Summary

- A **regularly parametrized curve** (RPC) is a smooth map $\gamma : I \to \mathbb{R}^n$ (where I is an open interval) such that, for all $t \in I$, $\gamma'(t) \neq 0$.
- Given a RPC γ, its **velocity** is γ', its **speed** is $|\gamma'|$ and its **acceleration** is γ''.
- The **tangent line** to γ at $t_0 \in I$ is

$$\widehat{\gamma}_{t_0} : \mathbb{R} \to \mathbb{R}^n, \qquad \widehat{\gamma}_{t_0}(t) = \gamma(t_0) + \gamma'(t_0)t.$$

- The **arclength function** based at $t_0 \in I$ is

$$\sigma_{t_0}(t) = \int_{t_0}^{t} |\gamma'(u)|du.$$

  Geometrically, this is the arclength along γ from $\gamma(t_0)$ to $\gamma(t)$ if $t \geq t_0$ (and minus the arclength if $t < t_0$).
- A **reparametrization** of a curve $\gamma : I \to \mathbb{R}^n$ is a curve $\gamma \circ h : J \to \mathbb{R}^n$ where $h : J \to I$ is smooth, surjective and has strictly positive derivative. If γ is a RPC, so is every reparametrization of γ.
- Arclength is unchanged by reparametrization.
- A **unit speed curve** (USC) is a curve with $|\gamma'(t)| = 1$ for all t.
- Every RPC has a reparametrization which is a USC. One can construct it, in principle, by reparametrizing with $h = \sigma_{t_0}^{-1}$.

Summary

- The curvature vector of a RPC measures how fast the tangent lines to the curve change **direction**.
- If γ is a unit speed curve, the **curvature vector** is $k(s) = \gamma''(s)$.
- In general

$$k(t) = \frac{1}{|\gamma'(t)|^2} \left\{ \gamma''(t) - \frac{\gamma''(t) \cdot \gamma'(t)}{|\gamma'(t)|^2} \gamma'(t) \right\}.$$

- The **unit tangent vector** along a curve $\gamma : I \to \mathbb{R}^n$ is

$$u(t) = \frac{\gamma'(t)}{|\gamma'(t)|}.$$

- The **normal projection** of $v : I \to \mathbb{R}^n$ is

$$v_\perp(t) = v(t) - [u(t) \cdot v(t)]u(t).$$

- An alternative formula for the curvature vector is

$$k(t) = \frac{\gamma''_\perp(t)}{|\gamma'(t)|^2}.$$

Summary

- For a planar curve $\gamma : I \to \mathbb{R}^2$ we can define the **unit normal vector**

$$n(t) = (-u_2(t), u_1(t)),$$

  where u is the unit tangent vector.

- Since the curvature vector is parallel to n, there is a scalar function $\kappa : I \to \mathbb{R}$ called the **signed curvature**, such that

$$k(t) = \kappa(t)n(t).$$

- A convenient formula for $\kappa(t)$ is

$$\kappa(t) = \frac{\gamma''(t) \cdot n(t)}{|\gamma'(t)|^2}.$$

- If $\kappa(t) > 0$, the curve is turning to the **left**. If $\kappa(t) < 0$, the curve is turning to the **right**.

- Given a function $\kappa(s)$, there is a planar USC $\gamma(s)$ whose signed curvature is $\kappa(s)$. This curve is unique up to rigid motions.

- Symmetries of κ imply symmetries of γ (and vice versa).

Summary

- The curve obtained from γ by tracing out the locus of its centres of curvature is called the **evolute** of γ. Explicitly

$$E_\gamma(t) = \gamma(t) + \frac{1}{\kappa(t)} n(t).$$

- The **involute** of γ based at $t_0 \in I$ is

$$I_\gamma(t) = \gamma(t) - \sigma_{t_0}(t) u(t).$$

- A **parallel** to γ is a curve

$$\gamma_\lambda(t) = \gamma(t) + \lambda n(t)$$

  where $\lambda \in \mathbb{R}$ is a constant.

- The evolute of an involute of γ is γ. Every involute of the evolute of γ is a parallel to γ.

- The regularity properties of evolutes and parallels can be analyzed in terms of the curvature properties of γ.

Summary

- Given a RPC $\gamma : I \to \mathbb{R}^3$ of nonvanishing curvature we define its **unit tangent vector** $u(t)$, **principal unit normal** $n(t)$ and **binormal** $b(t)$ by

$$u(t) = \frac{\gamma'(t)}{|\gamma'(t)|}, \qquad n(t) = \frac{k(t)}{|k(t)|}, \qquad b(t) = u(t) \times n(t).$$

  This triple of vectors forms an orthonormal basis for $\mathbb{R}^3$ called the **Frenet frame**.

- The curvature $|k(t)|$ is usually denoted $\kappa(t)$. Note $\kappa \geq 0$.

- For a **unit speed** curve $\gamma : I \to \mathbb{R}^3$ of nonvanishing curvature we define the **torsion** $\tau : I \to \mathbb{R}$ by

$$b'(s) = -\tau(s)n(s).$$

- A USCNVC is **planar** if and only if $\tau \equiv 0$.

- The rate of change of the Frenet frame as one travels along a USCNVC is determined by the torsion and curvature according to the **Frenet formulae**

$$\begin{array}{rcccc} u'(s) & = & & \kappa(s)n(s) & \\ n'(s) & = & -\kappa(s)u(s) & & +\tau(s)b(s) \\ b'(s) & = & & -\tau(s)n(s) & \end{array}$$

- A USCNVC is uniquely determined (up to rigid motions) by its curvature and torsion.

Summary

- A function $f : M \to \mathbb{R}$ is **smooth** if its coordinate expression $\widehat{f} = f \circ \phi : U \to \mathbb{R}$ is smooth.
- Given a smooth function $f : M \to \mathbb{R}$ and a tangent vector $y \in T_pM$, the **directional derivative** of f with respect to (or along) y is

$$\nabla_y f = (f \circ \alpha)'(0)$$

  where $\alpha(t)$ is any generating curve for y.
- The directional derivative is linear, that is

$$\nabla_{ax+by} f = a\nabla_x f + b\nabla_y f, \qquad \nabla_y(af + bg) = a\nabla_y f + b\nabla_y g.$$

- Directional derivatives along coordinate basis vectors reduce to partial derivatives

$$\nabla_{\phi_u} f = \frac{\partial \widehat{f}}{\partial u}, \qquad \nabla_{\phi_v} f = \frac{\partial \widehat{f}}{\partial v}.$$

- Vector **fields** are smooth maps $X : M \to \mathbb{R}^3$.
- We can extend the definition of directional derivative to vector fields. If X is a vector field and Y is a **tangent** vector field then the directional derivative $\nabla_Y X$ is another vector field.

Summary

- An **oriented** surface is a RPS together with a choice of unit normal vector field N. Usually we choose
$$N = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|}.$$
- The **shape operator** of an oriented surface is
$$S_p : T_pM \to T_pM, \qquad S_p(x) = -\nabla_x N.$$
The shape operator is **linear**,
$$S_p(ax + by) = aS_p(x) + bS_p(y) \quad \forall a, b \in \mathbb{R},\ x, y \in T_pM,$$
and **self adjoint**,
$$x \cdot S_p(y) = y \cdot S_p(x) \quad \forall x, y \in T_pM.$$
- The **principal curvatures** of M at p are the eigenvalues κ_1, κ_2 of S_p. The **principal curvature directions** are the corresponding (normalized) eigenvectors.
- The **normal curvature** of a **unit vector** $x \in T_pM$ is
$$k_p(x) = x \cdot S_p(x).$$
This coincides with $k(0) \cdot N(p)$ where $k(t)$ is the curvature vector of any generating curve for x. The principal curvatures are the maximum and minimum values of $k_p(x)$ as x takes all values in the unit tangent space at p.
- The **mean curvature** at p is
$$H = \frac{1}{2}(\kappa_1 + \kappa_2).$$
- The **Gauss curvature** at p is
$$K = \kappa_1 \kappa_2.$$
The sign of K has intrinsic meaning, independent of the choice of N: if $K(p) > 0$ then either all curves in M through p curve towards $N(p)$, or they all curve away from $N(p)$; if $K(p) < 0$ then some curves curve towards $N(p)$ and some curve away.
- Both H and K can be computed directly from any matrix $\widehat{S}_p$ representing S_p:
$$H = \frac{1}{2}\operatorname{tr} \widehat{S}_p, \qquad K = \det \widehat{S}_p.$$

This question paper consists of
4 printed pages, each of which
is identified by the reference MATH205101.

All calculators must carry
an approval sticker issued
by the School of Mathematics.

School of Mathematics

January 2018

MATH205101

Geometry of Curves and Surfaces

Time Allowed: 2 hours

You must attempt to answer 4 questions. If you answer more than 4 questions, only your best 4 answers will be counted towards your final mark for this exam.

1. (a) Determine whether the mapping $\alpha : \mathbb{R} \to \mathbb{R}^2$, $\alpha(t) = (\cos^2 t, \sin t)$, is a regularly parametrized curve. Clearly explain your reasoning.

(b) Consider the regularly parametrized curve $\beta : \mathbb{R} \to \mathbb{R}^4$, $\beta(t) = (t, \cos 2t, \sin 2t, 7)$.

(i) Compute $\sigma_0 : \mathbb{R} \to \mathbb{R}$, its signed arclength function based at $t_0 = 0$.

(ii) Hence, or otherwise, construct a unit speed reparametrization of β.

(c) Consider the regularly parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^3$, $\beta(t) = (t, t^2, \frac{1}{3}t^3)$.

(i) Construct $\widehat{\gamma}_1$, its tangent line at $t_0 = 1$.

(ii) Compute $k(1)$, its curvature vector at time $t = 1$.

(iii) Construct $[u(1), n(1), b(1)]$, the Frenet frame for γ at time $t = 1$.

(iv) How many points on γ lie exactly distance $\sqrt{2}$ from $(0, 0, 0)$? Briefly explain your reasoning.

2. Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be the parametrized curve with $\gamma(0) = (1, 0, 0)$ and $\gamma'(0) = (0, 1, 0)$ which satisfies the ordinary differential equation

$$\gamma''(s) = \gamma(s) \times \gamma'(s).$$

Denote by $\kappa : \mathbb{R} \to [0, \infty)$ its scalar curvature.

(a) Show that γ is a unit speed curve.

(b) Compute $\kappa(0)$.

(c) Show that $\gamma(s) \cdot \gamma'(s) = s$.

(d) Show that $|\gamma(s)|^2 = s^2 + 1$.

(e) Deduce that κ is constant.
(You may use without proof the vector identity $|b \times c|^2 = |b|^2|c|^2 - (b \cdot c)^2$.)

(f) Show that the binormal vector of γ is $b(s) = \gamma(s) - s\gamma'(s)$.
(You may use without proof the vector identity $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$.)

(g) Deduce a formula for $\tau : \mathbb{R} \to \mathbb{R}$, the torsion of γ.

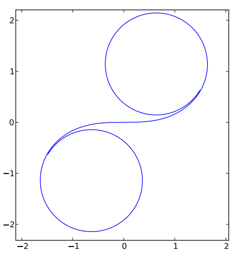
3. Given a prescribed function $\kappa : \mathbb{R} \to \mathbb{R}$, there exists a unique unit speed curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ with $\gamma(0) = (0,0)$, $\gamma'(0) = (1,0)$ and signed curvature κ. The curves corresponding to the signed curvature functions

$$\kappa_1(s) = \frac{6s}{1+s^4}, \quad \kappa_2(s) = \tanh s, \quad \kappa_3(s) = \frac{1}{4} + \cos s, \quad \kappa_4(s) = 1 + \tanh s,$$

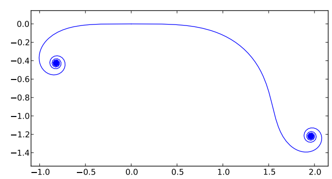
$$\kappa_5(s) = 4s\sin(s^2), \quad \kappa_6(s) = s^2 - 4, \quad \kappa_7(s) = s^3 - 2s^2,$$

are depicted below in the wrong order, figures (A) to (G):

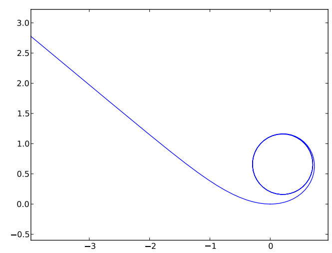
(A)



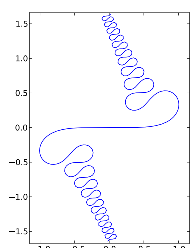
(B)



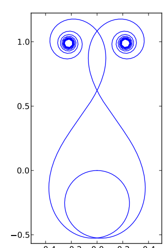
(C)



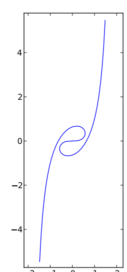
(D)



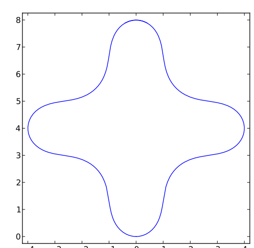
(E)



(F)



(G)



(a) Determine which curve corresponds to which signed curvature. In each case, briefly explain your reasoning. (Unexplained answers will not receive full credit.)

(b) One, and only one, of the curves depicted has a globally defined evolute. Identify this curve, explaining your reasoning. Is the evolute regular?

(c) Consider a general parallel curve

$$\gamma_\lambda(t) = \gamma(t) + \lambda n(t), \qquad \lambda \in \mathbb{R} \text{ a constant,}$$

to the curve γ with signed curvature κ_2. Determine the set of values of λ for which γ_λ is regular.

(d) What is the total arclength around one circuit of the closed curve labelled (G)?

4. (a) Define the following terms:

(i) An *open disk* in $\mathbb{R}^2$.

(ii) An *open subset* U of $\mathbb{R}^2$.

(iii) A *regular mapping* $M : U \to \mathbb{R}^3$.

(iv) A *regularly parametrized surface* $M : U \to \mathbb{R}^3$.

(b) Let M denote the mapping

$$M : \mathbb{R}^2 \to \mathbb{R}^3, \qquad M(x_1, x_2) = (x_1, x_1^2 + x_2^3, x_2 e^{x_1}),$$

$p = (0, 8, 2)$ and $v = (-2, 12, -3)$.

(i) Show that M is a regularly parametrized surface.

(ii) Show that p lies on M and write down its local coordinates.

(iii) Construct bases for the tangent space T_pM and normal space N_pM to M at p.

(iv) Show that v is tangent to M at p.

(v) Consider the function $f : M \to \mathbb{R}$, $f(y_1, y_2, y_3) = y_1 + y_2 + y_3$. Compute the directional derivative $v[f]$.

(vi) Construct a non-zero vector in T_pM which is orthogonal to v.

5. Let M denote the mapping $M : \mathbb{R}^2 \to \mathbb{R}^3$, $M(x_1, x_2) = (x_1, x_2, \sin(x_1 x_2))$ and $p = (0, \pi, 0)$. You are given that M is a regularly parametrized surface and p is a point on M. Let $\varepsilon_1, \varepsilon_2$ denote the coordinate basis vectors for M, and $S_p : T_pM \to T_pM$ denote the shape operator for M at p.

(a) Construct the canonical unit normal $N(x_1, x_2)$ on M.

(b) Show that $S_p(\varepsilon_1) = \dfrac{1}{\sqrt{1+\pi^2}}\varepsilon_2$.

(c) You are given that $S_p(\varepsilon_2) = \dfrac{1}{(1+\pi^2)^{3/2}}\varepsilon_1$. Construct the matrix $\widehat{S}_p$ representing the linear map S_p with respect to the basis $\varepsilon_1, \varepsilon_2$.

(d) Compute the mean curvature $H(p)$ and the Gauss curvature $K(p)$ of M at p.

(e) Compute the principal curvatures of M at p.

(f) You are given that u_1, the principal curvature direction corresponding to κ_1, the smaller of the principal curvatures, is

$$u_1 = \pm \frac{\left(1, -\sqrt{1+\pi^2}, \pi\right)}{\sqrt{2}\sqrt{1+\pi^2}}.$$

Deduce the other principal curvature direction, u_2.

(g) In each of the following cases, *either* construct a vector in T_pM with the specified properties *or* explain why no such vector exists:

(i) A unit vector v such that $v \cdot S_p(v) = 0$.

(ii) A unit vector v such that $v \cdot S_p(v) = 1$.

End.

Module Title: Geometry of curves and surfaces

School of Mathematics **Semester One 201819**

Calculator instructions:

- You are allowed to use a calculator which has had an approval sticker issued by the School of Mathematics.

Exam information:

- There are 4 pages to this exam.
- There will be **2 hours** to complete this exam.
- Answer all questions.
- All questions are worth equal marks.

1. (a) Say whether each of the following is a regularly parametrised curve. Support your answer with reasons.

i. $\gamma : \mathbb{R} \to \mathbb{R}^2, \quad \gamma(t) = (t^2, t^3 - t^2)$.

ii. $\gamma : \mathbb{R} \to \mathbb{R}^2, \quad \gamma(t) = (t^2, t^3 - t)$.

(b) Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be the regularly parametrised curve $\gamma(t) = (t, \cosh t)$.

i. Calculate the signed arclength function $\sigma_0 : \mathbb{R} \to \mathbb{R}$ for this curve.

ii. Calculate the tangent line $\widehat{\gamma}_1 : \mathbb{R} \to \mathbb{R}^2$ for this curve at $t = 1$.

(c) Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve with non-vanishing curvature.

i. Define the **unit tangent vector** $u(s)$, the **principal unit normal vector** $n(s)$, and the **binormal vector** $b(s)$.

ii. State the **Frenet formulae** for this curve.

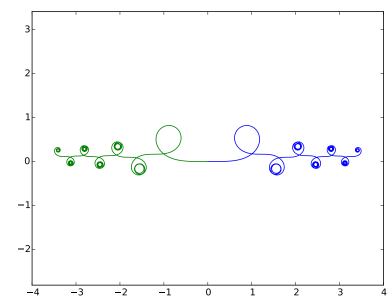
iii. Suppose that $\tau = 0$. Show that the curve γ is planar.

2. (a) The curves corresponding to the following signed curvature functions:

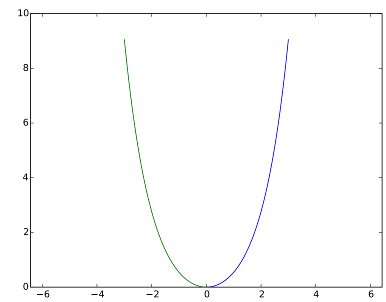
$$\kappa_1(s) = 2s\sin(s) \qquad \kappa_2(s) = \tanh(s)$$
$$\kappa_3(s) = 4s - s^3 \qquad \kappa_4(s) = \frac{1}{1+s^2}$$

are depicted below in the wrong order, figures (A) to (D).

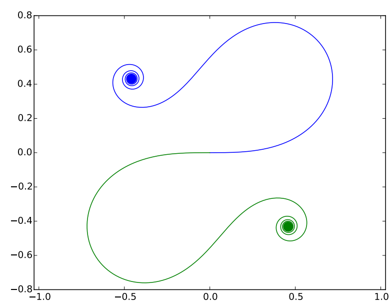
(A)



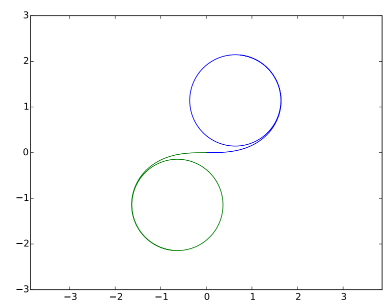
(B)



(C)



(D)



Construct a table determining whether the functions $\kappa_1, \ldots, \kappa_4$ are even or odd, the resulting symmetry of the curve, and the number of inflection points. Hence determine which curve corresponds to which signed curvature.

(b) Let $\gamma : I \longrightarrow \mathbb{R}^2$ be a planar curve with curvature vector $k : I \longrightarrow \mathbb{R}^2$.

i. Define the following:
- the **unit tangent vector** $u(t)$ of γ
- the **unit normal vector** $n(t)$ of γ
- the **signed curvature** $\kappa(t)$ of γ
- the **evolute** $E_\gamma : I \longrightarrow \mathbb{R}^2$ of γ.

ii. Show that $n' = -\kappa\gamma'$.

iii. Show that
$$E_\gamma'(t) = -\frac{\kappa'(t)}{\kappa(t)^2}n(t).$$

iv. Assuming that $\kappa'(t) < 0$, show that the arclength along E_γ from t_1 to t_2 is
$$\frac{1}{\kappa(t_2)} - \frac{1}{\kappa(t_1)}.$$

3. (a) Let $M : U \to \mathbb{R}^3$ be a smooth mapping. Say what is meant by

i. $(x_1, x_2) \in U$ is a **regular point of** M;

ii. M is a **regularly parametrised surface**.

(b) Let $M : U \to \mathbb{R}^3$ be a regularly parametrised surface. Let $V : M \to \mathbb{R}^3$ be a vector field and let $y \in M$. Say what is meant by

i. the **tangent space** T_yM **at** y,

ii. V is a **tangential vector field**,

iii. $\nabla_w V$ is the **directional derivative of** V **with respect to** w, where $w \in T_yM$,

iv. $\nabla_W V$ is the **directional derivative of** V **with respect to** W, where W is a tangential vector field.

(c) Let $M : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$M(x_1, x_2) = (x_2\, ,\, x_1 - x_2^2\, ,\, x_1^3 + x_1x_2)$$

i. Find the coordinate vector fields ϵ_1, ϵ_2 in terms of x_1, x_2.

ii. Show that M is a regularly parametrised surface.

iii. Compute $\nabla_{\epsilon_1}\epsilon_1$, $\nabla_{\epsilon_1}\epsilon_2$, $\nabla_{\epsilon_2}\epsilon_1$ and $\nabla_{\epsilon_2}\epsilon_2$.

iv. Consider the vector field V on M with coordinate expression

$$\hat{V}(x_1, x_2) = (1\, ,\, 1 - 2x_2\, ,\, 3x_1^2 + x_1 + x_2).$$

Show that V is a tangential vector field and compute the vector field $\nabla_V V$. [You may use standard properties of ∇.]

4. (a) Let $M : U \to \mathbb{R}^3$ be a regularly parametrized surface and let y be a point of M. Define what is meant by

i. an **orientation** on M;

ii. the **shape operator** at y;

iii. the **principal curvatures** at y;

iv. the **principal curvature directions** at y;

v. the **Gauss curvature** at y;

vi. the **mean curvature** at y.

(b) The following mapping $M : \mathbb{R}^2 \to \mathbb{R}^3$ defines a regularly parametrised surface:

$$M(x_1, x_2) = \left(x_1, x_2, \frac{1}{2}(x_1^2 + x_2^2) \right).$$

Let y be the point with coordinates $(x_1, x_2) = (1, 0)$.

i. Calculate the matrix of the shape operator at y.

ii. Calculate the principal curvatures, the Gauss curvature and the mean curvature at y.

iii. Does there exist a regular parametrised curve in M passing through y whose curvature vector $k = (k_1, k_2, k_3)$ at y satisfies $k_1 - k_3 = 0$? Justify your answer.

Check Sheet

1. (a) No it isn't, since $\alpha'(\pi/2) = (0,0)$.

(b) (i) $\sigma_0(t) = \sqrt{5}t$.

(ii)

(c) (i) $\widehat{\gamma}_1(t) = (1,1,\frac{1}{3}) + t(1,2,1)$.

(ii)

$$k(1) = \frac{1}{6}\left((0,2,2) - \frac{6}{6}(1,2,1)\right) = \frac{1}{6}(-1,0,1)$$

(iii) $u(1) = \gamma'(1)/|\gamma'(1)| = \frac{1}{\sqrt{6}}(1,2,1)$
$n(1) = k(1)/|k(1)| = \frac{1}{\sqrt{2}}(-1,0,1)$
$b(1) = u(1) \times n(1) = \frac{1}{\sqrt{3}}(1,-1,1)$

(iv) Two

2. (a)

(b) 1.

(c)

(d)

(e)

(f)

(g) $\tau(s) = s$

3. (a)

κ_1	κ_2	κ_3	κ_4	κ_5	κ_6	κ_7
F	A	G	C	D	E	B

A	B	C	D	E	F	G
κ_2	κ_7	κ_4	κ_5	κ_6	κ_1	κ_3

(b) (C). Yes, the evolute is regular.

(c) γ_λ is regular if and only if $\lambda \in [-1,1]$.

(d) 8π

4. (a) (i)

(ii)

(iii)

(iv)

(b) (i)

(ii) $(0,2)$

(iii) T_pM is spanned by $\{\varepsilon_1, \varepsilon_2\} = \{(1,0,2),(0,12,1)\}$
N_pM is spanned by $\nu = \varepsilon_1 \times \varepsilon_2 = (-24,-1,12)$

(iv)

(v) $v[f] = 7$

(vi) Any non-zero multiple of $\nu \times v = (-141,-96,-290)$ will do

5. (a) $N = \dfrac{(-x_2 \cos x_1 x_2, -x_1 \cos x_1 x_2, 1)}{\sqrt{1 + (x_1^2 + x_2^2) \cos^2 x_1 x_2}}$

(b)

(c) $\widehat{S}_p = \begin{pmatrix} 0 & (1+\pi^2)^{-3/2} \\ (1+\pi^2)^{-1/2} & 0 \end{pmatrix}$

(d)

$$\begin{aligned} H(p) &= \frac{1}{2} \operatorname{tr} \widehat{S}_p = 0 \\ K(p) &= \det \widehat{S}_p = -\frac{1}{(1+\pi^2)^2}. \end{aligned}$$

(e) $\kappa_1 = -\dfrac{1}{1+\pi^2}$, $\kappa_2 = \dfrac{1}{1+\pi^2}$.

(f) $u_2 = \pm N \times u_1 = \pm \dfrac{\left(1, \sqrt{1+\pi^2}, \pi\right)}{\sqrt{2}\sqrt{1+\pi^2}}$.

(g) (i) $v = \dfrac{1}{\sqrt{2}}(u_1 \pm u_2) = \dfrac{(1, 0, \pi)}{\sqrt{1+\pi^2}}$ or $(0, 1, 0)$, or minus these

(ii) No such v exists

Check sheet: Geometry of curves and surfaces

School of Mathematics **Semester One 201819**

1. (a) i. This is NOT a RPC.
ii. This IS a RPC.

(b) i. $\sigma_0(t) = \sinh t$
ii. $\widehat{\gamma}_1(t)(1, \cosh 1) + t(1, \sinh 1)$

(c) bookwork

2. (a)

κ_1	even	reflection	infinitely many	(A)
κ_2	odd	$180°$ rotation	one	(D)
κ_3	odd	$180°$ rotation	three	(C)
κ_4	even	reflection	none	(B)

(b) bookwork

3. (a) bookwork

(b) bookwork

(c) i.

$$\epsilon_1 = (0, 1, 3x_1^2 + x_2)$$
$$\epsilon_2 = (1, -2x_2, x_1)$$

ii. proof – omitted
iii.

$$\nabla_{\epsilon_1}\epsilon_1 = (0, 0, 6x_1)$$
$$\nabla_{\epsilon_1}\epsilon_2 = (0, 0, 1)$$
$$\nabla_{\epsilon_2}\epsilon_1 = (0, 0, 1)$$
$$\nabla_{\epsilon_2}\epsilon_2 = (0, -2, 0)$$

iv. $V = \epsilon_1 + \epsilon_2$. $\nabla_V V = (0, -2, 2 + 6x_1)$.

4. (a) bookwork

(b) i. In the standard coordinate basis ϵ_1, ϵ_2 the matrix is $\begin{pmatrix} -2^{-\frac{3}{2}} & 0 \\ 0 & -2^{-\frac{1}{2}} \end{pmatrix}$. (You might get a different answer if you chose a different basis.)
ii. $\kappa_1 = -2^{-\frac{1}{2}}$, $\kappa_2 = -2^{-\frac{3}{2}}$, Gauss curvature $= \frac{1}{4}$, mean curvature $= -3/2^{\frac{5}{2}}$.
iii. No.

 End.