{pong}^n(c))$$

Proof. Let us first examine the case when we fix some $(T, c) \in I_{(k),q}$ for k an odd positive integer. Then we know that $c = \binom{q}{j}2^{\text{flip}(T)}$. Let us consider $(\text{pong}(T), \text{pong}(c))$. By Corollary 2.3.1, since we have $l(\lambda) = 1$,

$$\text{pong}(c) = \binom{q}{j}2^{\text{flip}(\text{pong}(T))+1} = \binom{q}{j}2^{\text{flip}(T)+1} = 2 * \binom{q}{j}2^{\text{flip}(T)} = 2c.$$

From here on, for any $m \in [j-1], \text{pong}^m(c) = 2c$, since $\text{pong}^{-1}(T)$ is now valid, by Corollary 2.3.1, $\text{flip}(\text{pong}^{m-1}(T)) = \text{flip}(\text{pong}^m(T))$. Next, consider $T' = \text{pong}^{j-1}(T)$. We know that T' is at most 1-pongable, and that inverse pong is valid. Thus, applying Theorem 2.4, we see that $\text{flip}(T') - 1 = \text{flip}(\text{pong}(T'))$, meaning $\text{pong}^j(c) = c$.

Next we consider the following cases.

Case 1.1: j is even. Then there are $j/2$ number of odd and even numbers in $[j]$. Thus, the sum on the LHS is $c + c + ((j/2) - 1)(2c)$, while the RHS is $(j/2)2c$. It is easy to verify that RHS= LHS.

Case 1.2: j is odd. Then let $j = 2l + 1$ for l some even number. Then there are $l/2$ even numbers and $l/2 + 1$ odd numbers in $[j]$. The sum on the LHS is $c + (l/2)2c$ while on the RHS it is $(l/2)2c + c$, which is equal.

Now let us examine the case when $(T, c) \in I_{\lambda,q}$ for $\lambda \vdash k$, λ distinct, $l(\lambda) = 2$ and for k an odd positive integer. We note that $c = \binom{q}{j}2^{\text{flip}(\text{pong}(T))+1}$. Now, let us consider

$(\text{pong}(T), \text{pong}(c))$. By assumption, T does not have a valid inverse pong action, nor does T have only one row, thus by Corollary 2.3.1,

$$\text{pong}(c) = \binom{q}{j} 2^{\text{flip}(\text{pong}(T))+1} = \binom{q}{j} 2^{\text{flip}(T)+2} = 2 * \binom{q}{j} 2^{\text{flip}(\text{pong}(T))+1} = 2c.$$

Now, the argument follows similarly to when $\lambda = (k)$. $\square$

Theorem 3.2. *Let k be an odd positive integer and $q \geq k$ a positive integer. Then*

$$\bigoplus_{(T,c)\in G_q} (T, c) = \bigoplus_{(T,c)\in H_q} (T, c)$$

Proof. Let k be an odd positive integer and $q \geq k$ a positive integer. For all λ, unless stated otherwise, assume that $\lambda \vdash k$, λ distinct, and $l(\lambda) \leq 2$.

Utilizing Lemma 3.1,

$$\sum_{\substack{T_j\in\text{RMMT}(\lambda)\\ \lambda_1 \text{ odd}\\ \text{pong}^{-1}(T_j) \text{ invalid}}} \bigoplus_{\substack{n\in[j]\cup\{0\},\\ n \text{ even}}} (\text{pong}^n(T_j), \text{pong}^n(c)) = \sum_{\substack{T_j\in\text{RMMT}(\lambda)\\ \lambda_1 \text{ odd}\\ \text{pong}^{-1}(T_j) \text{ invalid}}} \bigoplus_{\substack{n\in[j],\\ n \text{ odd}}} (\text{pong}^n(T), \text{pong}^n(c)) \tag{3}$$

$$\sum_{\substack{T_j\in\text{RMMT}(\lambda)\\ \lambda_1 \text{ even}\\ \text{pong}^{-1}(T_j) \text{ invalid}}} \bigoplus_{\substack{n\in[j]\cup\{0\},\\ n \text{ even}}} (\text{pong}^n(T_j), \text{pong}^n(c)) = \sum_{\substack{T_j\in\text{RMMT}(\lambda)\\ \lambda_1 \text{ even}\\ \text{pong}^{-1}(T_j) \text{ invalid}}} \bigoplus_{\substack{n\in[j],\\ n \text{ odd}}} (\text{pong}^n(T), \text{pong}^n(c)) \tag{4}$$

For Equation 3, note that the LHS will be the $\oplus$ sum of (T, c) where the first row of each T is exclusively of odd length. The RHS, similarly, is the $\oplus$ sum of (T, c) where the first row of each T is exclusively of even length. Furthermore, for each addend (T, c) on the RHS, each T will have a valid inverse pong action, whereas only some on the LHS will. Equation 4 is similar to Equation 3 except the LHS will have exclusively even and the RHS will have exclusively odd.

Next, we note that any given T will either be 'minimally ponged', i.e. $\text{pong}^{-1}(T)$ will not be valid, or T is k-inverse pongable for some $k \in \mathbb{N}$. Fix any T. then, T will appear as an addend in either the LHS or RHS in either Equation 3 or Equation 4.

The LHS of Equation 3 and the RHS of Equation 4 together form the $\oplus$ sum over the set of all (T, c) whose T first row is odd. Likewise, the RHS of Equation 3 and the LHS of Equation 4 together form the $\oplus$ sum over the set of all (T, c) whose T first row is even. Therefore,

$$\sum_{\substack{T_j\in\text{RMMT}(\lambda)\\ \lambda_1 \text{ odd}\\ \text{pong}^{-1}(T_j) \text{ invalid}}} \bigoplus_{\substack{n\in[j]\cup\{0\},\\ n \text{ even}}} (\text{pong}^n(T_j), \text{pong}^n(c)) + \sum_{\substack{T_j\in\text{RMMT}(\lambda)\\ \lambda_1 \text{ even}\\ \text{pong}^{-1}(T_j) \text{ invalid}}} \bigoplus_{\substack{n\in[j],\\ n \text{ odd}}} (\text{pong}^n(T), \text{pong}^n(c))$$

$$= \bigoplus_{(T,c)\in G_q} (T, c)$$

and

$$\sum_{\substack{T_j \in \mathrm{RMMT}(\lambda) \\ \lambda_1 \text{ even} \\ \mathrm{pong}^{-1}(T_j) \text{ invalid}}} \bigoplus_{\substack{n \in [j] \cup \{0\}, \\ n \text{ even}}} (\mathrm{pong}^n(T_j), \mathrm{pong}^n(c)) + \sum_{\substack{T_j \in \mathrm{RMMT}(\lambda) \\ \lambda_1 \text{ odd} \\ \mathrm{pong}^{-1}(T_j) \text{ invalid}}} \bigoplus_{\substack{n \in [j], \\ n \text{ odd}}} (\mathrm{pong}^n(T), \mathrm{pong}^n(c))$$

$$= \bigoplus_{(T,c) \in H_q} (T, c).$$

Since we are adding Equation 3 and Equation 4, we find that

$$\bigoplus_{(T,c) \in G_q} (T, c) = \bigoplus_{(T,c) \in H_q} (T, c).$$

□

Let

$$I'_{\lambda,q} = \left\{ (T, 2^{\mathrm{flip}(T) + [l(\lambda)=2]}) : T \in \mathrm{MMST}_q(\lambda) \right\}.$$

Define

$$R_q = \bigcup_{\substack{\lambda \vdash k \\ \lambda \text{ distinct} \\ l(\lambda) \le 2 \\ \lambda_1 \text{ odd}}} I'_{\lambda,q}, \qquad S_q = \{(T, 1) : T \in \mathrm{CYT}_q(k)\} \cup \bigcup_{\substack{\lambda \vdash k \\ \lambda \text{ distinct} \\ l(\lambda) = 2 \\ \lambda_1 \text{ odd}}} I'_{\lambda,q}.$$

Corollary 3.2.1. *Let k be an odd positive integer and $q \ge k$ a positive integer. Then*

$$\bigoplus_{(T,c) \in R_q} (T, c) = \bigoplus_{(T,c) \in S_q} (T, c)$$

Proof. The proof follows naturally from Theorem 3.2 the fact that there are $\binom{q}{j}$ many minimally marked tableaux that are structurally the same as a given tableau $T \in \mathrm{RMMT}(\lambda)$. □

4 Power sum expansion into Schur's P

Let p_λ be the power sum symmetric function indexed by $\lambda \vdash k$. Let P_λ where $\lambda \vdash k$ and λ a distinct partition.

Theorem 4.1. *For $k \in \mathbb{P}$ and k odd,*

$$p_k = P_{(k)} + \sum_{\substack{\lambda \vdash k \\ \lambda \text{ distinct} \\ l(\lambda)=2}} (-1)^{[\lambda_1 \text{ is even}]} \cdot 2P_\lambda \tag{5}$$

where we use Iverson bracket notation for $[\lambda_1$ is even$]$.

Proof. Let X_q denote the set of variables $\{x_1, x_2, ..., x_q\}$. Consider the equation of symmetric functions

$$\begin{aligned} U(X_q) &= P_{(k)}(X_q) + \sum_{\substack{\lambda \vdash k \\ \lambda \text{ distinct} \\ l(\lambda)=2 \\ \lambda_1 \text{ is odd}}} 2P_\lambda(X_q). \\ V(X_q) &= p_{(k)}(X_q) + \sum_{\substack{\lambda \vdash k \\ \lambda \text{ distinct} \\ l(\lambda)=2 \\ \lambda_1 \text{ is even}}} 2P_\lambda(X_q), \end{aligned} \tag{6}$$

We note that each symmetric function can be expanded into the sum of monomials weighted by certain tableaux. We know that $p_{(k)}(X_q) = \sum_{T\in\mathrm{CYT}_q(k)} x^T$ and $P_\lambda(X_q) = \sum_{T\in\mathrm{SSMT}_q(\lambda)} x^T$. Thus we can imagine each symmetric function as a multiset of shifted marked tableaux. The symmetric function $U(X_q)$ is assigned the multiset

$$U_q = \{T \in \mathrm{SSMT}_q(k)\} \cup \left(\bigcup_{\substack{\lambda\vdash k \\ \lambda \text{ distinct} \\ l(\lambda)=2 \\ \lambda_1 \text{ is odd}}} \{T \in \mathrm{SSMT}_q(\lambda)\} \right)^2.$$

We denote the union squared to represent the union taken twice, to account for the factor of 2 in the original equation. Likewide, the $V(X_q)$ is assigned the multiset

$$V_q = \{T \in \mathrm{CYT}_q(\lambda)\} \cup \left(\bigcup_{\substack{\lambda\vdash k \\ \lambda \text{ distinct} \\ l(\lambda)=2 \\ \lambda_1 \text{ is even}}} \{T \in \mathrm{SSMT}_q(\lambda)\} \right)^2.$$

For the sets U_q, V_q, we define an equivalence relation on their elements. For $T, T' \in U$ or V, $T \sim_\gamma T'$ if $x^T = x^{T'}$. This forms an equivalence class partition of the sets U, V where each equivalence class can be identified by a minimally marked shifted tableau. This is akin to expanding each of the symmetric functions and collapsing all like terms, since the cardinality of each equivalence class becomes the coefficient of each monomial term. Thus, if the cardinality of U_q and V_q are equal, our equality holds. In order to do that, we will rewrite them as new sets U'_q, V'_q with an operation that respects the cardinality. The sets are as follows.

$$U'_q = \bigcup_{\substack{\lambda\vdash k \\ \lambda \text{ distinct} \\ l(\lambda)\leq 2 \\ \lambda_1 \text{ is odd}}} \{(T, |[T]|) : T \in \mathrm{MMST}_q(\lambda)\}$$

and

$$V'_q = \{(T,1) : T \in \mathrm{CYT}_q(\lambda)\} \cup \bigcup_{\substack{\lambda\vdash k \\ \lambda \text{ distinct} \\ l(\lambda)=2 \\ \lambda_1 \text{ is even}}} \{(T, |[T]|) : T \in \mathrm{MMST}_q(\lambda)\}$$

Where $[T]$ is the equivalence class on the relation $\sim_\gamma$. Our previously defined $\oplus$ operation in fact preserves the cardinality well. We see that

$$\bigoplus_{(T,c)\in U'_q} (T,c) = |U_q|, \qquad \bigoplus_{(T,c)\in V'_q} (T,c) = |V_q|. \tag{7}$$

Applying Corollary 1.2.1, we know $|[T]| = 2^{\mathrm{flip}(T)+[l(\lambda)=2]}$. Then, applying Corollary 3.2.1 to Equation 7,

$$\bigoplus_{(T,c)\in U'_q} (T,c) = \bigoplus_{(T,c)\in V'_q} (T,c).$$

Thus $U(X_q) = V(X_q)$. Since our argument was for arbitrary q, as we take the limit as q approaches infinity our claim is proven. $\square$

Corollary 4.1.1. *For $k \in \mathbb{P}$ and k odd,*

$$p_k = P_{(k)} + 2 \sum_{j=1}^{\frac{k-1}{2}} (-1)^j P_{(k-j,j)}$$

Remark. For $k = 1$, the sum becomes 0.