1 Regularly Parametrized Curves

1.1 Basic definitions

Let $I \subseteq \mathbb{R}$ be an open interval,

$$\text{e.g.} \qquad (0, \pi), \qquad (-\infty, 1), \qquad \mathbb{R} \quad \text{etc.}$$

Recall that a function $f : I \to \mathbb{R}$ is **smooth** if all its derivatives $f'(t)$, $f''(t)$, $f'''(t), \ldots$ exist for all t (shorthand: $\forall\, t \in I$).

E.g. polynomials, trigonometric functions (sin, cos etc.), exponentials, logarithms, hyperbolic trig functions (sinh, cosh etc.) are all smooth.

$f : \mathbb{R} \to \mathbb{R}$ such that $f(t) = t^{\frac{4}{3}}$ is **not** smooth. Check:

$$f'(t) = \frac{4}{3} t^{\frac{1}{3}} \qquad \Rightarrow \qquad f''(t) = \frac{4}{9} t^{-\frac{2}{3}}$$

so $f''(0)$ does not exist.

We can extend this definition of smoothness to maps $\gamma : I \to \mathbb{R}^n$ where $n \geq 2$. Such a map is a rule which assigns to each "time" t a vector

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)) \in \mathbb{R}^n.$$

We say that the map γ is smooth if every one of its component functions $\gamma_i : I \to \mathbb{R}$, $i = 1, 2, \ldots, n$ is smooth in the usual sense (all derivatives exist everywhere). We may think of γ as describing the trajectory of a point particle moving in $\mathbb{R}^n$. This leads us to:

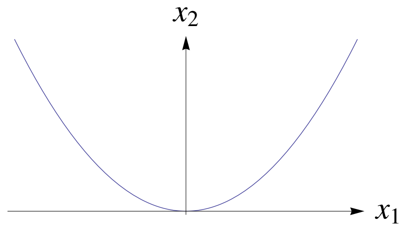
Definition 1 A **parametrized curve** (PC) in $\mathbb{R}^n$ is a smooth map $\gamma : I \to \mathbb{R}^n$. A time $t \in I$ is a **regular point** of γ if $\gamma'(t) \neq 0$. If $\gamma'(t) = 0$, then $t \in I$ is a **singular point** of γ. If every $t \in I$ is regular then γ is said to be a **regularly parametrized curve** (RPC). In other words, a PC is a RPC if and only if

$$\boxed{\text{there does not exist a time } t \in I \text{ such that } \gamma'(t) = 0 = (0, 0, \ldots, 0).}$$

The **image set** of a curve γ is the range of the mapping, that is,

$$\gamma(I) = \{\gamma(t) \in \mathbb{R}^n \,|\, t \in I\} \subset \mathbb{R}^n \qquad \square$$

Example 2 Consider the parabola $x_2 = x_1^2$. There are infinitely many PCs whose image set is this parabola.



Two examples:

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = (t, t^2)$$
$$\delta : \mathbb{R} \to \mathbb{R}^2, \qquad \delta(t) = (t^3, t^6).$$

γ is a **regularly** parametrized curve:

$$\gamma'(t) = (1, 2t) \neq (0, 0) \ \forall t \text{ because } 1 \neq 0$$

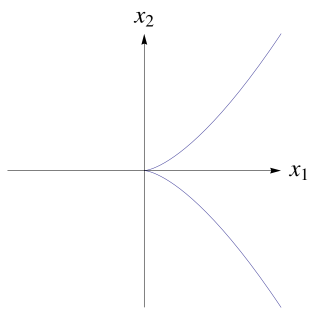
But δ is **not**:

$$\delta'(t) = (3t^2, 6t^5) = (0, 0) \text{ when } t = 0$$

. □

So Definition 1 concerns the *parametrization* of the curve, not just its image set $\gamma(I) \subset \mathbb{R}^n$. Why? A PC is a *smooth* map $\gamma : I \to \mathbb{R}^n$, but it does not necessarily represent a *"smooth"* curve! A **R**PC does, however.

Example 3 $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (t^2, t^3)$ is a smooth map, hence a PC. But its image set is not "smooth" – it has a cusp.



Note that γ is not a **R**PC:

$$\gamma'(t) = (2t, 3t^2) = (0, 0) \text{ when } t = 0$$

Note also that the nasty point in $\gamma(I)$ occurs precisely where $\gamma'(t) = 0$, that is, at the *singular point* of γ. □

Definition 4 Given a PC $\gamma : I \to \mathbb{R}^n$, its **velocity** is $\gamma' : I \to \mathbb{R}^n$, its **acceleration** is $\gamma'' : I \to \mathbb{R}^n$ and its **speed** is $|\gamma'| : I \to [0, \infty)$. □

Notes:

- $|v|$ denotes the Euclidean norm of a vector $v \in \mathbb{R}^n$, that is,

$$|v| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \geq 0.$$

- It's important to distinguish between *vector* and *scalar* quantities.

  Velocity is a *vector* . Acceleration is a *vector* . Speed is a *scalar* .

- We can rephrase definition 1 as follows:

$$\boxed{\text{PC } \gamma \text{ is a } \underline{\mathbf{R}}\text{PC} \iff \text{its velocity (or its speed) never vanishes}}$$

Example 5 (straight line) Simple but important. For any $v \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ one has the PC

$$\gamma : \mathbb{R} \to \mathbb{R}^n, \qquad \gamma(t) = x + vt.$$

Note that $\gamma'(t) = v$, constant, so γ is a RPC unless $v = 0$.
Note also that the *direction* of the straight line is determined solely by v. $\square$

A RPC has a well-defined tangent line at each $t_0 \in I$:

Definition 6 Let $\gamma : I \to \mathbb{R}^n$ be a RPC. Then its **tangent line** at $t_0 \in I$ is the PC

$$\widehat{\gamma}_{t_0} : \mathbb{R} \to \mathbb{R}^n, \qquad \widehat{\gamma}_{t_0}(t) = \gamma(t_0) + t\gamma'(t_0).$$

Note that $\widehat{\gamma}'_{t_0}(t) = 0 + \gamma'(t_0) \neq 0$ since γ is a RPC. Hence every tangent line $\widehat{\gamma}_{t_0}$ is a RPC too. $\square$

Example 7 Consider the PC $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (t^3 - t, t^2 - 1)$.

(A) Is it a RPC?

(B) Are any of its tangent lines vertical?

(A) Just check whether its velocity ever vanishes. Assume it does at time t. Then:

$$(0, 0) = \gamma'(t) = (3t^2 - t, 2t).$$

The equation $2t = 0$ implies that $t = 0$. Substituting in gives

$$\gamma'(0) = (-1, 0),$$

so the first component of γ' does not vanish. Therefore $\gamma'(t)$ is never zero.

Hence γ is a RPC.

(B) Tangent line to γ at $t_0 \in \mathbb{R}$ is

$$\widehat{\gamma}_{t_0}(t) = (t_0^3 - t_0, t_0^2 - 1) + t(3t_0^2 - 1, 2t_0)$$

The direction of the tangent line is given by its (constant) velocity vector

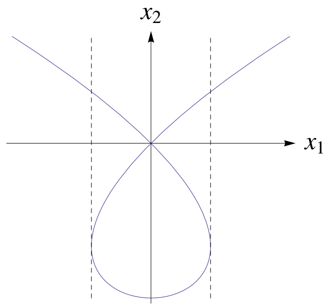
$$\widehat{\gamma}'_{t_0}(t) = (3t_0^2 - 1, 2t_0) = \gamma'(t_0).$$

The tangent line is *vertical* if the horizontal component (the x_1 component) of this vector is 0. Hence $\widehat{\gamma}_{t_0}$ is vertical if and only if

$$3t_0^2 - 1 = 0 \iff t_0 = \pm\frac{1}{\sqrt{3}}$$

so $\widehat{\gamma}_{\frac{1}{\sqrt{3}}}$ and $\widehat{\gamma}_{-\frac{1}{\sqrt{3}}}$ are vertical.

Here's a picture of γ, together with the two vertical tangent lines:



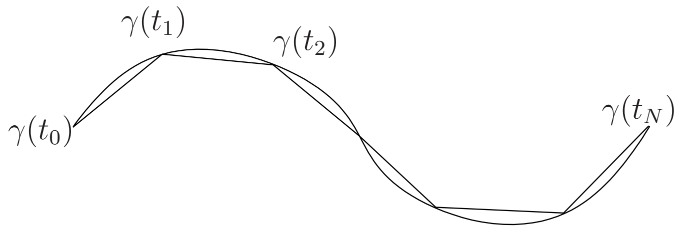
Note that the curve intersects itself exactly once. A self-intersection point is one where $\gamma(t_1) = \gamma(t_2)$ but $t_1 \neq t_2$, which implies, in this case,

$$\begin{aligned}
& t_1^3 - t_1 = t_2^3 - t_2, \qquad \text{and} \qquad t_1^2 - 1 = t_2^2 - 1 \\
\Rightarrow \quad & t_1^2 = t_2^2 \\
\Rightarrow \quad & t_2 = -t_1 \\
\Rightarrow \quad & t_1^3 - t_1 = -t_1^3 + t_1 = -(t_1^3 - t_1) \\
\Rightarrow \quad & 0 = t_1^3 - t_1 = t_1(t_1^2 - 1) \\
\Rightarrow \quad & t_1 = 0 \qquad \text{or} \qquad t_1 = \pm 1
\end{aligned}$$

If $t_1 = 0$ then $t_2 = -0 = t_1$, so this doesn't give a self-intersection point. Likewise $t_1 = 1$ and $t_1 = -1$ give the same self-intersection point, $\gamma(1) = \gamma(-1) = (0, 0)$.

Another question: what is the **length** of the segment of γ from $t = 0$ to $t = 1$? $\square$

1.2 Arc length



Given a RPC $\gamma : I \to \mathbb{R}^n$, what is the length of the curve segment from $t = a$ to $t = b$ say? Partition $[a, b]$ into N pieces $[t_{n-1}, t_n]$, $n = 1, 2, \ldots, N$ (with $t_0 = a$, $t_N = b$) of equal length $\delta t = \frac{b-a}{N}$. If N is large, then δt is small. The length of the straight line segment from $\gamma(t_{n-1})$ to $\gamma(t_n)$ is

$$\delta s_n = |\gamma(t_n) - \gamma(t_{n-1})| = |\gamma(t_{n-1} + \delta t) - \gamma(t_{n-1})| \approx |\gamma'(t_{n-1})|\delta t.$$

So the total length of the piecewise straight line from $\gamma(a)$ to $\gamma(b)$ is

$$s_N = \sum_{n=1}^{N} \delta s_n \approx \sum_{n=1}^{N} |\gamma'(t_{n-1})|\delta t.$$

In the limit $N \to \infty$, $\delta t \to 0$ and the piecewise straight line tends to the real curve γ. So the total length of the curve segment is

$$s = \lim_{N \to \infty} s_N = \int_a^b |\gamma'(t)|\, dt.$$

This motivates the following definition:

Definition 8 Let $\gamma : I \to \mathbb{R}^n$ be a RPC. The **arc length** along γ from $t = t_0$ to $t = t_1$ is

$$s = \int_{t_0}^{t_1} |\gamma'(t)|\, dt. \qquad \square$$

More informally, $\boxed{\text{distance travelled} = \text{integral of speed.}}$

Example 9 $\gamma : (0, \infty) \to \mathbb{R}^2$, $\gamma(t) = (t, \frac{2}{3}t^{\frac{3}{2}})$.
What is the arc length from $t = 3$ to $t = 15$?

$$\begin{aligned}
\gamma'(t) &= (1, t^{\frac{1}{2}}) \\
|\gamma'(t)| &= \sqrt{1+t} \\
s &= \int_3^{15} (1+t)^{\frac{1}{2}} dt = \frac{2}{3}\left[(1+t)^{\frac{3}{2}}\right]_3^{15} = 56
\end{aligned}$$

$\square$

WARNING! This example was cooked up to be easy. It's usually impossible to compute s in practice.

Example 7 (revisited) $\gamma(t) = (t^3 - t, t^2 - 1)$. What is the arc length from $t = 0$ to $t = 1$?
$\gamma'(t) = (3t^2 - 1, 2t)$ so

$$\begin{aligned}
s &= \int_0^1 \sqrt{(3t^2-1)^2 + (2t)^2} dt \\
&= \int_0^1 \sqrt{9t^4 - 2t^2 + 1} dt \\
&= ??? \text{ (1.36 to 2d.p.)}
\end{aligned}$$

$\square$

Note that we can use Definition 8 even if $t_1 < t_0$:

$$s = \int_{t_0}^{t_1} \underbrace{|\gamma'(t)|}_{\text{positive}} dt = -\int_{t_1}^{t_0} |\gamma'(t)| \, dt < 0$$

So s can be positive or negative. For this reason it is sometimes called the **signed** arc length.
Once we've chosen a base point $t_0 \in I$, at every other time we can assign a unique signed arc length.

Definition 10 Let $\gamma : I \to \mathbb{R}^n$ be a RPC. The **arc length function** based at $t_0 \in I$ is

$$\sigma_{t_0} : I \to \mathbb{R}, \qquad \sigma_{t_0}(t) = \int_{t_0}^{t} |\gamma'(u)| \, du. \qquad \square$$

Example 9 (revisited) The arc length function based at $t_0 = 1$ for $\gamma(t) = (t, \frac{2}{3}t^{\frac{3}{2}})$ is

$$\begin{aligned} \sigma_1(t) &= \int_1^t \sqrt{1+u}du \\ &= \frac{2}{3}\left((1+t)^{\frac{3}{2}} - 2^{\frac{3}{2}}\right) \end{aligned}$$

Trick questions: (A) What is $\sigma_1(1)$?
(B) What is $\sigma_1'(t)$? $\square$

$\sigma_{t_0}(t)$ is very hard (usually impossible) to compute explicitly in practice. But the fact that it *exists* is crucial to the theory of curves. We need to understand its properties.

Remark 11 The arc length function $\sigma_{t_0} : I \to \mathbb{R}$ has the following properties:

(a) $\sigma_{t_0}(t_0) = \displaystyle\int_{t_0}^{t_0} |\gamma'(q)|\, dq = 0.$

(b) σ_{t_0} is smooth, and its first derivative is strictly positive:

$$\sigma_{t_0}'(t) = \frac{d}{dt}\int_{t_0}^t |\gamma'(a)|\, da = |\gamma'(t)| > 0$$

for all $t \in I$ since γ is a **R**PC.

(c) It follows, by the Mean Value Theorem, that σ_{t_0} is *strictly increasing* (if $t_2 > t_1$ then $\sigma_{t_0}(t_2) > \sigma_{t_0}(t_1)$), and hence is *injective* (one-to-one).

(d) Let $J \subseteq \mathbb{R}$ denote the *range* of σ_{t_0}. It's another (possibly unbounded) open interval. Given (c), there exists an *inverse function* to σ_{t_0}, let's call it $\tau_{t_0} : J \to I$, so that

$$\begin{aligned} \tau_{t_0}(\sigma_{t_0}(t)) &= t \qquad \text{for all } t \in I, \text{ and} \\ \sigma_{t_0}(\tau_{t_0}(s)) &= s \qquad \text{for all } s \in J \qquad (\clubsuit) \end{aligned}$$

$$\begin{array}{ccc} I & \overset{\sigma_{t_0}}{\rightarrow} & J \\ \text{time} & & \text{arc length} \\ t & \underset{\tau_{t_0}}{\leftarrow} & s \end{array}$$

Trick question: what is $\tau_{t_0}(0)$?

(e) The inverse function τ_{t_0} is also strictly increasing. To see this, differentiate (♣) with respect to s using the chain rule:

$$\begin{aligned} \sigma'_{t_0}(\tau_{t_0}(s))\tau'_{t_0}(s) &= 1 \\ \Rightarrow \quad \tau'_{t_0}(s) &= \frac{1}{\sigma'_{t_0}(\tau_{t_0}(s))} = \frac{1}{|\gamma'(\tau_{t_0}(s))|} > 0 \end{aligned}$$

$\square$

Example 9 (revisited) $\gamma : (0, \infty) \to \mathbb{R}^2$ $\gamma(t) = (t, \frac{2}{3}t^{\frac{3}{2}})$. Recall

$$\sigma_1 : (0, \infty) \to \mathbb{R} \qquad \sigma_1(t) = \frac{2}{3}((1+t)^{\frac{3}{2}} - 2^{\frac{3}{2}}).$$

What is τ_1? Domain of τ_1 = range of σ_1. But since σ_1 is increasing, this is just the interval $J = (a, b)$, where

$$\begin{aligned} a &= \lim_{t\to 0} \sigma_1(t) = -\frac{2}{3}(2^{\frac{3}{2}} - 1) \quad \text{and} \\ b &= \lim_{t\to\infty} \sigma_1(t) = \infty. \end{aligned}$$

To find a formula for $\tau_1(s)$, we must solve $s = \sigma_1(t)$ to find t as a function of s:

$$\begin{aligned} s &= \frac{2}{3}\left[(1+t)^{\frac{3}{2}} - 2^{\frac{3}{2}}\right] \\ \Rightarrow \quad \left(\frac{3}{2}s + 2^{\frac{3}{2}}\right)^{\frac{2}{3}} &= 1 + t \\ \Rightarrow \quad \tau_1(s) &= \left(\frac{3}{2}s + 2^{\frac{3}{2}}\right)^{\frac{2}{3}} - 1. \end{aligned}$$

[Check: $\tau_1(0) = (0 + 2^{\frac{3}{2}})^{\frac{2}{3}} - 1 = 2^0 - 1 = 1$ as expected]

What is $\tau_1'(s)$?

$$\tau_1'(s) = \frac{1}{\sigma_1'(\tau_1(s))} = \frac{1}{|\gamma'(\tau_1(s))|} = \frac{1}{\sqrt{1 + \tau_1(s)}} = \left(\frac{3}{2}s + 2^{\frac{3}{2}}\right)^{-\frac{1}{3}} \quad \square$$

1.3 Reparametrization

A reparametrization of a RPC $\gamma : I \to \mathbb{R}^n$ is a redefinition of "time":

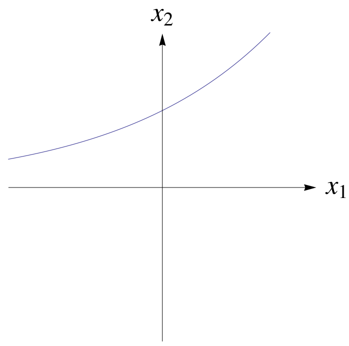
$$\beta : J \to \mathbb{R}^n, \qquad \beta(u) = \gamma(h(u)) = (\gamma \circ h)(u)$$

for some interval J and function $h : J \to I$.

Example 12 Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ such that $\gamma(t) = (t, e^t)$. Let $h : (0, \infty) \to \mathbb{R}$ such that $h(u) = \log u$. This gives the reparametrization

$$\beta : (0, \infty) \to \mathbb{R}^2,$$

$$\beta(u) = \gamma(h(u)) = (\log u, u)$$



Note that although γ and β are different functions their *image sets* are *identical*: $\beta((0, \infty)) = \gamma(\mathbb{R})$. We've just changed the way we *label* the points on the curve.
Note also that both γ and β are **R**PCs in this case:

$$\gamma'(t) = (1, e^t) \neq 0 \ \forall t \in \mathbb{R} \qquad \beta'(u) = (u^{-1}, 1) \neq 0 \ \forall 0 < u < \infty$$

But we can't allow h to be any function $J \to \mathbb{R}$ if we want β to be a RPC. For example, let $h : \mathbb{R} \to \mathbb{R}$ such that $h(u) = \sin u$. Then

$$\begin{aligned} \beta(u) &= \gamma(h(u)) = (\sin u, e^{\sin u}) \\ \beta'(u) &= (\cos u, \cos u e^{\sin u}) \end{aligned}$$

so $\beta'(\frac{\pi}{2}) = (0, 0)$, and β is not regular! This is an example of a bad redefinition of time. We want to exclude things like this from our formal definition of reparametrization.
$\square$

Definition 13 A **reparametrization** of a PC $\gamma : I \to \mathbb{R}^n$ is a map $\beta : J \to \mathbb{R}^n$ defined by $\beta(u) = \gamma(h(u))$, where J is an open interval, and $h : J \to I$ is smooth, surjective and has $h'(u) > 0$ for all $u \in J$. $\square$

Notes:

- We require h to be smooth so that β is smooth (by the Chain Rule), hence a PC.
- We require h to be surjective so that $\beta(J) = \gamma(I)$. In other words, this ensures that β covers all of γ, not just part of it.
- Since h is strictly increasing, it is injective. Hence for each time $t \in I$ there is one (h surjective) and only one (h injective) corresponding new time $u \in J$.

Lemma 14 *Any reparametrization of a RPC is also a RPC.*

Proof: Let $\beta(u) = \gamma(h(u))$ where γ is a RPC. Then

$$\begin{aligned} \beta'(u) = \gamma'(h(u))h'(u) &= 0 \quad \Rightarrow \\ \gamma'(h(u)) &= 0 \qquad \text{(impossible since } \gamma \text{ is regular)} \\ \text{or} \quad h'(u) &= 0 \qquad \text{(impossible since } h' > 0\text{).} \end{aligned}$$

Hence β is a RPC. $\square$

Note that reparametrization preserves the *direction* and *sense* of the velocity vector, but not its length:

$$\text{new velocity} = \beta'(u) = h'(u)\gamma'(h(u)) = \text{positive number} \times \text{old velocity}.$$

However:

Lemma 15 *Reparametrization preserves arc length.*

Proof: Let γ be a PC and s be the arc length along γ from $\gamma(t_0)$ to $\gamma(t_1)$. Let $\beta = \gamma \circ h$ be a reparametrization of γ where $h(u_0) = t_0$ and $h(u_1) = t_1$. We must show that s is also the arc length along β from $\beta(u_0)$ to $\beta(u_1)$. In fact

$$\begin{aligned} s &= \int_{t_0}^{t_1} |\gamma'(t)|\, dt \qquad \text{Substitution: } t = h(u),\ dt = h'(u)du \\ &= \int_{h^{-1}(t_0)}^{h^{-1}(t_1)} |\gamma'(h(u))|\, h'(u)du \\ &= \int_{u_0}^{u_1} |\gamma'(h(u))h'(u)|\, du \qquad \text{since } h'(u) > 0 \\ &= \int_{u_0}^{u_1} |\beta'(u)|\, du \end{aligned}$$

as required. $\square$

Definition 16 A **unit speed curve** (USC) is a smooth map $\gamma : I \to \mathbb{R}^n$ such that $|\gamma'(s)| = 1$ for all $s \in I$ $\square$

Notes:

- Clearly USC $\Rightarrow$ RPC.
- It's conventional to denote the "time" parameter of a unit speed curve by s rather than t, because the parameter *is* signed arc length (up to a constant):

$$\int_{s_0}^{s} |\gamma'(u)|du = \int_{s_0}^{s} 1\, du = s - s_0. \quad (\clubsuit)$$

Unit speed curves may seem very special, but in fact they are, in a sense, completely universal:

Theorem 17 *Every RPC $\gamma : I \to \mathbb{R}^n$ has a unit speed reparametrization (USR) $\beta : J \to \mathbb{R}^n$. This USR is unique up to "time" translation. More precisely, if $\delta : K \to \mathbb{R}^n$ is another USR of γ, then there exists a constant $c \in \mathbb{R}$ such that*

$$\beta(s) = \delta(s - c).$$

Proof:
(A) Existence:
Choose $t_0 \in I$ and let $\sigma_{t_0} : I \to J$ be the signed arc length function of γ based at t_0. Recall that there exists a smooth, well-defined inverse function $\tau_{t_0} : J \to I$ which has positive derivative (Remark 11(d) and (e)). So $h = \tau_{t_0}$ gives a reparametrization of γ, according to Definition 13: $\beta : J \to \mathbb{R}^n$, $\beta(s) = \gamma(\tau_{t_0}(s))$. But

$$\beta'(s) = \gamma'(\tau_{t_0}(s))\tau_{t_0}'(s) = \gamma'(\tau_{t_0}(s))\frac{1}{|\gamma'(\tau_{t_0}(s))|}$$

by Remark 11(e). Hence $|\beta'(s)| = 1$ for all $s \in J$, and β is a USC as required.

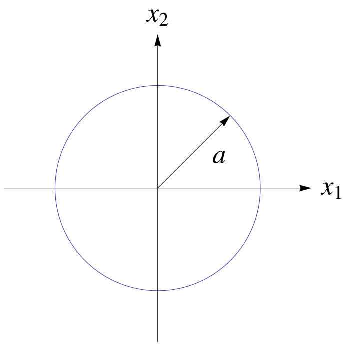
(B) Uniqueness:
Let $\delta : K \to \mathbb{R}^n$ be another USR of γ, σ be the arc length along γ from $\gamma(t_0) = \beta(s_0) = \delta(\widetilde{s}_0)$ to $\gamma(t) = \beta(s) = \delta(\widetilde{s})$. By Lemma 15 and ($\clubsuit$),

$$\begin{aligned} \sigma &= s - s_0 = \widetilde{s} - \widetilde{s}_0 \\ \Rightarrow \quad \widetilde{s} &= s - (s_0 - \widetilde{s}_0) \end{aligned}$$

Hence $\qquad \beta(s) = \delta(\widetilde{s}) = \delta(s - (s_0 - \widetilde{s}_0)).$ $\square$

Example 18 Circle $\gamma(t) = a(\cos t, \sin t)$, or radius $a > 0$. To find a USR of γ we compute the arclength function and invert it, then use τ_0 to reparametrize γ:

$$\begin{aligned} \gamma'(t) &= a(-\sin t, \cos t) \\ |\gamma'(t)| &= a\sqrt{\sin^2 t + \cos^2 t} = a \\ \sigma_0(t) &= \int_0^t a\, du = at \\ \tau_0(s) &= \frac{s}{a} \\ \beta(s) &= \gamma(\tau_0(s)) = a\left(\cos\left(\frac{s}{a}\right), \sin\left(\frac{s}{a}\right)\right) \end{aligned}$$



$\square$

Although Theorem 17 ensures a USR exists, we may not be able to construct it explicitly. For example, for the curve in example 7 we were not able to find the arclength function explicitly, so certainly wouldn't be able to find a USR.

Summary

- A **regularly parametrized curve** (RPC) is a smooth map $\gamma : I \to \mathbb{R}^n$ (where I is an open interval) such that, for all $t \in I$, $\gamma'(t) \neq 0$.
- Given a RPC γ, its **velocity** is γ', its **speed** is $|\gamma'|$ and its **acceleration** is γ''.
- The **tangent line** to γ at $t_0 \in I$ is

$$\widehat{\gamma}_{t_0} : \mathbb{R} \to \mathbb{R}^n, \qquad \widehat{\gamma}_{t_0}(t) = \gamma(t_0) + \gamma'(t_0)t.$$

- The **arclength function** based at $t_0 \in I$ is

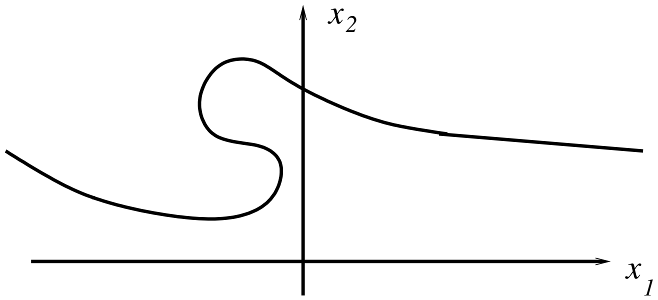
$$\sigma_{t_0}(t) = \int_{t_0}^{t} |\gamma'(u)| du.$$

  Geometrically, this is the arclength along γ from $\gamma(t_0)$ to $\gamma(t)$ if $t \geq t_0$ (and minus the arclength if $t < t_0$).
- A **reparametrization** of a curve $\gamma : I \to \mathbb{R}^n$ is a curve $\gamma \circ h : J \to \mathbb{R}^n$ where $h : J \to I$ is smooth, surjective and has strictly positive derivative. If γ is a RPC, so is every reparametrization of γ.
- Arclength is unchanged by reparametrization.
- A **unit speed curve** (USC) is a curve with $|\gamma'(t)| = 1$ for all t.
- Every RPC has a reparametrization which is a USC. One can construct it, in principle, by reparametrizing with $h = \sigma_{t_0}^{-1}$.

2 Curvature of a parametrized curve

2.1 Basic definition

In this section we will develop a measure of the *curvature* of a RPC.



What distinguishes a region of high curvature from one of low curvature?

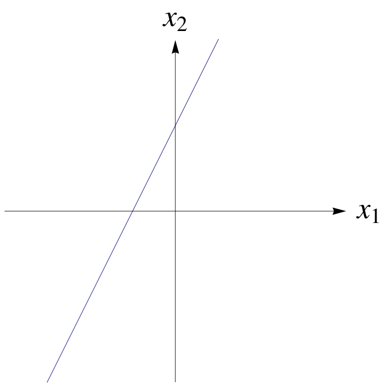
High curvature:	tangent lines change direction very rapidly
Low curvature:	tangent lines change direction only slowly
No curvature (straight line):	tangent lines don't change direction at all

So we require a measure of the rate of change of direction of the curve's tangent lines. Recall that the tangent line at $\gamma(t_0)$ of a RPC γ is

$$\widehat{\gamma}_{t_0}(u) = \gamma(t_0) \quad + \quad u\,\underbrace{\gamma'(t_0)}_{\substack{\text{determines}\\ \text{direction}}}$$

Perhaps we can use $\dfrac{d}{dt_0}(\gamma'(t_0)) = \gamma''(t_0)$ to measure the curvature of γ at $t = t_0$? Not quite: γ'' tells us about rate of change of *length* of γ' as well as rate of change of *direction.*

Example 19 $\gamma(t) = (\log t, 2\log t + 1)$ is a straight line, and hence is not curved.



However

$$\begin{aligned} \gamma'(t) &= \left(\frac{1}{t}, \frac{2}{t}\right) \\ \gamma''(t) &= \left(-\frac{1}{t}, -\frac{2}{t^2}\right) \end{aligned}$$

so γ'' is not zero. $\square$

But what if γ happens to be a unit speed curve, $\gamma(s)$? Then $|\gamma'(s)| = 1$ for all s, so $\gamma''(s)$ *does* tell us only about the rate of change of the *direction* of the tangent lines.

Definition 20 (a) Let $\gamma : I \to \mathbb{R}^n$ be a USC. Then the **curvature vector** of γ is $k : I \to \mathbb{R}^n$ where

$$k(s) = \gamma''(s).$$

Note that k is a *vector* quantity. We shall refer to the norm of k, $|k| : I \to [0, \infty)$ as the curvature of γ.

(b) If a RPC $\gamma : I \to \mathbb{R}^n$ is *not* a USC, then Theorem 17 says that it has a unit speed reparametrization $\beta : J \to \mathbb{R}^n$, $\beta = \gamma \circ h$. In that case, we define the curvature vector of γ at $t = h(s)$ to be the curvature vector of β at s, as in part (a), that is, $\beta''(s)$. In other words, $k : I \to \mathbb{R}^n$ such that

$$k = \beta'' \circ h^{-1}.$$

Note: To make sense, this definition should be independent of the choice of unit speed reparametrization β of γ. It is. Recall, by Theorem 17, that any pair of USRs of γ differ only by shifting the origin of the new time coordinate s. But such a shift has no effect on the second (or indeed the first) derivative of β. $\square$

Example 21 Helix $\gamma : \mathbb{R} \to \mathbb{R}^3$, $\gamma(s) = \dfrac{1}{\sqrt{2}}(\cos s, \sin s, s)$.

$$\begin{aligned}
\gamma'(s) &= \frac{1}{\sqrt{2}}(-\sin s, \cos s, 1) \\
|\gamma'(s)| &= \frac{1}{\sqrt{2}}\left(\cos^2 s + \sin^2 s + 1\right)^{\frac{1}{2}} = 1 \\
k(s) = \gamma''(s) &= \frac{1}{\sqrt{2}}(-\cos s, -\sin s, 0) \\
|k(s)| &= \frac{1}{\sqrt{2}}\left(\cos^2 s + \sin^2 s\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}.
\end{aligned}$$

$\square$

Example 22 Circle $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (2\cos t, 2\sin t)$

$$\begin{aligned}
\gamma'(t) &= (-2\sin t, 2\cos t) \\
|\gamma'(t)| &= 2
\end{aligned}$$

so *not* a USC. But we can find a USR of γ. How?

$$\text{Arclength function:} \quad \sigma_0(t) = \int_0^t 2dt = 2t$$

$$\text{Inverse function:} \quad \tau_0(s) = \tfrac{1}{2}s$$

$$\text{USR:} \quad \beta(s) = \gamma(\tau_0(s)) = \left(2\cos\tfrac{s}{2}, 2\sin\tfrac{s}{2}\right)$$

Now compute curvature vector of β:

$$\begin{aligned}
\beta'(s) &= \left(-\sin\frac{s}{2}, \cos\frac{s}{2}\right) \\
\beta''(s) &= \left(-\frac{1}{2}\cos\frac{s}{2}, -\frac{1}{2}\sin\frac{s}{2}\right)
\end{aligned}$$

Change back to old parametrization, $s = \sigma_0(t) = 2t$

$$k(t) = \beta''(\sigma_0(t)) = \left(-\frac{1}{2}\cos t, -\frac{1}{2}\sin t\right).$$

□

In general this process is rather clumsy. In particular, it's usually impossible to write down β, the USR of γ, explicitly (recall Example 7 (revisted)).

So let's calculate k once and for all for a general RPC γ, to obtain a more user-friendly version of Definition 20. Two preliminary observations:

(A) Given two vectors $u, v \in \mathbb{R}^n$, we define their scalar product $u \cdot v \in \mathbb{R}$ by

$$u \cdot v = u_1v_1 + u_2v_2 + \cdots u_nv_n.$$

In particular, the norm of v may be rewritten

$$|v| = \sqrt{v \cdot v}.$$

(B) Given two vector valued functions $u, v : I \to \mathbb{R}^n$, we have a product rule for differention,

$$\frac{d}{dt}(u(t) \cdot v(t)) = u'(t) \cdot v(t) + u(t) \cdot v'(t).$$

In particular,

$$\frac{d}{dt}|u(t)| = \frac{d}{dt}(u \cdot u)^{\frac{1}{2}} = \frac{u \cdot u' + u' \cdot u}{2(u \cdot u)^{\frac{1}{2}}} = \frac{u(t) \cdot u'(t)}{|u(t)|}.$$

Let $\gamma(t)$ be a RPC and $\beta(s) = \gamma(h(s))$ be a USR of γ. Then

$$\beta'(s) = \frac{d}{ds}(\gamma(t)) = \frac{d\gamma}{dt}\frac{dt}{ds} = \frac{\gamma'(t)}{ds/dt}.$$

But $|\beta'(s)| = 1$, so $ds/dt = |\gamma'(t)|$, and

$$\beta'(s) = \frac{\gamma'(t)}{|\gamma'(t)|}.$$

Differentiating this equation w.r.t. s once again yields

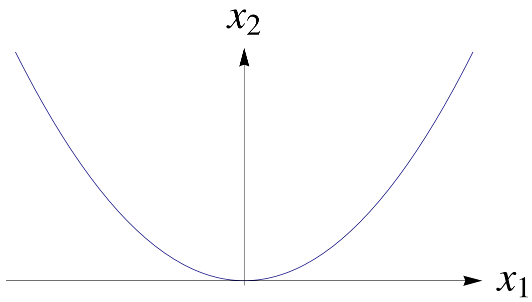
$$\begin{aligned}
\beta''(s) &= \frac{d}{ds}\left(\frac{\gamma'(t)}{|\gamma'(t)|}\right) = \frac{d}{dt}\left(\frac{\gamma'(t)}{|\gamma'(t)|}\right)\frac{dt}{ds} \\
&= \left\{\frac{\gamma''(t)}{|\gamma'(t)|} - \frac{\gamma'(t)}{|\gamma'(t)|^2}\frac{d}{dt}|\gamma'(t)|\right\}\frac{1}{|\gamma'(t)|} \\
&= \left\{\frac{\gamma''(t)}{|\gamma'(t)|} - \frac{\gamma'(t)}{|\gamma'(t)|^2}\left(\frac{\gamma'(t)\cdot\gamma''(t)}{|\gamma'(t)|}\right)\right\}\frac{1}{|\gamma'(t)|} \\
&= \frac{1}{|\gamma'(t)|^2}\left\{\gamma''(t) - \left(\frac{\gamma'(t)\cdot\gamma''(t)}{|\gamma'(t)|^2}\right)\gamma'(t)\right\},
\end{aligned}$$

where we have used observation (B) above. This leads us to

Definition 20 (*) The **curvature vector** of a RPC $\gamma : I \to \mathbb{R}^n$ is $k : I \to \mathbb{R}^n$, where

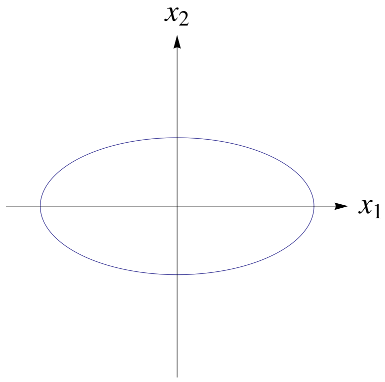
$$k(t) = \frac{1}{|\gamma'(t)|^2}\left\{\gamma''(t) - \left(\frac{\gamma'(t)\cdot\gamma''(t)}{|\gamma'(t)|^2}\right)\gamma'(t)\right\}. \qquad \square$$

Example 23 Parabola $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (t, t^2)$.



$$\begin{aligned}
\gamma'(t) &= (1, 2t) \\
|\gamma'(t)|^2 &= 1 + 4t^2 \\
\gamma''(t) &= (0, 2) \\
\gamma'(t)\cdot\gamma''(t) &= 4t \\
k(t) &= \frac{1}{1+4t^2}\left\{(0,2) - \frac{4t}{1+4t^2}(1,2t)\right\} = \frac{1}{(1+4t^2)^2}(-4t, 2)
\end{aligned} \qquad \square$$

Example 24 Ellipse $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (2\cos t, \sin t)$.



$$
\begin{aligned}
\gamma'(t) &= (-2\sin t, \cos t) \\
|\gamma'(t)|^2 &= 4\sin^2 t + \cos^2 t \\
\gamma''(t) &= (-2\cos t, -\sin t) \\
\gamma'(t)\cdot\gamma''(t) &= 3\sin t\cos t \\
k(t) &= \frac{1}{4\sin^2 t + \cos^2 t}\left\{(-2\cos t, -\sin t) - \frac{3\sin t\cos t}{4\sin^2 t + \cos^2 t}(-2\sin t, \cos t)\right\} \\
&= \frac{1}{(4\sin^2 t + \cos^2 t)^2}(-8\sin^2 t\cos t - 2\cos^3 t + 6\sin^2 t\cos t, \\
&\quad -4\sin^3 t - \sin t\cos^2 t - 3\sin t\cos^2 t) \\
&= \frac{1}{(4\sin^2 t + \cos^2 t)^2}(-2\cos t, -4\sin t) \\
|k(t)| &= \frac{2}{(4\sin^2 t + \cos^2 t)^2}\sqrt{\cos^2 t + 4\sin^2 t} = \frac{2}{(4\sin^2 t + \cos^2 t)^{\frac{3}{2}}}
\end{aligned}
$$

Definition 20(*) is much easier to use than definition 20, but it's not so memorable. Can we improve it?

2.2 The unit tangent vector and normal projection

We start with a simple observation about the curvature vector:

Fact 25 The curvature vector k of a RPC γ is always orthogonal to its velocity, $k(t) \cdot \gamma'(t) = 0$. This follows directly from Definition 20(*):

$$k(t)\cdot\gamma'(t) = \frac{1}{|\gamma'(t)|^2}\left\{\gamma''(t)\cdot\gamma'(t) - \left(\frac{\gamma'(t)\cdot\gamma''(t)}{|\gamma'(t)|^2}\right)\gamma'(t)\cdot\gamma'(t)\right\} = 0. \qquad \square$$

Looking back at examples 21, 22, 24 one sees several illustrations of this:

Helix	$\gamma(s) = \frac{1}{\sqrt{2}}(\cos s, \sin s, s)$	$\gamma'(s) = \frac{1}{\sqrt{2}}(-\sin s, \cos s, 1)$	$k(s) = \frac{1}{\sqrt{2}}(-\cos s, -\sin s, 0)$
Circle	$\gamma(t) = (2\cos t, 2\sin t)$	$\gamma'(t) = (-2\sin t, 2\cos t)$	$k(t) = \frac{1}{2}(-\cos t, -\sin t)$
Ellipse	$\gamma(t) = (2\cos t, \sin t)$	$\gamma'(t) = (-2\sin t, \cos t)$	$k(t) = \frac{-2}{(1+3\sin^2 t)^2}(\cos t, 2\sin t)$

We can use this fact to give a more memorable version of Definition 20(*). In preparation for this, we need:

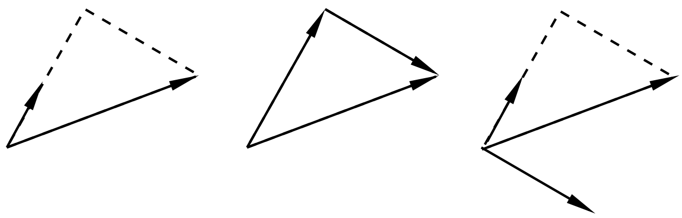
Definition 26 Let $\gamma : I \to \mathbb{R}^n$ be a RPC. Its **unit tangent vector** $u : I \to \mathbb{R}^n$ is

$$u(t) = \frac{\gamma'(t)}{|\gamma'(t)|}.$$

Note that u is well defined because γ is a **R**PC (so $|\gamma'(t)| > 0$ for all t). Note also that $|u(t)| = 1$ for all t by construction.

Given any other vector valued function $v : I \to \mathbb{R}^n$, we define its **normal projection**, $v_\perp : I \to \mathbb{R}^n$ by

$$v_\perp(t) = v(t) - [v(t)\cdot u(t)]u(t).$$



We can think of $v_\perp$ as that part of v left over after we have subtracted off the component of v in the direction of u. $\square$

Example 27 $\gamma : \mathbb{R} \to \mathbb{R}^3$, $\gamma(t) = (\frac{t^2}{2}, \sin t, \cos t)$. What are u and $\gamma''_\perp$?

$$\begin{aligned}
\gamma'(t) &= (t, \cos t, -\sin t) \\
|\gamma'(t)| &= \sqrt{t^2 + \cos^2 t + \sin^2 t} = \sqrt{t^2+1} \\
u(t) &= \frac{1}{\sqrt{t^2+1}}(t, \cos t, -\sin t) \\
\gamma''(t) &= (1, -\sin t, -\cos t) \\
\gamma''(t) \cdot u(t) &= \frac{1}{\sqrt{t^2+1}}(t - \sin t \cos t + \sin t \cos t) = \frac{t}{\sqrt{t^2+1}} \\
\gamma''_\perp(t) &= (1, -\sin t, -\cos t) - \frac{t}{t^2+1}(t, \cos t, -\sin t) \\
&= \left(\frac{1}{t^2+1}, -\sin t - \frac{t}{t^2+1} \cos t, -\cos t + \frac{t}{t^2+1} \sin t \right)
\end{aligned}$$

□

The normal projection of the acceleration vector $\gamma''_\perp : I \to \mathbb{R}^n$ is of particular interest, because it occurs in Definition 20(*):

$$\begin{aligned}
k(t) &= \frac{1}{|\gamma'(t)|^2} \left\{ \gamma''(t) - \left(\frac{\gamma'(t)}{|\gamma'(t)|} \cdot \gamma''(t) \right) \frac{\gamma'(t)}{|\gamma'(t)|} \right\} \\
&= \frac{1}{|\gamma'(t)|^2} \{\gamma''(t) - [u(t) \cdot \gamma''(t)] u(t)\} = \frac{\gamma''_\perp(t)}{|\gamma'(t)|^2}
\end{aligned}$$

Definition 20 (**) The **curvature vector** of a RPC $\gamma : I \to \mathbb{R}^n$ is $k : I \to \mathbb{R}^n$,

$$k(t) = \frac{\gamma''_\perp(t)}{|\gamma'(t)|^2}$$

□

Example 27 (revisited) $\gamma(t) = (\frac{t^2}{2}, \sin t, \cos t)$ has curvature vector

$$k(t) = \frac{1}{t^2+1} \left(\frac{1}{t^2+1}, -\sin t - \frac{t}{t^2+1} \cos t, -\cos t + \frac{t}{t^2+1} \sin t \right)$$

□

Summary

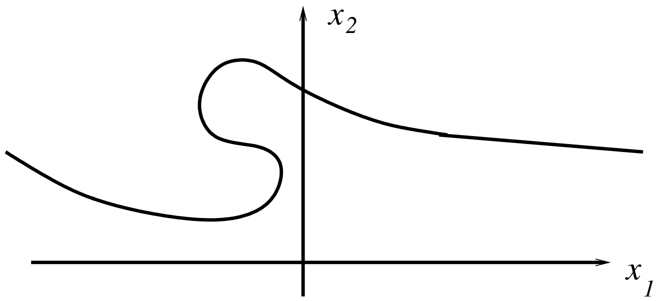
- The curvature vector of a RPC measures how fast the tangent lines to the curve change **direction**.
- If γ is a unit speed curve, the **curvature vector** is $k(s) = \gamma''(s)$.
- In general
$$k(t) = \frac{1}{|\gamma'(t)|^2} \left\{ \gamma''(t) - \frac{\gamma''(t) \cdot \gamma'(t)}{|\gamma'(t)|^2} \gamma'(t) \right\}.$$
- The **unit tangent vector** along a curve $\gamma : I \to \mathbb{R}^n$ is
$$u(t) = \frac{\gamma'(t)}{|\gamma'(t)|}.$$
- The **normal projection** of $v : I \to \mathbb{R}^n$ is
$$v_\perp(t) = v(t) - [u(t) \cdot v(t)]u(t).$$
- An alternative formula for the curvature vector is
$$k(t) = \frac{\gamma''_\perp(t)}{|\gamma'(t)|^2}.$$

3 Planar curves

3.1 Signed curvature of a planar curve

The theory of curvature can be developed further in the special case of *planar curves*, that is, RPCs $\gamma : I \to \mathbb{R}^2$ in two dimensions. These are special because all the curvature information associated with γ may be encoded in a single real-valued function $\kappa : I \to \mathbb{R}$, called the *signed curvature.* How?

Recall that k in this case is a 2-vector, so it consists of a *pair* of real functions $k(t) = (k_1(t), k_2(t))$ say. However, Fact 25 states that k is always orthogonal to the unit tangent vector u. So in fact we already know the *direction* of k. The only extra information we need to provide is the length of k, and its *sense*: whether it points to the left or the right of u.



Definition 28 Let $\gamma : I \to \mathbb{R}^2$ be a RPC with unit tangent vector u (recall $u = \gamma'/|\gamma'|$). The **unit normal vector** of γ is $n : I \to \mathbb{R}^2$,

$$n(t) = (-u_2(t), u_1(t)).$$

The **signed curvature** of γ is $\kappa : I \to \mathbb{R}$,

$$\kappa(t) = k(t) \cdot n(t)$$

where $k : I \to \mathbb{R}^2$ is the curvature vector of γ, as in Definition 20(**). □

Notes:

(a) n is orthogonal to u ($n(t) \cdot u(t) = 0$) and has unit length ($|n|^2 = u_2^2 + u_1^2 = |u|^2 = 1$) by construction. In fact, n is the vector obtained by rotating u 90° anticlockwise.

(b) Since both n and k are orthogonal to u, they must be parallel. In fact we can re-interpret the above definition as follows: given that $k(t)$ and $n(t)$ are parallel, we define $\kappa(t)$ to be the constant of proportionality,

$$k(t) = \kappa(t) n(t).$$

It follows that $|\kappa(t)| = |k(t)|$. However, κ contains more information than the (unsigned) curvature $|k|$ — its sign tells us the "sense" of k.

(c) Recall that $k = \gamma''_\perp / |\gamma'|^2$, so

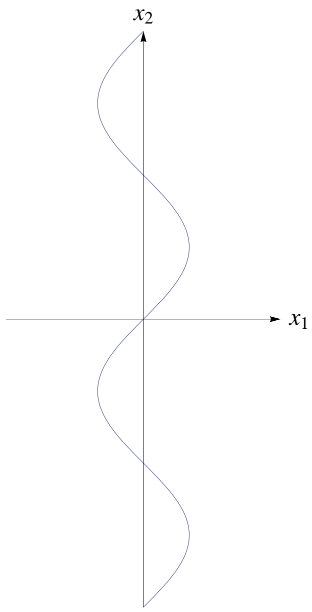
$$\kappa(t) = \frac{\gamma''_\perp(t) \cdot n(t)}{|\gamma'(t)|^2}.$$

However,
$$\gamma''_{\perp} \cdot n = (\gamma'' - (\gamma'' \cdot u)u) \cdot n = \gamma'' \cdot n,$$
so we can give a slightly more convenient definition of κ:

Definition 28 (*) Let $\gamma : I \to \mathbb{R}^2$ be a RPC, $u = \gamma'/|\gamma'|$ be its unit tangent vector and $n = (-u_2, u_1)$ be its unit normal. Then its **signed curvature** $\kappa : I \to \mathbb{R}$ is
$$\kappa(t) = \frac{\gamma''(t) \cdot n(t)}{|\gamma'(t)|^2}. \qquad \square$$

Example 29 A sinusoidal curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (\sin t, t)$

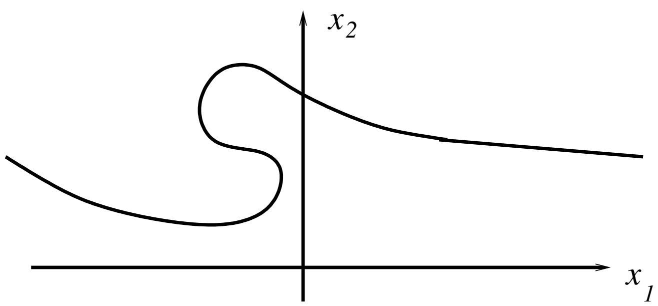
$$\begin{array}{rcl} \gamma'(t) & = & (\cos t, 1) \\ |\gamma'(t)| & = & \sqrt{1+\cos^2 t} \\ u(t) & = & \dfrac{1}{\sqrt{1+\cos^2 t}}(\cos t, 1) \\ n(t) & = & \dfrac{1}{\sqrt{1+\cos^2 t}}(-1, \cos t) \\ \gamma''(t) & = & (-\sin t, 0) \\ \kappa(t) & = & \dfrac{\sin t}{(1+\cos^2 t)^{\frac{3}{2}}} \end{array}$$



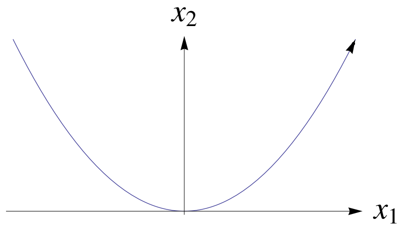
. $\square$

Note that

$\kappa > 0$	$\Rightarrow$	k points in same sense as n	$\Rightarrow$	curve is turning **left**
$\kappa < 0$	$\Rightarrow$	k points in opposite sense to n	$\Rightarrow$	curve is turning **right**



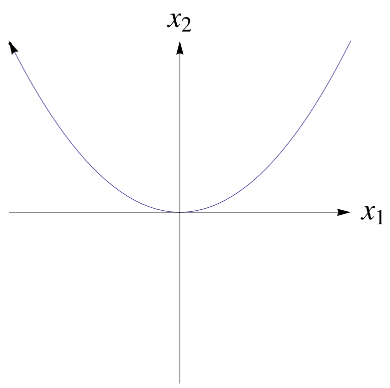
Example 30 Looking at the parabola $\gamma(t) = (t, t^2)$ it's immediately clear that $\kappa(t)$ is always positive. Let's check:

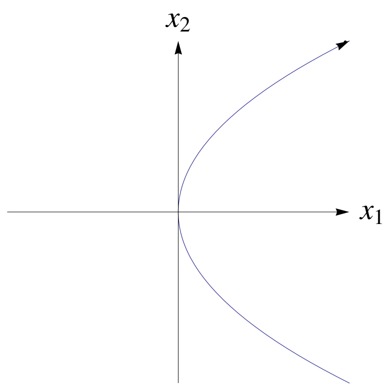


$$\begin{aligned}
\gamma'(t) &= (1, 2t) \\
u(t) &= \frac{1}{\sqrt{1+4t^2}}(1, 2t) \\
n(t) &= \frac{1}{\sqrt{1+4t^2}}(-2t, 1) \\
\gamma''(t) &= (0, 2) \\
\kappa(t) &= \frac{2}{(1+4t^2)^{\frac{3}{2}}}
\end{aligned}$$

Note that this observation depends crucially on the *orientation* of the curve, that is, the direction in which it is traversed. For example, for both of the parabolae below, the signed curvature is always negative:

$\gamma(t) = (-t, t^2)$ $\gamma(t) = (t^2, t)$





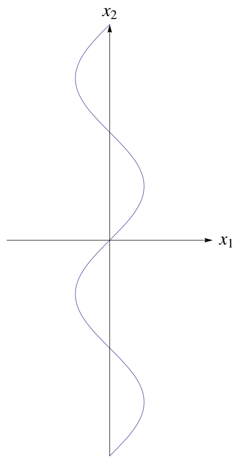
. □

Points on a curve where the signed curvature changes sign have a special name:

Definition 31 Let $\gamma : I \to \mathbb{R}^2$ be a RPC and $\kappa : I \to \mathbb{R}$ be its signed curvature. If there exists a time $t_* \in I$ such that $\kappa(t_*) = 0$ and κ changes sign at $t = t_*$, then $\gamma(t_*)$ is an **inflexion point** of γ. □

Example 29 (revisited) What are the inflexion points of the sinusoidal curve $\gamma(t) = (\sin t, t)$? Recall that its signed curvature function is

$$\kappa(t) = \frac{\sin t}{(1 + \cos^2 t)^{\frac{3}{2}}}$$



So $\kappa(t) = 0$ if and only if $t = N\pi$, where $N \in \mathbb{Z}$. Further, the *sign* of $\kappa(t)$ changes at each such time. Hence for all $N \in \mathbb{Z}$,

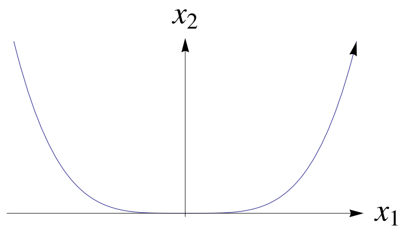
$$\gamma(N\pi) = (0, N\pi)$$

is an inflexion point.

WARNING!
$\kappa(t_*) = 0$ does NOT imply that $\gamma(t_*)$ is an inflexion point!
κ must CHANGE SIGN at $t = t_*$ too!

Counterexample 32 The quartic curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (t, t^4)$ has $\kappa(0) = 0$. However, it clearly has no inflexion points ($\kappa(t) \geq 0$, since the curve never turns right),

$$\begin{aligned}
\gamma'(t) &= (1, 4t^3) \\
\gamma''(t) &= (0, 12t^2) \\
\gamma''(0) &= (0, 0) \\
\kappa(0) &= \frac{\gamma''(0) \cdot n(0)}{|\gamma'(0)|^2} = 0
\end{aligned}$$



Exercise: show that

$$\kappa(t) = \frac{12t^2}{(1 + 16t^6)^{\frac{3}{2}}}. \qquad \square$$

3.2 Planar curves of prescribed curvature

In this section we will consider only planar unit speed curves (PUSCs) $\gamma(s)$ ($|\gamma'(s)| = 1$ for all s). Note this entails no loss of generality by Theorem 17.

So far, given a PUSC $\gamma(s)$ we can construct its signed curvature $\kappa(s)$. Can we go the other way? That is, given a function $\kappa(s)$, can we reconstruct the PUSC $\gamma(s)$ whose curvature is κ? Yes, provided we also specify an initial position $\gamma(0)$ and tangent vector $\gamma'(0)$.

How? Note $|\gamma'(s)| = 1$ so each velocity vector is determined by just its *direction*. That is, there exists smooth $\theta : I \to \mathbb{R}$ such that

$$\gamma'(s) = (\cos\theta(s), \sin\theta(s)) \qquad (A).$$

Note that $u(s) = \gamma'(s)$ and hence the unit normal vector is

$$n(s) = (-\sin\theta(s), \cos\theta(s)).$$

Now, for a USC, curvature $k(s) = \gamma''(s) = (-\sin\theta(s), \cos\theta(s))\theta'(s)$, and hence the signed curvature is

$$\kappa(s) = n(s) \cdot k(s) = \theta'(s) \qquad (B).$$

We may collect (A), (B) into a coupled system of 3 nonlinear ordinary differential equations (ODEs):

$$(*) \begin{cases} \dfrac{d\theta}{ds} = \kappa(s) & (1) \\[2ex] \dfrac{d\gamma_1}{ds} = \cos\theta & (2) \\[2ex] \dfrac{d\gamma_2}{ds} = \sin\theta & (3) \end{cases}$$

So any PUSC of curvature κ is a solution of system (*), and *vice versa*. Given a *prescribed* $\kappa(s)$, we can solve (*) for the curve $\gamma(s)$.

Theorem 33 *Given any smooth $\kappa : I \to \mathbb{R}$, $0 \in I$, and constants $\theta_0 \in \mathbb{R}$ and $\gamma_0 \in \mathbb{R}^2$, there exists a unique USC $\gamma : I \to \mathbb{R}^2$ with $\gamma(0) = \gamma_0$, $\gamma'(0) = (\cos\theta_0, \sin\theta_0)$ and signed curvature κ.*

Proof: We must prove existence of a unique global solution $(\theta(s), \gamma_1(s), \gamma_2(s))$ of the initial value problem [IVP] $(\theta(0), \gamma_1(0), \gamma_2(0)) = (\theta_0, a, b)$ for system (*) [where $\gamma_0 = (a, b)$].

In fact, (*) is separable. First consider IVP (1):

$$\frac{d\theta}{ds} = \kappa(s), \qquad \theta(0) = \theta_0.$$

This has solution

$$\theta(s) = f(s) := \theta_0 + \int_0^s \kappa(\alpha)d\alpha.$$

[Note that $f'(s) = \kappa(s)$ by the Fundamental Theorem of the Calculus, and $f(0) = \theta_0$, so $\theta = f$ is a solution with the right initial data.]
Is this solution unique? Yes, by the Mean Value Theorem (MVT).
[Assume it is not unique. Then there exists another solution $\theta(s) = g(s)$ with $g(0) = \theta_0$ but $g \neq f$, that is, there exists $s_0 \in I$ such that $g(s_0) \neq f(s_0)$. Consider the

function $F(s) = f(s) - g(s)$. Clearly $F(0) = 0$, $F(s_0) \neq 0$ and F is differentiable on I. Hence, by the MVT, there exists s_* between 0 and s_0 such that

$$F'(s_*) = \frac{F(s_0) - F(0)}{s_0 - 0} = \frac{F(s_0)}{s_0} \neq 0.$$

But f and g both solve (1), so $F'(s) = \kappa(s) - \kappa(s) = 0$ for all s, a contradiction.]

Now substitute this unique solution ($\theta = f$) into IVP (2):

$$\frac{d\gamma_1}{ds} = \cos f(s) \qquad \gamma_1(0) = a.$$

This has unique solution

$$\gamma_1(s) = a + \int_0^s \cos f(\alpha)\, d\alpha$$

by an identical argument. Similarly, substituting $\theta = f$ into IVP (3),

$$\frac{d\gamma_2}{ds} = \sin f(s) \qquad \gamma_2(0) = b,$$

one has the unique solution

$$\gamma_2(s) = b + \int_0^s \sin f(\alpha)\, d\alpha. \qquad \square$$

Note that Theorem 34 doesn't just prove existence and uniqueness of the curve $\gamma(s)$, it also gives a *formula* for it:

$$\boxed{\gamma(s) = \gamma_0 + \left(\int_0^s \cos\theta(\alpha)\, d\alpha, \int_0^s \sin\theta(\alpha)\, d\alpha\right), \quad \text{where} \quad \theta(\alpha) = \theta_0 + \int_0^\alpha \kappa(\beta)\, d\beta. \qquad (C)}$$

Example 34 Curves of constant curvature: $\kappa(s) = \kappa_0 \neq 0$, constant. Let's always choose $\gamma(0) = (0, 0)$, $\theta(0) = 0$ henceforth. Then formula (C) implies

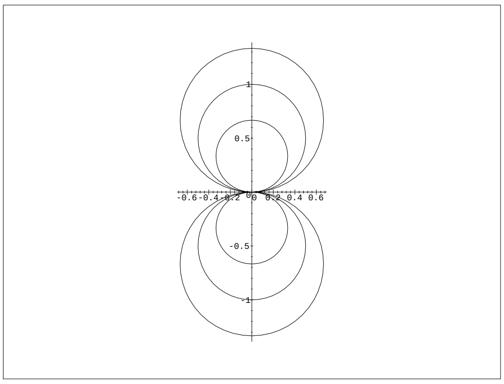
$$\begin{aligned}
\theta(\alpha) &= \int_0^\alpha \kappa_0 d\beta = \kappa_0\alpha \\
\Rightarrow \gamma_1(s) &= \int_0^s \cos(\kappa_0\alpha) d\alpha = \frac{1}{\kappa_0}\sin(\kappa_0 s) \\
\gamma_2(s) &= \int_0^s \sin(\kappa_0\alpha) d\alpha = \frac{1}{\kappa_0}(1 - \cos(\kappa_0 s)) \\
\Rightarrow \gamma(s) &= \frac{1}{\kappa_0}\left(\sin(\kappa_0 s), 1 - \cos(\kappa_0 s)\right)
\end{aligned}$$

This curve is a circle of radius $1/|\kappa_0|$ centred on $(0, 1/\kappa_0)$. Question: what happens when $\kappa_0 = 0$?

If we go back and use formula (C) with $\kappa(\beta) \equiv 0$ we find that

$$\kappa_0 = 0 \quad \Rightarrow \quad \gamma(s) = (s, 0),$$

that is, the solution degenerates to a horizontal straight line.



Example 35 $\kappa(s) = \dfrac{1}{1+s^2}$. Note $\kappa(s) > 0$ for all s, so curve always turns left and has no inflexion points. Applying formula (C) again:

$$\begin{aligned}
\theta(\alpha) &= \int_0^\alpha \frac{1}{1+\beta^2} d\beta = \arctan\alpha \\
\Rightarrow \cos\theta(\alpha) &= \frac{1}{\sec\theta(\alpha)} = \frac{1}{\sqrt{1+\tan^2\theta(\alpha)}} = \frac{1}{\sqrt{1+\alpha^2}} \\
\sin\theta(\alpha) &= \tan\theta(\alpha)\cos\theta(\alpha) = \frac{\alpha}{\sqrt{1+\alpha^2}} \\
\gamma_1(s) &= \int_0^s \frac{1}{\sqrt{1+\alpha^2}} d\alpha = \operatorname{arcsinh}(s) \\
\gamma_2(s) &= \int_0^s \frac{\alpha}{\sqrt{1+\alpha^2}} d\alpha = \sqrt{1+s^2} - 1 \\
\Rightarrow \gamma(s) &= \left(\operatorname{arcsinh}(s), \sqrt{1+s^2} - 1\right).
\end{aligned}$$

In fact, using the reparametrization $s = \sigma(t) = \sinh t$, this becomes

$$\widetilde{\gamma}(t) = (\gamma \circ \sigma)(t) = (t, \cosh t - 1),$$

the graph of the cosh function shifted down one unit. $\square$

It's actually quite difficult to cook up curvature functions $\kappa(s)$ for which the integrals in formula (C) are explicitly calculable. Even a seemingly simple choice such as $\kappa(s) = s$ turns out to be intractable:

$$\begin{aligned}
\theta(\alpha) &= \int_0^\alpha \beta\, d\beta = \frac{1}{2}\alpha^2 \\
\gamma_1(s) &= \int_0^s \cos\frac{\alpha^2}{2}\, d\alpha = ??? \\
\gamma_2(s) &= \int_0^s \sin\frac{\alpha^2}{2}\, d\alpha = ???
\end{aligned}$$

What can we say about the geometry of this curve?

Reminder: A function $f : I \to \mathbb{R}$ is

$$\begin{array}{c}\textbf{even if } f(-t) = f(t) \text{ for all } t \in I \\ \textbf{odd if } f(-t) = -f(t) \text{ for all } t \in I.\end{array}$$

Proposition 36 *Let $\gamma(s)$ be the PUSC of curvature $\kappa(s)$ with $\gamma(0) = 0$, $\gamma'(0) = (1,0)$.*

(a) If κ is even, γ is symmetric under reflexion in the x_2 axis.

(b) If κ is odd, γ is symmetric under rotation by 180 degrees about $(0,0)$.

Proof: From formula (C) one sees that

$$\theta(-\alpha) = \int_0^{-\alpha} \kappa(\beta)d\beta = -\int_0^{\alpha} \kappa(-\xi)d\xi$$

where $\xi := -\beta$. Hence κ even implies θ odd, while κ odd implies θ even. Similarly,

$$\gamma_1(-s) = \int_0^{-s} \cos(\theta(\alpha))d\alpha = -\int_0^{s} \cos(\theta(-\eta))d\eta$$

so if κ is odd or even (meaning θ is even or odd) then γ_1 is odd. Further,

$$\gamma_2(-s) = \int_0^{-s} \sin(\theta(\alpha))d\alpha = -\int_0^{s} \sin(\theta(-\eta))d\eta$$

so if κ is even (θ odd) γ_2 is even while if κ is odd (θ even) γ_2 is odd. Summarizing:

$$\begin{aligned}\kappa \text{ even} \quad &\Rightarrow \quad \gamma(-s) \equiv (-\gamma_1(s), \gamma_2(s)) \\ \kappa \text{ odd} \quad &\Rightarrow \quad \gamma(-s) \equiv (-\gamma_1(s), -\gamma_2(s))\end{aligned}$$

and hence γ has the symmetry claimed. □

Applying Proposition 36 to $\kappa(s) = s$, an odd function, we see that the corresponding curve γ must have rotational symmetry about the origin. Also, γ has one and only one inflexion point: $\gamma(0) = (0,0)$. For $s > 0$, $\kappa > 0$ meaning the curve always turns leftwards, and as s grows this turning gets tighter and tighter ($|\kappa|$ is unbounded). The behaviour for $s < 0$ is determined by that for $s > 0$ by the symmetry property.

To get an idea of the specific shape of the curve γ, we can solve system (*) approximately using an ODE solver package, in Python for example. Given a function κ and an interval $I = (s_0, s_1)$ with $s_0 < 0 < s_1$, the following Python code computes the curve $\gamma : I \to \mathbb{R}^2$ of curvature κ ($\gamma(0) = 0$, $\gamma'(0) = (1,0)$) and then plots it. We begin (as always) by importing some useful Python modules:

```
>>> import numpy as np
>>> import matplotlib.pyplot as plt
>>> from scipy.integrate import odeint
```

Numpy (which we abbreviate to np) gives us access to the trig functions sin and cos, and to numerical vectors; matplotlib is a plotting library; scipy.integrate has the ODE solver we will use. We next define a Python function which solves the ODEs and plots the solution:

```
>>> def show_curve(kappa,s0,s1):
...     Y0=[0,0,0]
...     xr=np.linspace(0,s1,200)
...     xl=np.linspace(0,s0,200)
...     def dY_ds(Y, s):
...             return [kappa(s), np.cos(Y[0]), np.sin(Y[0])]
...     Yr=odeint(dY_ds,Y0,xr)
...     Yl=odeint(dY_ds,Y0,xl)
...     yr1=Yr[:,1]
...     yr2=Yr[:,2]
...     yl1=Yl[:,1]
...     yl2=Yl[:,2]
...     plt.plot(yr1,yr2,yl1,yl2)
...     plt.axis('equal')
...     plt.show()
...
```

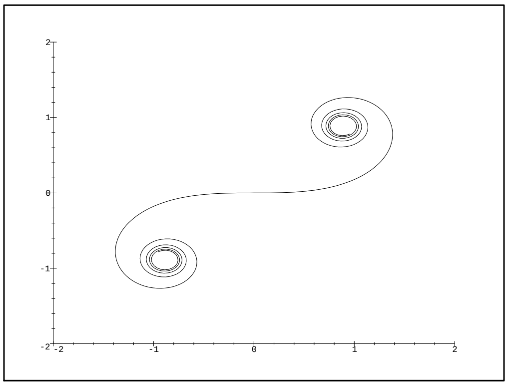
To apply this to our case ($\kappa(s) = s$) one defines

```
>>> def kappa(s):
...     return s
...
```

and then executes (for example)

```
>>> show_curve(kappa,-8,8)
```

The result is:

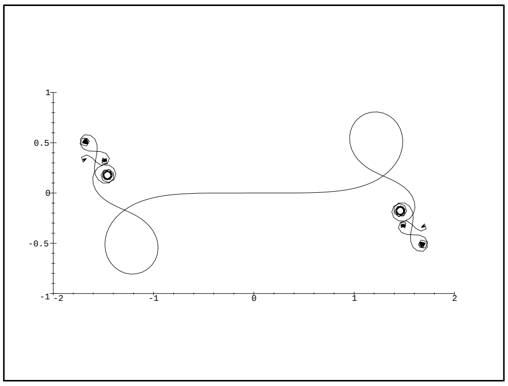


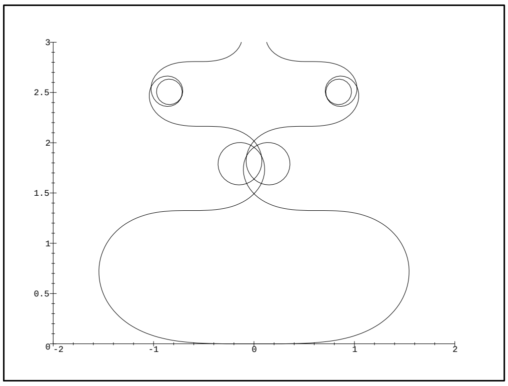
Note the curve has the predicted turning behaviour and symmetry. It is now straightforward to turn the program loose on just about any curvature function. The results can be quite entertaining.

Example 37 Let $\kappa(s) = s^2 \sin s$. That is:

```
>>> def kappa(s):
...     return s**2*np.sin(s)
...
```

Note that κ is again odd, so the corresponding curve must have rotational symmetry about the origin. Note also that $\gamma(s)$ has infinitely many inflexion points, since κ changes sign at every $s = m\pi$, where $m \in \mathbb{Z}$. Executing `show_curve(kappa,-15,15);` one obtains (below left):





Compare with $\kappa(s) = s \sin s$, an even function:

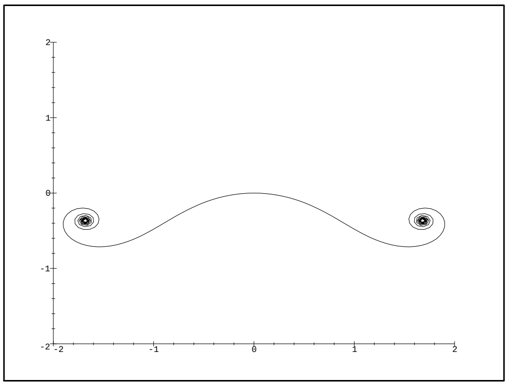
```
>>> def kappa(s):
...     return s*np.sin(s)
...
>>> show_curve(kappa,-8,8)
```

The corresponding curve is depicted above, right. Note the reflexion symmetry in the x_2 axis. $\square$

Example 38 Let $\kappa(s) = s^2 - 1$ Note that κ is __even__, so the corresponding curve must have __reflection__ symmetry. Note also that $\gamma(s)$ has exactly __two__ inflexion points, since κ changes sign at $s =$ __± 1__. Executing

```
>>> def kappa(s):
...     return s**2-1.0
...
>>> show_curve(kappa,-6,6)
```

one obtains:



It's not hard to identify the inflexion points on the curve above.

Trick question: What's the arc length along the curve from one inflexion point to the next?
Answer:______________ □

Go forth and experiment

You are encouraged to experiment in Python with the function `show_curve`. To this end, you will find a copy of its defining program linked from the course Minerva page. You should be able to cut and paste the program directly into a terminal running Python, or a Python notebook. If you experience problems, please let me know.

The value of experimenting with `show_curve` is that it will help develop your intuition about the geometric relationship between the function κ and the corresponding curve γ. Of course, in the final exam for this module, Python will not be available to you. Hopefully the intuition you've developed will. In particular, relying heavily on `show_curve` to complete Problem Sheet 2 would be a mistake. You have been warned...

Summary

- For a planar curve $\gamma : I \to \mathbb{R}^2$ we can define the **unit normal vector**
$$n(t) = (-u_2(t), u_1(t)),$$
where u is the unit tangent vector.

- Since the curvature vector is parallel to n, there is a scalar function $\kappa : I \to \mathbb{R}$ called the **signed curvature**, such that
$$k(t) = \kappa(t)n(t).$$

- A convenient formula for $\kappa(t)$ is
$$\kappa(t) = \frac{\gamma''(t) \cdot n(t)}{|\gamma'(t)|^2}.$$

- If $\kappa(t) > 0$, the curve is turning to the **left**. If $\kappa(t) < 0$, the curve is turning to the **right**.

- Given a function $\kappa(s)$, there is a planar USC $\gamma(s)$ whose signed curvature is $\kappa(s)$. This curve is unique up to rigid motions.

- Symmetries of κ imply symmetries of γ (and vice versa).

4 New plane curves from old: evolutes, involutes and parallels

Given one planar curve, there are a number of geometrically interesting ways to generate new curves from it. In this section we will study three of these, showing how their geometric properties are realted to those of the parent curve, and how the curves themselves are related to one another.

4.1 The evolute of a plane curve

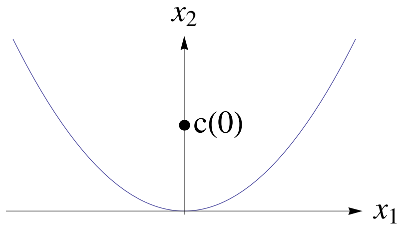
Throughout this section, $\gamma : I \to \mathbb{R}^2$ will denote a planar RPC, not necessarily unit speed, $u : I \to \mathbb{R}^2$ will denote its unit tangent vector, $n : I \to \mathbb{R}^2$ its unit normal vector and $\kappa : I \to \mathbb{R}$ its signed curvature. First we need:

Definition 39 Given $t_0 \in I$, the **centre of curvature** of γ at $t = t_0$ is

$$c(t_0) = \gamma(t_0) + \frac{1}{\kappa(t_0)} n(t_0).$$

Note this definition only makes sense if $\kappa(t_0) \neq 0$. □

Example 40 We saw in example 30 that the parabola $\gamma(t) = (t, t^2)$ has, at $t = 0$,



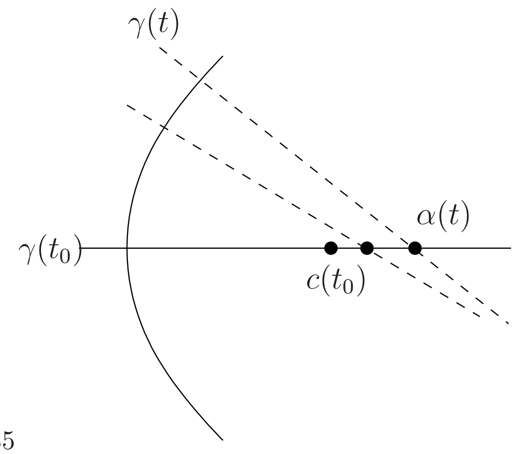
$$\begin{aligned} n(0) &= (0,1) \\ \kappa(0) &= 2 \\ \Rightarrow c(0) &= \left(0, \frac{1}{2}\right) \end{aligned}$$

. □

We can give a nice geometric interpretation of the centre of curvature in terms of the behaviour of the **normal lines** to the curve γ.

Theorem 41 *For $t_0 \in I$ fixed and $t \in I$ variable, consider the normal lines to γ through $\gamma(t_0)$ and $\gamma(t)$. If $\kappa(t_0) \neq 0$ and $|t - t_0|$ is sufficiently small then these normals intersect at some point $\alpha(t) \in \mathbb{R}^2$. Then*

$$\lim_{t \to t_0} \alpha(t) = c(t_0),$$



the centre of curvature of γ at t_0. .

To prove this, we'll need a useful lemma:

Lemma 42 *For all $t \in I$,*

$$(a) \quad n'(t) = -\kappa(t)\gamma'(t), \qquad (b) \quad u'(t) = \kappa(t)|\gamma'(t)|n(t).$$

Proof: Since $[u(t), n(t)]$ is an *orthonormal* pair of vectors, they form a (t dependent) basis for $\mathbb{R}^2$. So we can always express any $\mathbb{R}^2$ valued function as a linear combination of $u(t)$ and $n(t)$. Applying this idea to $n'(t)$, we see that there must exist smooth functions $\lambda : I \to \mathbb{R}$ and $\mu : I \to \mathbb{R}$ such that

$$n'(t) = \lambda(t)u(t) + \mu(t)n(t).$$

Taking the scalar product of both sides of this equation with $u(t)$ gives

$$u \cdot n' = \lambda \underbrace{u \cdot u}_{1} + \mu \underbrace{u \cdot n}_{0} = \lambda \quad \text{(orthonormality)}$$

Now

$$u \cdot n = 0 \quad \Rightarrow \quad u' \cdot n + u \cdot n' = 0 \quad \Rightarrow \quad u \cdot n' = -n \cdot u'.$$

Hence

$$\lambda = -n \cdot \frac{d}{dt}\left(\frac{\gamma'}{|\gamma'|}\right) = -n \cdot \left(\frac{\gamma''}{|\gamma'|} - \gamma' \frac{d|\gamma'|^{-1}}{dt}\right) = -\frac{n \cdot \gamma''}{|\gamma'|} = -\kappa|\gamma'|$$

Similarly,

$$n \cdot n' = \lambda \underbrace{n \cdot u}_{0} + \mu \underbrace{n \cdot n}_{1} = \mu \quad \text{(orthonormality)}$$

and

$$n \cdot n = 1 \quad \Rightarrow \quad n' \cdot n + n \cdot n' = 0 \quad \Rightarrow \quad n \cdot n' = 0$$

so it follows that $\mu(t) = 0$. Hence,

$$n'(t) = -\kappa(t)|\gamma'(t)|u(t) = -\kappa(t)\gamma'(t)$$

which proves part (a).

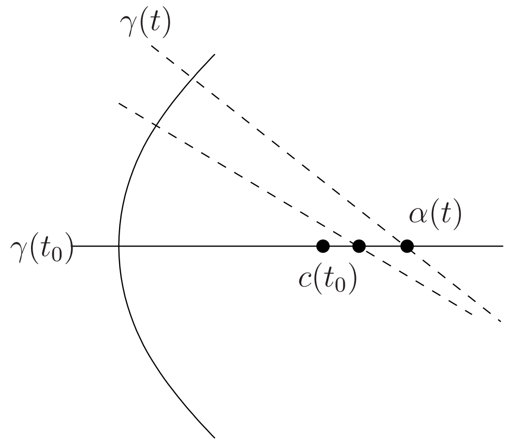
Part (b) follows from similar reasoning, so we shall be more brief.

$$u' = (u \cdot u')u + (n \cdot u')n = 0u - (u \cdot n')n = \kappa|\gamma'|n$$

by part (a). □

We may now give a *Proof of Theorem 41:* We may give the normals through $\gamma(t)$, $\gamma(t_0)$ the (unit speed) parametrizations (with parameters s and s_0 respectively)

$$\gamma(t) + sn(t), \qquad \gamma(t_0) + s_0 n(t_0).$$



Hence, their intersection point, which will depend on t, is

$$\alpha(t) = \gamma(t) + s(t)n(t) = \gamma(t_0) + s_0(t)n(t_0) \tag{$\clubsuit$}$$

where $s(t)$ and $s_0(t)$ are two unknown functions of t. Differentiating ($\clubsuit$) with respect to t and taking the limit $t \to t_0$, we find that

$$\gamma'(t_0) + s'(t_0)n(t_0) + s(t_0)n'(t_0) = s_0'(t_0)n(t_0). \tag{$\spadesuit$}$$

Taking the scalar product of ($\spadesuit$) with $u(t_0)$ yields

$$\begin{aligned} |\gamma'(t_0)| + 0 + s(t_0)n'(t_0) \cdot u(t_0) &= 0 \\ \Rightarrow \quad |\gamma'(t_0)| - s(t_0)\kappa(t_0)\gamma'(t_0) \cdot u(t_0) &= 0 \\ \Rightarrow \quad s(t_0) &= \frac{1}{\kappa(t_0)} \end{aligned}$$

by Lemma 42. Hence

$$\lim_{t \to t_0} \alpha(t) = \gamma(t_0) + s(t_0)n(t_0) = \gamma(t_0) + \frac{1}{\kappa(t_0)}n(t_0)$$

as was to be proved. □

Definition 43 The **evolute** of the planar curve γ is $E_\gamma : I \to \mathbb{R}^2$, defined such that

$$E_\gamma(t) = \gamma(t) + \frac{1}{\kappa(t)}n(t),$$

in other words, it is the curve of centres of curvature of the curve γ. □

Notes:

- E_γ is only well defined (on the whole of I) provided γ has nonvanishing signed curvature κ. In particular, γ must have no inflexion points, or else E_γ "escapes to infinity."
- Even if $\kappa(t) \neq 0$ for all $t \in I$, the evolute may not be a **R**PC, as we will now show.

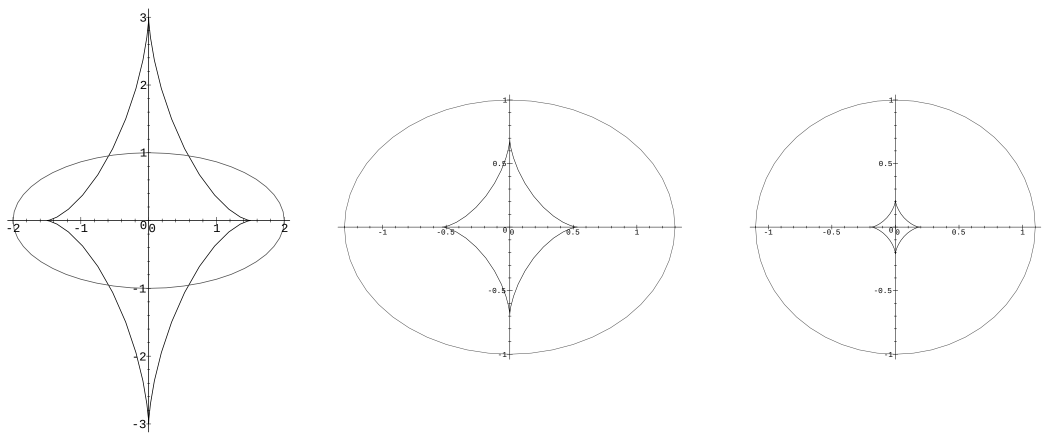
$$
\begin{aligned}
E'_\gamma(t) &= \gamma'(t) - \frac{\kappa'(t)}{\kappa(t)^2}n(t) + \frac{1}{\kappa(t)}n'(t) \\
&= \gamma'(t) - \frac{\kappa'(t)}{\kappa(t)^2}n(t) - \frac{1}{\kappa(t)}\kappa(t)\gamma'(t) \\
&= -\frac{\kappa'(t)}{\kappa(t)^2}n(t) \qquad \diamondsuit
\end{aligned}
$$

by Lemma 42. So $E'_\gamma(t) = (0,0)$ if and only if $\kappa'(t) = 0$. Hence, if κ has critical points, then E_γ is not regular. At such points, E_γ may exhibit "cusps".

Example 44 (ellipse) $\gamma(t) = (a\cos t, \sin t)$

$$
\begin{aligned}
\gamma'(t) &= (-a\sin t, \cos t) \\
n(t) &= \frac{1}{\sqrt{\cos^2 t + a^2\sin^2 t}}(-\cos t, -a\sin t) \\
\gamma''(t) &= (-a\cos t, -\sin t) \\
\kappa(t) &= \frac{a}{(\cos^2 t + a^2\sin^2 t)^{\frac{3}{2}}} \\
E_\gamma(t) &= (a\cos t, \sin t) - \frac{\cos^2 t + a^2\sin^2 t}{a}(\cos t, a\sin t) \\
&= (a\cos t - a\sin^2 t\cos t - \frac{1}{a}\cos^3 t, \sin t - \cos^2 t\sin t - a^2\sin^3 t) \\
&= (a\cos^3 t - \frac{1}{a}\cos^3 t, \sin^3 t - a^2\sin^3 t) \\
&= \left(a^2 - 1\right)\left(\frac{1}{a}\cos^3 t, -\sin^3 t\right)
\end{aligned}
$$

$a = 2$ $\qquad$ $a = 1.3$ $\qquad$ $a = 1.1$



In the limit $a \to 1$, the ellipse degenerates to a circle, and its evolute collapses to a single point $(0,0)$. $\square$

We can relate the geometric quantities associated with E_γ to those of γ.

Theorem 45 *Let u^E, n^E, $\sigma^E_{t_0}$ denote the unit tangent, unit normal and arclength function of E_γ respectively. If $\kappa'(t) < 0$ for all $t \in I$, then*

$$(a) \quad u^E(t) = n(t), \qquad (b) \quad n^E(t) = -u(t), \qquad (c) \quad \sigma^E_{t_0}(t) = \frac{1}{\kappa(t)} - \frac{1}{\kappa(t_0)}.$$

Proof: (a) From ($\diamondsuit$), we know that

$$E'_\gamma(t) = -\frac{\kappa'(t)}{\kappa(t)^2} n(t)$$

so if $\kappa'(t) < 0$ then

$$|E'_\gamma(t)| = -\frac{\kappa'(t)}{\kappa(t)^2}.$$

Hence $u^E = E'_\gamma / |E'_\gamma| = n$.
(b) $n^E = (-u^E_2, u^E_1) = (-n_2, n_1) = (-u_1, -u_2)$ by part (a).
(c) Arclength is the integral of speed, which by part (a) is

$$\sigma^E_{t_0}(t) = -\int_{t_0}^t \frac{\kappa'(q)}{\kappa(q)^2} dq = \int_{t_0}^t \left(\frac{1}{\kappa}\right)'(q) dq = \frac{1}{\kappa(t)} - \frac{1}{\kappa(t_0)}$$

which was to be proved. $\square$

4.2 Involutes and parallels of a planar curve

Once again, let $\gamma : I \to \mathbb{R}^2$ be a RPC, and let u, n, κ denote its unit tangent, unit normal and signed curvature respectively. In this section we describe a different way to generate new planar curves from γ which is in some sense the inverse of taking the evolute.

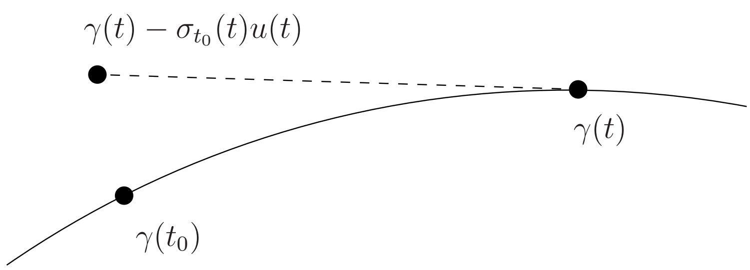
Definition 46 The **involute** of γ starting at $t_0 \in I$ is $I_\gamma : I \to \mathbb{R}^2$, defined by

$$I_\gamma(t) = \gamma(t) - \sigma_{t_0}(t) u(t).$$

Note that if $\gamma(t)$ is a USC then $\sigma_{t_0}(t) = t - t_0$ so

$$I_\gamma(t) = \gamma(t) - (t - t_0)\gamma'(t).$$

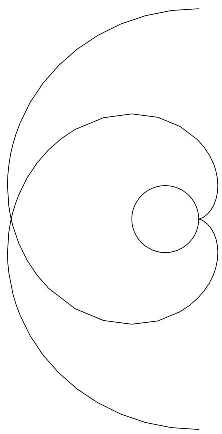
$\square$



In the picture, the dashed line is tangent to γ at t. The point $I_\gamma(t) = \gamma(t) - \sigma_{t_0}(t)u(t)$ is the point on this line at distance $\sigma_{t_0}(t)$ from $\gamma(t)$.

As an example of an involute, imagine that you have a spool with cotton thread wrapped around. Imagine tying a pencil to the end of the thread, and unwinding the thread, with the thread pulled taut at all times. The curve drawn by the end of the pencil is the involute of the circle.

Example 47 A circle $\gamma(t) = (\cos t, \sin t)$. This is a USC, so the involute of γ based at $t = 0$ is



$$\begin{aligned} I_\gamma(t) &= (\cos t, \sin t) - t(-\sin t, \cos t) \\ &= (\cos t + t \sin t, \sin t - t \cos t) \end{aligned}$$

.
$\square$

Note that I_γ has a cusp at $t = 0$ in this case. Accident? No: I_γ is **never** a RPC.

$$\begin{aligned} & I_\gamma(t) = \gamma(t) - \sigma_{t_0}(t)u(t) \\ \Rightarrow \quad & I'_\gamma(t) = \gamma'(t) - \sigma'_{t_0}(t)u(t) - \sigma_{t_0}(t)u'(t) \\ & \quad = \gamma'(t) - |\gamma'(t)|u(t) - \sigma_{t_0}(t)\kappa(t)|\gamma'(t)|n(t) \\ & \quad = -\sigma_{t_0}(t)\kappa(t)|\gamma'(t)|n(t) \qquad (\spadesuit) \end{aligned}$$

where we have used Lemma 42. Hence $I'_\gamma(t) = (0,0)$ if and only if $\sigma_{t_0}(t) = 0$ or $\kappa(t) = 0$ (since γ is a RPC, $|\gamma'(t)|$ is never 0). So $I'_\gamma(t) = (0,0)$ when $t = t_0$ and whenever $\kappa(t) = 0$ (for example, where the contact point is an inflexion point of γ).

A useful analogy:

function $f(x)$

derivative $\swarrow$ $\qquad$ $\searrow$ integral

$f'(x)$ $\qquad$ $\int_{x_0}^x f(q)dq$

curve γ

evolute $\swarrow$ $\qquad$ $\searrow$ involute

E_γ $\qquad$ I_γ

Fundamental Theorem of Calculus: derivative of integral is the original function f.

Theorem 48 *Let I_γ be an involute of γ. Then the evolute E_I of I_γ is γ.*

The strategy of the proof is simple: we construct the evolute of I_γ, the involute of γ starting at t_0. To do this, we will need the unit normal and signed curvature of I_γ, so we begin by proving:

Lemma 49 *Let γ be a unit speed curve and I be its involute based at time t_0. Then I has unit normal*

$$n^I(t) = \frac{(t-t_0)\kappa}{|t-t_0||\kappa(t)|}u(t),$$

and signed curvature

$$\kappa^I(t) = \frac{\kappa(t)}{|\kappa(t)|}\frac{1}{|t-t_0|},$$

where κ, u are the signed curvature and unit tangent vector of γ

Proof: Since γ has unit speed,

$$I'(t) = -(t-t_0)\kappa(t)n(t) = -(t-t_0)k(t) \qquad (\clubsuit)$$

by equation ($\spadesuit$). Hence I has unit tangent

$$u^I = \frac{I'}{|I'|} = -\frac{(t-t_0)\kappa}{|t-t_0||\kappa|}n$$

and so, rotating this 90 degrees anticlockwise yields

$$n^I = (-u_2^I, u_1^I) = \frac{(t-t_0)\kappa}{|t-t_0||\kappa|}u.$$

Differentiating ($\clubsuit$) gives

$$I'' = -k - (t-t_0)k',$$

whence the signed curvature of I is

$$\begin{aligned} \kappa^I &= \frac{I''\cdot n^I}{|I_\gamma'|^2} = -\frac{(t-t_0)\kappa}{|t-t_0|^3|\kappa|^3}[k+(t-t_0)k']\cdot u \\ &= -\frac{\kappa}{|t-t_0||\kappa|^3}k'\cdot u \end{aligned}$$

since $k\cdot u = 0$ by Fact 25. But

$$k\cdot u = 0 \quad \Rightarrow \quad k'\cdot u + k\cdot u' = 0 \quad \Rightarrow \quad k'\cdot u = -k\cdot u' = -|k|^2 = -\kappa^2$$

since γ is a USC (so $u = \gamma'$ and $u' = \gamma'' = k$). Hence

$$\kappa^I = \frac{\kappa}{|\kappa|}\frac{1}{|t-t_0|},$$

as was claimed. $\square$

Theorem 48 now follows immediately:

Proof of Theorem 48: By Theorem 17 we may assume without loss of generality that γ is a USC. Then its involute I based at t_0 has evolute

$$\begin{aligned} E_I(t) &= I(t) + \frac{1}{\kappa^I(t)} n^I(t) \\ &= [\gamma(t) - (t-t_0)u(t)] + \left[\frac{|\kappa(t)|}{\kappa(t)}|t-t_0|\right] \times \left[\frac{(t-t_0)\kappa(t)}{|t-t_0||\kappa(t)|}u(t)\right] \\ &= \gamma(t) \end{aligned}$$

by Lemma 49. $\square$

So taking the evolute "undoes" the involute, just as differentiation "undoes" integration. What about the converse? If we first differentiate $f(x)$, then take a definite integral of f', we don't necessarily get the same function $f(x)$ back again – it could be shifted by a constant c. For example:

$$\begin{aligned} & f(x) = \cos x \qquad \Rightarrow \qquad f'(x) = -\sin x \\ \Rightarrow \qquad & \int_0^x f'(q)dq = -\int_0^x \sin q \, dq = \cos x - 1 = f(x) - 1 \end{aligned}$$

Thinking of the graphs of the functions, we could say that integrating the derivative in general gives a shifted, or "parallel", function. An analogous statement holds for curves too, i.e. if we take an involute of the evolute of γ, the result is a curve "parallel" to γ, in the following precise sense:

Definition 50 Given a RPC $\gamma : I \to \mathbb{R}^2$ and a constant $\lambda \in \mathbb{R}$, the curve $\gamma_\lambda : I \to \mathbb{R}^2$ defined by

$$\gamma_\lambda(t) = \gamma(t) + \lambda n(t)$$

is a **parallel curve** to γ. $\square$

Note the similarity between γ_λ and the evolute of γ,

$$E_\gamma(t) = \gamma(t) + \kappa(t)^{-1} n(t).$$

Under what circumstances is γ_λ regular?

Lemma 51 *The parallel γ_λ to γ is a RPC if and only if $\kappa(t) \neq 1/\lambda$ for all $t \in I$. In other words, $1/\lambda$ must lie outside the range of the function κ.*

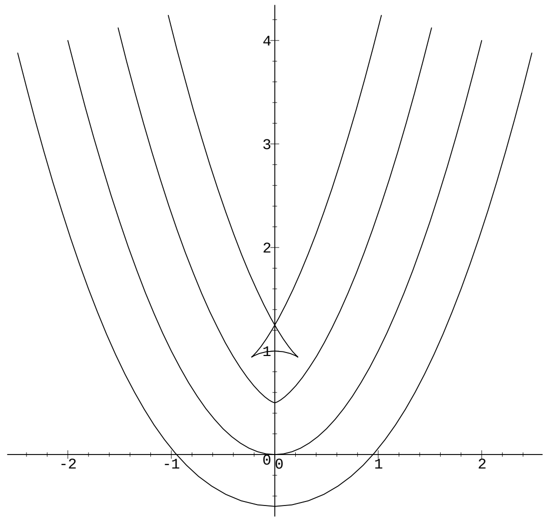
Proof:

$$\gamma_\lambda'(t) = \gamma'(t) + \lambda n'(t) = \gamma'(t) - \lambda\kappa(t)\gamma'(t)$$

by Lemma 42. Hence $\gamma_\lambda'(t) = (0,0)$ if and only if $\kappa(t) = 1/\lambda$. $\square$

Example 52 Let's construct the general parallel curve to the parabola $\gamma(t) = (t, t^2)$:

$$
\begin{aligned}
\gamma'(t) &= (1, 2t) \\
n(t) &= \frac{1}{\sqrt{1+4t^2}}(-2t, 1) \\
\gamma_\lambda(t) &= (t, t^2) + \frac{\lambda}{\sqrt{1+4t^2}}(-2t, 1)
\end{aligned}
$$



Which values of the constant λ give *regular* parallels? Example 30 $\Rightarrow$

$$\kappa(t) = \frac{2}{(1+4t^2)^{\frac{3}{2}}}.$$

Hence κ has range $(0, 2]$, so γ_λ is regular if and only if $1/\lambda < 0$ (note $1/\lambda$ cannot *equal* 0) or $1/\lambda > 2$. That is, γ_λ is regular if and only if $\lambda < \frac{1}{2}$.

$\square$

Theorem 53 *Let γ have evolute E_γ. Then every involute I_E of E_γ is a parallel curve to γ.*

Proof: We simply construct the involute of E_γ starting at $t_0 \in I$. To do this we will need the arclength function $\sigma^E_{t_0}(t)$ and the unit tangent $u^E(t)$ of E_γ. We shall assume that $\kappa'(t) < 0$ (a similar argument works for the case $\kappa'(t) > 0$). Recall that in this case,

$$\sigma_{t_0}(t) = \frac{1}{\kappa(t)} - \frac{1}{\kappa(t_0)} \quad \text{and} \quad u^E(t) = n(t)$$

by Theorem 45, parts (c) and (a). Hence,

$$
\begin{aligned}
I_E(t) &= E_\gamma(t) - \sigma^E(t) u^E(t) \\
&= \gamma(t) + \frac{1}{\kappa(t)} n(t) - \left(\frac{1}{\kappa(t)} - \frac{1}{\kappa(t_0)} \right) n(t) \\
&= \gamma(t) + \frac{1}{\kappa(t_0)} n(t).
\end{aligned}
$$

But this is just γ_λ where $\lambda = 1/\kappa(t_0)$. $\square$

Summary

- The curve obtained from γ by tracing out the locus of its centres of curvature is called the **evolute** of γ. Explicitly

$$E_\gamma(t) = \gamma(t) + \frac{1}{\kappa(t)} n(t).$$

- The **involute** of γ based at $t_0 \in I$ is

$$I_\gamma(t) = \gamma(t) - \sigma_{t_0}(t) u(t).$$

- A **parallel** to γ is a curve

$$\gamma_\lambda(t) = \gamma(t) + \lambda n(t)$$

  where $\lambda \in \mathbb{R}$ is a constant.

- The evolute of an involute of γ is γ. Every involute of the evolute of γ is a parallel to γ.

- The regularity properties of evolutes and parallels can be analyzed in terms of the curvature properties of γ.

5 Curves in $\mathbb{R}^3$ and the Frenet frame

5.1 The Frenet frame

Planar curves ($\gamma : I \to \mathbb{R}^2$) are special because given one vector in $\mathbb{R}^2$, the unit tangent vector $u = \gamma'/|\gamma'|$) say, one can uniquely determine an orthogonal one by rotating 90° anticlockwise, the unit normal n in this case.
Curves in $\mathbb{R}^3$ ($\gamma : I \to \mathbb{R}^3$) are also special: given an *ordered pair* of vectors in $\mathbb{R}^3$, e.g. $u = \gamma'/|\gamma'|$, $k = \gamma''_\perp/|\gamma'|^2$, one can uniquely determine a third orthogonal to both by using the **vector product** ($u \times k$).

Reminder 54 Given vectors $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{R}^3$, their **vector product** is

$$u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

It has the following properties:

(a) $v \times u = -u \times v$

(b) For all $\lambda, \mu \in \mathbb{R}$, $(\lambda u + \mu v) \times w = \lambda(u \times w) + \mu(v \times w)$

(c) If u is parallel to v (i.e. $u = \lambda v$) then $u \times v = 0$ [follows from (a), (b)]

(d) $|u \times v|^2 = |u|^2|v|^2 - (u \cdot v)^2$.

(e) $u \cdot (v \times w) = w \cdot (u \times v) = v \cdot (w \times u)$

(f) $u \times v$ is orthogonal to both u and v [follows from (e)] □

Definition 55 Let $\gamma : I \to \mathbb{R}^3$ be a RPC whose *curvature never vanishes* (i.e. for all $t \in I$, $|k(t)| \neq 0$). Then in addition to the **unit tangent vector**

$$u(t) := \frac{\gamma'(t)}{|\gamma'(t)|}$$

one defines the **principal unit normal vector**

$$n(t) := \frac{k(t)}{|k(t)|}$$

and the **binormal vector**

$$b(t) := u(t) \times n(t).$$

The ordered triplet $[u(t), n(t), b(t)]$ is called the **Frenet frame** of the curve. γ □

Note: we call $n(t)$ the **principal** unit normal to distinguish it from the infinitely many other unit vectors lying in the plane orthogonal to $u(t)$. This is only possible if $k(t) \neq 0$.

Lemma 56 *The Frenet frame is orthonormal (the vectors u, n, b are mutually orthogonal and each has unit length).*

Proof: $|u(t)| = 1$ and $|n(t)| = 1$ for all t by definition. Also $n(t)$ is parallel to $k(t)$ which is orthogonal to $u(t)$ by Fact 25. It remains to show that (i) $|b(t)| = 1$ and (ii) $b(t)$ is orthogonal to $u(t)$ and $n(t)$. But (i) follows from Reminder 54(d),

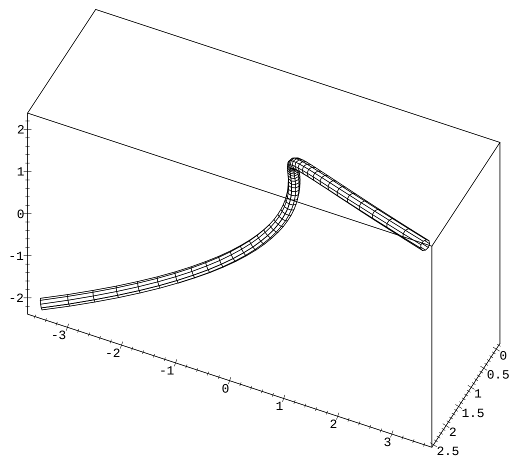
$$|b|^2 = |u \times n|^2 = |u|^2|n|^2 - (u \cdot n)^2 = 1 \times 1 - 0^2 = 1,$$

and (ii) follows directly from Reminder 54(f). $\square$

So given any regularly parametrized curve of nonvanishing curvature (RPCNVC) $\gamma : I \to \mathbb{R}^3$, the Frenet frame $[u, n, b]$ forms an orthonormal basis for the vector space $\mathbb{R}^3$.

Example 57 Construct the Frenet frame at $t = 0$ for the curve $\gamma : \mathbb{R} \to \mathbb{R}^3$, $\gamma(t) = (\frac{t^2}{2}, \frac{t^3}{3}, t)$. Note: this is only possible provided $k(0) \neq 0$!

$$\begin{aligned}
\gamma'(t) &= (t, t^2, 1) \\
\gamma''(t) &= (1, 2t, 0) \\
\gamma'(0) &= (0, 0, 1) \\
\gamma''(0) &= (1, 0, 0) \\
\gamma''_\perp(0) &= (1, 0, 0) \\
k(0) &= \frac{\gamma''_\perp(0)}{|\gamma'(0)|^2} = (1, 0, 0)
\end{aligned}$$



which is nonzero, so the Frenet frame is well-defined. So

$$\begin{aligned}
u(0) &= \frac{\gamma'(0)}{|\gamma'(0)|} = (0, 0, 1) \\
n(0) &= \frac{k(0)}{|k(0)|} = (1, 0, 0) \\
b(0) &= u(0) \times n(0) \\
&= (0, 1, 0)
\end{aligned}$$

$\square$

5.2 Torsion of a unit speed curve in $\mathbb{R}^3$

Our aim is to describe the geometry of curves in $\mathbb{R}^3$ by analysing the time dependence of the Frenet frame. This process simplifies greatly if the curve under consideration is a unit speed curve, so henceforth, we will consider only USCs. Note that this entails no loss of generality by Theorem 17: every RPC has a unit speed reparametrization, unique up to time translation.

The construction of the Frenet frame for a USC $\gamma : I \to \mathbb{R}^3$ simplifies somewhat because γ' is already a unit vector, and the curvature $k = \gamma''$. Hence,

$$u(s) = \gamma'(s), \qquad n(s) = \frac{u'(s)}{|u'(s)|}, \qquad b(s) = u(s) \times n(s).$$

It is conventional in the context of curves in $\mathbb{R}^3$ to denote the **scalar** curvature by $\kappa(s)$ rather than $|k(s)|$. This should not be confused with the **signed** curvature of a curve in $\mathbb{R}^2$. Here $\kappa(s)$ just means the length of the curvature vector $k(s)$, which of course is never negative. Noting that $k(s) = u'(s)$ for a USC, we may re-write the definition of the principal unit normal as

$$\boxed{u'(s) = \kappa(s)n(s)} \tag{1}$$

Since the Frenet frame $[u(s), n(s), b(s)]$ spans $\mathbb{R}^3$ we should be able to find similar formulae for $n'(s)$ and $b'(s)$. The coefficient functions we extract should tell us about the geometry of γ, just as κ does. We start with $b'(s)$. Since it's a 3-vector, there must exist smooth functions $\lambda, \mu, \nu : I \to \mathbb{R}$ such that

$$b'(s) = \lambda(s)u(s) + \mu(s)n(s) + \nu(s)b(s)$$

Taking the scalar product of both sides with $b(s)$ gives:

$$b \cdot b' = \lambda \underbrace{b \cdot u}_{0} + \mu \underbrace{b \cdot n}_{0} + \nu \underbrace{b \cdot b}_{1} = \nu \quad \text{(orthonormality)}$$

But

$$b \cdot b = 1 \quad \Rightarrow \quad b' \cdot b + b \cdot b' = 0 \quad \Rightarrow \quad b' \cdot b = 0$$

and hence $\nu = 0$. Similarly,

$$u \cdot b' = \lambda \underbrace{u \cdot u}_{1} + \mu \underbrace{u \cdot n}_{0} + \nu \underbrace{u \cdot b}_{0} = \lambda \quad \text{(orthonormality)}$$

But

$$\begin{aligned} u \cdot b = 0 \quad \Rightarrow \quad u' \cdot b + u \cdot b' = 0 \quad \Rightarrow \quad u \cdot b' &= -u' \cdot b \\ &= -\kappa\, n \cdot b \qquad \text{(by eqn. (1))} \\ &= 0. \end{aligned}$$

Hence $\lambda = 0$ also. It follows that

$$b'(s) = \mu(s)n(s)$$

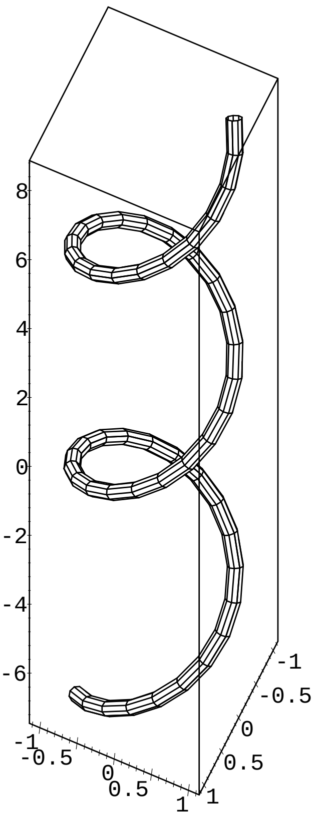
for some function μ. We call $-\mu(s)$ the *torsion* of the curve.

Definition 58 Let $\gamma : I \to \mathbb{R}^3$ be a USCNVC. The **torsion** of the curve is that function $\tau : I \to \mathbb{R}$ defined by the equation

$$\boxed{b'(s) = -\tau(s)n(s)} \tag{2}$$

Alternatively, $\tau(s) = -b'(s) \cdot n(s)$. $\square$

Example 59 Construct the Frenet frame and torsion for the helix $\gamma : \mathbb{R} \to \mathbb{R}^3$, $\gamma(s) = (\cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}})$.



First check it's a USCNVC:

$$\begin{aligned}
\gamma'(s) &= \left(-\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\
|\gamma'(s)| &= \frac{1}{\sqrt{2}}\left(\cos^2\frac{s}{\sqrt{2}} + \sin^2\frac{s}{\sqrt{2}}, 0\right)^{\frac{1}{2}} = 1 \\
k(s) &= \gamma''(s) = \left(-\frac{1}{2}\cos\frac{s}{\sqrt{2}}, -\frac{1}{2}\sin\frac{s}{\sqrt{2}}, 0\right)
\end{aligned}$$

OK. Now compute $[u, n, b]$:

$$\begin{aligned}
u(s) &= \gamma'(s) = \frac{1}{\sqrt{2}}\left(-\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, 1\right) \\
n(s) &= \frac{u'(s)}{|u'(s)|} = \left(-\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0\right) \\
b(s) &= u(s) \times n(s) = \frac{1}{\sqrt{2}}\left(\sin\frac{s}{\sqrt{2}}, -\cos\frac{s}{\sqrt{2}}, 1\right)
\end{aligned}$$

To find $\tau(s)$, we compute $b'(s)$ and compare with $n(s)$:

$$\begin{aligned}
b'(s) &= \frac{1}{2}\left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, 0\right) \\
\tau(s) &= \frac{1}{2}
\end{aligned}$$

$\square$

The rest of the material in section 5.2 is non-examinable, and included purely for your delight and edification.

What is the geometric meaning of torsion? Recall that the curvature $\kappa = |u'|$ measures the rate of change of the direction of the tangent line to the curve. Torsion has a similar interpretation, but in terms of *planes* rather than lines.

At the point $\gamma(s_0)$ we define the **osculating plane** of the curve to be that plane through $\gamma(s_0)$ spanned by the orthonormal pair $[u(s_0), n(s_0)]$, or equivalently, by the velocity $\gamma'(s_0)$ and acceleration $\gamma''(s_0)$ of the curve. The orientation of this plane is uniquely determined by any vector normal to it, for example the binormal vector $b(s_0) = u(s_0) \times n(s_0)$. Consider the Taylor expansion of the curve $\gamma(s)$ based at the time s_0:

$$\gamma(s) = \underbrace{\gamma(s_0) + \gamma'(s_0)(s - s_0) + \frac{1}{2}\gamma''(s_0)(s - s_0)^2}_{\text{stays in osculating plane}} + \frac{1}{6}\gamma'''(s_0)(s - s_0)^3 + \cdots$$

The failure of $\gamma(s)$ to stay in the osculating plane is controlled (locally) by $\gamma'''(s_0)$. The osculating plane divides $\mathbb{R}^3$ in half: we can classify any vector v **not** in the plane as **positive** if $v \cdot b(s_0) > 0$ and **negative** if $v \cdot b(s_0) < 0$. Thinking of the osculating plane as horizontal, with $b(s_0)$ pointing "up", positive vectors point on the upside of the plane, negative vectors on the downside. So if $\gamma'''(s_0) \cdot b(s_0) > 0$, the curve punctures the osculating plane upwards as it passes through $\gamma(s_0)$, while if $\gamma'''(s_0) \cdot b(s_0) < 0$, it punctures the osculating plane downwards. What does this have to do with torsion?

Proposition 60 *Let $\gamma : I \to \mathbb{R}^3$ be a USCNVC. Then $\tau(s) = (\gamma'''(s) \cdot b(s))/\kappa(s)$.*

Proof: Recall $b = u \times n = \gamma' \times (\gamma''/|\gamma''|)$. Hence

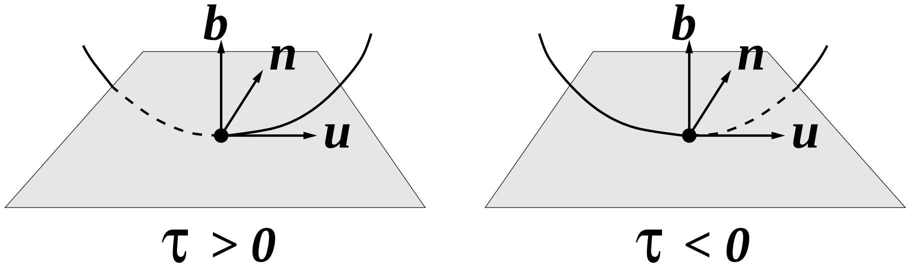
$$\begin{aligned} b' &= \gamma'' \times \left(\frac{\gamma''}{|\gamma''|}\right) + \gamma' \times \left[\frac{\gamma'''}{|\gamma''|} - \frac{\gamma''' \cdot \gamma''}{|\gamma''|^2}\gamma''\right] \\ &= 0 + u \times \left[\frac{\gamma'''}{\kappa} - (\gamma''' \cdot n)n\right]. \end{aligned}$$

But $b' = -\tau n$, so

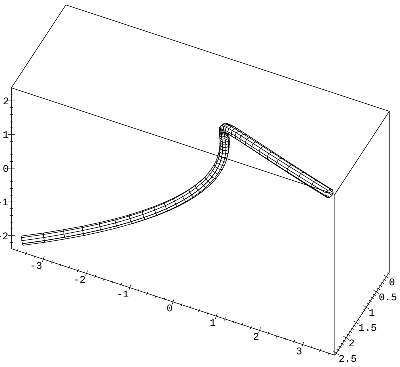
$$\tau = -n \cdot b' = -n \cdot \left[u \times \frac{\gamma'''}{\kappa}\right] = \frac{\gamma'''}{\kappa} \cdot (u \times n) = \frac{\gamma'''}{\kappa} \cdot b.$$

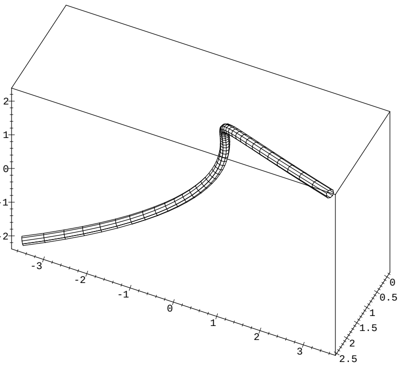
$\square$

So if $\tau(s_0) > 0$, the curve punctures its osculating plane at $\gamma(s_0)$ upwards, and if $\tau(s_0) < 0$ it punctures downwards.

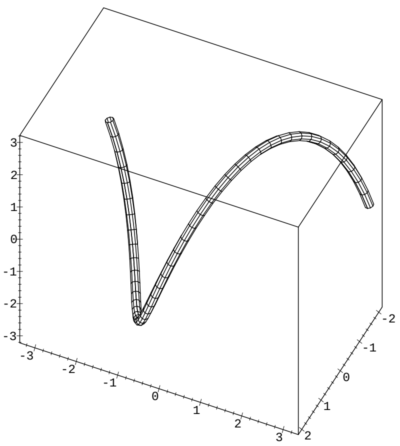


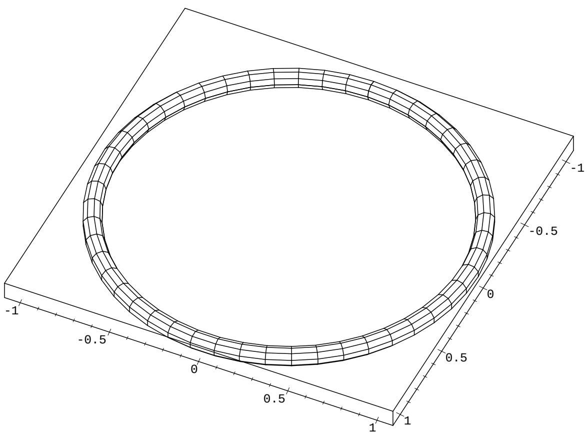
To test your understanding, see if you can tell whether the torsion of the following curves is positive, negative or zero. (Just ask me if you want to know the answers.)





Note: the sign of τ does *not* depend on the *orientation* of the curve, that is, the direction in which it is traversed.





5.3 The Frenet formulae

Recall that the curvature κ and torsion τ of a USCNVC in $\mathbb{R}^3$ can be extracted by decomposing $u'(s)$ and $b'(s)$ relative to the Frenet frame $[u(s), n(s), b(s)]$:

$$\boxed{u'(s) = \kappa(s)n(s)} \text{ — (1)} \qquad\qquad \boxed{b'(s) = -\tau(s)n(s)} \text{ — (2)}$$

What about $n'(s)$? Does this give us yet another scalar quantity analogous to κ and τ? In fact, it does not, as we shall now show.

As before, since $n' : I \to \mathbb{R}^3$ is a smooth, vector valued function, there must exist smooth scalar functions $\lambda, \mu, \nu : I \to \mathbb{R}$ such that

$$n'(s) = \lambda(s)u(s) + \mu(s)n(s) + \nu(s)b(s),$$

and, since $[u, n, b]$ is orthonormal (Lemma 56), we may extract the coefficients by taking scalar products:

$$\lambda(s) = u(s) \cdot n'(s), \qquad \mu(s) = n(s) \cdot n'(s), \qquad \nu(s) = b(s) \cdot n'(s).$$

- Now $u(s) \cdot n(s) = 0$ for all s, which upon differentiating with respect to s yields

  $$0 = u'(s) \cdot n(s) + u(s) \cdot n'(s) = \kappa(s)n(s) \cdot n(s) + u(s) \cdot n'(s) = \kappa(s) + \lambda(s),$$

  using equation (1). Hence $\lambda(s) = -\kappa(s)$.

- Similarly, $n(s) \cdot n(s) = 1$ for all s, and differentiating yields

  $$0 = n'(s) \cdot n(s) + n(s) \cdot n'(s) = 2\mu(s).$$

  Hence $\mu(s) = 0$.

- Finally, $b(s) \cdot n(s) = 0$ implies

  $$0 = b'(s) \cdot n(s) + b(s) \cdot n'(s) = -\tau(s)n(s) \cdot n(s) + b(s) \cdot n'(s) = -\tau(s) + \nu(s),$$

  using equation (2). Hence $\nu(s) = \tau(s)$.

Assembling the pieces, we have the formula

$$\boxed{n'(s) = -\kappa(s)u(s) + \tau(s)b(s)} \text{ — (3)}$$

Taking (1),(2),(3) together, we have just proved:

Theorem 61 (The Frenet formulae) *Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve of nonvanishing curvature. Then its Frenet frame satisfies the formulae*

$$\begin{array}{rcccc} u'(s) & = & & \kappa(s)n(s) & \\ n'(s) & = & -\kappa(s)u(s) & & +\tau(s)b(s) \\ b'(s) & = & & -\tau(s)n(s) & \end{array}$$

□

We will use the Frenet formulae to prove two fundamental results:

- That a USCNVC in $\mathbb{R}^3$ is *planar* if and only if its torsion is zero.

- That a USCNVC in $\mathbb{R}^3$ is uniquely determined by its scalar curvature and torsion (almost).

Definition 62 A **plane** $P \subset \mathbb{R}^3$ is the set of points $x \in \mathbb{R}^3$ satisfying the equation

$$B \cdot x = \nu,$$

where $B \in \mathbb{R}^3$ and $\nu \in \mathbb{R}$ are constants, and B is a unit vector ($|B| = 1$). Geometrically, B determines the orientation of the plane P, while $|\nu|$ is its distance from the origin, $(0, 0, 0)$. [Note that the pairs (B, ν) and $(-B, -\nu)$ describe the same plane.]

A curve $\gamma : I \to \mathbb{R}^3$ is **planar** if its image is contained in some plane P, that is, if there exist constants $B \in \mathbb{R}^3$, $|B| = 1$ and $\nu \in \mathbb{R}$, such that

$$B \cdot \gamma(s) = \nu$$

for all $s \in I$

Example 63 The USCNVC $\gamma(s) = \frac{1}{2}(1+2\sin s, \sqrt{3}+\cos s, 1-\sqrt{3}\cos s)$ is a **planar curve**. $\square$

How on earth do I know this? One way to settle the issue is to use:

Theorem 64 *Let $\gamma : I \to \mathbb{R}^3$ be a USCNVC. Then γ is planar if and only if $\tau(s) = 0$ for all $s \in I$.*

Proof: We must prove:

$$\begin{array}{llcl} \text{(A)} & \gamma \text{ planar} & \Rightarrow & \tau = 0 \\ \text{(B)} & \tau = 0 & \Rightarrow & \gamma \text{ planar} \end{array}$$

(A) Assume γ is planar. Then there exist constants $B \in \mathbb{R}^3$ and $\nu \in \mathbb{R}$, as in definition 62 such that, for all $s \in I$,

$$\begin{array}{rrcl} & \gamma(s) \cdot B & = & \nu \\ \text{Diff. w.r.t. } s\text{:} & u(s) \cdot B & = & 0 \\ \text{Diff. w.r.t. } s \text{ again:} & k(s) \cdot B & = & 0 \\ \Rightarrow & \kappa(s) n(s) \cdot B & = & 0 \\ \Rightarrow & n(s) \cdot B & = & 0 \end{array}$$

since γ has nonvanishing curvature. So B is a unit vector, and is orthogonal to both $u(s)$ and $n(s)$ for all s. Compare this with $b(s) = u(s) \times n(s)$. This has precisely the same properties. Furthermore, in $\mathbb{R}^3$, the unit vector orthogonal to both $u(s)$ and $n(s)$ is *unique* up to sign. Hence

$$b(s) = \pm B,$$

a *constant* vector. Hence $b'(s) = 0$ and so $\tau(s) = 0$ for all $s \in I$ by the last Frenet formula.

(B) Assume $\tau \equiv 0$. Then $b' \equiv 0$ and hence $b(s) = b_0$, a constant unit vector (by the Mean Value Theorem). But then

$$\frac{d}{ds}(\gamma(s) \cdot b_0) = u(s) \cdot b_0 = u(s) \cdot b(s) = 0.$$

Hence $\gamma(s) \cdot b_0 = a$, some constant (by the Mean Value Theorem again). So γ lies in the plane determined by the unit vector $B = b_0$ and the constant $\nu = a$. $\square$
[Note: we only used the first and last Frenet formulae in this proof.]

Example 63 (revisited) Let's use theorem 64 to show that

$$\gamma(s) = \frac{1}{2}(1 + 2\sin s, \sqrt{3} + \cos s, 1 - \sqrt{3}\cos s)$$

is planar. First, check it's a USCNVC:

$$\begin{aligned}
\gamma'(s) &= \frac{1}{2}(2\cos s, -\sin s, \sqrt{3}\sin s) \\
|\gamma'(s)| &= \frac{1}{2}(4\cos^2 s + \sin^2 s + 3\sin^2 s)^{\frac{1}{2}} = 1 \\
k(s) = \gamma''(s) &= \frac{1}{2}(-2\sin s, -\cos s, \sqrt{3}\cos s) \\
\kappa(s) = |k(s)| &= \frac{1}{2}(4\sin^2 s + \cos^2 s + 3\cos^2 s)^{\frac{1}{2}} = 1
\end{aligned}$$

Now construct its Frenet frame, compute $b'(s)$ and extract τ:

$$\begin{aligned}
u(s) = \gamma'(s) &= \frac{1}{2}(2\cos s, -\sin s, \sqrt{3}\sin s) \\
n(s) = \frac{k(s)}{|k(s)|} &= \frac{1}{2}(-2\sin s, -\cos s, \sqrt{3}\cos s) \\
b(s) = u(s) \times n(s) &= \frac{1}{4}(0, -2\sqrt{3}, -2) \\
&= \frac{1}{2}(0, -\sqrt{3}, -1) \\
b'(s) &= (0, 0, 0) \\
\tau(s) &= 0
\end{aligned}$$

and hence γ is planar.

So what kind of curve is this?

- It's planar.
- It has constant curvature.

Is it, perhaps, a **circle**? The answer is **yes**, but we will need to develop another piece of theory to demonstrate this.

Recall that a USC in $\mathbb{R}^2$ is uniquely determined by its signed curvature κ and its initial position γ_0 and initial velocity $(\cos\theta_0, \sin\theta_0)$ (Theorem 33). It turns out that a similar result holds for a USCNVC in $\mathbb{R}^3$, except now we have to specify the scalar curvature κ, the torsion τ, the initial position γ_0, the initial velocity u_0 **and** the initial normal vector n_0.

Theorem 65 *Given smooth functions $\kappa : I \to (0,\infty)$, $\tau : I \to \mathbb{R}$ ($0 \in I$) and constants $\gamma_0, u_0, n_0 \in \mathbb{R}^3$, $|u_0| = |n_0| = 1$, $u_0 \cdot n_0 = 0$, there exists a unique unit speed curve $\gamma : I \to \mathbb{R}^3$ with*
(a) $\gamma(0) = \gamma_0$ (b) $\gamma'(0) = u_0$ (c) $n(0) = n_0$ (d) curvature κ (e) torsion τ.

Partial proof: Existence – too hard. Uniqueness: Let γ, $\widetilde{\gamma}$ be any pair of curves satisfying (a)–(e). We will show that $\gamma(s) = \widetilde{\gamma}(s)$ for all $s \in I$. Since both curves have identical torsion and (nonvanishing) curvature, their Frenet frames, call them $[u, n, b]$ and $[\widetilde{u}, \widetilde{n}, \widetilde{b}]$ respectively, both satisfy the Frenet formulae for κ and τ. It follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\left\{|u-\widetilde{u}|^2 + |n-\widetilde{n}|^2 + |b-\widetilde{b}|^2\right\} \\
&\quad = (u-\widetilde{u})\cdot(u'-\widetilde{u}') + (n-\widetilde{n})\cdot(n'-\widetilde{n}') + (b-\widetilde{b})\cdot(b'-\widetilde{b}') \\
&\quad = -[u\cdot\widetilde{u}' + u'\cdot\widetilde{u} + n\cdot\widetilde{n}' + n'\cdot\widetilde{n} + b\cdot\widetilde{b}' + b'\cdot\widetilde{b}] \\
&\quad = -[u\cdot\kappa\widetilde{n} + \kappa n\cdot\widetilde{u} + n\cdot(-\kappa\widetilde{u} + \tau\widetilde{b}) + (-\kappa u + \tau b)\cdot\widetilde{n} + b\cdot(-\tau\widetilde{n}) - \tau n\cdot\widetilde{b}] \\
&\quad = 0
\end{aligned}$$

by the Frenet formulae. Hence

$$|u-\widetilde{u}|^2 + |n-\widetilde{n}|^2 + |b-\widetilde{b}|^2 = C \tag{1}$$

some constant. Substituting $s = 0$ in (1) and using properties (b) and (c), one sees that $C = 0$. Since all terms on the left hand side of (1) are non-negative, it follows that each is identically zero. Hence $|u(s) - \widetilde{u}(s)| = 0$ for all s. It follows that for all s,

$$\begin{aligned}
\frac{d}{ds}(\gamma(s) - \widetilde{\gamma}(s)) &= 0 \\
\Rightarrow \gamma(s) - \widetilde{\gamma}(s) &= v
\end{aligned}$$

a constant vector, which by property (a) must be 0. Hence $\gamma \equiv \widetilde{\gamma}$. $\square$

A less formal way of stating this result is that, up to rigid motions (translations and rotations of $\mathbb{R}^3$), a USCNVC is uniquely determined by $\kappa(s)$ and $\tau(s)$.

So, since a circle has constant scalar curvature (Example 34) and zero torsion (it's planar), Example 63 **must** be a circle, by Theorem 65.

Summary

- Given a RPC $\gamma : I \to \mathbb{R}^3$ of nonvanishing curvature we define its **unit tangent vector** $u(t)$, **principal unit normal** $n(t)$ and **binormal** $b(t)$ by

$$u(t) = \frac{\gamma'(t)}{|\gamma'(t)|}, \qquad n(t) = \frac{k(t)}{|k(t)|}, \qquad b(t) = u(t) \times n(t).$$

  This triple of vectors forms an orthonormal basis for $\mathbb{R}^3$ called the **Frenet frame**.

- The curvature $|k(t)|$ is usually denoted $\kappa(t)$. Note $\kappa \geq 0$.

- For a **unit speed** curve $\gamma : I \to \mathbb{R}^3$ of nonvanishing curvature we define the **torsion** $\tau : I \to \mathbb{R}$ by

$$b'(s) = -\tau(s)n(s).$$

- A USCNVC is **planar** if and only if $\tau \equiv 0$.

- The rate of change of the Frenet frame as one travels along a USCNVC is determined by the torsion and curvature according to the **Frenet formulae**

$$\begin{array}{rclccc} u'(s) & = & & \kappa(s)n(s) & \\ n'(s) & = & -\kappa(s)u(s) & & +\tau(s)b(s) \\ b'(s) & = & & -\tau(s)n(s) & \end{array}$$

- A USCNVC is uniquely determined (up to rigid motions) by its curvature and torsion.

6 Regularly parametrized surfaces

6.1 The basic definition

Our approach to studying the geometry of **curves** in $\mathbb{R}^n$ was to think of them as smooth maps $\gamma : I \to \mathbb{R}^n$. Recall that not every such map gives a nice, smooth, regular curve. We needed to impose a constraint on the derivative of γ, namely $\gamma'(t)$ should never vanish, in order to guarantee the curve was well behaved.

We now want to develop a similar method for studying **surfaces**, roughly speaking, nice smooth **two-dimensional** sets in $\mathbb{R}^3$. Actually, the techniques we will study generalize quite easily to work for $(n-1)$-dimensional surfaces in $\mathbb{R}^n$ for any n, but we will stick to two-dimensional surfaces in $\mathbb{R}^3$ because these are easiest to visualize. Again, we will think of surfaces as smooth maps $\phi : U \to \mathbb{R}^3$, where U is (a subset of) $\mathbb{R}^2$, so a surface will be a $\mathbb{R}^3$-valued function of **two** variables, $\phi(u, v)$. Once again, not every such map gives a nice smooth surface, and we need to impose constraints on the behaviour of $\phi(u, v)$ to guarantee its image set is well behaved. Just as for curves, the constraints amount to the requirement that the map $\phi : U \to \mathbb{R}^3$ be **regular**, but to make this notion precise requires a little more work.

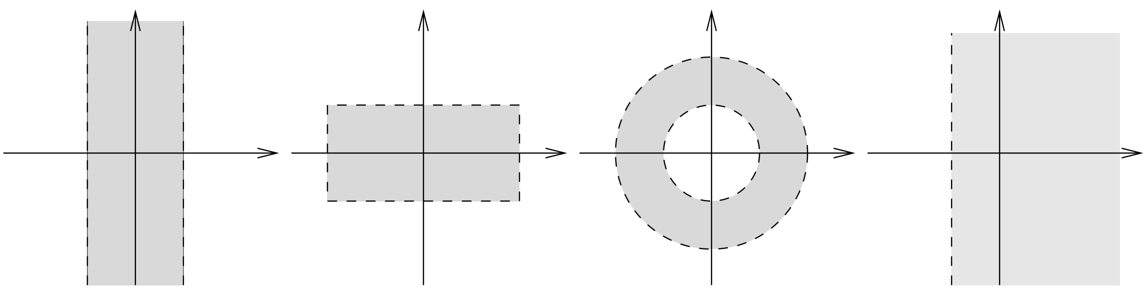
First, we identify the class of **domains** U we will be using.

Definition 66 For each $x \in \mathbb{R}^2$ and $r > 0$ let $B_r(x) = \{y \in \mathbb{R}^2 : |y - x| < r\}$ be the **disk of radius r centred at x.** Then a subset $U \subseteq \mathbb{R}^2$ is **open** if for all $x \in U$ there exists $\delta > 0$ such that $B_\delta(x) \subseteq U$. $\square$

The idea is that no point in U lies at the edge of U, because every point can be surrounded by some disk in U. This is a generalization of the idea of an open interval in $\mathbb{R}$ (one which has no endpoints).

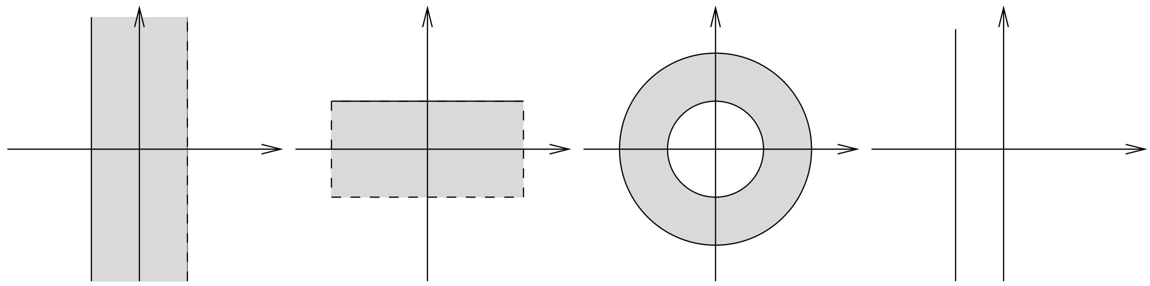
Example 67 The sets

$$(-1,1)\times\mathbb{R}, \quad (-\pi,\pi)\times(-1,1), \quad \{x \in \mathbb{R}^2 : 1 < |x| < 2\}, \quad \{x \in \mathbb{R}^2 : x_1 > -1\}$$



are all open sets, while

$$[-1,1)\times\mathbb{R}, \quad (-\pi,\pi)\times(-1,1], \quad \{x \in \mathbb{R}^2 : 1 \le |x| \le 2\}, \quad \{x \in \mathbb{R}^2 : x_1 = -1\}$$



are not. $\square$

Definition 68 Given a smooth map $\phi : U \to \mathbb{R}^3$, where U is an open subset of $\mathbb{R}^2$, the **coordinate basis vectors** $\phi_u, \phi_v : U \to \mathbb{R}^3$ are

$$\phi_u(u,v) = \frac{\partial \phi}{\partial u}(u,v), \qquad \phi_v(u,v) = \frac{\partial \phi}{\partial v}(u,v).$$

A point $(u,v) \in U$ is a **regular point** of ϕ if the vectors $\phi_u(u,v)$ and $\phi_v(u,v)$ are linearly independent. The map ϕ is **regular** if every $(u,v) \in U$ is a regular point. $\square$

Since $\phi_u(u,v)$, $\phi_v(u,v)$ are just **two** vectors, they are linearly independent if and only if they are not **parallel**. Two vectors in $\mathbb{R}^3$ are parallel if and only if their vector product vanishes, so another (more convenient) way to state Definition 68 is

Definition 68 (*) $\phi : U \to \mathbb{R}^3$ is **regular** if for all $(u,v) \in U$,

$$\phi_u(u,v) \times \phi_v(u,v) \neq (0,0,0).$$

Definition 69 A **regularly parametrized surface** (RPS) is an injective (i.e. one-to-one), regular map $\phi : U \to \mathbb{R}^3$. $\square$

We will often denote the **image set** of the map by M:

$$M := \phi(U) = \{\phi(u,v) \in \mathbb{R}^3 \ : \ (u,v) \in U\}.$$

Example 70 (The graph of a function) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be *any* smooth function. Then

$$\phi : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \phi(u,v) = (u, v, f(u,v))$$

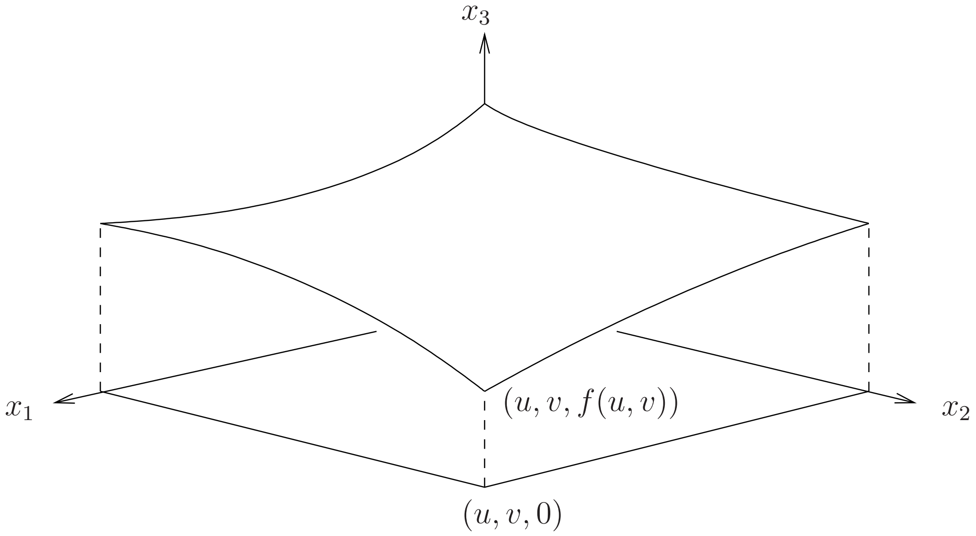
is a RPS. Let's check it:

$$\begin{aligned} \phi(u,v) = \phi(u',v') \quad &\Longrightarrow \quad (u,v,f(u,v)) = (u',v',f(u',v')) \\ &\Longrightarrow \quad u = u' \text{ (from first component) and} \\ &\qquad\;\; v = v' \text{ (from second component).} \end{aligned}$$

So ϕ is injective. Furthermore

$$\begin{aligned}
\phi_u &= \left(1, 0, \frac{\partial f}{\partial u}\right) \\
\phi_v &= \left(0, 1, \frac{\partial f}{\partial v}\right) \\
\Rightarrow \phi_u \times \phi_v &= \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right).
\end{aligned}$$

The third component never vanishes, so $\phi_u \times \phi_v$ never vanishes and ϕ is regular. Hence ϕ is a RPS. It is known as the *graph* of the function f.



Example 71 (A sphere (almost)) Let $U = (0, \pi) \times (0, 2\pi)$. Note this is an open subset of $\mathbb{R}^2$ (it is a rectangle without boundary). Consider the mapping

$$\phi : U \to \mathbb{R}^3, \qquad \phi(u, v) = (\sin u \cos v, \sin u \sin v, \cos u)$$

I claim this also is a RPS. Let's check:

(A) Is ϕ injective?
Suppose $\phi(u, v) = \phi(u', v')$. Then

$$(1)\ \sin u \cos v = \sin u' \cos v' \quad (2)\ \sin u \sin v = \sin u' \sin v' \quad (3)\ \cos u = \cos u'$$

(3) implies that $u = u'$, because $0 < u, u' < \pi$ and cos is strictly decreasing on this interval. Then $\sin u = \sin u' \neq 0$, so we can divide through in (1) and (2) to obtain

$$(1')\ \cos v = \cos v' \quad (2')\ \sin v = \sin v'$$

These imply that $v' = v + 2n\pi$ for some $n \in \mathbb{Z}$. However, $0 < v, v' < 2\pi$ so in fact $v = v'$.

(B) Is ϕ regular?

$$\begin{aligned}
\phi_u(u,v) &= (\cos u \cos v, \cos u \sin v, -\sin u) \\
\phi_v(u,v) &= (-\sin u \sin v, \sin u \cos v, 0) \\
\phi_u \times \phi_v &= (\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u) \\
&= \sin u(\sin u \cos v, \sin u \sin v, \cos u) \\
|\phi_u \times \phi_v|^2 &= \sin^2 u(\sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u) \\
&= \sin^2 u(\sin^2 v + \cos^2 v) \\
&= \sin^2 u
\end{aligned}$$

Since $\sin^2 u \neq 0$ for $0 < u < \pi$, $\phi_u \times \phi_v$ never vanishes.

What does the image $M = \phi(U)$ of this RPS look like? Notice that

$$|\phi(u,v)|^2 = \sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u = \sin^2 u + \cos^2 u = 1.$$

So the image of ϕ is a subset of the sphere of radius 1. Is the image the whole of the sphere?

Consider fixing $u = u_0 \in (0,\pi)$ and varying v. Doing so traces out a curve in $\mathbb{R}^3$. The third component of $\phi(u,v)$ doesn't depend on v, so this curve is contained in the horizontal plane $x_3 = \cos u_0$. The first two components are $(\sin u_0 \cos v, \sin u_0 \sin v)$, and these trace out a circle of radius $\sin u_0$ as v varies in the interval $(0, 2\pi)$. However, v is not allowed to equal 0 or 2π, so the point

$$(\sin u_0 \cos 0, \sin u_0 \sin 0, \cos u_0) = (\sin u_0, 0, \cos u_0)$$

is missing from the circle.

Now consider fixing $v = v_0 \in (0, 2\pi)$ and varying u. This traces out a vertical semicircular arc on the surface of the sphere as u varies in the interval $(0,\pi)$. However the points with $u = 0$ and $u = \pi$ are missing, and these are

$$(0,0,1) \text{ and } (0,0,-1).$$

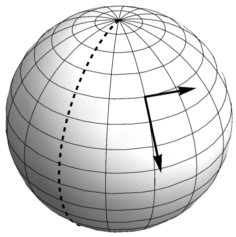
Putting things together, the image of ϕ is the whole of the two-sphere except for the semi-circular arc described by

$$\{(\sin u, 0, \cos u) \ : \ 0 \leq u \leq \pi\}.$$

Consider again the curve $\gamma(v) = \phi(u_0, v)$ for fixed u_0. The tangent vector to this curve is just

$$\gamma'(v) = \frac{\partial \phi}{\partial v}(u_0, v) = \phi_v(u_0, v).$$

So ϕ_v can be interpreted as the velocity of a curve obtained by varying v with u held constant. This observation should help you to visualise ϕ_v. Similarly, ϕ_u is the velocity of a curve obtained by varying u with v constant.



This picture of the sphere illustrates the preceding discussion. The curves drawn on the surface are the curves of constant u and of constant v. The dashed line is the semi-circular arc that is missing from our parametrization. The two arrows are the vectors ϕ_u and ϕ_v at a particular point $\phi(u,v)$.

Remark 72 A similar idea applies to any RPS $M : U \to \mathbb{R}^3$. If we fix $v = b$, some constant, we get a curve $\alpha_b(u) = \phi(u, b)$ along which u varies. Changing the fixed value b generates a family of curves which sweep out the surface $M = \phi(U)$. Similarly if we fix $u = a$, some constant, we get a curve $\beta_a(v) = \phi(a, v)$ along which v varies. Changing the fixed value a generates a family of curves which sweep out the surface M. Furthermore

$$\alpha_b'(u) = \frac{\partial}{\partial u}\phi(u,b) = \phi_u(u,b), \qquad \beta_a'(v) = \frac{\partial}{\partial v}\phi(a,v) = \phi_v(a,v)$$

which gives a geometric interpretation of the coordinate basis vectors.

Remark 73 Since a RPS $\phi : U \to \mathbb{R}^3$ is one-to-one, given a point $p = (p_1, p_2, p_3)$ on $M = \phi(U)$, there is one and only one point $(u, v) \in U$ such that $\phi(u, v) = (p_1, p_2, p_3)$. We call the numbers (u, v) the **local coordinates** of the point p. In this way, we can specify a point on M by giving two numbers (its local coordinates) rather than three (its coordinates in $\mathbb{R}^3$), in much the same way that we can specify a point on a curve $\gamma(t)$ by giving the single number t.

Example 70 (revisited) For the sphere

$$\phi : (0,\pi) \times (0, 2\pi) \to \mathbb{R}^3, \qquad \phi(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$$

what are the local coordinates of the point $p = (0, 1, 0)$? We seek $u \in (0, \pi)$ and $v \in (0, 2\pi)$ such that

$$(0,1,0) = \phi(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$$

3rd component $\implies \cos u = 0 \implies u = \frac{\pi}{2}$.
2nd component $\implies \sin v = 1 \implies v = \frac{\pi}{2}$.
1st component: check $\sin u \cos v = \sin \frac{\pi}{2} \cos \frac{\pi}{2} = 0$ as required.
So $(u, v) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

6.2 The tangent and normal spaces

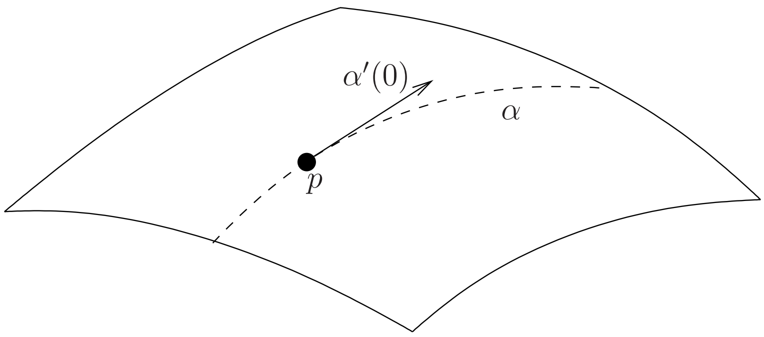
Recall that every point on a regularly parametrized curve has a well defined *tangent line*. The generalization of this to regularly parametrized surfaces is called the *tangent space*:

Definition 74 Let $p \in M$ be a point on a RPS $M = \phi(U)$ where $\phi : U \to \mathbb{R}^3$. Then a **curve in** M **through** p is a smooth map $\alpha : I \to M$ $(0 \in I)$ with $\alpha(0) = p$. The **tangent space** to M at $p \in M$ is

$$T_pM = \{x \in \mathbb{R}^3 : \text{there exists a curve } \alpha \text{ in } M \text{ through } p \text{ with } \alpha'(0) = x\}.$$

Any $x \in T_pM$ is called a **tangent vector** to M at p. $\square$

So the tangent space at p is the set of all possible velocity vectors $\alpha'(0) \in \mathbb{R}^3$ of curves α passing through p. Note that the curve α is *not* assumed to be regular.



Example 75 (paraboloid) Let $\phi : \mathbb{R}^2 \to \mathbb{R}^3$, $\phi(u, v) = (u, v, u^2 + v^2)$ and $p = (2, 1, 5) = \phi(2, 1)$. This the graph of the smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(u, v) = u^2 + v^2$, and hence is a RPS (it's a particular case of Example 70). The easiest way to define a curve through p in M is to write it as

$$\alpha(t) = \phi(u(t), v(t)).$$

The function $\widehat{\alpha} : I \to \mathbb{R}^2$ $\widehat{\alpha}(t) = (u(t), v(t))$ is known as the *coordinate expression* for α. In order to be a curve *through* p, this must satisfy

$$\widehat{\alpha}(0) = (2, 1)$$

where $(2, 1)$ are the coordinates of p. For example,

$$\widehat{\alpha}(t) = (2, 1 + t) \quad \Rightarrow \quad \alpha(t) = \phi(\widehat{\alpha}(t)) \;=\; (2, 1 + t, 5 + 2t + t^2)$$

$$\widehat{\beta}(t) = (2 + t, 1 - t) \quad \Rightarrow \quad \beta(t) = \phi(\widehat{\beta}(t)) \;=\; (2 + t, 1 - t, 5 + 2t + 2t^2)$$

$$\widehat{\gamma}(t) = (2 + t^2, 1 + \cos t) \quad \Rightarrow \quad \gamma(t) = \phi(\widehat{\alpha}(t)) \;=\; (2 + t^2, 1 + \cos t, (2 + t^2)^2 + (1 + \cos t)^2)$$

all give curves through p in M. The corresponding *tangent vectors* are:

$$\begin{aligned}\alpha'(0) &= (0,1,2) \\ \beta'(0) &= (1,-1,2) \\ \gamma'(0) &= (0,0,0).\end{aligned}$$

[Note that γ is *not* a RPC]. $\square$

A handy way to calculate with tangent vectors is to express them in terms of the coordinate basis vectors ϕ_u, ϕ_v. If α is a curve through $p = \phi(\bar{u}, \bar{v})$, then

$$\begin{aligned}\alpha(t) &= \phi(u(t), v(t)) \\ \Rightarrow \quad \alpha'(0) &= u'(0)\left.\frac{\partial\phi}{\partial u}\right|_{(u(0),v(0))} + v'(0)\left.\frac{\partial\phi}{\partial v}\right|_{(u(0),v(0))} = u'(0)\phi_u(\bar{u},\bar{v}) + v'(0)\phi_v(\bar{u},\bar{v}).\end{aligned}$$

So every tangent vector can be expressed as a linear combination of $\phi_u(\bar{u},\bar{v}), \phi_v(\bar{u},\bar{v})$. Furthermore, every linear combination of $\phi_u(\bar{u},\bar{v}), \phi_v(\bar{x}\bar{u},\bar{v})$ is a tangent vector, as we shall now prove. Let $x = a\phi_u(\bar{u},\bar{v}) + b\phi_v(\bar{u},\bar{v})$ where $a, b \in \mathbb{R}$. I claim that $x \in T_pM$, that is, that there exists a curve through p whose velocity vector at time $t = 0$ is x. One such curve is

$$\alpha(t) = \phi(\bar{u} + at, \bar{v} + bt).$$

Check:

$$\begin{aligned}\alpha'(0) &= a\left.\frac{\partial\phi}{\partial u}\right|_{(u(0),v(0))} + b\left.\frac{\partial\phi}{\partial v}\right|_{(u(0),v(0))} \\ &= a\phi_u(\bar{u},\bar{v}) + b\phi_v(\bar{u},\bar{v}) = x\end{aligned}$$

Hence

$$T_pM = \{a\phi_u(\bar{u},\bar{v}) + b\phi_v(\bar{u},\bar{v}) \,|\, a, b \in \mathbb{R}\}.$$

It follows that the **subset** $T_pM \subset \mathbb{R}^3$ is closed under vector addition and scalar multiplication: T_pM is a **subspace** of $\mathbb{R}^3$. Further, $\{\phi_u, \phi_v\}$ is a spanning set for this subspace. Since M is regular, this spanning set is linearly independent, hence a **basis** for T_pM. This explains why we call ϕ_u, ϕ_v the coordinate basis vectors of the surface.

To summarize, we have proved the following:

Theorem 76 *T_pM is a vector space of dimension 2 spanned by $\{\phi_u(\bar{u},\bar{v}), \phi_v(\bar{u},\bar{v})\}$.* $\square$

Definition 77 The **normal space** at $p \in M$ is

$$N_pM = \{y \in \mathbb{R}^3 : y \cdot x = 0 \text{ for all } x \in T_pM\}.$$

Any $y \in N_pM$ is said to be **normal to M at** p. $\square$

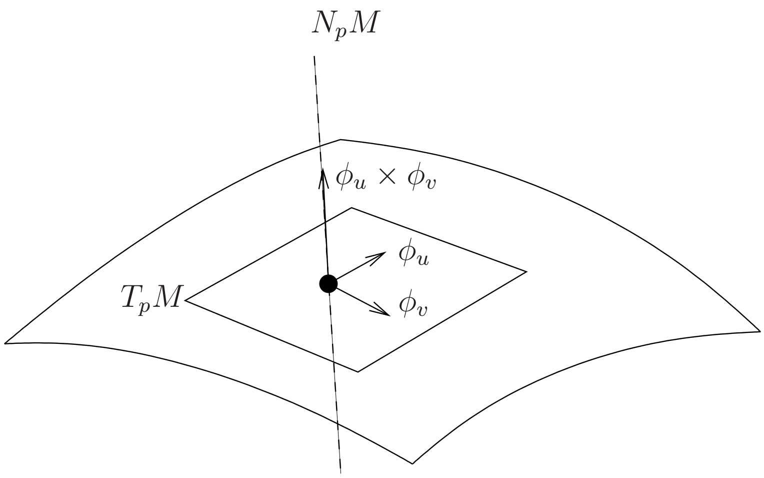
Remark 78 By its definition, N_pM is also a vector space, that is, it is closed under vector addition and scalar multiplication: let $y, z \in N_pM$ and $a, b \in \mathbb{R}$. Then for all $x \in T_pM$,

$$x \cdot (ay + bz) = a(x \cdot y) + b(x \cdot z) = 0 + 0$$

so $ay + bz \in N_pM$. Clearly N_pM is one-dimensional, so any non-zero normal vector, for example $\phi_u \times \phi_v$, is a basis for N_pM.

Furthermore, the *tangent space* T_pM is precisely the two dimensional space of vectors orthogonal to N_pM, or equivalently, to any nonzero vector in N_pM. Hence

$$x \in T_pM \quad \Leftrightarrow \quad x \cdot (\phi_u \times \phi_v) = 0.$$



This gives us a sneaky way of checking whether a given vector is a tangent vector.

Example 79 (saddle surface) Let $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$\phi(u, v) = (u, v, uv).$$

Again, this is certainly an RPS since it's a particular case of Example 70 (for $f(u, v) = uv$).

The coordinate basis vectors are

$$\phi_u = (1, 0, v) \qquad \phi_v = (0, 1, u).$$

The point $p = (3, -2, -6) \in M$ has local coordinates $(3, -2)$, so the coordinate basis for T_pM is

$$\phi_u(3, -2) = (1, 0, -2) \qquad \phi_v(3, -2) = (0, 1, 3)$$

and N_pM is spanned by

$$\phi_u \times \phi_v = (1, 0, -2) \times (0, 1, 3) = (2, -3, 1)$$

Let's determine whether the following vectors are in T_pM, N_pM or neither:

$$x = (1, 2, 4), \qquad y = (2, 1, 1), \qquad z = (-4, 6, -2).$$

$\boxed{x:}$ $u \cdot (\phi_u \times \phi_v) = 2 - 6 + 4 = 0$

So $x \in T_pM$. Hence $(1,2,4) = a\phi_u + b\phi_v$ for some $a, b \in \mathbb{R}$. In fact

$(1,2,4) = (1, 0-2) + 2(0,1,3)$

so $a = 1$ and $b = 2$.

$\boxed{y:}$ $y \cdot (\phi_u \times \phi_v) = 4 - 3 + 1 = 2$

So $y \notin T_pM$. What about N_pM?

$$y \cdot \phi_u = 2 + 0 - 2 = 0 \quad y \cdot \phi_v = 0 + 1 + 3 = 4$$

So $y \notin N_pM$ either.

$\boxed{z:}$ $z \cdot (\phi_u \times \phi_v) = -8 - 18 - 2 = -28$

So $z \notin T_pM$. What about N_pM?

$$z \cdot \phi_u = -4 + 0 + 4 = 0 \quad z \cdot \phi_v = 0 + 6 - 6$$

So $z \in N_pM$. Hence $z = c\,\phi_u \times \phi_v$ for some $c \in \mathbb{R}$. In fact

$$(-4, 6, -2) = -2(2, -3, 1)$$

so $c = -2$. □

Summary

- Given a smooth mapping $\phi : U \to \mathbb{R}^3$, where $U \subseteq \mathbb{R}^2$ is an open set, its **coordinate basis vectors** are
$$\phi_u = \frac{\partial \phi}{\partial u}, \qquad \phi_v = \frac{\partial \phi}{\partial v}.$$
ϕ is **regular** if for all $(u,v) \in U$, $\phi_u(u,v) \times \phi_v(u,v) \neq 0$.
- A **regularly parametrized surface** is a regular, one-to-one map $\phi : U \to \mathbb{R}^3$.
- The **tangent space** T_pM at $p = \phi(\bar{u}, \bar{v})$ is the set of all velocity vectors of curves in the surface $M = \phi(U)$ passing through the point p. It is a two-dimensional vector space spanned by $\{\phi_u(\bar{u},\bar{v}), \phi_v(\bar{u},\bar{v})\}$.
- The **normal space** N_pM at $p = M(\bar{u},\bar{v})$ is the set of vectors in $\mathbb{R}^3$ orthogonal to T_pM. It is a one-dimensional vector space spanned by $\phi_u(\bar{u},\bar{v}) \times \phi_v(\bar{u},\bar{v})$.

7 Calculus on surfaces

7.1 Directional derivatives

A mapping $f : \mathbb{R}^3 \to \mathbb{R}$ is *smooth* if all its partial derivatives (of all orders) exist everywhere. Given a regularly parametrized surface $M \subset \mathbb{R}^3$ we can define functions $f : M \to \mathbb{R}$: they are mappings which assign a real number $f(p)$ to each point p on the surface M. What does it mean to say that such a function is smooth?

Definition 80 Let $\phi : U \to \mathbb{R}^3$ be a regularly parametrized surface with $M = \phi(U)$ and let $f : M \to \mathbb{R}$ be a function on M. We say that f is **smooth** if $f \circ \phi : U \to \mathbb{R}$ is smooth in the usual sense. We refer to $\widehat{f} = f \circ \phi$

$$\widehat{f}(u, v) = f(\phi(u, v)).$$

as the **coordinate expression** of f.

Example 81 Consider the RPS defined in Example 71, whose image is (almost all of) the unit sphere:

$$\phi : (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3, \quad \phi(u, v) = (\sin u \cos v, \sin u \sin v, \cos u).$$

The following maps $M \to \mathbb{R}$

$$f : (y_1, y_2, y_3) \mapsto y_1 + y_2 + y_3, \quad g : (y_1, y_2, y_3) \mapsto y_1^2 + y_2^2 - y_3^2$$

have coordinate expressions

$$\begin{aligned} \widehat{f}(u, v) = f(\phi(u, v)) &= \sin u \cos v + \sin u \sin v + \cos u \\ \widehat{g}(u, v) = g(\phi(u, v)) &= \sin^2 u \cos^2 v + \sin^2 u \sin^2 v - \cos^2 u = \sin^2 u - \cos^2 u = -\cos(2u). \end{aligned}$$

These are both smooth functions of u and v, so f, g are smooth functions $M \to \mathbb{R}$.

Given a smooth function $f : M \to \mathbb{R}$ on a regularly parametrized surface M, and a tangent vector $x \in T_pM$, we can ask "what is the rate of change of f at p in the direction of x?" The answer is given (almost) by the following definition:

Definition 82 Let $f : M \to \mathbb{R}$ be a smooth function and $x \in T_pM$. The **directional derivative** of f along x is

$$\nabla_x f = (f \circ \alpha)'(0) = \left. \frac{d}{dt} f(\alpha(t)) \right|_{t=0}$$

where α is a generating curve[1] for x. [Such a curve exists by the definition of T_pM, Definition 74.] Note that $\nabla_x f$ is a single number, associated with the point p, not a function on M. $\square$

[1]Recall this is a curve $\alpha(t)$ in M with $\alpha(0) = p$ and $\alpha'(0) = x$

Example 83 Let M be the unit sphere (parametrized as in Example 71 say), let $p = (0,1,0)$ and $x = (0,0,1)$. Then $x \in T_pM$ (**check it!**). Consider the function $f : M \to \mathbb{R}$ given by

$$f(y_1, y_2, y_3) = y_3$$

which assigns to each point on the sphere its "height" above the (y_1, y_2) plane. What is $\nabla_x f$?

From the definition, we must choose a curve through $(0,1,0)$ in M whose initial velocity is $(0,0,1)$. One obvious choice is

$$\alpha(t) = (0, \cos t, \sin t).$$

$$\begin{aligned} \text{Check:} \quad |\alpha(t)| &= \sqrt{\cos^2 + \sin^2 t} = 1 \\ \alpha(0) &= (0,1,0) \\ \alpha'(0) &= (0, -\sin 0, \cos 0) = (0,0,1) \end{aligned}$$

Therefore α is a generating curve for x. Now

$$\begin{aligned} (f \circ \alpha)(t) &= \sin t \\ (f \circ \alpha)'(t) &= \cos t \\ \text{so} \quad \nabla_x f &= (f \circ \alpha)'(0) = 1 \end{aligned}$$

Note that $\nabla_x f$ isn't quite just the "rate of change of f in the **direction** of x," because it depends on the **length** of x as well as its direction. For example, let $y = 2x = (0,0,2) \in T_pM$. Then this points in the same direction as x, but is the initial velocity vector of the curve

$$\beta(t) = (0, \cos 2t, \sin 2t)$$

through p.

$$\begin{aligned} \text{Now:} \quad (f \circ \beta)(t) &= \sin 2t \\ (f \circ \beta)'(t) &= 2\cos 2t \\ \text{So:} \quad \nabla_y f &= 2 \end{aligned}$$

which is different from $\nabla_x f$. In fact $\nabla_y f = 2\nabla_x f$. Coincidence? □

One nice thing about Definition 82 is that it doesn't use the specific parametrization of the surface involved (we never actually had to worry about the defining map

$\phi(u, v)$ for the unit sphere in the above example). There is something a bit worrying about it, however. To compute $\nabla_x f$ we have to choose a generating curve α for x. There are infinitely many such curves. In the example above, we could equally well have chosen

$$\alpha(t) = (0, \sqrt{1-t^2}, t), \quad \text{or} \quad \alpha(t) = (t^2, \sqrt{1-t^4-\sin^2 t}, \sin t)$$

for example. How do we know that the answer doesn't depend on our choice?

Lemma 84 $\nabla_x f$ *is independent of the choice of generating curve α for the tangent vector x.*

Proof: Let $\alpha_1, \alpha_2 : I \to M$ be two curves through $p = \phi(\bar{u}, \bar{v})$ with $\alpha_1'(0) = \alpha_2'(0) = x$. We can write

$$\alpha_i(t) = \phi(u_i(t), v_i(t)), \quad \text{for } i = 1, 2.$$

Then

$$\begin{aligned} x = \alpha_1'(0) &= u_1'(0)\frac{\partial \phi}{\partial u}(u_1(0), v_1(u)) + v_1'(0)\frac{\partial \phi}{\partial v}(u_1(0), v_1(u)) \\ &= u_1'(0)\phi_u(\bar{u}, \bar{v}) + v_1'(0)\phi_v(\bar{u}, \bar{v}) \end{aligned}$$

Similarly,

$$x = \alpha_2'(0) = u_2'(0)\phi_u(\bar{u}, \bar{v}) + v_2'(0)\phi_v(\bar{u}, \bar{v})$$

Since $\phi_u(\bar{u}, \bar{v})$ and $\phi_v(\bar{u}, \bar{v})$ are linearly independent, $u_1'(0) = u_2'(0)$ and $v_1'(0) = v_2'(0)$.

For $i = 1$ or 2,

$$f \circ \alpha_i(t) = f(\phi(u_i(t), v_i(t))) = f \circ \phi(u_i(t), v_i(t)) = \widehat{f}(u_i(t), v_i(t)).$$

So

$$\begin{aligned} (f \circ \alpha_i)'(0) &= u_i'(0)\frac{\partial \widehat{f}}{\partial u}(u_i(0), v_i(0)) + v_i'(0)\frac{\partial \widehat{f}}{\partial v}(u_i(0), v_i(0)) \\ &= u_i'(0)\frac{\partial \widehat{f}}{\partial u}(\bar{u}, \bar{v}) + v_i'(0)\frac{\partial \widehat{f}}{\partial v}(\bar{u}, \bar{v}). \end{aligned}$$

Since $u_1'(0) = u_2'(0)$ and $v_1'(0) = v_2'(0)$,

$$(f \circ \alpha_1)'(0) = (f \circ \alpha_2)'(0).$$

□

This is reassuring. However, Definition 82 is rather cumbersome to use in practice because it's tiresome to have to keep inventing generating curves for tangent vectors. Recall that we may also think of T_pM as

$$T_pM = \{a\,\phi_u + b\,\phi_v \;:\; a, b, \in \mathbb{R}\}.$$

The next Lemma will allow us to reduce the calculation of directional derivatives to straightforward partial differentiation with respect to the local coordinates (x_1, x_2).

Lemma 85 $\nabla_x f$ *is linear with respect to both x and f. That is, for all $x, y \in T_pM$, $a, b \in \mathbb{R}$ and $f, g : M \to \mathbb{R}$,*

(A) $\nabla_{ax+by} f = a\nabla_x f + b\nabla_y f$

(B) $\nabla_x(af + bg) = a\nabla_x f + b\nabla_x g$

Proof:
(A) Let $\alpha_1, \alpha_2 : I \to M$ be curves through $p = \phi(\bar{u}, \bar{v})$ with $\alpha_1'(0) = x$, $\alpha_2'(0) = y$. As before we can write these as

$$\alpha_i(t) = \phi(u_i(t), v_i(t)).$$

Consider the curve

$$\begin{aligned} \alpha_3(t) &= \phi(u_3(t), v_3(t)), \\ (u_3(t), v_3(t)) &= a(u_1(t), v_1(t)) + b(u_2(t), v_2(t)) - (\bar{u}, \bar{v}). \end{aligned}$$

Then $\alpha_3(0) = p$ and $\alpha_3'(0) = ax + by$. [**Check it!**]

Now we try to calculate $\nabla_{ax+by} f = (f \circ \alpha_3)'(0)$. We have that, for $i = 1, 2, 3$,

$$(f \circ \alpha_i)(t) = f(\phi(u_i(t), v_i(t))) = \widehat{f}(u_i(t), v_i(t)),$$

so

$$\begin{aligned} (f \circ \alpha_i)'(0) &= u_i'(0)\frac{\partial \widehat{f}}{\partial u}(u_i(0), v_i(0)) + v_i'(0)\frac{\partial \widehat{f}}{\partial v}(u_i(0), v_i(0)) \\ &= u_i'(0)\frac{\partial \widehat{f}}{\partial u}(\bar{u}, \bar{v}) + v_i'(0)\frac{\partial \widehat{f}}{\partial v}(\bar{u}, \bar{v}). \end{aligned}$$

Therefore

$$\begin{aligned} \nabla_{ax+by} f &= (f \circ \alpha_3)'(0) \\ &= (au_1'(0) + bu_2'(0))\frac{\partial \widehat{f}}{\partial u}(\bar{u}, \bar{v}) + (av_1'(0) + bv_2'(0))\frac{\partial \widehat{f}}{\partial v}(\bar{u}, \bar{v}) \\ &= a\left(u_1'(0)\frac{\partial \widehat{f}}{\partial u} + v_1'(0)\frac{\partial \widehat{f}}{\partial v}\right) + b\left(u_2'(0)\frac{\partial \widehat{f}}{\partial u} + v_2'(0)\frac{\partial \widehat{f}}{\partial v}\right) \\ &= a(f \circ \alpha_1)'(0) + b(f \circ \alpha_2)'(0) \\ &= a\nabla_x f + b\nabla_y f. \end{aligned}$$

(B) Follows immediately from the definition: if α is a generating curve for x then

$$\begin{aligned} \nabla_x(af + bg) &= ((af + bg) \circ \alpha)'(0) = (af \circ \alpha + bg \circ \alpha)'(0) \\ &= a(f \circ \alpha)'(0) + b(g \circ \alpha)'(0) = a\nabla_x f + b\nabla_x g. \end{aligned}$$ □

The upshot of Lemma 85 is that if we know the directional derivatives $\nabla_{\phi_u} f$ and $\nabla_{\phi_v} f$ with respect to the coordinate basis vectors, then we know $\nabla_x f$ for *all* tangent vectors $x \in T_pM$. For any $x \in T_pM$ may be written as

$$x = a\phi_u + b\phi_v$$

so applying Lemma 85 part (A) gives

$$\nabla_x f = a\nabla_{\phi_u} f + b\nabla_{\phi_v} f. \qquad (\spadesuit)$$

But $\phi_u(\bar u, \bar v)$ and $\phi_v(\bar u, \bar v)$ may be represented by the curves

$$\phi(\bar u + t, \bar v) \text{ and } \phi(\bar u, \bar v + t)$$

respectively. Hence

$$\nabla_{\phi_u} f = \frac{\partial \widehat{f}}{\partial u}, \qquad \nabla_{\phi_v} = \frac{\partial \widehat{f}}{\partial v}.$$

Example 83 (revisited) Let's re-do Example 83 using the trick of reducing directional derivatives to partial differentiation. Recall M is the unit sphere parametrized by

$$\phi(u, v) = (\sin u \cos v, \sin u \sin v, \cos u).$$

The function is $f(p_1, p_2, p_3) = p_3$ and we wish to compute $\nabla_x f$ where $p = (0, 1, 0)$ and $x = (0, 0, 1) \in T_pM$.

- The local coordinates of p are $(u, v) = (\ \pi/2\ ,\ \pi/2\ )$
  [Check: $\phi(\ \pi/2\ ,\ \pi/2\ ) = (0, 1, 0)$.
- The local coordinate expression for f is

$$\widehat{f}(u, v) = \cos(u)$$

- The coordinate basis vectors are

$$\begin{aligned} \phi_u(u, v) &= (\cos u \cos v, \cos u \sin v, -\sin u) \\ \phi_v(u, v) &= (-\sin u \sin v, \sin u \cos v, 0). \end{aligned}$$

  At p they are

$$\begin{aligned} \phi_u(\ \pi/2\ ,\ \pi/2\ ) &= (0, 0, -1) \\ \phi_v(\ \pi/2\ ,\ \pi/2\ ) &= (-1, 0, 0). \end{aligned}$$

- Hence, $x = a\phi_u + b\phi_v$ where

$$a = -1, b = 0$$

- Finally,

$$\begin{aligned} \nabla_x f &= a\nabla_{\phi_u} f + b\nabla_{\phi_v} f = a\frac{\partial \widehat{f}}{\partial u}(\ \pi/2\ ,\ \pi/2\ ) + b\frac{\partial \widehat{f}}{\partial v}(\ \pi/2\ ,\ \pi/2\ ) \\ &= (-1)(-\sin(\pi/2)) + 0 = 1 \end{aligned}$$

We got the same answer! Which calculation was easier, Example 83 or Example 83 (revisited)?

The fact that we can reduce the calculation of $\nabla_x f$ to partial differentiation is theoretically convenient, too. For example, we know that partial derivatives obey the product (or Leibniz) rule

$$\frac{\partial}{\partial u}(\widehat{f}\widehat{g}) = \widehat{f}\frac{\partial \widehat{g}}{\partial u} + \widehat{g}\frac{\partial \widehat{f}}{\partial u}$$

and similarly for $\partial/\partial v$. We leave it as an **exercise** for you to use this to prove that

Corollary 86 *For all $f, g : M \to \mathbb{R}$ and $x \in T_pM$,*

$$\nabla_x(fg) = f(p)\nabla_x g + g(p)\nabla_x f$$

□

7.2 Vector fields

As usual, let U be an open subset of $\mathbb{R}^2$, let $\phi : U \to \mathbb{R}^3$ be a RPS and let $M = \phi(U)$.

Definition 87 A **vector field** on a RPS M is a smooth map $X : M \to \mathbb{R}^3$ (where smooth means that each of its component functions $X_1, X_2, X_3 : M \to \mathbb{R}$ is smooth). If $X(p) \in T_pM$ for all $p \in M$ then X is called a **tangent vector field**. If $X(p) \in N_pM$ for all $p \in M$ then X is a **normal vector field**.

Remark 88 Just as for functions, we define the **coordinate expression** of a vector field $X : M \to \mathbb{R}^3$ to be

$$\widehat{X} : U \to \mathbb{R}^3, \qquad \widehat{X}(u, v) = V(\phi(u, v)).$$

It follows from Theorem 76 that the coordinate expression of any **tangent** vector field X takes the form

$$\widehat{X}(u, v) = f(u, v)\phi_u(u, v) + g(u, v)\phi_v(u, v)$$

where f, g are smooth real-valued functions on U. Similarly, it follows from Remark 78 that any **normal** vector field $X : M \to \mathbb{R}^3$ takes the form

$$\widehat{X}(u, v) = f(u, v)\, \phi_u(u, v) \times \phi_v(u, v)$$

where f is a smooth real-valued function on U.

We will often refer to $\phi_u(u, v), \phi_v(u, v)$ as tangent vector fields, even though they are really coordinate expressions for tangent vector fields. Similarly, we will normally refer to $\phi_u \times \phi_v$ as a normal vector field (even though it's actually a coordinate expression for a normal vector field).

We may define directional derivatives of vector fields along tangent vectors exactly as we defined directional derivatives of functions:

Definition 89 Let $X : M \to \mathbb{R}^3$ be a vector field on M and $y \in T_pM$. Then the **directional derivative** of X with respect to y is

$$\nabla_y X = (X \circ \alpha)'(0)$$

where $\alpha : I \to M$ is a curve through $p \in M$ with $\alpha'(0) = x$. □

Notes:

- X is a vector field, but y and $\nabla_y X$ are just vectors.
- X can be any vector field, but y must be a **tangent** vector.
- Even if X is a tangent vector field, there is no reason to expect $\nabla_y X$ to be a tangent vector.
- We can write $X = (X_1, X_2, X_3)$ where $X_1, X_2, X_3 : M \to \mathbb{R}$ are ordinary functions. Then

$$\nabla_y X = ((X_1 \circ \alpha)'(0), (X_2 \circ \alpha)'(0), (X_3 \circ \alpha)'(0)) = (\nabla_y X_1, \nabla_y X_2, \nabla_y X_3). \quad (\diamondsuit)$$

  So we can write $\nabla_y X$ in terms of the directional derivatives of the functions X_1, X_2, X_3. Since $\nabla_y X_1, \nabla_y X_2, \nabla_y X_3$ are independent of the choice of curve α (lemma 85), $\nabla_y X$ is also independent of the choice of curve.

This observation allows us to transfer our knowledge about directional derivatives of functions to directional derivatives of vector fields:

Lemma 90 *Let X, Y be vector fields on M, $z, w \in T_pM$, f be a smooth function on M and $a, b \in \mathbb{R}$. Then*

(a) $\nabla_{az+bw} X = a\nabla_z X + b\nabla_w X$.

(b) $\nabla_z(aX + bY) = a\nabla_z X + b\nabla_z Y$.

(c) $\nabla_z(fX) = (\nabla_z f)X(p) + f(p)\nabla_z X$.

(d) $\nabla_z(X \cdot Y) = (\nabla_z X) \cdot Y(p) + X(p) \cdot (\nabla_z Y)$.

Proof: We prove (d) as follows. Write $X = (X_1, X_2, X_3)$ and $Y = (Y_1, Y_2, Y_3)$ for functions X_i, Y_i. Then

$$\begin{aligned}
\nabla_z(X \cdot Y) &= \nabla_z(X_1Y_1 + X_2Y_2 + X_3Y_3) \\
&= \nabla_z(X_1Y_1) + \nabla_z(X_2Y_2) + \nabla_z(X_3Y_3) && \text{(Lemma 85)} \\
&= (\nabla_z X_1)Y_1(p) + X_1(p)(\nabla_z Y_1) + (\nabla_z X_2)Y_2(p) \\
&\quad + X_2(p)(\nabla_z Y_2) + (\nabla_z X_3)Y_3(p) + X_3(p)(\nabla_z Y_3) && \text{(Corollary 86)} \\
&= (\nabla_z X_1, \nabla_z X_2, \nabla_z X_3) \cdot (Y_1(p), Y_2(p), Y_3(p)) \\
&\quad + (X_1(p), X_2(p), X_3(p)) \cdot (\nabla_z Y_1, \nabla_z Y_2, \nabla_z Y_3) \\
&= (\nabla_z X) \cdot Y(p) + X(p) \cdot (\nabla_z Y).
\end{aligned}$$

We leave it as an **exercise** to prove parts (a), (b) and (c) using Lemma 85, Corollary 86 and equation ($\diamondsuit$). $\square$.

As with directional derivatives of functions, we can use Lemma 90 to reduce calculation of $\nabla_y X$ to partial differentiation with respect to (u, v). We just need to express y in terms of the coordinate basis and find the coordinate expression for X.

Example 91 Let $\phi(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$ be our standard parametrisation for the unit sphere M. Let $p = (0,1,0) \in M$, $y = (1,0,0) \in T_pM$, and let X be the vector field $X(q) = (q_2, -q_1, 0)$. Calculate $\nabla_y X$.

Rather than working directly from definition (which involves finding a generating curve for y), we calculate $\nabla_y X$ by expressing y in terms of ϕ_u and ϕ_v. The coordinates of p are $(\pi/2, \pi/2)$. We calculate:

$$\begin{aligned}
\phi_u(u,v) &= (\cos u \cos v, -\cos u \sin v, \sin u) \\
\phi_v(u,v) &= (-\sin u \sin v, \sin u \cos v, 0) \\
\phi_u(\pi/2,\pi/2) &= (0,0,1) \\
\phi_v(\pi/2,\pi/2) &= (-1,0,0).
\end{aligned}$$

So $y = a\phi_u + b\phi_v$ with

$$a = 0 \quad , b = -1 \quad .$$

Now

$$\begin{aligned}
\widehat{X}(u,v) &= (\sin u \sin v, -\sin u \cos v, 0) \\
\frac{\partial \widehat{X}}{\partial u}(u,v) &= (\cos u \sin v, -\cos u \cos v, 0) \\
\frac{\partial \widehat{X}}{\partial v}(u,v) &= (\sin u \cos v, \sin u \sin v, 0) \\
\frac{\partial \widehat{X}}{\partial u}(\pi/2,\pi/2) &= (0,0,0) \\
\frac{\partial \widehat{X}}{\partial v}(\pi/2,\pi/2) &= (0,1,0) \quad .
\end{aligned}$$

So

$$\nabla_y X = 0\frac{\partial \widehat{X}}{\partial u} + (-1)\frac{\partial \widehat{X}}{\partial v} = (0,-1,0) \qquad .$$

$\square$

Given a tangent vector **field** $Y : M \to \mathbb{R}^3$ and another vector field $X : M \to \mathbb{R}^3$ (tangent, normal or neither), we can define a 3rd vector **field** $Z : M \to \mathbb{R}^3$ as follows: at each $p \in M$,

$$Z(p) = \nabla_{Y(p)} X.$$

We shall denote this vector field $Z = \nabla_Y X$.

Example 92 Consider again the unit sphere M with the parametrization from example 91. Let $X : M \to \mathbb{R}^3$ be the tangent vector field $X(q) = (q_3 q_1, q_3 q_2, q_3^2 - 1)$. We will calculate the vector field $\nabla_X X$.

We calculate

$$\begin{aligned}
\phi_u(u,v) &= (\cos u\cos v, \cos u\sin v, -\sin u) \\
\phi_v(u,v) &= (-\sin u\sin v, \sin u\cos v, 0) \\
\widehat{X}(u,v) &= (\sin u\cos u\cos v, \sin u\cos u\cos v, -\sin^2 u) \qquad .
\end{aligned}$$

Since we were told that X is a tangent vector field, we should be able to find functions $\widehat{f}(u,v)$, $\widehat{g}(u,v)$ such that $\widehat{X} = \widehat{f}\phi_u + \widehat{g}\phi_v$. We find

$$\widehat{f}(u,v) = \sin u \quad , \widehat{g}(u,v) = 0 \quad .$$

So at the point p with coordinates (u,v),

$$\begin{aligned}
\nabla_{X(p)}X &= \sin u\frac{\partial\widehat{X}}{\partial u} + 0 \times \frac{\partial\widehat{X}}{\partial v} \\
&= \sin u((\cos^2 u - \sin^2 u)\cos v, (\cos^2 u - \sin^2 u)\sin v, -2\sin u\cos u) \\
&= \sin u(\cos 2u\cos v, \cos 2u\sin v, \sin 2u) \quad .
\end{aligned}$$

This formula gives us the coordinate expression for $\nabla_X X$.

In the particular case that Y is the coordinate vector fields ϕ_u, $\nabla_Y X$ is easy to evaluate: since $Y = 1 \times \phi_u + 0 \times \phi_v$ the coordinate expression for $\nabla_{\phi_u}X$ is

$$\widehat{\nabla_{\phi_u}X} = 1 \times \frac{\partial\widehat{X}}{\partial u}(u,v) + 0 \times \frac{\partial\widehat{X}}{\partial v}(u,v) = \frac{\partial\widehat{X}}{\partial u}(u,v).$$

Similarly ∇_{ϕ_v} corresponds to taking the partial derivative in v.

Lemma 93

$$\nabla_{\phi_u}\phi_v = \nabla_{\phi_v}\phi_u.$$

Proof: The coordinate expression for ϕ_v is $\partial\phi/\partial v$, so the coordinate expression for $\nabla_{\phi_u}\phi_v$ is

$$\widehat{\nabla_{\phi_u}\phi_v} = \frac{\partial^2\phi}{\partial u\partial v}$$

Similarly,

$$\widehat{\nabla_{\phi_v}\phi_u} = \frac{\partial^2\phi}{\partial v\partial u}.$$

Since partial derivative commute, these two expressions must be equal. $\square$

Summary

- A function $f : M \to \mathbb{R}$ is **smooth** if its coordinate expression $\widehat{f} = f \circ \phi : U \to \mathbb{R}$ is smooth.
- Given a smooth function $f : M \to \mathbb{R}$ and a tangent vector $y \in T_pM$, the **directional derivative** of f with respect to (or along) y is

$$\nabla_y f = (f \circ \alpha)'(0)$$

where $\alpha(t)$ is any generating curve for y.

- The directional derivative is linear, that is

$$\nabla_{ax+by} f = a\nabla_x f + b\nabla_y f, \qquad \nabla_y(af + bg) = a\nabla_y f + b\nabla_y g.$$

- Directional derivatives along coordinate basis vectors reduce to partial derivatives

$$\nabla_{\phi_u} f = \frac{\partial \widehat{f}}{\partial u}, \qquad \nabla_{\phi_v} f = \frac{\partial \widehat{f}}{\partial v}.$$

- Vector **fields** are smooth maps $X : M \to \mathbb{R}^3$.
- We can extend the definition of directional derivative to vector fields. If X is a vector field and Y is a **tangent** vector field then the directional derivative $\nabla_Y X$ is another vector field.

8 Curvature of an oriented surface

8.1 The shape operator on an oriented surface

Definition 94 An **orientation** on a RPS $\phi : U \to \mathbb{R}^3$ with image $M = \phi(U)$ is a choice of unit normal vector field N. In other words, $N : M \to \mathbb{R}^3$ must satisfy $|N(p)| = 1$ and $N(p) \in N_pM$ for all $p \in M$. □

For example, if M is the plane $z = 0$ then $N(p) = (0, 0, 1)$ is a unit normal vector field, so is an orientation for M. Another choice of orientation could be $N(p) = (0, 0, -1)$.

If M is the unit sphere (i.e. the set of vectors p of unit length) then we could choose $N(p) = p$. This is a vector field that points radially outwards. Then for any $p \in M$, $|N(p)| = |p| = 1$ and $N(p)$ is normal to M at p, so N is an orientation. Can you think of another orientation of the sphere?

In general, we can choose an orientation for an RPS as follows: recall that $\phi_u \times \phi_v$ is a normal vector field which never vanishes. It follows that

$$N = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|}$$

is a **unit** normal vector field. This N is known as the **canonical orientation** of ϕ. The only other possible choice of orientation is

$$\tilde{N} = -\frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|}.$$

The key point about oriented surfaces is that, as we noted in Remark 78, the unit normal at p, $N(p)$, determines the tangent space at p:

$$T_pM = \{x \in \mathbb{R}^3 : x \cdot N(p) = 0\}.$$

So we can glean information about how the tangent space varies with p by computing directional derivatives $\nabla_x N$ where $x \in T_pM$. Just as with the curvature of a regularly parametrized curve, it's essential that N is a *unit* normal vector field – if it weren't then $\nabla_x N$ would contain information about the rate of change of the *length* of N in the direction u, as well as the rate of change of the direction of N.

Lemma 95 *Let $\phi : U \to \mathbb{R}^3$ be an RPS and let N be a an orientation for $M = \phi(U)$. Then for any $p \in M$ and $x \in T_pM$, $\nabla_x N \in T_pM$ also.*

Proof: It suffices to show that $N(p) \cdot (\nabla_x N) = 0$. Now N is a unit vector field so $N \cdot N = 1$. Taking the directional derivative of this (constant) function with respect to x gives

$$0 = \nabla_x(N \cdot N) = (\nabla_x N) \cdot N(p) + N(p) \cdot (\nabla_x N) = 2N(p) \cdot (\nabla_x N),$$

where we have used Lemma 90 part (d). The result immediately follows. □

Reminder 96 A map $L : V \to V$, where V is a vector space, is said to be **linear** if for all $a, b \in \mathbb{R}$ and $x, y \in V$,

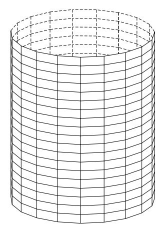
$$L(ax + by) = aL(x) + bL(y).$$

Definition 97 Let $\phi : U \to \mathbb{R}^3$ and let N be an orientation of $M = \phi(U)$. The **shape operator** at $p \in M$ is the map

$$S_p : T_pM \to T_pM, \qquad S_p(x) = -\nabla_x N.$$

S_p really does map T_pM to itself, by Lemma 95, and is a linear map by Lemma 90. It is often called the **Weingarten map** in honour of its discoverer. □

Example 98 (a cylinder) Let $R > 0$ and let $\phi : (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$ be the map



$$\phi(u, v) = (R\cos u, R\sin u, v).$$

This is a cylinder of radius $R > 0$. Let's calculate the shape operator of ϕ. First we find the canonical orientation:

$$\begin{aligned}
\phi_u &= (-R\sin u, R\cos u, 0) \\
\phi_v &= (0, 0, 1) \\
\phi_u \times \phi_v &= (R\cos u, R\sin u, 0) \\
|\phi_u \times \phi_v|^2 &= R^2\cos^2 u + R^2\sin^2 u = R^2 \\
N &= \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} = (\cos u, \sin u, 0)
\end{aligned}$$

Now we find the shape operator S_p. We start by calculating $S_p(\phi_u)$ and $S_p(\phi_v)$:

$$\begin{aligned}
S_p(\phi_u) &= -\nabla_{\phi_u} N = (\sin u, -\cos u, 0) \\
&= -\frac{1}{R}\phi_u + \quad 0\phi_v \\
S_p(\phi_v) &= -\nabla_{\phi_v} N = (0, 0, 0) \\
&= 0\phi_u + \quad 0\phi_v.
\end{aligned}$$

If we wanted to calculate $S_p(x)$ for any other $x \in T_pM$ we could start by finding a, b such that $x = a\phi_u + b\phi_v$. Then we would know that

$$S_p(x) = S_p(a\phi_u + b\phi_v) = aS_p(\phi_u) + bS_p(\phi_v) = -\frac{a}{R}\phi_u$$

because S_p is linear. So $S_p(x)$ is easily worked out from $S_p(\phi_u)$ and $S_p(\phi_v)$. As is usually done with linear maps, we can completely describe S_p by writing down its matrix $\widehat{S}_p$ with respect to the basis ϕ_u, ϕ_v for T_pM:

$$\widehat{S_p} = \begin{pmatrix} \uparrow & \uparrow \\ S_p(\phi_u) & S_p(\phi_v) \\ \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}.$$

Then applying S_p to $x = a\phi_u + b\phi_v$ corresponds to multiplying the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ with $\widehat{S}_p$. □

Example 99 (a plane) Let $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ such that $\phi(u, v) = (u, v, 0)$. The image is the plane $y_3 = 0$. As discussed above, $N(u, v) = (0, 0, 1)$ is a unit normal. Then N is constant, so $\nabla_x N = 0$ for any $x \in T_pM$. Therefore S_p sends every tangent vector x to zero.
Question: what is the matrix of S_p in this case? □

Example 100 (unit sphere) Let's calculate the shape operator for the unit sphere M, parametrized as usual by $\phi(u, v) = (\sin u \cos v, \sin u \sin v, \cos u)$. Then

$$\begin{aligned} \phi_u &= (\cos u \cos v, \cos u \sin v, -\sin u) \\ \phi_v &= (-\sin u \sin v, \sin u \cos v, 0) \\ \phi_u \times \phi_v &= (\sin^2 u \cos v, \sin^2 \sin v, \sin u \cos u) \\ &= \sin u(\sin u \cos v, \sin u \sin v, \cos u) \\ |\phi_u \times \phi_v| &= \sin u \\ N(u, v) &= \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} = (\sin u \cos v, \sin u \sin v, \cos u). \end{aligned}$$

So

$$\begin{aligned} S_p(\phi_u) &= -\nabla_{\phi_u} N = -(\cos u \cos v, \cos u \sin v, -\sin u) = -\phi_u + 0\phi_v \\ S_p(\phi_v) &= -\nabla_{\phi_v} N = -(-\sin u \sin v, \sin u \cos v, 0)) = 0\phi_u - \phi_v. \end{aligned}$$

So the matrix of S_p is

$$\widehat{S}_p = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For any $x \in T_pM$, $S_p(x) = -x$, so $S_p = -\mathrm{Id}$, where $\mathrm{Id} : T_pM \to T_pM$ is the identity map.

Note that we could have taken a shortcut to get this answer: as stated earlier, we can choose the orientation to be $N(p) = p$, and this has coordinate expression $\widehat{N}(u, v) = \phi(u, v)$. Then $\nabla_{\phi_u} N = -\partial\widehat{N}/\partial u = -\partial\phi/\partial u = -\phi_u$, and similarly for ϕ_v.

Our measure(s) of the curvature of a surface will be defined in terms of the *eigenvalues* of the linear map $S_p : T_pM \to T_pM$. It turns out to be crucial that S_p has a property called *self-adjointness*.

Definition 101 A linear map $L : T_pM \to T_pM$ is **self adjoint** if for all $x, y \in T_pM$, $x \cdot L(y) = y \cdot L(x)$. $\square$

Theorem 102 *Let M be an oriented surface and $S_p : T_pM \to T_pM$ be its shape operator at the point p. Then S_p is self adjoint.*

Proof: Let $x = a\phi_u + b\phi_v$ and $y = c\phi_u + d\phi_v$. Then

$$\begin{aligned} x.S_p(y) - y.S_p(x) &= ac\phi_u \cdot S_p(\phi_u) + ad\phi_u \cdot S_p(\phi_v) + bc\phi_v \cdot S_p(\phi_u) + bd\phi_v \cdot S_p(\phi_v) \\ &\quad -ac\phi_u \cdot S_p(\phi_u) - ad\phi_v \cdot S_p(\phi_u) - bc\phi_u \cdot S_p(\phi_v) - bd\phi_v \cdot S_p(\phi_v) \\ &= (ad - bc)(\phi_u \cdot S_p(\phi_v) - \phi_v \cdot S_p(\phi_u)). \end{aligned}$$

So we just need to show that $\phi_u \cdot S_p(\phi_v) = \phi_v \cdot S_p(\phi_u)$, i.e. that $\phi_u \cdot \nabla_{\phi_v} N = \phi_v \cdot \nabla_{\phi_u} N$.

Since ϕ_u is a tangent vector field and N is a normal vector field, $\phi_u \cdot N = 0$, so

$$0 = \nabla_{\phi_v}(\phi_u \cdot N) = (\nabla_{\phi_u}\phi_v) \cdot N + \phi_v \cdot (\nabla_{\phi_u} N)$$

by lemma 90(d). Similarly, $(\nabla_{\phi_v}\phi_u) \cdot N + \phi_u \cdot (\nabla_{\phi_v} N) = 0$. So

$$\phi_u \cdot S_p(\phi_v) - \phi_v \cdot S_p(\phi_u) = -\phi_u \cdot (\nabla_{\phi_v} N) + \phi_v \cdot (\nabla_{\phi_u} N) = (\nabla_{\phi_v}\phi_u) \cdot N - (\nabla_{\phi_u}\phi_v) \cdot N = 0$$

by lemma 93. $\square$

Example 103 (a saddle surface) Let $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ be the map $\phi(u, v) = (u, v, uv)$. This is a RPS, as we have observed previously. Let's construct its shape operator

$S_p : T_pM \to T_pM$ at the point $p = \phi(1,0) = (1,0,0)$.

$$\begin{aligned}
\phi_u &= (1,0,v) \\
\phi_v &= (0,1,u) \\
\phi_u \times \phi_v &= (-v,-u,1) \\
|\phi_u \times \phi_v| &= \sqrt{1+u^2+v^2} \\
N &= \frac{1}{\sqrt{1+u^2+v^2}}(-v,-u,1) \\
\frac{\partial}{\partial u}\frac{1}{\sqrt{1+u^2+v^2}} &= \frac{-u}{(1+u^2+v^2)^{3/2}} \\
\frac{\partial}{\partial v}\frac{1}{\sqrt{1+u^2+v^2}} &= \frac{-v}{(1+u^2+v^2)^{3/2}} \\
S_p(\phi_u) &= -\nabla_{\phi_u}N = \frac{u}{(1+u^2+v^2)^{3/2}}(-v,-u,1) + \frac{1}{\sqrt{1+u^2+v^2}}(0,1,0) \\
&= \frac{1}{2^{3/2}}(0,-1,1) + \frac{1}{2^{1/2}}(0,1,0) \qquad \text{at } (u,v) = (1,0) \\
&= 2^{-3/2}(0,1,1) \\
S_p(\phi_v) &= -\nabla_{\phi_v}N = \frac{v}{(1+u^2+v^2)^{3/2}}(-v,-u,1) + \frac{1}{\sqrt{1+u^2+v^2}}(1,0,0) \\
&= 2^{-1/2}(1,0,0) \qquad \text{at } (u,v) = (1,0)
\end{aligned}$$

Now we express these in terms of the basis ϕ_u, ϕ_v at p:

$$\begin{aligned}
\phi_u(1,0) &= (1,0,0) \\
\phi_v(1,0) &= (0,1,1) \\
S_p(\phi_u) &= 0\phi_u + 2^{-3/2}\phi_v \\
S_p(\phi_v) &= 2^{-1/2}\phi_u + 0\phi_v.
\end{aligned}$$

So the matrix of the shape operator is

$$\widehat{S}_p = \begin{pmatrix} \uparrow & \uparrow \\ S_p(\phi_u) & S_p(\phi_v) \\ \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} 0 & 2^{-1/2} \\ 2^{-3/2} & 0 \end{pmatrix}.$$

Let's check that S_p is self-adjoint at $p = \phi(1,0)$:

$$\begin{aligned}
\phi_u \cdot \phi_u &= 1^2 + 0^2 + 0^2 = 1 \\
\phi_v \cdot \phi_v &= 0^2 + 1^2 + 1^2 = 2 \\
\phi_u \cdot S_p(\phi_v) &= 2^{-1/2}\phi_u \cdot \phi_u = 2^{-1/2} \\
\phi_v \cdot S_p(\phi_u) &= 2^{-3/2}\phi_v \cdot \phi_v = 2^{-3/2} \times 2 = 2^{-1/2}
\end{aligned}$$

8.2 The principal curvatures of an oriented surface

Let V be a vector space (e.g. $V = T_pM$) and $L : V \to V$ be a linear map (e.g. $S_p : T_pM \to T_pM$). An **eigenvalue** of L is a number λ such that

$$L(x) = \lambda x \qquad (\diamondsuit)$$

for some $x \in V$, $x \neq 0$. The vector x is called an **eigenvector** corresponding to the eigenvalue λ. Clearly, given one such x any multiple ax ($a \neq 0$) is also an eigenvector, so we are free to choose our eigenvectors to have *unit length*. Let $\mathrm{Id} : V \to V$ be the identity map on V ($\mathrm{Id}(x) = x$). Then we may re-arrange ($\diamondsuit$) to read

$$(L - \lambda\,\mathrm{Id})(x) = 0, \qquad x \neq 0,$$

so λ is an eigenvalue if and only if the linear map $L - \lambda\,\mathrm{Id}$ *fails to be injective* (both $u \neq 0$ and 0 get mapped to 0 by $L - \lambda\,\mathrm{Id}$), that is, if and only if $L - \lambda\,\mathrm{Id}$ is *not invertible*. Choose a basis $e_1, \ldots, e_m$ for V and let $\widehat{L}$ be the $m \times m$ matrix representing L. Now Id is represented by the $m \times m$ identity matrix $\mathbb{I}_m$, so the linear map $L - \lambda\mathrm{Id}$ is represented by the square matrix $\widehat{L} - \lambda\mathbb{I}_m$. Hence, it fails to be invertible if and only if

$$\det(\widehat{L} - \lambda\mathbb{I}_m) = 0. \qquad (\clubsuit)$$

Equation ($\clubsuit$) is a degree m polynomial equation in λ, called the **characteristic equation** of the linear map L. Although it *looks* like it depends on the matrix $\widehat{L}$ chosen to represent L (i.e. the choice of *basis* for V), in fact it doesn't. The equation has exactly m solutions, counted with multiplicity. However, even though the coefficients of the polynomial are all real, these solutions may, in general, be **complex**. In this case, the solution λ is still called an eigenvalue of L, but its interpretation is rather subtle.

Example 104 On $\mathbb{R}^2$ consider the linear map $L : (a,b) \mapsto (-b,a)$. Relative to the standard basis $e_1 = (1,0)$, $e_2 = (0,1)$ this has matrix representation

$$\widehat{L} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

so the characteristic equation is

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0.$$

So L has only complex eigenvalues, namely $\pm i$. $\square$

Luckily, this can never happen for the shape operator $S_p : T_pM \to T_pM$ because it is self-adjoint.

Theorem 105 *Let $L : T_pM \to T_pM$ be a self-adjoint linear map. Then*

(A) *its eigenvalues are all real, and*

(B) *its eigenvectors form an orthonormal basis for T_pM.*

Proof: (A) Let e_1, e_2 be an orthonormal basis for T_pM and

$$\widehat{L} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

be the matrix representing L relative to this basis. Then, by definition,

$$e_1 \cdot L(e_2) = e_1 \cdot (L_{12}e_1 + L_{22}e_2) = L_{12},$$

and

$$e_2 \cdot L(e_1) = e_2 \cdot (L_{11}e_1 + L_{21}e_2) = L_{21}.$$

But L is self adjoint, so $L_{12} = L_{21}$. Hence, the characteristic equation of L is

$$0 = \det(\widehat{L} - \lambda \mathbb{I}_2) = \begin{vmatrix} L_{11} - \lambda & L_{12} \\ L_{12} & L_{22} - \lambda \end{vmatrix} = \lambda^2 - (L_{11} + L_{22})\lambda + L_{11}L_{22} - L_{12}^2. \qquad (\spadesuit)$$

The discriminant of this quadratic polynomial is

$$\text{“}b^2 - 4ac\text{”} = (L_{11} + L_{22})^2 - 4L_{11}L_{22} + 4L_{12}^2 = (L_{11} - L_{22})^2 + 4L_{12}^2 \geq 0.$$

Hence ($\spadesuit$) has two real solutions.
(B) Let these real eigenvalues be λ_1, λ_2 and denote their corresponding eigenvectors u_1, u_2. If $\lambda_1 = \lambda_2$, equation ($\spadesuit$) has a repeated root, hence its discriminant vanishes, so $L_{11} = L_{22} = \lambda$ and $L_{12} = 0$. But then $\widehat{L} = \lambda \mathbb{I}_2 \quad \Rightarrow \quad L = \lambda\,\mathrm{Id}$, so *every* vector $u \in T_pM$ is an eigenvector corresponding to eigenvalue λ. So we may choose $u_1 = e_1$ and $u_2 = e_2$, which are orthonormal.

If $\lambda_1 \neq \lambda_2$ then

$$0 = u_1 \cdot L(u_2) - u_2 \cdot L(u_1) = (\lambda_2 - \lambda_1)u_1 \cdot u_2$$

so $u_1 \cdot u_2 = 0$. $\square$

This allows us to make the following definition:

Definition 106 Let M be an oriented surface, $S_p : T_pM \to T_pM$ be its shape operator at $p \in M$. Then the **principal curvatures** of M at p are κ_1, κ_2, the eigenvalues of S_p. The **principal curvature directions** of M at p are the corresponding eigenvectors e_1, e_2 (normalized to have unit length). By Theorem 105, κ_1, κ_2 are real and the eigenvectors e_1, e_2 form an orthonormal basis for T_pM. $\square$

Example 107 (saddle surface) Recall (Example 103) that the saddle surface

$$\phi(u,v) = (u, v, uv)$$

at $p = \phi(1,0) = (1,0,0)$ has shape operator

$$\widehat{S}_p = \begin{pmatrix} 0 & 2^{-\frac{1}{2}} \\ 2^{-3/2} & 0 \end{pmatrix}$$

relative to the coordinate basis $\phi_u = (1,0,0), \phi_v = (0,1,1)$ for $T_{(1,0,0)}M$. Note this matrix is not symmetric, because the coordinate basis is not orthonormal. Nonetheless, its eigenvalues must still be real, and its eigenvectors orthonormal. Let's check. The characteristic equation is

$$0 = \begin{vmatrix} -\lambda & 2^{-\frac{1}{2}} \\ 2^{-\frac{3}{2}} & -\lambda \end{vmatrix} = \lambda^2 - \frac{1}{4}$$

so the principal curvatures are:

$$\kappa_1 = -\frac{1}{2} \quad , \kappa_2 = \frac{1}{2} \quad .$$

To find e_1, solve the linear system $(\widehat{S}_p - \kappa_1 \mathbb{I}_2)e_1 = 0$:

$$\begin{pmatrix} 2^{-1} & 2^{-\frac{1}{2}} \\ 2^{-\frac{3}{2}} & 2^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff a + b\sqrt{2} = 0 \text{ so choose } e_1 = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$$

We won't worry about the length of e_1 yet. Similarly for e_2:

$$\begin{pmatrix} -2^{-1} & 2^{-\frac{1}{2}} \\ 2^{-\frac{3}{2}} & -2^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff a - b\sqrt{2} = 0 \text{ so choose } e_2 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$

Are these vectors orthogonal? They certainly don't *look* orthogonal. But you must remember that these 2×1 column vectors represent e_1 and e_2 with respect to the basis ϕ_u, ϕ_v. The vectors e_1, e_2 themselves lie in $T_pM \subset \mathbb{R}^3$, that is, they are 3-dimensional vectors. In fact,

$$\begin{aligned} e_1 &= \sqrt{2}(1,0,0) - (0,1,1) = (\sqrt{2}, -1, -1) \\ e_2 &= \sqrt{2}(1,0,0) + (0,1,1) = (\sqrt{2}, 1, 1) \end{aligned}$$

whence we see that $e_1 \cdot e_2 = 0$ as expected. Note that when we normalize them to have unit length, we must again think of them as vectors in $\mathbb{R}^3$, not 2×1 column matrices:

$$|(\sqrt{2}, -1, -1)| = |(\sqrt{2}, 1, 1)| = 2 \implies e_1 = \frac{1}{2}(\sqrt{2}, -1, -1), \; e_2 = \frac{1}{2}(\sqrt{2}, 1, 1)$$

$\square$

Example 108 (a cylinder) Recall (Example 98) that the shape operator S_p cylinder of radius R has matrix

$$\widehat{S}_p = \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}$$

This matrix obviously has eigenvalues $\kappa_1 = -\frac{1}{R}$ and eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The principle curvature directions at $p = \phi(u, v)$ are therefore

$$1 \times \phi_u + 0 \times \phi_v = (-R \sin u, R \cos u, 0), \quad 0 \times \phi_u + 1 \times \phi_v = (0, 0, 1).$$

We normalise these to get

$$e_1 = (-\sin u, \cos u, 0), \quad e_2 = (0, 0, 1).$$

Notice that as R grows large, $\kappa_1 = -\frac{1}{R}$ approaches zero. In this sense wide cylinders are "more curved" than narrow cylinders. □

Example 109 (unit sphere) Recall (Example 100) that in the case of the unit sphere, we found that $S_p(x) = -x$ for any $x \in T_pM$. Thus *any* non-zero vector $x \in T_pM$ is an eigenvector of S_p with eigenvalue -1. So S_p has a single eigenvalue -1, and $\kappa_1 = \kappa_2 = -1$. In this situation there are no preferred principal curvature directions. □

8.3 Normal curvature

Definition 110 Let M be a regularly parametrized surface. The **unit tangent space** at $p \in M$ is

$$U_pM = \{x \in T_pM : |x| = 1\},$$

the set of unit vectors in T_pM. Geometrically, we can think of U_pM as the unit circle in T_pM. □

Note that U_pM is *not* a vector space.

Definition 111 Let M be an oriented surface. The **normal curvature function** of M at $p \in M$ is

$$k_p : U_pM \to \mathbb{R}, \qquad k_p(x) = x \cdot S_p(x)$$

where S_p denotes the shape operator as usual. □

Why call this "normal curvature"?

Lemma 112 *Let M be a surface, let N be an orientation for M, and $x \in U_pM$. Let $\alpha : I \to M$ be any unit speed curve in M through p generating x. Then the component of the curvature vector of α in the direction of $N(p)$ is $k_p(x)$.*

Proof: We have a curve α in M with $\alpha(0) = p$, $\alpha'(0) = x$. Since α is a unit speed curve, its curvature vector at p is $k(0) = \alpha''(0)$. Now α stays in M, so its velocity is always tangent to M, and hence

$$\alpha'(t) \cdot N(\alpha(t)) = 0 \qquad . \qquad ((\clubsuit))$$

for all $t \in I$. Differentiate (♣) with respect to t and set $t = 0$:

$$\begin{aligned} \alpha''(0) \cdot N(\alpha(0)) + \alpha'(0) \cdot (N \circ \alpha)'(0) &= 0 \\ \implies k(0) \cdot N(p) + x \cdot (N \circ \alpha)'(0) &= 0. \end{aligned}$$

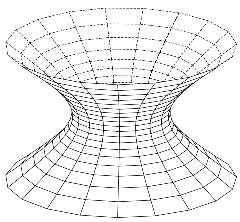
But $(N \circ \alpha)'(0) = \nabla_x N$ by the definition of directional derivatives (Definition 89). Hence

$$k(0) \cdot N(p) = -u \cdot \nabla_x N = u \cdot S_p(x) = k_p(x)$$

as was to be proved. □

Example 113 (hyperboloid of one sheet)
The hyperboloid

$$M = \{y \in \mathbb{R}^3 : y_1^2 + y_2^2 - y_3^2 = 1\}$$



may be parametrized by

$$\phi : (-\pi, \pi) \times \mathbb{R} \to \mathbb{R}^3,$$
$$\phi(u, v) = (\cosh v \cos u, \cosh v \sin u, \sinh v).$$

Then

$$\begin{aligned} \phi_u &= (-\cosh v \sin u, \cosh v \cos u, 0) \\ \phi_v &= (\sinh v \cos u, \sinh v \sin u, \cosh v) \\ \phi_u \times \phi_v &= (\cosh^2 v \cos u, \cosh^2 v \sin u, -\sinh v \cosh v) \\ |\phi_u \times \phi_v| &= \cosh v \sqrt{\cosh^2 v + \sinh^2 v} \\ &= \cosh v \sqrt{2\cosh^2 v - 1} \\ N &= \frac{1}{\sqrt{2\cosh^2 v - 1}} (\cosh v \cos u, \cosh v \sin u, -\sinh v) \end{aligned}$$

Let's calculate the shape operator at $p = \phi(0,0) = (1,0,0)$:

$$\begin{aligned}
\phi_u(0,0) &= (0,1,0) \\
\phi_v(0,0) &= (0,0,1) \\
S_p(\phi_u) &= -\nabla_{\phi_u} N(0,0) = \frac{1}{\sqrt{2-1}}(-\cosh 0 \sin 0, \cosh 0 \cos 0, -\sinh 0) \\
&= (0,1,0) \\
&= \phi_u \\
S_p(\phi_v) &= -\nabla_{\phi_u} N(0,0) = \frac{1}{\sqrt{2-1}}(\sinh 0 \cos 0, \sinh 0 \sin 0, -\cosh 0) + \sinh 0 \times \ldots \\
&= (0,0,-1) \\
&= -\phi_v
\end{aligned}$$

Now ϕ_u and ϕ_v are both unit vectors, so both belong to U_0M. Let's calculate their normal curvature:

$$\begin{aligned}
k_p(\phi_u) &= \phi_u \cdot S_p(\phi_u) = \phi_u \cdot \phi_u = (0,1,0) \cdot (0,1,0) = 1 \\
k_p(\phi_v) &= \phi_v \cdot S_p(\phi_v) = -\phi_v \cdot \phi_v = -(0,0,1) \cdot (0,0,1) = -1
\end{aligned}$$

In general, if $x = \cos\theta\phi_u + \sin\theta\phi_v = (0, \cos\theta, \sin\theta)$,

$$k_p(x) = x \cdot (0, \cos\theta, -\sin\theta) = \cos^2\theta - \sin^2\theta = \cos 2\theta$$

Notice that the maximum and minimum values of k_p are $\pm 1 = \kappa_1, \kappa_2 \ldots$ $\square$

Normal curvature allows us to give a new interpretation of the principal curvatures of a surface.

Theorem 114 *Let M be an oriented surface and κ_1, κ_2 be its principal curvatures at $p \in M$, ordered so that $\kappa_1 \leq \kappa_2$. Then*

$$\begin{aligned}\kappa_1 &= \min\{k_p(x) : x \in U_pM\} \\ \kappa_2 &= \max\{k_p(x) : x \in U_pM\}.\end{aligned}$$

Proof: Let e_1, e_2 be the orthonormal pair of eigenvectors corresponding to κ_1, κ_2 (see Theorem 105). These span T_pM so any unit vector $x \in U_pM$ may be written

$$x = \cos\theta\, e_1 + \sin\theta\, e_2$$

for some angle θ. The normal curvature of x is

$$\begin{aligned}k_p(x) &= (\cos\theta\, e_1 + \sin\theta\, e_2) \cdot S_p(\cos\theta\, e_1 + \sin\theta\, e_2) \\ &= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta \qquad (\spadesuit) \\ &= \kappa_1 + (\kappa_2 - \kappa_1)\sin^2\theta.\end{aligned}$$

Now $\kappa_2 - \kappa_1 \geq 0$, and the maximum and minimum values of $\sin^2\theta$ are 1 and 0, so the maximum and minimum values of $k_p(x)$ are κ_2 and κ_1 respectively. $\square$

Example 115 (hyperboloid of one sheet) For M and p as in Example 113, construct vectors $x \in U_pM$ with the following properties:

$$\text{(a)} \quad k_p(x) = 0, \quad \text{(b)} \quad k_p(x) = 2, \quad \text{(c)} \quad S_p(x) = 0.$$

(a) Recall that we calculated $k_p(0, \cos\theta, \sin\theta) = \cos 2\theta$. So $k_p = 0$ is solved by

$$2\theta = \frac{\pi}{2} + n\pi \iff \theta = \frac{\pi}{4} + \frac{n\pi}{2} = \ldots, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots.$$

So we obtain four solutions:

$$x = \frac{1}{\sqrt{2}}(0,1,1),\ \frac{1}{\sqrt{2}}(0,-1,1),\ \frac{1}{\sqrt{2}}(0,-1,-1),\ \frac{1}{\sqrt{2}}(0,1,-1).$$

(b) No such u can exist by Theorem 114. For all $x \in U_pM$, $-1 \leq k_p(x) \leq 1$.

(c) Again, no such x can exist. If it did, then since $x \neq 0$, it follows that 0 is an eigenvalue of S_p, and hence a principle curvature. But the principal curvatures are -1 and 1, not 0. $\square$

8.4 Mean and Gauss curvatures

Definition 116 The **mean curvature** of M at $p \in M$ is

$$H(p) = \frac{1}{2}(\kappa_1 + \kappa_2)$$

where κ_1, κ_2 are the principal curvatures of M at p. The **Gauss curvature** of M at p is

$$K(p) = \kappa_1\kappa_2 \qquad \square$$

One need not solve the eigenvalue problem for S_p to compute $H(p)$ and $K(p)$:

Proposition 117 *For all $p \in M$,*

$$H(p) = \frac{1}{2}\mathrm{tr}\,\widehat{S}_p, \quad K(p) = \det \widehat{S}_p,$$

where $\widehat{S}_p$ is the matrix representing $S_p : T_pM \to T_pM$ relative to some choice of basis for T_pM.

[Recall that the **trace** of a square matrix L is the sum of its diagonal elements.]

Proof: Let $\widehat{S}_p = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ relative to our chosen basis. Then κ_1, κ_2 are roots of the polynomial

$$p(\lambda) = \begin{vmatrix} S_{11} - \lambda & S_{12} \\ S_{21} & S_{22} - \lambda \end{vmatrix} = \lambda^2 - (S_{11} + S_{22})\lambda + (S_{11}S_{22} - S_{12}S_{21}) = \lambda^2 - \mathrm{tr}\,\widehat{S}_y\,\lambda + \det \widehat{S}_y. \quad (\clubsuit)$$

But $p(\kappa_1) = p(\kappa_2) = 0$ and the coefficient of λ^2 is one, so

$$p(\lambda) = (\lambda - \kappa_1)(\lambda - \kappa_2) = \lambda^2 - (\kappa_1 + \kappa_2)\lambda + \kappa_1\kappa_2. \quad (\spadesuit)$$

Comparing coefficients of λ and unity in ($\spadesuit$) and ($\clubsuit$), we see that $\kappa_1 + \kappa_2 = \mathrm{tr}\,\widehat{S}_p$ while $\kappa_1\kappa_2 = \det \widehat{S}_p$. $\square$

Example 118 (a) The unit sphere has shape operator $S_p : v \mapsto -v$ at every point p (see Example 100). Hence, relative to *any* basis for T_pM,

$$\widehat{S}_p = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It follows that

$$\begin{aligned} H(p) &= \frac{1}{2}(-1 \,+\, -1) = -1 \\ K(p) &= (-1)(-1) = 1 \end{aligned}$$

for all $p \in M$.

(b) For the cylinder of radius R (Example 98), the shape operator has matrix $\widehat{S}_p = \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}$. So at any point p,

$$H(p) = \frac{1}{2}\left(-\frac{1}{R} + 0\right) = -\frac{1}{2R}, \quad K(p) = -\frac{1}{R} \times 0 = 0$$

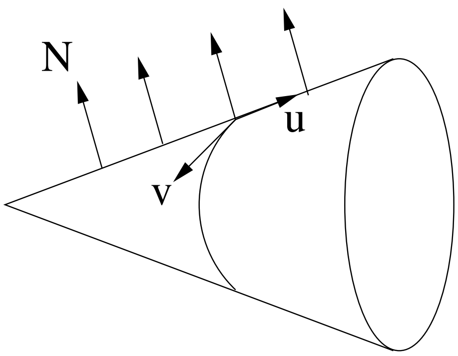
The cylinder has the interesting property that it can be made by rolling up a sheet of paper. It is a deep and amazing fact that a surface can be made by bending a flat sheet of paper paper **if and only if** its Gauss curvature vanishes everywhere. This is a consequence of Gauss' Theorema Egregrium. You can learn more about this remarkable theorem by studying MATH3113 Differential Geometry or MATH5113 Advanced Differential Geometry.

(c) For the hyperboloid (Example 113), the principle curvatures at $p = (1, 0, 0)$ where $\kappa_1 = -1$ and $\kappa_2 = 1$. So

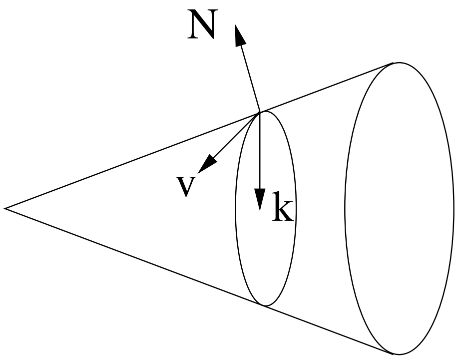
$$H(1,0,0) = \frac{1}{2}(-1+1) = 0, \quad K(1,0,0) = -1 \times 1 = -1$$

$\square$

Example 119 Let M = a cone, outwardly oriented. What can we deduce about K and H just by *looking* at the surface?



Pick orthonormal pair u, v at any point $p \in M$. Then N is constant along the straight line in M generating u $\Rightarrow$ $\nabla_u N = 0$ $\Rightarrow$ $S_p(u) = 0$. Hence, u is an eigenvector of S_p, with eigenvalue 0, so at least one principal curvature $\kappa_2 = 0$, and u is the corresponding principal curvature direction. Hence $K(p) = \kappa_1\kappa_2 = 0$.



The other principal curvature direction is orthogonal to u, so must be v (or $-v$). Hence, v is an eigenvector of S_p with eigenvalue κ_2, the other principal curvature. So

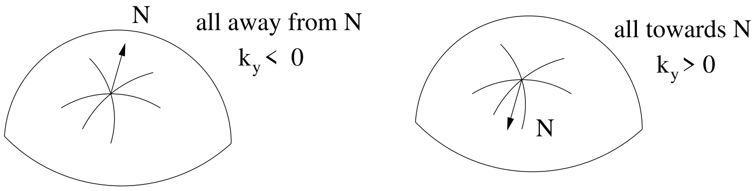
$$k_p(v) = v \cdot S_p(v) = v \cdot (\kappa_1 v) = \kappa_1$$

But $k_p(v) = k \cdot N < 0$ since the circle generating v curves *away* from the outward unit normal. Hence $\kappa_2 < 0$, so $H(p) = (\kappa_1 + \kappa_2)/2 = \kappa_2/2 < 0$ also. $\square$

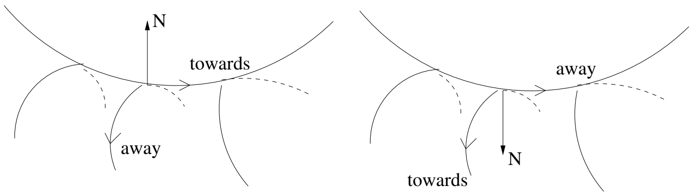
Fact 120 If we change orientation $N \mapsto \widetilde{N} = -N$ then $S_p \mapsto \widetilde{S}_p = -S_p \quad \Rightarrow \quad \kappa_i \mapsto \widetilde{\kappa}_i = -\kappa_i \quad \Rightarrow \quad H \mapsto \widetilde{H} = -H$ **but** $K \mapsto \widetilde{K} = K$. The only measure of curvature we've introduced so far which is independent of the choice of orientation on M is the *Gauss curvature*. For all other types of curvature (including normal curvature), the *sign* of the curvature has no intrinsic meaning (i.e. no meaning independent of our more or less arbitrary choice of orientation).

The sign of K **does** have an intrinsic geometric meaning!

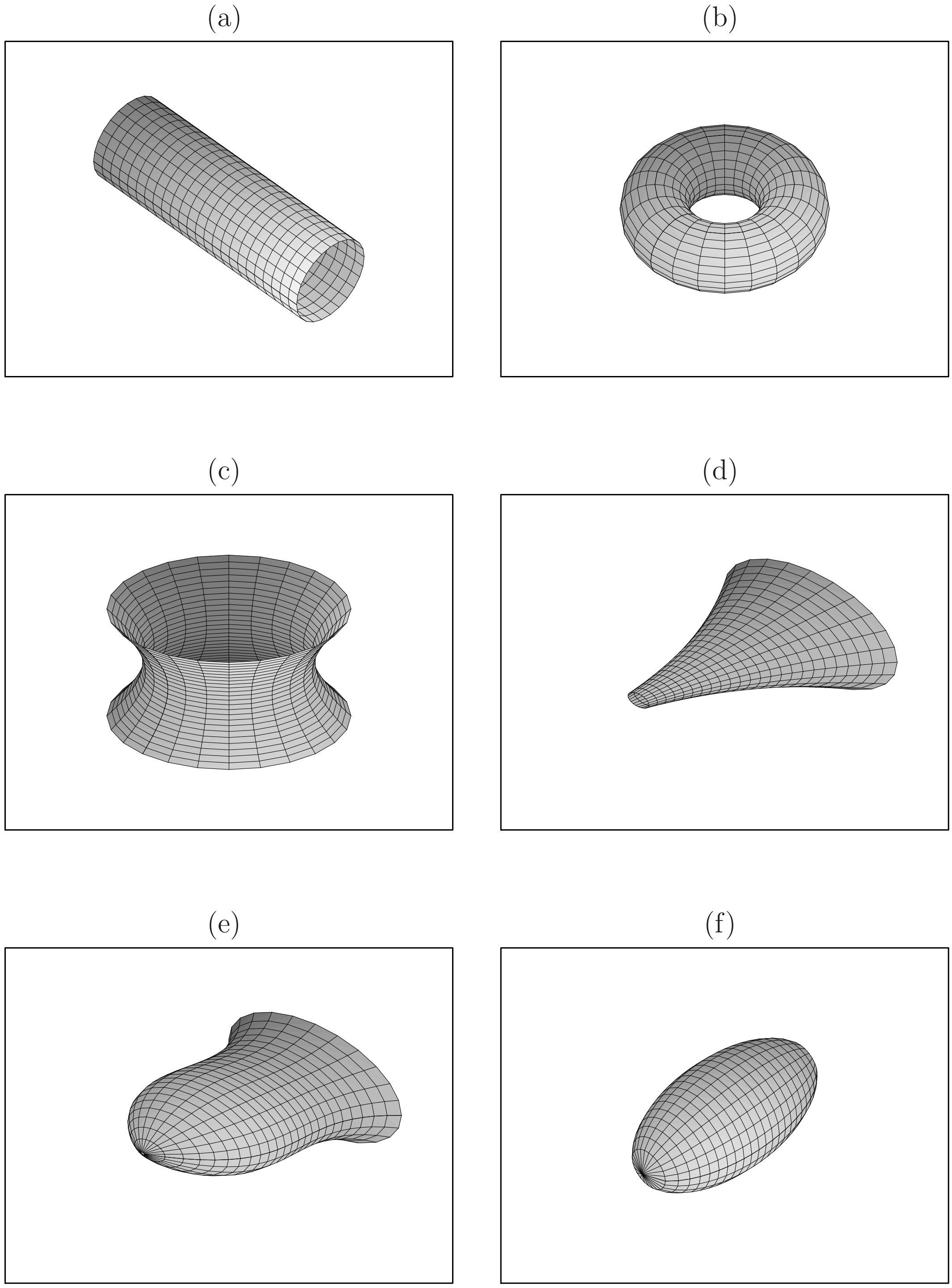
$K(p) > 0 \Rightarrow \kappa_1, \kappa_2$ have same sign
$\Rightarrow$ $k_p(u)$ strictly positive (if $\kappa_1, \kappa_2 > 0$) <u>or</u>
$k_p(u)$ strictly negative (if $\kappa_1, \kappa_2 < 0$)
$\Rightarrow$ all curves through p curve **towards** $N(p)$ <u>or</u>
all curves through p curve **away from** $N(p)$
$\Rightarrow$ all curves curve in the same "sense".



$$
\begin{aligned}
K(p) < 0 \quad &\Rightarrow \quad \kappa_1, \kappa_2 \text{ differ in sign} \\
&\Rightarrow \quad k_p(u) \text{ takes both positive \underline{and} negative values} \\
&\Rightarrow \quad \exists \text{ curves through } p \text{ which curve in opposite senses.}
\end{aligned}
$$



Example 121 Let's test our intuition on the following surfaces:


(a)
(b)
(c)
(d)
(e)
(f)

Summary

- An **oriented** surface is a RPS together with a choice of unit normal vector field N. Usually we choose
$$N = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|}.$$

- The **shape operator** of an oriented surface is
$$S_p : T_pM \to T_pM, \qquad S_p(x) = -\nabla_x N.$$
The shape operator is **linear**,
$$S_p(ax + by) = aS_p(x) + bS_p(y) \quad \forall a, b \in \mathbb{R},\ x, y \in T_pM,$$
and **self adjoint**,
$$x \cdot S_p(y) = y \cdot S_p(x) \quad \forall x, y \in T_pM.$$

- The **principal curvatures** of M at p are the eigenvalues κ_1, κ_2 of S_p. The **principal curvature directions** are the corresponding (normalized) eigenvectors.

- The **normal curvature** of a **unit vector** $x \in T_pM$ is
$$k_p(x) = x \cdot S_p(x).$$
This coincides with $k(0) \cdot N(p)$ where $k(t)$ is the curvature vector of any generating curve for x. The principal curvatures are the maximum and minimum values of $k_p(x)$ as x takes all values in the unit tangent space at p.

- The **mean curvature** at p is
$$H = \frac{1}{2}(\kappa_1 + \kappa_2).$$

- The **Gauss curvature** at p is
$$K = \kappa_1 \kappa_2.$$
The sign of K has intrinsic meaning, independent of the choice of N: if $K(p) > 0$ then either all curves in M through p curve towards $N(p)$, or they all curve away from $N(p)$; if $K(p) < 0$ then some curves curve towards $N(p)$ and some curve away.

- Both H and K can be computed directly from any matrix $\widehat{S}_p$ representing S_p:
$$H = \frac{1}{2}\operatorname{tr} \widehat{S}_p, \qquad K = \det \widehat{S}_p.$$

This question paper consists of
4 printed pages, each of which
is identified by the reference MATH205101.

All calculators must carry
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School of Mathematics

January 2018

MATH205101

Geometry of Curves and Surfaces

Time Allowed: 2 hours

You must attempt to answer 4 questions. If you answer more than 4 questions, only your best 4 answers will be counted towards your final mark for this exam.

1. (a) Determine whether the mapping $\alpha : \mathbb{R} \to \mathbb{R}^2$, $\alpha(t) = (\cos^2 t, \sin t)$, is a regularly parametrized curve. Clearly explain your reasoning.

(b) Consider the regularly parametrized curve $\beta : \mathbb{R} \to \mathbb{R}^4$, $\beta(t) = (t, \cos 2t, \sin 2t, 7)$.

(i) Compute $\sigma_0 : \mathbb{R} \to \mathbb{R}$, its signed arclength function based at $t_0 = 0$.

(ii) Hence, or otherwise, construct a unit speed reparametrization of β.

(c) Consider the regularly parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^3$, $\beta(t) = (t, t^2, \frac{1}{3}t^3)$.

(i) Construct $\widehat{\gamma}_1$, its tangent line at $t_0 = 1$.

(ii) Compute $k(1)$, its curvature vector at time $t = 1$.

(iii) Construct $[u(1), n(1), b(1)]$, the Frenet frame for γ at time $t = 1$.

(iv) How many points on γ lie exactly distance $\sqrt{2}$ from $(0, 0, 0)$? Briefly explain your reasoning.

2. Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be the parametrized curve with $\gamma(0) = (1, 0, 0)$ and $\gamma'(0) = (0, 1, 0)$ which satisfies the ordinary differential equation

$$\gamma''(s) = \gamma(s) \times \gamma'(s).$$

Denote by $\kappa : \mathbb{R} \to [0, \infty)$ its scalar curvature.

(a) Show that γ is a unit speed curve.

(b) Compute $\kappa(0)$.

(c) Show that $\gamma(s) \cdot \gamma'(s) = s$.

(d) Show that $|\gamma(s)|^2 = s^2 + 1$.

(e) Deduce that κ is constant.
(You may use without proof the vector identity $|b \times c|^2 = |b|^2|c|^2 - (b \cdot c)^2$.)

(f) Show that the binormal vector of γ is $b(s) = \gamma(s) - s\gamma'(s)$.
(You may use without proof the vector identity $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$.)

(g) Deduce a formula for $\tau : \mathbb{R} \to \mathbb{R}$, the torsion of γ.

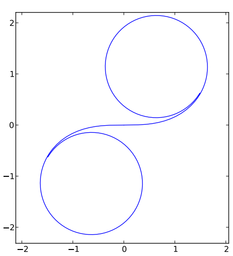
3. Given a prescribed function $\kappa : \mathbb{R} \to \mathbb{R}$, there exists a unique unit speed curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ with $\gamma(0) = (0,0)$, $\gamma'(0) = (1,0)$ and signed curvature κ. The curves corresponding to the signed curvature functions

$$\kappa_1(s) = \frac{6s}{1+s^4}, \quad \kappa_2(s) = \tanh s, \quad \kappa_3(s) = \frac{1}{4} + \cos s, \quad \kappa_4(s) = 1 + \tanh s,$$

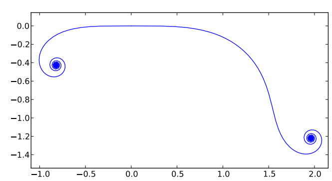
$$\kappa_5(s) = 4s\sin(s^2), \quad \kappa_6(s) = s^2 - 4, \quad \kappa_7(s) = s^3 - 2s^2,$$

are depicted below in the wrong order, figures (A) to (G):

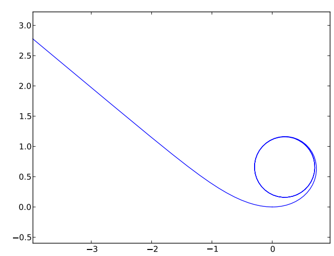
(A)



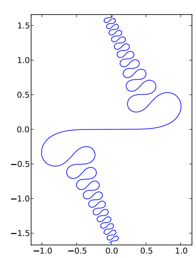
(B)



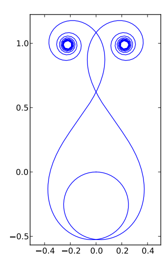
(C)



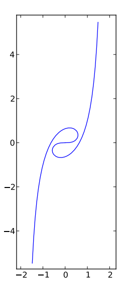
(D)



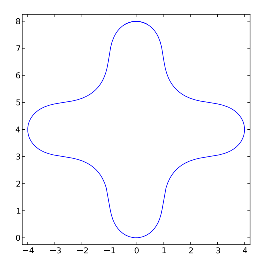
(E)



(F)



(G)



(a) Determine which curve corresponds to which signed curvature. In each case, briefly explain your reasoning. (Unexplained answers will not receive full credit.)

(b) One, and only one, of the curves depicted has a globally defined evolute. Identify this curve, explaining your reasoning. Is the evolute regular?

(c) Consider a general parallel curve

$$\gamma_\lambda(t) = \gamma(t) + \lambda n(t), \qquad \lambda \in \mathbb{R} \text{ a constant},$$

to the curve γ with signed curvature κ_2. Determine the set of values of λ for which γ_λ is regular.

(d) What is the total arclength around one circuit of the closed curve labelled (G)?

4. (a) Define the following terms:

(i) An *open disk* in $\mathbb{R}^2$.

(ii) An *open subset* U of $\mathbb{R}^2$.

(iii) A *regular mapping* $M : U \to \mathbb{R}^3$.

(iv) A *regularly parametrized surface* $M : U \to \mathbb{R}^3$.

(b) Let M denote the mapping

$$M : \mathbb{R}^2 \to \mathbb{R}^3, \qquad M(x_1, x_2) = (x_1, x_1^2 + x_2^3, x_2 e^{x_1}),$$

$p = (0, 8, 2)$ and $v = (-2, 12, -3)$.

(i) Show that M is a regularly parametrized surface.

(ii) Show that p lies on M and write down its local coordinates.

(iii) Construct bases for the tangent space T_pM and normal space N_pM to M at p.

(iv) Show that v is tangent to M at p.

(v) Consider the function $f : M \to \mathbb{R}$, $f(y_1, y_2, y_3) = y_1 + y_2 + y_3$. Compute the directional derivative $v[f]$.

(vi) Construct a non-zero vector in T_pM which is orthogonal to v.

5. Let M denote the mapping $M : \mathbb{R}^2 \to \mathbb{R}^3$, $M(x_1, x_2) = (x_1, x_2, \sin(x_1 x_2))$ and $p = (0, \pi, 0)$. You are given that M is a regularly parametrized surface and p is a point on M. Let $\varepsilon_1, \varepsilon_2$ denote the coordinate basis vectors for M, and $S_p : T_pM \to T_pM$ denote the shape operator for M at p.

(a) Construct the canonical unit normal $N(x_1, x_2)$ on M.

(b) Show that $S_p(\varepsilon_1) = \dfrac{1}{\sqrt{1+\pi^2}} \varepsilon_2$.

(c) You are given that $S_p(\varepsilon_2) = \dfrac{1}{(1+\pi^2)^{3/2}} \varepsilon_1$. Construct the matrix $\widehat{S}_p$ representing the linear map S_p with respect to the basis $\varepsilon_1, \varepsilon_2$.

(d) Compute the mean curvature $H(p)$ and the Gauss curvature $K(p)$ of M at p.

(e) Compute the principal curvatures of M at p.

(f) You are given that u_1, the principal curvature direction corresponding to κ_1, the smaller of the principal curvatures, is

$$u_1 = \pm \frac{\left(1, -\sqrt{1+\pi^2}, \pi\right)}{\sqrt{2}\sqrt{1+\pi^2}}.$$

Deduce the other principal curvature direction, u_2.

(g) In each of the following cases, *either* construct a vector in T_pM with the specified properties *or* explain why no such vector exists:

(i) A unit vector v such that $v \cdot S_p(v) = 0$.

(ii) A unit vector v such that $v \cdot S_p(v) = 1$.

End.

Module Title: Geometry of curves and surfaces

School of Mathematics **Semester One 201819**

Calculator instructions:

- You are allowed to use a calculator which has had an approval sticker issued by the School of Mathematics.

Exam information:

- There are 4 pages to this exam.
- There will be **2 hours** to complete this exam.
- Answer all questions.
- All questions are worth equal marks.

1. (a) Say whether each of the following is a regularly parametrised curve. Support your answer with reasons.

i. $\gamma : \mathbb{R} \to \mathbb{R}^2, \quad \gamma(t) = (t^2, t^3 - t^2)$.

ii. $\gamma : \mathbb{R} \to \mathbb{R}^2, \quad \gamma(t) = (t^2, t^3 - t)$.

(b) Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be the regularly parametrised curve $\gamma(t) = (t, \cosh t)$.

i. Calculate the signed arclength function $\sigma_0 : \mathbb{R} \to \mathbb{R}$ for this curve.

ii. Calculate the tangent line $\widehat{\gamma}_1 : \mathbb{R} \to \mathbb{R}^2$ for this curve at $t = 1$.

(c) Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve with non-vanishing curvature.

i. Define the **unit tangent vector** $u(s)$, the **principal unit normal vector** $n(s)$, and the **binormal vector** $b(s)$.

ii. State the **Frenet formulae** for this curve.

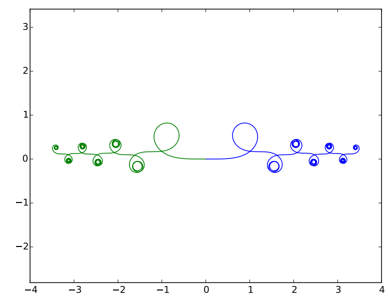
iii. Suppose that $\tau = 0$. Show that the curve γ is planar.

2. (a) The curves corresponding to the following signed curvature functions:

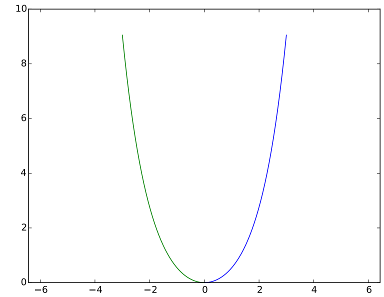
$$\kappa_1(s) = 2s\sin(s) \qquad \kappa_2(s) = \tanh(s)$$
$$\kappa_3(s) = 4s - s^3 \qquad \kappa_4(s) = \frac{1}{1+s^2}$$

are depicted below in the wrong order, figures (A) to (D).

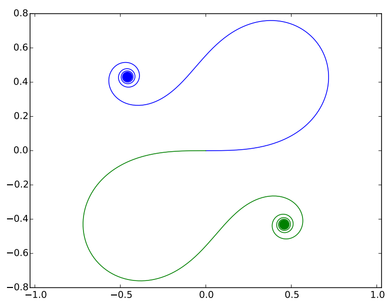
(A)



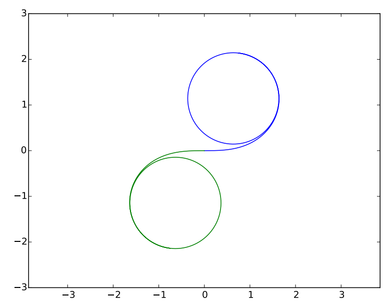
(B)



(C)



(D)



Construct a table determining whether the functions $\kappa_1, \dots, \kappa_4$ are even or odd, the resulting symmetry of the curve, and the number of inflection points. Hence determine which curve corresponds to which signed curvature.

(b) Let $\gamma : I \longrightarrow \mathbb{R}^2$ be a planar curve with curvature vector $k : I \longrightarrow \mathbb{R}^2$.

i. Define the following:
   - the **unit tangent vector** $u(t)$ of γ
   - the **unit normal vector** $n(t)$ of γ
   - the **signed curvature** $\kappa(t)$ of γ
   - the **evolute** $E_\gamma : I \longrightarrow \mathbb{R}^2$ of γ.

ii. Show that $n' = -\kappa\gamma'$.

iii. Show that
$$E'_\gamma(t) = -\frac{\kappa'(t)}{\kappa(t)^2}n(t).$$

iv. Assuming that $\kappa'(t) < 0$, show that the arclength along E_γ from t_1 to t_2 is
$$\frac{1}{\kappa(t_2)} - \frac{1}{\kappa(t_1)}.$$

3. (a) Let $M : U \to \mathbb{R}^3$ be a smooth mapping. Say what is meant by

i. $(x_1, x_2) \in U$ is a **regular point of** M;

ii. M is a **regularly parametrised surface**.

(b) Let $M : U \to \mathbb{R}^3$ be a regularly parametrised surface. Let $V : M \to \mathbb{R}^3$ be a vector field and let $y \in M$. Say what is meant by

i. the **tangent space** $T_y M$ **at** y,

ii. V is a **tangential vector field**,

iii. $\nabla_w V$ is the **directional derivative of** V **with respect to** w, where $w \in T_y M$,

iv. $\nabla_W V$ is the **directional derivative of** V **with respect to** W, where W is a tangential vector field.

(c) Let $M : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$M(x_1, x_2) = (x_2 \, , \, x_1 - x_2^2 \, , \, x_1^3 + x_1 x_2)$$

i. Find the coordinate vector fields ϵ_1, ϵ_2 in terms of x_1, x_2.

ii. Show that M is a regularly parametrised surface.

iii. Compute $\nabla_{\epsilon_1}\epsilon_1$, $\nabla_{\epsilon_1}\epsilon_2$, $\nabla_{\epsilon_2}\epsilon_1$ and $\nabla_{\epsilon_2}\epsilon_2$.

iv. Consider the vector field V on M with coordinate expression

$$\hat{V}(x_1, x_2) = (1 \, , \, 1 - 2x_2 \, , \, 3x_1^2 + x_1 + x_2).$$

Show that V is a tangential vector field and compute the vector field $\nabla_V V$. [You may use standard properties of ∇.]

4. (a) Let $M : U \to \mathbb{R}^3$ be a regularly parametrized surface and let y be a point of M. Define what is meant by

i. an **orientation** on M;

ii. the **shape operator** at y;

iii. the **principal curvatures** at y;

iv. the **principal curvature directions** at y;

v. the **Gauss curvature** at y;

vi. the **mean curvature** at y.

(b) The following mapping $M : \mathbb{R}^2 \to \mathbb{R}^3$ defines a regularly parametrised surface:

$$M(x_1, x_2) = \left(x_1, x_2, \frac{1}{2}(x_1^2 + x_2^2) \right).$$

Let y be the point with coordinates $(x_1, x_2) = (1, 0)$.

i. Calculate the matrix of the shape operator at y.

ii. Calculate the principal curvatures, the Gauss curvature and the mean curvature at y.

iii. Does there exist a regular parametrised curve in M passing through y whose curvature vector $k = (k_1, k_2, k_3)$ at y satisfies $k_1 - k_3 = 0$? Justify your answer.

Check Sheet

1. (a) No it isn't, since $\alpha'(\pi/2) = (0,0)$.

(b) (i) $\sigma_0(t) = \sqrt{5}t$.

(ii)

(c) (i) $\widehat{\gamma}_1(t) = (1,1,\frac{1}{3}) + t(1,2,1)$.

(ii)

$$k(1) = \frac{1}{6}\left((0,2,2) - \frac{6}{6}(1,2,1)\right) = \frac{1}{6}(-1,0,1)$$

(iii) $u(1) = \gamma'(1)/|\gamma'(1)| = \frac{1}{\sqrt{6}}(1,2,1)$
$n(1) = k(1)/|k(1)| = \frac{1}{\sqrt{2}}(-1,0,1)$
$b(1) = u(1) \times n(1) = \frac{1}{\sqrt{3}}(1,-1,1)$

(iv) Two

2. (a)

(b) 1.

(c)

(d)

(e)

(f)

(g) $\tau(s) = s$

3. (a)

κ_1	κ_2	κ_3	κ_4	κ_5	κ_6	κ_7
F	A	G	C	D	E	B

A	B	C	D	E	F	G
κ_2	κ_7	κ_4	κ_5	κ_6	κ_1	κ_3

(b) (C). Yes, the evolute is regular.

(c) γ_λ is regular if and only if $\lambda \in [-1,1]$.

(d) 8π

4. (a) (i)

(ii)

(iii)

(iv)

(b) (i)

(ii) $(0,2)$

(iii) T_pM is spanned by $\{\varepsilon_1, \varepsilon_2\} = \{(1,0,2),(0,12,1)\}$
N_pM is spanned by $\nu = \varepsilon_1 \times \varepsilon_2 = (-24,-1,12)$

(iv)

(v) $v[f] = 7$

(vi) Any non-zero multiple of $\nu \times v = (-141,-96,-290)$ will do

5. (a) $N = \dfrac{(-x_2 \cos x_1 x_2, -x_1 \cos x_1 x_2, 1)}{\sqrt{1 + (x_1^2 + x_2^2)\cos^2 x_1 x_2}}$

(b)

(c) $\widehat{S}_p = \begin{pmatrix} 0 & (1+\pi^2)^{-3/2} \\ (1+\pi^2)^{-1/2} & 0 \end{pmatrix}$

(d)

$$\begin{aligned} H(p) &= \frac{1}{2}\mathrm{tr}\, \widehat{S}_p = 0 \\ K(p) &= \det \widehat{S}_p = -\frac{1}{(1+\pi^2)^2}. \end{aligned}$$

(e) $\kappa_1 = -\dfrac{1}{1+\pi^2}$, $\kappa_2 = \dfrac{1}{1+\pi^2}$.

(f) $u_2 = \pm N \times u_1 = \pm \dfrac{\left(1, \sqrt{1+\pi^2}, \pi\right)}{\sqrt{2}\sqrt{1+\pi^2}}$.

(g) (i) $v = \dfrac{1}{\sqrt{2}}(u_1 \pm u_2) = \dfrac{(1, 0, \pi)}{\sqrt{1+\pi^2}}$ or $(0, 1, 0)$, or minus these

(ii) No such v exists

Check sheet: Geometry of curves and surfaces

School of Mathematics **Semester One 201819**

1. (a) i. This is NOT a RPC.

ii. This IS a RPC.

(b) i. $\sigma_0(t) = \sinh t$

ii. $\widehat{\gamma}_1(t)(1, \cosh 1) + t(1, \sinh 1)$

(c) bookwork

2. (a)

κ_1	even	reflection	infinitely many	(A)
κ_2	odd	$180°$ rotation	one	(D)
κ_3	odd	$180°$ rotation	three	(C)
κ_4	even	reflection	none	(B)

(b) bookwork

3. (a) bookwork

(b) bookwork

(c) i.

$$\epsilon_1 = (0, 1, 3x_1^2 + x_2)$$
$$\epsilon_2 = (1, -2x_2, x_1)$$

ii. proof – omitted

iii.

$$\nabla_{\epsilon_1}\epsilon_1 = (0, 0, 6x_1)$$
$$\nabla_{\epsilon_1}\epsilon_2 = (0, 0, 1)$$
$$\nabla_{\epsilon_2}\epsilon_1 = (0, 0, 1)$$
$$\nabla_{\epsilon_2}\epsilon_2 = (0, -2, 0)$$

iv. $V = \epsilon_1 + \epsilon_2$. $\nabla_V V = (0, -2, 2 + 6x_1)$.

4. (a) bookwork

(b) i. In the standard coordinate basis ϵ_1, ϵ_2 the matrix is $\begin{pmatrix} -2^{-\frac{3}{2}} & 0 \\ 0 & -2^{-\frac{1}{2}} \end{pmatrix}$. (You might get a different answer if you chose a different basis.)

ii. $\kappa_1 = -2^{-\frac{1}{2}}$, $\kappa_2 = -2^{-\frac{3}{2}}$, Gauss curvature $= \frac{1}{4}$, mean curvature $= -3/2^{\frac{5}{2}}$.

iii. No.

 End.