Geometry of Curves and Surfaces Summary Notes:

January 21, 2020

0 Preamble:

These notes are using my own thoughts on what could come up. This is not exhaustive list of the module, just what I believe is necessary for the exam. A full pdf (Including 2 papers with check sheets) of the notes can be found on my git repository . The one labelled (Full Notes) contains all the notes, while the other only contains the end of chapter summaries.

1 Question 1:

Question one usually asks for a set of definitions, ones that have come up so far are (Note, this section also covers Question 2 in 2018 paper):

1.1 RPCs

Definition 1 A **parametrized curve** (PC) in $\mathbb{R}^n$ is a smooth map $\gamma : I \to \mathbb{R}^n$. A time $t \in I$ is a **regular point** of γ if $\gamma'(t) \neq 0$. If $\gamma'(t) = 0$, then $t \in I$ is a **singular point** of γ. If every $t \in I$ is regular then γ is said to be a **regularly parametrized curve** (RPC). In other words, a PC is a RPC if and only if

$$\boxed{\text{there does not exist a time } t \in I \text{ such that } \gamma'(t) = 0 = (0, 0, \dots, 0).}$$

The **image set** of a curve γ is the range of the mapping, that is,

$$\gamma(I) = \{\gamma(t) \in \mathbb{R}^n \,|\, t \in I\} \subset \mathbb{R}^n$$

□

Definition 2 Let $\gamma : I \to \mathbb{R}^n$ be a RPC. Then its **tangent line** at $t_0 \in I$ is the PC

$$\widehat{\gamma}_{t_0} : \mathbb{R} \to \mathbb{R}^n, \qquad \widehat{\gamma}_{t_0}(t) = \gamma(t_0) + t\gamma'(t_0).$$

Note that $\widehat{\gamma}'_{t_0}(t) = 0 + \gamma'(t_0) \neq 0$ since γ is a RPC. Hence every tangent line $\widehat{\gamma}_{t_0}$ is a RPC too. □

Definition 3 Let $\gamma : I \to \mathbb{R}^n$ be a RPC. The **arc length** along γ from $t = t_0$ to $t = t_1$ is

$$s = \int_{t_0}^{t_1} |\gamma'(t)|\, dt. \qquad \square$$

Definition 4 Let $\gamma : I \to \mathbb{R}^n$ be a RPC. The **arc length function** based at $t_0 \in I$ is

$$\sigma_{t_0} : I \to \mathbb{R}, \qquad \sigma_{t_0}(t) = \int_{t_0}^{t} |\gamma'(u)|\, du. \qquad \square$$

Definition 5 A **reparametrization** of a PC $\gamma : I \to \mathbb{R}^n$ is a map $\beta : J \to \mathbb{R}^n$ defined by $\beta(u) = \gamma(h(u))$, where J is an open interval, and $h : J \to I$ is smooth, surjective and has $h'(u) > 0$ for all $u \in J$. $\square$

Definition 6 A **unit speed curve** (USC) is a smooth map $\gamma : I \to \mathbb{R}^n$ such that $|\gamma'(s)| = 1$ for all $s \in I$ $\square$

Theorem 7 *Every RPC $\gamma : I \to \mathbb{R}^n$ has a unit speed reparametrization (USR) $\beta : J \to \mathbb{R}^n$. This USR is unique up to "time" translation. More precisely, if $\delta : K \to \mathbb{R}^n$ is another USR of γ, then there exists a constant $c \in \mathbb{R}$ such that*

$$\beta(s) = \delta(s - c).$$

1.2 Curvature

Definition 8 (a) Let $\gamma : I \to \mathbb{R}^n$ be a USC. Then the **curvature vector** of γ is $k : I \to \mathbb{R}^n$ where

$$k(s) = \gamma''(s).$$

Note that k is a *vector* quantity. We shall refer to the norm of k, $|k| : I \to [0, \infty)$ as the curvature of γ. (b) If a RPC $\gamma : I \to \mathbb{R}^n$ is *not* a USC, then Theorem 7 says that it has a unit speed reparametrization $\beta : J \to \mathbb{R}^n$, $\beta = \gamma \circ h$. In that case, we define the curvature vector of γ at $t = h(s)$ to be the curvature vector of β at s, as in part (a), that is, $\beta''(s)$. In other words, $k : I \to \mathbb{R}^n$ such that

$$k = \beta'' \circ h^{-1}.$$

Note: To make sense, this definition should be independent of the choice of unit speed reparametrization β of γ. It is. Recall, by Theorem 7, that any pair of USRs of γ differ only by shifting the origin of the new time coordinate s. But such a shift has no effect on the second (or indeed the first) derivative of β. $\square$

Definition 9 [*] The **curvature vector** of a RPC $\gamma : I \to \mathbb{R}^n$ is $k : I \to \mathbb{R}^n$, where

$$k(t) = \frac{1}{|\gamma'(t)|^2} \left\{ \gamma''(t) - \left(\frac{\gamma'(t) \cdot \gamma''(t)}{|\gamma'(t)|^2} \right) \gamma'(t) \right\}. \qquad \square$$

Definition 10 Let $\gamma : I \to \mathbb{R}^n$ be a RPC. Its **unit tangent vector** $u : I \to \mathbb{R}^n$ is

$$u(t) = \frac{\gamma'(t)}{|\gamma'(t)|}.$$

Note that u is well defined because γ is a **R**PC (so $|\gamma'(t)| > 0$ for all t). Note also that $|u(t)| = 1$ for all t by construction. Given any other vector valued function $v : I \to \mathbb{R}^n$, we define its **normal projection**, $v_\perp : I \to \mathbb{R}^n$ by

$$v_\perp(t) = v(t) - [v(t) \cdot u(t)]u(t).$$

We can think of $v_\perp$ as that part of v left over after we have subtracted off the component of v in the direction of u. □

Definition 11 The **curvature vector** of a RPC $\gamma : I \to \mathbb{R}^n$ is $k : I \to \mathbb{R}^n$,

$$k(t) = \frac{\gamma''_\perp(t)}{|\gamma'(t)|^2} \qquad \square$$

1.3 Frenet Frame

Definition 12 Let $\gamma : I \to \mathbb{R}^2$ be a RPC with unit tangent vector u (recall $u = \gamma'/|\gamma'|$). The **unit normal vector** of γ is $n : I \to \mathbb{R}^2$,

$$n(t) = (-u_2(t), u_1(t)).$$

The **signed curvature** of γ is $\kappa : I \to \mathbb{R}$,

$$\kappa(t) = k(t) \cdot n(t)$$

where $k : I \to \mathbb{R}^2$ is the curvature vector of γ, as in Definition 8(**).

Definition 13 Let $\gamma : I \to \mathbb{R}^3$ be a RPC whose *curvature never vanishes* (i.e. for all $t \in I$, $|k(t)| \neq 0$). Then in addition to the **unit tangent vector**

$$u(t) := \frac{\gamma'(t)}{|\gamma'(t)|}$$

one defines the **principal unit normal vector**

$$n(t) := \frac{k(t)}{|k(t)|}$$

and the **binormal vector**

$$b(t) := u(t) \times n(t).$$

The ordered triplet $[u(t), n(t), b(t)]$ is called the **Frenet frame** of the curve. γ □

Theorem 14 *[The Frenet formulae] Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve of nonvanishing curvature. Then its Frenet frame satisfies the formulae*

$$\begin{array}{rcl} u'(s) & = & \kappa(s)n(s) \\ n'(s) & = & -\kappa(s)u(s) \quad +\tau(s)b(s) \\ b'(s) & = & -\tau(s)n(s) \end{array}$$

□

Definition 15 Let $\gamma : I \to \mathbb{R}^3$ be a USCNVC. The **torsion** of the curve is that function $\tau : I \to \mathbb{R}$ defined by the equation

$$\boxed{b'(s) = -\tau(s)n(s)} \tag{2}$$

Alternatively, $\tau(s) = -b'(s) \cdot n(s)$. □

□

2 Question 2

This question begins with finding a USC from a graph of its curvature. We know this will be a larger question than it was in 2019 from Derek's revision guide video. So I'll briefly sum up the method:

- First a reminder, the *parity* of an equation is either even if $f(-t) = f(t) \forall t \in I$ or odd if $f(-t) = -f(t) \forall t \in I$. A function can also have no parity.

- Next, a proposition from the notes (see Full Notes page 30 for proof, however I doubt this would come up as a proof)

  Proposition 1 *Let $\gamma(s)$ be the PUSC of curvature $\kappa(s)$ with $\gamma(0) = 0$, $\gamma'(0) = (1, 0)$. If κ is even, γ is symmetric under reflexion in the x_2 axis. If κ is odd, γ is symmetric under rotation by 180 degrees about $(0, 0)$.*

- Now, for a question you are given a set of curvatures $\{\kappa_1, \cdots, \kappa_i\}$ as well as images of i curves. The first step I'd do is set out this table:

	Parity	Symmetry	Inflexion	$+\infty$	$-\infty$
κ_1					
κ_2					
$\vdots$					
κ_i					

- Next, from the curves I'd fill in their details. Parity is simple to check and then corresponds with the Symmetry field. The Inflexion category is for how many times the curve changes direction, this is given by how many s^* there are such that $\kappa(s^*) = 0$. Next is behaviour as $s \to +\infty$ and $s \to -\infty$. We can seperate the behaviours into a set of cases:

$$s \to \pm\infty \begin{cases} \kappa \to 0 & \text{The curve straightens} \\ \kappa \to \infty & \text{The curve spirals} \\ \kappa \text{ oscillates and is bounded} & \text{The curve wiggles, i.e.:}\kappa = \sin(s) \\ \kappa \text{ oscillates and is unbounded} & \text{The curve has alternating spirals, i.e.:}\kappa = s\cos(s) \end{cases}$$

- Finally, check which of the curveatures corresponds to each given curve.

The rest of this question usually goes into Evolutes and Involutes, so relevant definitions are:

Definition 2 The **evolute** of the planar curve γ is $E_\gamma : I \to \mathbb{R}^2$, defined such that

$$E_\gamma(t) = \gamma(t) + \frac{1}{\kappa(t)} n(t),$$

in other words, it is the curve of centres of curvature of the curve γ. □

Definition 3 The **involute** of γ starting at $t_0 \in I$ is $I_\gamma : I \to \mathbb{R}^2$, defined by

$$I_\gamma(t) = \gamma(t) - \sigma_{t_0}(t) u(t).$$

Note that if $\gamma(t)$ is a USC then $\sigma_{t_0}(t) = t - t_0$ so

$$I_\gamma(t) = \gamma(t) - (t - t_0)\gamma'(t).$$

Definition 4 Given a RPC $\gamma : I \to \mathbb{R}^2$ and a constant $\lambda \in \mathbb{R}$, the curve $\gamma_\lambda : I \to \mathbb{R}^2$ defined by

$$\gamma_\lambda(t) = \gamma(t) + \lambda n(t)$$

is a **parallel curve** to γ. □

Theorem 5 *Let I_γ be an involute of γ. Then the evolute E_I of I_γ is γ.*

Lemma 6 *Let γ be a unit speed curve and I be its involute based at time t_0. Then I has unit normal*

$$n^I(t) = \frac{(t - t_0)\kappa}{|t - t_0||\kappa(t)|} u(t),$$

and signed curvature

$$\kappa^I(t) = \frac{\kappa(t)}{|\kappa(t)|} \frac{1}{|t - t_0|},$$

where κ, u are the signed curvature and unit tangent vector of γ

Theorem 7 *Let γ have evolute E_γ. Then every involute I_E of E_γ is a parallel curve to γ.*

3 Question 3:

This question will be on Regularly Parametrised surfaces:

3.1 RPS

Definition 1 For each $x \in \mathbb{R}^2$ and $r > 0$ let $B_r(x) = \{y \in \mathbb{R}^2 : |y - x| < r\}$ be the **disk of radius r centred at x.** Then a subset $U \subseteq \mathbb{R}^2$ is **open** if for all $x \in U$ there exists $\delta > 0$ such that $B_\delta(x) \subseteq U$. $\square$

Definition 2 Given a smooth map $\phi : U \to \mathbb{R}^3$, where U is an open subset of $\mathbb{R}^2$, the **coordinate basis vectors** $\phi_u, \phi_v : U \to \mathbb{R}^3$ are

$$\phi_u(u,v) = \frac{\partial \phi}{\partial u}(u,v), \qquad \phi_v(u,v) = \frac{\partial \phi}{\partial v}(u,v).$$

A point $(u,v) \in U$ is a **regular point** of ϕ if the vectors $\phi_u(u,v)$ and $\phi_v(u,v)$ are linearly independent. The map ϕ is **regular** if every $(u,v) \in U$ is a regular point. $\square$

Definition 3 [*] $\phi : U \to \mathbb{R}^3$ is **regular** if for all $(u,v) \in U$,

$$\phi_u(u,v) \times \phi_v(u,v) \neq (0,0,0).$$

Definition 4 A **regularly parametrized surface** (RPS) is an injective (i.e. one-to-one), regular map $\phi : U \to \mathbb{R}^3$. $\square$

Remark 5 Since a RPS $\phi : U \to \mathbb{R}^3$ is one-to-one, given a point $p = (p_1, p_2, p_3)$ on $M = \phi(U)$, there is one and only one point $(u,v) \in U$ such that $\phi(u,v) = (p_1, p_2, p_3)$. We call the numbers (u,v) the **local coordinates** of the point p. In this way, we can specify a point on M by giving two numbers (its local coordinates) rather than three (its coordinates in $\mathbb{R}^3$), in much the same way that we can specify a point on a curve $\gamma(t)$ by giving the single number t.

3.2 Tangent and Normal spaces

Definition 6 Let $p \in M$ be a point on a RPS $M = \phi(U)$ where $\phi : U \to \mathbb{R}^3$. Then a **curve in M through p** is a smooth map $\alpha : I \to M$ $(0 \in I)$ with $\alpha(0) = p$. The **tangent space** to M at $p \in M$ is

$$T_pM = \{x \in \mathbb{R}^3 : \text{there exists a curve } \alpha \text{ in } M \text{ through } p \text{ with } \alpha'(0) = x\}.$$

Any $x \in T_pM$ is called a **tangent vector** to M at p. $\square$

Definition 7 The **normal space** at $p \in M$ is

$$N_pM = \{y \in \mathbb{R}^3 : y \cdot x = 0 \text{ for all } x \in T_pM\}.$$

Any $y \in N_pM$ is said to be **normal to M at** p. $\square$

Theorem 8 *T_pM is a vector space of dimension 2 spanned by $\{\phi_u(\bar{u},\bar{v}),\phi_v(\bar{u},\bar{v})\}$.*$\square$

Remark 9 By its definition, N_pM is also a vector space, that is, it is closed under vector addition and scalar multiplication: let $y,z \in N_pM$ and $a,b \in \mathbb{R}$. Then for all $x \in T_pM$,

$$x \cdot (ay+bz) = a(x \cdot y) + b(x \cdot z) = 0 + 0$$

so $ay + bz \in N_pM$. Clearly N_pM is one-dimensional, so any non-zero normal vector, for example $\phi_u \times \phi_v$, is a basis for N_pM.
Furthermore, the *tangent space* T_pM is precisely the two dimensional space of vectors orthogonal to N_pM, or equivalently, to any nonzero vector in N_pM. Hence

$$x \in T_pM \quad \Leftrightarrow \quad x \cdot (\phi_u \times \phi_v) = 0.$$

This gives us a sneaky way of checking whether a given vector is a tangent vector.

3.3 Co-ordinate expression and directional derivatives

Definition 10 Let $\phi : U \to \mathbb{R}^3$ be a regularly parametrized surface with $M = \phi(U)$ and let $f : M \to \mathbb{R}$ be a function on M. We say that f is **smooth** if $f \circ \phi : U \to \mathbb{R}$ is smooth in the usual sense. We refer to $\widehat{f} = f \circ \phi$

$$\widehat{f}(u,v) = f(\phi(u,v)).$$

as the **coordinate expression** of f.

Definition 11 Let $f : M \to \mathbb{R}$ be a smooth function and $x \in T_pM$. The **directional derivative** of f along x is

$$\nabla_x f = (f \circ \alpha)'(0) = \frac{d}{dt} f(\alpha(t))\bigg|_{t=0}$$

where α is a generating curve[1] for x. [Such a curve exists by the definition of T_pM, Definition 6.] Note that $\nabla_x f$ is a single number, associated with the point p, not a function on M. $\square$

Lemma 12 *$\nabla_x f$ is linear with respect to both x and f. That is, for all $x,y \in T_pM$, $a,b \in \mathbb{R}$ and $f,g : M \to \mathbb{R}$,*

(A) $\nabla_{ax+by} f = a\nabla_x f + b\nabla_y f$

(B) $\nabla_x(af + bg) = a\nabla_x f + b\nabla_x g$

Directional derivatives along coordinate basis vectors reduce to partial derivatives

$$\nabla_{\phi_u} f = \frac{\partial \widehat{f}}{\partial u}, \qquad \nabla_{\phi_v} f = \frac{\partial \widehat{f}}{\partial v}.$$

[1] Recall this is a curve $\alpha(t)$ in M with $\alpha(0) = p$ and $\alpha'(0) = x$

Definition 13 A **vector field** on a RPS M is a smooth map $X : M \to \mathbb{R}^3$ (where smooth means that each of its component functions $X_1, X_2, X_3 : M \to \mathbb{R}$ is smooth). If $X(p) \in T_pM$ for all $p \in M$ then X is called a **tangent vector field**. If $X(p) \in N_pM$ for all $p \in M$ then X is a **normal vector field**.

Remark 14 Just as for functions, we define the **coordinate expression** of a vector field $X : M \to \mathbb{R}^3$ to be

$$\widehat{X} : U \to \mathbb{R}^3, \qquad \widehat{X}(u,v) = V(\phi(u,v)).$$

It follows from Theorem 8 that the coordinate expression of any **tangent** vector field X takes the form

$$\widehat{X}(u,v) = f(u,v)\phi_u(u,v) + g(u,v)\phi_v(u,v)$$

where f, g are smooth real-valued functions on U. Similarly, it follows from Remark 9 that any **normal** vector field $X : M \to \mathbb{R}^3$ takes the form 12l-valued function on U.

We will often refer to $\phi_u(u,v), \phi_v(u,v)$ as tangent vector fields, even though they are really coordinate expressions for tangent vector fields. Similarly, we will normally refer to $\phi_u \times \phi_v$ as a normal vector field (even though it's actually a coordinate expression for a normal vector field).

Definition 15 Let $X : M \to \mathbb{R}^3$ be a vector field on M and $y \in T_pM$. Then the **directional derivative** of X with respect to y is

$$\nabla_y X = (X \circ \alpha)'(0)$$

where $\alpha : I \to M$ is a curve through $p \in M$ with $\alpha'(0) = x$. $\square$

Lemma 16 *Let X, Y be vector fields on M, $z, w \in T_pM$, f be a smooth function on M and $a, b \in \mathbb{R}$. Then*

(a) $\nabla_{az+bw}X = a\nabla_z X + b\nabla_w X$.

(b) $\nabla_z(aX + bY) = a\nabla_z X + b\nabla_z Y$.

(c) $\nabla_z(fX) = (\nabla_z f)X(p) + f(p)\nabla_z X$.

(d) $\nabla_z(X \cdot Y) = (\nabla_z X) \cdot Y(p) + X(p) \cdot (\nabla_z Y)$.

- *Note, These reduce as with 12*

4 Question 4:

This question relates to the last section of notes which is *Heavily reduced* due to the strike action. If you're revising from both this and the main notes (as I'd suggest) then ignore section 8.3 as well as Definitions 101 to 105. Relevant definitions for this section:

4.1 Orientation

Definition 1 An **orientation** on a RPS $\phi : U \to \mathbb{R}^3$ with image $M = \phi(U)$ is a choice of unit normal vector field N. In other words, $N : M \to \mathbb{R}^3$ must satisfy $|N(p)| = 1$ and $N(p) \in N_pM$ for all $p \in M$. □

In general, we can choose an orientation for an RPS as follows: recall that $\phi_u \times \phi_v$ is a normal vector field which never vanishes. It follows that

$$N = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|}$$

is a **unit** normal vector field. This N is known as the **canonical orientation** of ϕ. The only other possible choice of orientation is

$$\tilde{N} = -\frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|}.$$

Lemma 2 *Let $\phi : U \to \mathbb{R}^3$ be an RPS and let N be an orientation for $M = \phi(U)$. Then for any $p \in M$ and $x \in T_pM$, $\nabla_x N \in T_pM$ also.*

Definition 3 Let $\phi : U \to \mathbb{R}^3$ and let N be an orientation of $M = \phi(U)$. The **shape operator** at $p \in M$ is the map

$$S_p : T_pM \to T_pM, \qquad S_p(x) = -\nabla_x N.$$

S_p really does map T_pM to itself, by Lemma 2, and is a linear map by Lemma 16. It is often called the **Weingarten map** in honour of its discoverer. □

4.2 Principal Curvature

Definition 4 Let M be an oriented surface, $S_p : T_pM \to T_pM$ be its shape operator at $p \in M$. Then the **principal curvatures** of M at p are κ_1, κ_2, the eigenvalues of S_p. The **principal curvature directions** of M at p are the corresponding eigenvectors e_1, e_2 (normalized to have unit length). By Theorem *REDACTED DUE TO STRIKES*, κ_1, κ_2 are real and the eigenvectors e_1, e_2 form an orthonormal basis for T_pM. □

4.3 Mean and Gauss curvatures

Definition 5 The **mean curvature** of M at $p \in M$ is

$$H(p) = \frac{1}{2}(\kappa_1 + \kappa_2)$$

where κ_1, κ_2 are the principal curvatures of M at p. The **Gauss curvature** of M at p is

$$K(p) = \kappa_1 \kappa_2 \qquad \square$$

The sign of K has intrinsic meaning, independent of the choice of N: if $K(p) > 0$ then either all curves in M through p curve towards $N(p)$, or they all curve away from $N(p)$; if $K(p) < 0$ then some curves curve towards $N(p)$ and some curve away.

Proposition 6 *For all $p \in M$,*

$$H(p) = \frac{1}{2}\mathrm{tr}\,\widehat{S}_p, \quad K(p) = \det \widehat{S}_p,$$

where $\widehat{S}_p$ is the matrix representing $S_p : T_pM \to T_pM$ relative to some choice of basis for T_pM.

[Recall that the **trace** of a square matrix L is the sum of its diagonal elements.]
NOTE, Normal Curvature also exists but is non-examinable due to strikes.

5 Final Notes

This module is pretty dense with info but hopefully you all should be fine. I believe in you all and I hope these notes help organise the info a little bit more for you, see you all on exam day!