Chapter 2: Vector Differentiation

Basic Vector Differentiation

(1) If $\boldsymbol{F} = \boldsymbol{F}(t) = (F_1, F_2, F_3)$ then

$$\frac{d\boldsymbol{F}}{dt} = \left(\frac{dF_1}{dt}, \frac{dF_2}{dt}, \frac{dF_3}{dt}\right)$$

(2) The unit tangent to the curve $\mathbf{r}(s)$ is given by

$$\frac{d\mathbf{r}/ds}{|d\mathbf{r}/ds|}$$

Grad, Div and Curl

(3) The *gradient* of a scalar field $f(x, y, z)$ $(= f(x_1, x_2, x_3))$ is given by

$$\text{grad} f = \boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$$

(4) $\boldsymbol{\nabla} f$ is the vector field with a direction perpendicular to the isosurfaces of f with a magnitude equal to the rate of change of f in that direction.

(5) The *directional derivative* of f in the direction of a unit vector $\hat{\boldsymbol{u}}$ is $(\boldsymbol{\nabla} f) \cdot \hat{\boldsymbol{u}}$

(6) $\boldsymbol{\nabla}$ pronounced *del* or *nabla* is a *vector differential operator*. It is possible to study the 'algebra of $\boldsymbol{\nabla}$'.

(7) The *divergence* of a vector field $\boldsymbol{F}$ is given by

$$\text{div}\, \boldsymbol{F} = \boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$$

(8) A vector field $\boldsymbol{F}$ is *solenoidal* if $\boldsymbol{\nabla}\cdot\boldsymbol{F} = 0$ everywhere.

(9) The *curl* of a vector field $\boldsymbol{F}$ is given by

$$\begin{aligned}
\text{curl}\, \boldsymbol{F} = \boldsymbol{\nabla}\times\boldsymbol{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{e}_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{e}_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{e}_3 \\
&= \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)\mathbf{e}_3 \\
&= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix}
\end{aligned}$$

(10) A vector field $\boldsymbol{F}$ is *irrotational* if $\boldsymbol{\nabla}\times\boldsymbol{F} = 0$ everywhere.

(11) $(\boldsymbol{F} \cdot \boldsymbol{\nabla})$ is a vector differential operator which can act on a scalar or a vector

$$(\boldsymbol{F} \cdot \boldsymbol{\nabla})\, f = F_1\frac{\partial f}{\partial x} + F_2\frac{\partial f}{\partial y} + F_3\frac{\partial f}{\partial z}$$

For Cartesian co-ordinates only

$$(\boldsymbol{F} \cdot \boldsymbol{\nabla})\boldsymbol{G} = ((\boldsymbol{F} \cdot \boldsymbol{\nabla})\, G_1\, ,\, (\boldsymbol{F} \cdot \boldsymbol{\nabla})\, G_2\, ,\, (\boldsymbol{F} \cdot \boldsymbol{\nabla})\, G_3)$$

(12) The *Laplacian* operator $\nabla^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$ is given, in Cartesian co-ordinates, by $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$.

Chapter 3 summary: Index Notation

(1) Any index may only appear *once* or *twice* in any term in an equation

(2) A index that appears just once is called a *free index*.
The free indices must be the same on both sides of the equation.
Free indices take the values 1, 2 and 3

(3) A index that appears twice is called a *dummy index*.
Summation Convention: Dummy indices are summed over from 1 to 3
The name of a dummy index is not important.

$$\mathbf{a}\cdot\mathbf{b} = a_j b_j = a_l b_l = a_p b_p = a_1 b_1 + a_2 b_2 + a_3 b_3$$

(4) The Kronecker Delta:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

or

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Kronecker Delta is *symmetric* $\delta_{ij} = \delta_{ji}$. If one index on δ_{ij} is free and the other dummy then the action of δ_{ij} is to substitute the dummy index with the free index

$$\delta_{ij} a_j = a_i$$

If both indices are dummies then the δ_{ij} acts as scalar product.

$$\delta_{ij} a_i b_j = \mathbf{a}\cdot\mathbf{b}$$

(5) The Alternating Tensor:

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any of } i,\ j \text{ or } k \text{ are equal,} \\ 1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) = (1,3,2),\ (3,2,1) \text{ or } (2,1,3) \end{cases}$$

The Alternating Tensor is *antisymmetric*:

$$\epsilon_{ijk} = -\epsilon_{jik}$$

The Alternating Tensor is invariant under cyclic permutations of the indices:

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$$

(6) The vector product:

$$(\mathbf{a}\times\mathbf{b})_i = \epsilon_{ijk}\, a_j b_k$$

(7) The relation between δ_{ij} and ϵ_{ijk}:

$$\epsilon_{ijk}\,\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

Grad, Div and Curl and index notation

$$(\boldsymbol{\nabla})_i = \frac{\partial}{\partial x_i} \equiv \partial_i$$

$$\text{grad} f = (\boldsymbol{\nabla} f)_i = \frac{\partial f}{\partial x_i} = \partial_i f$$

$$\text{div}\, \boldsymbol{F} = \boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{\partial F_j}{\partial x_j} = \partial_j\, F_j$$

$$(\text{curl}\, \boldsymbol{F})_i = (\boldsymbol{\nabla}\times\boldsymbol{F})_i = \epsilon_{ijk} \frac{\partial F_k}{\partial x_j} = \epsilon_{ijk}\, \partial_j\, F_k$$

$$(\boldsymbol{F} \cdot \boldsymbol{\nabla}) = F_j \frac{\partial}{\partial x_j} = F_j\, \partial_j$$

Note: Here you cannot move the ∂_j around as it acts on everything that follows it.

Vector Differential Identities.
If $\boldsymbol{F}$ and $\boldsymbol{G}$ are vector fields and f and g are scalar fields then

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} f) = \nabla^2 f$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{F}) = 0$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla} f) = \boldsymbol{0}$$

$$\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f$$

$$\boldsymbol{\nabla}\cdot(f\boldsymbol{F}) = f\boldsymbol{\nabla}\cdot\boldsymbol{F} + \boldsymbol{F}\cdot\boldsymbol{\nabla} f$$

$$\boldsymbol{\nabla}\times(f\boldsymbol{F}) = f\boldsymbol{\nabla}\times\boldsymbol{F} + \boldsymbol{\nabla} f \times \boldsymbol{F}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) - \nabla^2\boldsymbol{F}$$

$$\boldsymbol{\nabla}(\boldsymbol{F}\cdot\boldsymbol{G}) = \boldsymbol{F}\times(\boldsymbol{\nabla}\times\boldsymbol{G}) + \boldsymbol{G}\times(\boldsymbol{\nabla}\times\boldsymbol{F}) + (\boldsymbol{F}\cdot\boldsymbol{\nabla})\boldsymbol{G} + (\boldsymbol{G}\cdot\boldsymbol{\nabla})\boldsymbol{F}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{F}\times\boldsymbol{G}) = \boldsymbol{G}\cdot(\boldsymbol{\nabla}\times\boldsymbol{F}) - \boldsymbol{F}\cdot(\boldsymbol{\nabla}\times\boldsymbol{G})$$

$$\boldsymbol{\nabla}\times(\boldsymbol{F}\times\boldsymbol{G}) = \boldsymbol{F}(\boldsymbol{\nabla}\cdot\boldsymbol{G}) - \boldsymbol{G}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) + (\boldsymbol{G}\cdot\boldsymbol{\nabla})\boldsymbol{F} - (\boldsymbol{F}\cdot\boldsymbol{\nabla})\boldsymbol{G}$$

These results can be proved using index notation.

Chapter 4 Summary: Multiple Integrals

(1) If a, b and c, d are constants then

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

(2) The double integral over a region $\mathcal{R}$ is defined by

$$\iint\limits_{\mathcal{R}} f(x,y)\,dA = \iint\limits_{\mathcal{R}} f(x,y)\,dy\,dx = \int_c^d \int_{r(y)}^{s(y)} f(x,y)\,dx\,dy = \int_a^b \int_{p(x)}^{q(x)} f(x,y)\,dy\,dx$$

where $\mathcal{R}$ is enclosed in the rectangle $a \le x \le b$, $c \le y \le d$ and the boundaries of $\mathcal{R}$ are given by $y = p(x), q(x)$ and $x = r(y), s(y)$.

(3) The triple/volume integral over a volume V is given by

$$\iiint\limits_V f(x,y,z)\,dx\,dy\,dz = \iiint\limits_V f(x,y,z)\,dV$$

Changing Variables:

(1) If there are variables such that $x = x(u,v)$, $y = y(u,v)$ then

$$\iint\limits_A f(x,y)\,dx\,dy = \iint\limits_{A'} F(u,v)\,|J|\,du\,dv$$

where $F(u,v) = f(x,y)$, A' is the region in (u,v)-plane corresponding to A and

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\[2ex] \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

(2) Plane Polars: In plane polars (R,ϕ) the transformation is

$$\iint\limits_A f(x,y)\,dA = \iint\limits_A f(x,y)\,dx\,dy = \iint\limits_{A'} F(R,\phi)\,R\,dR\,d\phi$$

(3) Volume Integrals: If there are variables such that $x = x(u, v, w)$, $y = y(u, v, w)$ and
$z = z(u, v, w)$ then

$$\iiint\limits_V f(x, y, z)\, dx\, dy\, dz = \iiint\limits_{V'} F(u, v, w)\, |J|\, du\, dv\, dw$$

where $F(u, v, w) = f(x, y, z)$, V' is the volume in (u, v, w)-space corresponding to V and

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\[2ex] \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\[2ex] \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

(4) Cylindrical Polar Coordinates: In cylindrical polars (R, ϕ, z),

$$x = R\cos\phi, \qquad y = R\sin\phi, \qquad z = z,$$

and volume element is given by $dV = dx\, dy\, dz = R\, dR\, d\phi\, dz$, so that

$$\iiint\limits_V f(x, y, z) dV = \iiint\limits_{V'} F(R, \phi, z)\, R\, dR\, d\phi\, dz$$

(5) Spherical Polar Coordinates: In spherical polar coordinates (r, θ, ϕ)

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

and volume element is given by $dV = dx\, dy\, dz = r^2\sin\theta\, dr\, d\theta\, d\phi$, so that

$$\iiint\limits_V f(x, y, z) dV = \iiint\limits_{V'} F(r, \theta, \phi)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

Chapter 5: Vector Integration

Part I: Curves and Line Integrals

(1) A *curve* may be represented parametrically by $\mathbf{x}(t)$ where $\mathbf{x}$ is the position vector of a point on the curve and t is the parameter that varies along the curve.

(2) The *line integral* of a scalar function $f(x,y,z) = f(\mathbf{x})$ along a curve C is

$$\int_C f(\mathbf{x})ds = \int_{t_1}^{t_2} f(\mathbf{x}(t))\frac{ds}{dt}\,dt = \int_{t_1}^{t_2} f(\mathbf{x}(t))\Big(\Big(\frac{dx}{dt}\Big)^2 + \Big(\frac{dy}{dt}\Big)^2 + \Big(\frac{dz}{dt}\Big)^2\Big)^{\frac{1}{2}}\,dt$$

where

$$ds = \Big(\Big(\frac{dx}{dt}\Big)^2 + \Big(\frac{dy}{dt}\Big)^2 + \Big(\frac{dz}{dt}\Big)^2\Big)^{\frac{1}{2}}\,dt$$

is arc length.

(3) The *integral around a closed curve C*

$$\oint_C f(\mathbf{x})ds$$

does not depend on the starting point of the integration.

(4) A *scalar line integral of the vector field*, $\boldsymbol{F}$, around a curve C is given by

$$\int\limits_C \boldsymbol{F}\cdot d\boldsymbol{x} = \int\limits_C \mathbf{F}\cdot\hat{\mathbf{t}}ds = \int F_1\,dx + F_2\,dy + F_3\,dz.$$

where $\hat{\mathbf{T}}$ is the unit vector directed along the curve.

(5) These line integrals are evaluated by parametrising the curve and writing

$$\int\limits_C \boldsymbol{F}\cdot d\boldsymbol{x} = \int_{t=t_{min}}^{t=t_{max}} \boldsymbol{F}\cdot\frac{d\boldsymbol{x}}{dt}dt$$

where

$$\frac{d\boldsymbol{x}}{dt} = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})$$

(6) The work done by a force $\boldsymbol{F}$ in moving a particle around a given curve C is given by $\int\limits_C \boldsymbol{F}\cdot d\boldsymbol{x}$

(7) A vector field $\boldsymbol{F}$ is said to be *conservative* if $\oint\limits_C \boldsymbol{F}\cdot d\boldsymbol{x} = 0$ around any curve C.

Equivalently $\boldsymbol{F}$ is said to be *conservative* if $\int\limits_P^Q \boldsymbol{F}\cdot d\boldsymbol{x}$ is independent of the path of integration between P and Q.

(8) The following 4 statements are equivalent:

(i) $\boldsymbol{\nabla}\times\boldsymbol{F} = 0$ at each point.

(ii) $\oint\limits_C \boldsymbol{F}\cdot d\boldsymbol{x} = 0$ around every closed curve in the region.

(iii) $\int\limits_P^Q \boldsymbol{F}\cdot d\boldsymbol{x}$ is independent of the path of integration from P to Q.

(iv) $\boldsymbol{F} = \boldsymbol{\nabla}\phi$ for some scalar ϕ which is single-valued in the region.

(9) Line integrals which yield a vector as an answer are $\int\limits_C \boldsymbol{F}\times d\boldsymbol{x}$ and $\int\limits_C f\,d\boldsymbol{x}$

Chapter 5: Vector Integration
Part II: Surface Integrals
Summary

(1) A surface is defined parametrically by

$$\mathbf{x} = \mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$$

where u and v are continuous parameters and x, y, z are continuous, single-valued functions of u and v.

(2) The unit normal to the surface $\mathbf{x} = \mathbf{x}(u, v)$ is given by

$$\hat{\mathbf{n}} = \frac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|}$$

where $\mathbf{t}_u = \dfrac{\partial \mathbf{x}}{\partial u}$ and $\mathbf{t}_v = \dfrac{\partial \mathbf{x}}{\partial v}$. This normal is not unique but by convention points out from the positive side of the surface.

(3) The scalar area element $\mathrm{d}S$ of a surface $\mathbf{x} = \mathbf{x}(u, v)$ is given by

$$\mathrm{d}S = |\mathbf{t}_u \times \mathbf{t}_v| \, \mathrm{d}u \, \mathrm{d}v = \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| \mathrm{d}u \, \mathrm{d}v$$

(4) The vector area element $\mathrm{d}\boldsymbol{S}$ of a surface $\mathbf{x} = \mathbf{x}(u, v)$ is equal to $\hat{\mathbf{n}} \, \mathrm{d}S$ and is given by

$$\mathrm{d}\boldsymbol{S} = (\mathbf{t}_u \times \mathbf{t}_v) \, \mathrm{d}u \, \mathrm{d}v = \left(\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right) \mathrm{d}u \, \mathrm{d}v$$

(5) There are two surface integrals which give scalars as answers. If S is parametrised by $\mathbf{x} = \mathbf{x}(u, v)$ and R is the region in (u, v) space that corresponds to S then

$$\begin{aligned} \iint_S f \, \mathrm{d}S &= \iint_R f(u, v) \, |\mathbf{t}_u \times \mathbf{t}_v| \, \mathrm{d}u \, \mathrm{d}v \\ \iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} &= \iint_S \boldsymbol{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \iint_R \boldsymbol{F}(u, v) \cdot (\mathbf{t}_u \times \mathbf{t}_v) \, \mathrm{d}u \, \mathrm{d}v \end{aligned}$$

(6) The *flux* Q of a vector field $\boldsymbol{F}$ through a surface S is defined to be $Q = \iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$.

(7) The following surface integrals give vectors and are calculated in the obvious way

$$\iint_S f \, \mathrm{d}\boldsymbol{S} \qquad \iint_S \boldsymbol{F} \, \mathrm{d}S, \qquad \iint_S \boldsymbol{F} \times \mathrm{d}\boldsymbol{S}.$$

(8) If a surface integral is evaluated over a closed surface S then it is written

$$\oiint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

Chapter 6: Vector Integral Theorems

Alternative Definitions of divergence and curl

(1) An alternative definition of *divergence* is given by

$$\boldsymbol{\nabla}\cdot\boldsymbol{F} = \lim_{\delta V \to 0} \frac{1}{\delta V} \oiint_{\delta S} \boldsymbol{F} \cdot d\boldsymbol{S}$$

where δV is a small volume bounded by a surface δS which has outward-pointing normal vector surface element $d\boldsymbol{S}$.

(2) An alternative definition of *curl* is given by

$$\boldsymbol{n} \cdot \boldsymbol{\nabla}\times\boldsymbol{F} = \lim_{\delta S \to 0} \frac{1}{\delta S} \oint_{\delta C} \boldsymbol{F} \cdot d\boldsymbol{x},$$

where δS is a small open surface bounded by a curve δC which is traversed in a right-handed sense with respect to the normal $\boldsymbol{n}$ to δS.

The Divergence and Stokes' Theorems

(3) The *divergence theorem* states that, for a vector field $\boldsymbol{F}$ that is continuously differentiable throughout a volume V,

$$\iiint_V \boldsymbol{\nabla}\cdot\boldsymbol{F} dV = \oiint_S \boldsymbol{F} \cdot d\boldsymbol{S},$$

where S is the closed surface enclosing the volume V and the normal points outwards from the surface.

(4) *Stokes's theorem* states that, for a vector field $\boldsymbol{F}$ that is continuously differentiable everywhere on a surface S,

$$\iint_S (\boldsymbol{\nabla}\times\boldsymbol{F}) \cdot d\boldsymbol{S} = \oint_C \boldsymbol{F} \cdot d\boldsymbol{x},$$

where the closed curve C is the boundary of S.

Chapter 7: Curvilinear Coordinates

(1) *Curvilinear coordinates* u, v, w are defined by a mapping $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$.

(2) The surfaces u_i =constant are called *coordinate surfaces.* The intersection of coordinate surfaces are called *coordinate curves*

(3) Unit vectors in the direction of the new coordinates are given by

$$\mathbf{e}_u = \mathbf{t}_u/h_u = \frac{\partial \mathbf{x}}{\partial u}/h_u, \quad \mathbf{e}_v = \mathbf{t}_v/h_v = \frac{\partial \mathbf{x}}{\partial v}/h_v, \quad \mathbf{e}_w = \mathbf{t}_w/h_w = \frac{\partial \mathbf{x}}{\partial w}/h_w,$$

where the *scale factors* h_u, h_v and h_w are given by

$$h_u = |\mathbf{t}_u| = \left|\frac{\partial \mathbf{x}}{\partial u}\right|, \quad h_v = |\mathbf{t}_v| = \left|\frac{\partial \mathbf{x}}{\partial v}\right|, \quad h_w = |\mathbf{t}_w| = \left|\frac{\partial \mathbf{x}}{\partial w}\right|.$$

(4) If $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ then the coordinate system is an *orthogonal curvilinear coordinate system* and these vectors form an *othonormal basis.* If, in addition, $\mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) = \epsilon_{ijk}$ then the system is *right-handed.*

(5) For orthogonal coordinates, the length of a line element ds is given by

$$ds^2 = h_u^2\, du^2 + h_v^2\, dv^2 + h_w^2\, dw^2.$$

(6) For right-handed orthogonal coordinates, the surface element dS on the u coordinate surface is given by

$$\mathbf{dS} = h_v\, h_w\, dv\, dw \mathbf{e}_v \times \mathbf{e}_w = h_v\, h_w\, dv\, dw \mathbf{e}_u.$$

(7) For orthogonal coordinates, the volume element dV is given by

$$dV = h_u h_v\, h_w\, du\, dv\, dw.$$

(8) For orthogonal coordinates, the gradient of a scalar is given by

$$\boldsymbol{\nabla} f = \frac{1}{h_u}\frac{\partial f}{\partial u}\mathbf{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\mathbf{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\mathbf{e}_w.$$

(9) For orthogonal coordinates, the divergence of a vector is given by

$$\boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(h_v h_w F_u) + \frac{\partial}{\partial v}(h_u h_w F_v) + \frac{\partial}{\partial w}(h_u h_v F_w)\right].$$

(10) For orthogonal coordinates, the curl of a vector is given by

$$\boldsymbol{\nabla}\times\boldsymbol{F} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\mathbf{e}_u & h_v\mathbf{e}_v & h_w\mathbf{e}_w \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}.$$

(11) For orthogonal coordinates, the Laplacian of a scalar is given by

$$\nabla^2 f = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\right].$$

Cylindrical and Spherical Polars

(12) For cylindrical polars (R, ϕ, z).

$$h_R = 1, \quad h_\phi = R, \quad h_z = 1$$

$$\mathbf{e}_R = (\cos\phi, \sin\phi, 0), \quad \mathbf{e}_\phi = (-\sin\phi, \cos\phi, 0), \quad \mathbf{e}_z = (0, 0, 1).$$

The formulas for grad, div, curl and the Laplacian in cylindrical coordinates are therefore,

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\partial f}{\partial\phi}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{e}_z,$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{1}{R}\frac{\partial}{\partial R}(RF_R) + \frac{1}{R}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z},$$

$$\boldsymbol{\nabla}\times\boldsymbol{F} = \left(\frac{1}{R}\frac{\partial F_z}{\partial\phi} - \frac{\partial F_\phi}{\partial z}\right)\mathbf{e}_R + \left(\frac{\partial F_R}{\partial z} - \frac{\partial F_z}{\partial R}\right)\mathbf{e}_\phi + \frac{1}{R}\left(\frac{\partial}{\partial R}(RF_\phi) - \frac{\partial F_R}{\partial\phi}\right)\mathbf{e}_z,$$

$$\boldsymbol{\nabla}^2 f = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial f}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}.$$

(13) For spherical polars (r, θ, ϕ), $h_r = 1, \quad h_\theta = r, \quad h_\phi = r\sin\theta$

$$\mathbf{e}_r = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad \mathbf{e}_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta),$$

$$\mathbf{e}_\phi = (-\sin\phi, \cos\phi, 0).$$

The formulas for grad, div, curl and $\nabla^2 f$ in spherical coordinates are therefore,

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\mathbf{e}_\phi$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{F} =& \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta F_\phi) - \frac{\partial F_\theta}{\partial\phi}\right)\mathbf{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial}{\partial r}(rF_\phi)\right)\mathbf{e}_\theta \\ &+ \frac{1}{r}\left(\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right)\mathbf{e}_\phi\end{aligned}$$

$$\boldsymbol{\nabla}^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}.$$

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4 printed pages, each of which
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All calculators must carry
an approval sticker issued
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School of Mathematics

January 2018

MATH236501

Vector Calculus

Time Allowed: 2 hours 30 minutes

You must attempt to answer 4 questions.
If you answer more than 4 questions, only your best 4 answers will be counted towards your final mark for this exam.
All questions carry equal marks.

1. (a) By reversing the order of integration, evaluate the integral

$$I \equiv \int_0^4 \int_{\frac{1}{2}\sqrt{y}}^{\sqrt{y}} f(x)\, \mathrm{d}x\, \mathrm{d}y,$$

where $f(x)$ is the function

$$f(x) = \begin{cases} \sin(x^3) & \text{for} \quad x \le 1 \\ \sin(12x - x^3) & \text{for} \quad x > 1. \end{cases}$$

(b) By evaluating some suitable line integrals, find the length of the perimeter of the region of integration in part (a). You may use the standard integral $\int \sqrt{1+u^2}\mathrm{d}u = \frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\text{arcsinh } u + \text{constant}$.

(c) By using the Jacobian of the transformation

$$\begin{aligned} x &= v^2/u, \\ y &= uv \end{aligned}$$

find the area of the (x, y)-plane enclosed by the curves $x = y^2$, $x = 8y^2$, $y = 1/x$ and $y = 8/x$. Clearly explain your reasoning and include a sketch of the region.

2. Recall that, for curvilinear coordinates (u, v, w) with scale factors h_u, h_v, h_w and basis unit vectors $\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w$, the following expressions apply for differential operators:

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\mathbf{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\mathbf{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\mathbf{e}_w$$

and

$$\nabla \cdot \mathbf{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(F_u h_v h_w) + \frac{\partial}{\partial v}(F_v h_u h_w) + \frac{\partial}{\partial w}(F_w h_u h_v)\right]$$

and that the scale factors for spherical polar coordinates are $h_r = 1$, $h_\theta = r$ and $h_\phi = r\sin\theta$.

(a) Find the divergence of the vector field given in spherical polar coordinates by $\mathbf{F}(r, \theta, \phi) = r^2\mathbf{e}_r + \sin^2\phi\, \mathbf{e}_\phi$.

(b) Find (in a spherical polar basis) the gradient of the scalar field given in spherical polar coordinates by $f(r, \theta, \phi) = \exp(3r)$ and sketch the resulting field (grad f) in the plane $\theta = \pi/2$. Describe surfaces of constant f.

(c) Prolate spheroidal coordinates (u, v, w) are given by:

$$x = \sinh u \sin v \cos w, \qquad y = \sinh u \sin v \sin w, \qquad z = \cosh u \cos v,$$

where $u \ge 0$, $0 \le v \le \pi$, $0 \le w \le 2\pi$.

For prolate spheroidal coordinates:

i. find the basis unit vectors $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_w$ and the associated scale factors h_u, h_v and h_w,

ii. show that the basis vectors are orthogonal.

3. (a) Use index notation to prove the identity

$$\text{div curl } \mathbf{A} \equiv 0$$

where $\mathbf{A}$ is a vector field.

(b) For the case $\mathbf{A} = (x^2y, -y^2x, z^2)$, evaluate the vector field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ and check that $\mathbf{B}$ is incompressible.

(c) Demonstrate that the gradient of the scalar field $\phi = xy$ is perpendicular to the field $\mathbf{B}$ found in part (b).

(d) If f and $\mathbf{G}$ are arbitrary scalar and vector fields respectively, use index notation and the product rule of differentiation to find an identity expressing the divergence of their product, div $(f\mathbf{G})$, in terms of each of the fields and their individual gradient and divergence.

(e) Explaining your reasoning, use your answers to parts (b), (c) and (d) to evaluate the divergence of the field $\mathbf{C} = \phi\mathbf{B}$ without any further detailed calculations.

(f) For the field defined in part (c), find the equation of the plane tangent to the surface $\phi = 6$ at the point $(2, 3, 4)$.

(g) Use index notation to prove the identity

$$\boldsymbol{\nabla} \times (f\mathbf{G}) \equiv f\boldsymbol{\nabla} \times \mathbf{G} - \mathbf{G} \times \boldsymbol{\nabla} f,$$

and confirm it by evaluating both sides of the equation for the case where $\mathbf{G} = \mathbf{A}$ and $f = \phi$, as defined in parts (b) and (c) respectively.

4. The vector field $\mathbf{A}$ is given in Cartesian coordinates by $\mathbf{A} = (y + yz, x + xz, xy + 2z)$.

(a) Define the term "irrotational" and find whether it applies to the field $\mathbf{A}$.

(b) Define the term "conservative" and argue why $\mathbf{A}$ can be expressed as the gradient of a potential Φ.

(c) Evaluate the potential of $\mathbf{A}$.

(d) A square-based pyramid stands with its base B on the region of the (x, y)-plane given by $B = \{(x, y, z) : -\frac{a}{2} \leq x \leq \frac{a}{2}, -\frac{a}{2} \leq y \leq \frac{a}{2}, z = 0\}$ and its apex at $(0, 0, h)$. Sketch the sections of the pyramid on the (x, z) and (y, z) planes and label your sketches with the equations of the pyramid's boundaries. Hence evaluate a triple-integral to confirm the formula for the volume of the pyramid,

$$\tfrac{1}{3}(\text{area of base}) \times \text{height}.$$

[You may find it easiest to integrate with respect to z last, i.e. as the outer integration.] Define "flux" and hence use the divergence theorem to find (without evaluation of any surface integrals), the total flux of $\mathbf{A}$ out of the surface of the pyramid.

5. (a) In terms of the components F_1 and F_2 (in the usual notation), find an expression for the curl of a vector field $\mathbf{F}$ that has no z component and no z dependence.

(b) State Stokes's theorem and apply it to the field $\mathbf{F}$ in part (a), on a flat surface S in the x, y plane bounded by the closed curve C, to derive Green's theorem,

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y = \oint_C (F_1 \mathrm{d}x + F_2 \mathrm{d}y)$$

and state the direction in which the line integral on C should be evaluated.

(c) Sketch the region S given by $x^3 \leq y \leq x$ with $0 \leq x \leq 1$ and $z = 0$. For this region S, confirm Green's theorem by evaluating both sides of the equation in part (b) in the case where $F_1 = xy + y^2$, $F_2 = x^2$.

End.

Vector Calculus

School of Mathematics

Semester One 201819

Calculator instructions:

- You are allowed to use a calculator which has had an approval sticker issued by the School of Mathematics.

Exam information:

- There are 3 pages to this exam.
- There will be **2 hours 30 minutes** to complete this exam.
- Answer all questions.
- All questions are worth equal marks.
- You must show all your calculations.

1. (a) For the vector field $\mathbf{A} = (x^2, y^2, z^2)$, evaluate the integral

$$\iint_H \mathbf{A} \cdot \mathrm{d}\mathbf{S}$$

where H is the curved surface of a hemisphere $x^2 + y^2 + z^2 = 1;\ z \geq 0$, demonstrating clearly how you derive the surface element $\mathrm{d}\mathbf{S}$. You may use the fact that

$$\int_0^{2\pi} \sin^3 \phi \, \mathrm{d}\phi = \int_0^{2\pi} \cos^3 \phi \, \mathrm{d}\phi = 0.$$

(b) Find div $\mathbf{A}$. Hence evaluate the integral

$$\iiint \boldsymbol{\nabla} \cdot \mathbf{A} \mathrm{d}V$$

over the volume enclosed by the (x, y)-plane and the hemisphere H. The volume element $\mathrm{d}V$ for your chosen coordinate system may be quoted without derivation if you wish.

(c) State the divergence theorem. By considering an appropriate direction for the unit normal to the disc $x^2 + y^2 + z^2 \leq 1;\ z = 0$, show that your results to parts (a) and (b) are consistent with the divergence theorem.

2. The vector field $\mathbf{F}(\mathbf{r})$ is given by

$$\mathbf{F} = \left(\frac{x}{1 + x^2 + y^2}, \frac{y}{1 + x^2 + y^2}, 2z \right).$$

(a) Sketch the field $\mathbf{F}$ in the (x, y)-plane.

(b) Showing your workings, explicitly evaluate the line integral

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

where C is a circle of radius 2 in the (x, y)-plane, traversed anti-clockwise.

(c) Evaluate curl $\mathbf{F}$ and state why your answer is consistent with the result of part (b).

(d) Find the scalar potential Φ of $\mathbf{F}$ and sketch some contours of Φ in the (x, y)-plane. You may find it useful to recall that the derivative of the natural logarithm of a function $h(t)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \log h = \frac{1}{h} \frac{\mathrm{d}h}{\mathrm{d}t}.$$

(e) Calculate the divergence of $\mathbf{F}$. Hence, without explicitly evaluating a surface integral, state whether J is positive, negative or zero, where J is defined as the integral

$$J \equiv \oiint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

over the surface of a sphere of radius 3, centred on the point $(1, 1, 1)$. Explain your reasoning by referring to a relevant theorem.

3. (a) Clearly explaining your method, prove that, for all scalar fields f,

$$\varepsilon_{ijk}\frac{\partial^2 f}{\partial x_j \partial x_k} \equiv 0$$

and express this identity in Gibbs (vector) notation.

(b) Using the identity in part (a) or otherwise, use index notation to prove

$$\boldsymbol{\nabla}\cdot(\mathbf{B}\times\boldsymbol{\nabla}f) \equiv (\boldsymbol{\nabla}f)\cdot\boldsymbol{\nabla}\times\mathbf{B}$$

for all vector fields $\mathbf{B}$.

(c) Express the identities in parts (a) and (b) using the notation grad, div and curl instead of the nabla ($\boldsymbol{\nabla}$) symbol. Make it clear which identity is which.

(d) Demonstrate the identity in part (b) by evaluating both sides for the fields $f = x\exp(yz)$ and $\mathbf{B} = (xy,\, -z,\, yz^2)$.

(e) Demonstrate the identity in part (b) by evaluating both sides for $\mathbf{B} = (y, 0, 0)$ while f remains arbitrary.

4. Recall that, for orthogonal curvilinear coordinates (u, v, w) with scale factors h_u, h_v, h_w and basis unit vectors $\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w$, the following expressions apply for differential operators:

$$\boldsymbol{\nabla}f = \frac{1}{h_u}\left(\frac{\partial f}{\partial u}\right)\mathbf{e}_u + \frac{1}{h_v}\left(\frac{\partial f}{\partial v}\right)\mathbf{e}_v + \frac{1}{h_w}\left(\frac{\partial f}{\partial w}\right)\mathbf{e}_w$$

and

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(F_u h_v h_w) + \frac{\partial}{\partial v}(F_v h_u h_w) + \frac{\partial}{\partial w}(F_w h_u h_v)\right].$$

Bipolar cylindrical coordinates (u, v, w) are defined by the transformation equations

$$x = \frac{a\sinh v}{\cosh v - \cos u}, \qquad y = \frac{a\sin u}{\cosh v - \cos u}, \qquad z = w$$

where a is a positive constant.

(a) Find the basis unit vectors $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_w$ for this coordinate system. Show that the scale factor for the u and v coordinates are given by

$$h_u = h_v = \frac{a}{\cosh v - \cos u}$$

and find the other scale factor. You may use the identities $\sinh^2 v \equiv \cosh^2 v - 1$ and $\mathrm{d}(\sinh t)/\mathrm{d}t = \cosh t$ and $\mathrm{d}(\cosh t)/\mathrm{d}t = \sinh t$.

(b) Show that the basis vectors for bipolar cylindrical coordinates are mutually orthogonal and that they form a right-handed basis in the order u, v, w.

(c) Give an expression for ∇f in bipolar cylindrical coordinates. Hence, using the identity $\operatorname{div}(\operatorname{grad} f) \equiv \nabla^2 f$, evaluate $\nabla^2 g$ for the scalar field $g = \sin u$.

 End.

School of Mathematics

January 2018

MATH236501

Vector Calculus

Check-list solutions

1. (a) $1-\cos 1+\frac{1}{3}(\cos 11-\cos 16)$

(b) $1+\sqrt{17}+\frac{1}{2}\sqrt{65}+\frac{1}{4}\text{arcsinh }4+\frac{1}{16}\text{arcsinh }8$

(c) $7\log 2$

2. (a) $4r+\frac{\sin 2\phi}{r\sin\theta}$

(b) $3e^{3r}\mathbf{e}_r$; spheres

(c)(i) $\mathbf{e}_u=(\cosh u\sin v\cos w,\cosh u\sin v\sin w,\sinh u\cos v)/\sqrt{\sinh^2 u+\sin^2 v}$,
$h_u=\sqrt{\sinh^2 u+\sin^2 v}=\sqrt{\cosh^2 u-\cos^2 v}$
$\mathbf{e}_v=(\sinh u\cos v\cos w,\sinh u\cos v\sin w,-\cosh u\sin v)/\sqrt{\sinh^2 u+\sin^2 v}$,
$h_v=h_u$
$\mathbf{e}_w=(-\sin w,\cos w,0)$,
$h_w=\sinh u\sin v$.

3 (b) $\mathbf{B}=(0,0,-x^2-y^2)$; $\nabla\cdot\mathbf{B}=0$.

(c) (grad $\phi)\cdot\mathbf{B}=0$

(d) div $(f\mathbf{G})\equiv f$div $\mathbf{G}+\mathbf{G}\cdot$grad f

(e) 0

(f) $3x+2y=13$

(g) Both sides $=(xz^2,-yz^2,-2xy^3-2x^3y)$

4 (a) Yes

(c) $xy+xyz+z^2$+constant

(d) $\frac{2}{3}a^2h$

5 (a) $\left(0,0,\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$

(c) $-2/35$

School of Mathematics

Semester 1 201819

MATH236501

Vector Calculus

Check-list solutions

1. (a) $\pi/2$

(b) div$\mathbf{A} = 2(x+y+z)$;

integral $= \pi/2$.

2. (b) 0.

(c) $(0,0,0)$.

(d) $\Phi = \frac{1}{2}\log(1+x^2+y^2)+z^2+$const.

(e) div $\mathbf{F} = 2 + \frac{2}{(1+x^2+y^2)^2}$;

J is positive.

3 (c) In (a), curl grad $f \equiv 0$;

in (b) div$(\mathbf{B} \times \text{grad} f) \equiv (\text{grad} f) \cdot$ curl $\mathbf{B}$.

(d) L. H. S. = R. H. S. $= (1 - x^2 y + z^2)e^{yz}$.

(e) L. H. S. = R. H. S. $= -\frac{\partial f}{\partial z}$.

4 (a)

$$\mathbf{e}_u = \frac{(-\sin u \sinh v, \cos u \cosh v - 1, 0)}{\cosh v - \cos u},$$
$$\mathbf{e}_v = \frac{(1 - \cos u \cosh v, -\sin u \sinh v, 0)}{\cosh v - \cos u},$$
$$\mathbf{e}_w = (0,0,1),$$
$$h_w = 1.$$

(c)

$$\nabla f = \frac{(\cosh v - \cos u)}{a}\left[\left(\frac{\partial f}{\partial u}\right)\mathbf{e}_u + \left(\frac{\partial f}{\partial v}\right)\mathbf{e}_v\right] + \left(\frac{\partial f}{\partial w}\right)\mathbf{e}_w,$$
$$\nabla^2 g = -\left(\frac{\cosh v - \cos u}{a}\right)^2 \sin u.$$