Proof of the Compactness Theorem via Tychonoff's Theorem

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Topology and Tychonoff

Let X an arbitrary set.

Def: A topology τ is such that $\tau \subset \mathcal{P}(X)$ and:
- $X, \varnothing \in \tau$.
- Arbitrary union of elements of τ belong in τ
- Finite intersections of elements in τ belong in τ.

An open set $U \in \tau$ is open. If U is open, then $X \setminus U$ is closed. If U is both open and closed, U is clopen.

If $\tau = \mathcal{P}(X)$ then τ is called the discrete topology.

Loose Definition: Let $\{ X_j \}_{j \in J}$ a collection of top spaces. The cartesian product $\prod X_j$ has the product topology which an element can be thought of as $U_1 \times U_2 \times ...$ where there are finitely many U_i which are NOT the whole space X_i.

Def: An open cover of X is a set $\{ U_i \} \subset \tau$ such that $\bigcup U_i = X$.

Def: A top space X is compact if for any open cover of X admits a finite subcover.

Prop. A top space X is compact iff for collection of closed sets of X, if any finite subset of the closed cover has nonempty intersection then

Theorem: (Tychonoff's Theorem) Let $\{ X_j \}_{j \in J}$ be an arbitrary collection of compact sets. Then $\prod X_j$ is compact.

Compactness Theorem Setup

Def: Truth functional language looks like variables and symbols called logical connectives arranged in a certain way, called our grammar. First order logic uses a set of variables $\mathcal{A} = \{ P, Q, R, S, ... \}$ and symbols

$$\neg, \wedge, \vee, \implies, \iff$$

For example,

$$\neg((P \wedge Q) \implies R) \iff P$$

is a sentence in first order propositional logic.

Def: An interpretation $I : \mathcal{A} \to \{ T, F \}$.

If ϕ is a sentence, then $I^+(\phi) \in \{ T, F \}$ depending on the rules of interpreting the logical connectives. Draw logical table for $P \implies Q$.

For example, if

$$\phi = P \implies Q$$

and $I(P) = T$ and $I(Q) = F$, then $I^+(\phi) = F$.

We say that an interpretation I^+ is a model for a sentence ϕ or some set of sentences Γ if $I^+(\phi) = T$ or $I^+(\psi)$ for all $\psi \in \Gamma$.

1 Prop and Proof

Prop. Let Γ be a, possibly infinite, set of sentences. Then Γ has a model if and only if for every subset S of Γ, there is a model of S.

Proof. The forward direction is clear since if Γ has a model, then any subset of sentences must have a model.

Let Γ be a set of sentences and let V be the set of variables used in Γ. Note these sets are allowed to be infinite.

Let $\{ T, F \}$ be endowed with the discrete topology and consider $\{ T, F \}^V$. We can consider this set to be the set of all interpretations over V. Consider the product topology $\{ T, F \}^V$. Since each $\{ T, F \}$ is compact (can check this easily) then by Tychonoff's theorem $\{ T, F \}^V$ is compact.

Let $W \subset V$ any finite subset of V. An 'open set' of $\{ T, F \}^V$ looks like $B^{p_1,\ldots,p_k}_{v_1,\ldots,v_k} = \{ I : I(p_i) = v_i \}$ for $p_i \in P$ and $v_i \in \{ T, F \}$. But note that each of these $B^{p_1,\ldots,p_k}_{v_1,\ldots,v_k}$ is also closed since it is the complement of $B^{p_1,\ldots,p_k}_{\neg v_1,\ldots,\neg v_k}$.

For every $\phi \in \Gamma$ let $M_\phi = \{ I \in \{ T, F \}^V : I^+(\phi) = T \}$. This looks exactly like an open set we looked at just before. Why? Because there is some B set such that each I belongs to it and those evaluations in the subscript are exactly what makes ϕ true. So it is closed.

Let $S = \{ M_\phi \}$ be a given finite set. Let $S' = \{ M_\phi : \phi \in \Gamma \}$. Since $\{ T, F \}^V$ is compact, then if $\bigcap S$ is nonempty then $\bigcap S' \neq \varnothing$. Hence, there is some I^+ belonging to this intersection and hence is a model of Γ. $\square$