SUPSI

B3209E
Algorithm Design

Chapter 01: Representative Problems
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Outline

- Five Representative Problems
- Stable Matching

Outline

- **Five Representative Problems**
- Stable Matching

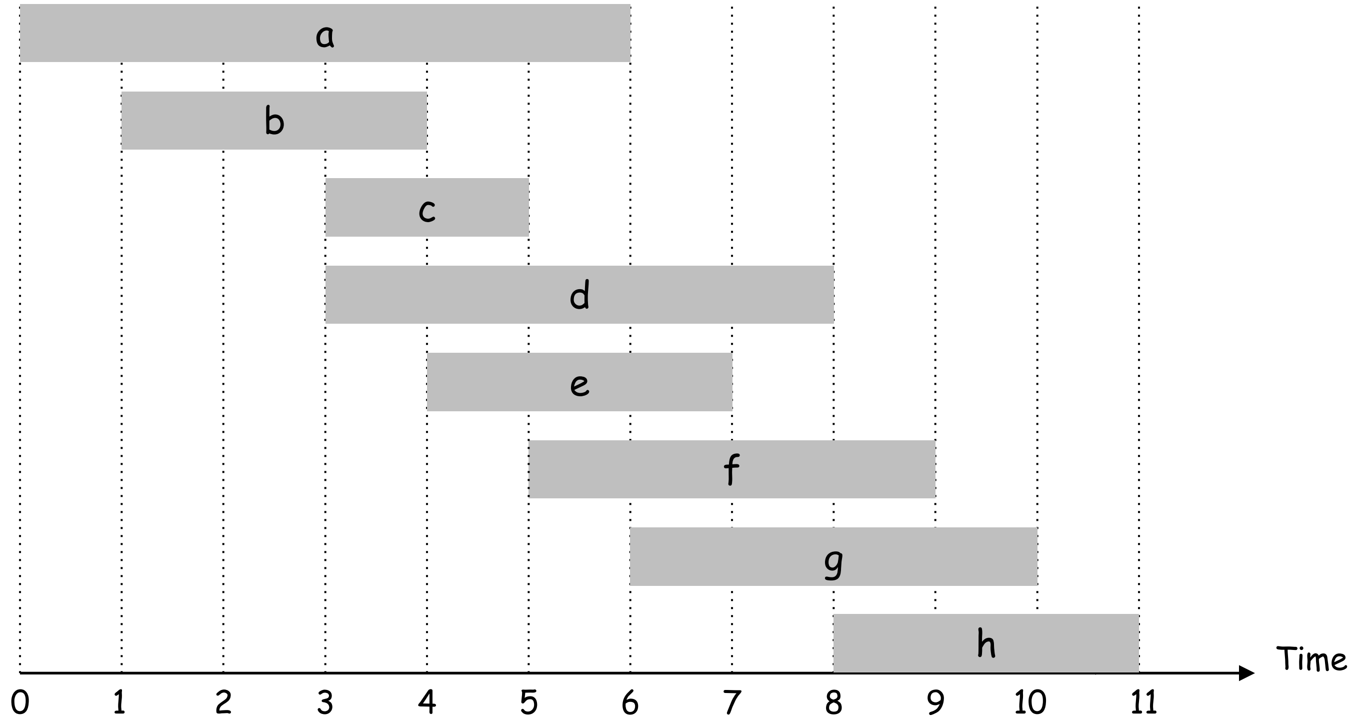
Five Representative Problems

- Interval Scheduling
- Weighted Interval Scheduling
- Bipartite Matching
- Independent Set
- Competitive Facility Location

Interval Scheduling

jobs don't overlap

- Input. Set of jobs with start times and finish times.
- Goal. Find maximum cardinality subset of mutually compatible jobs.



Interval Scheduling

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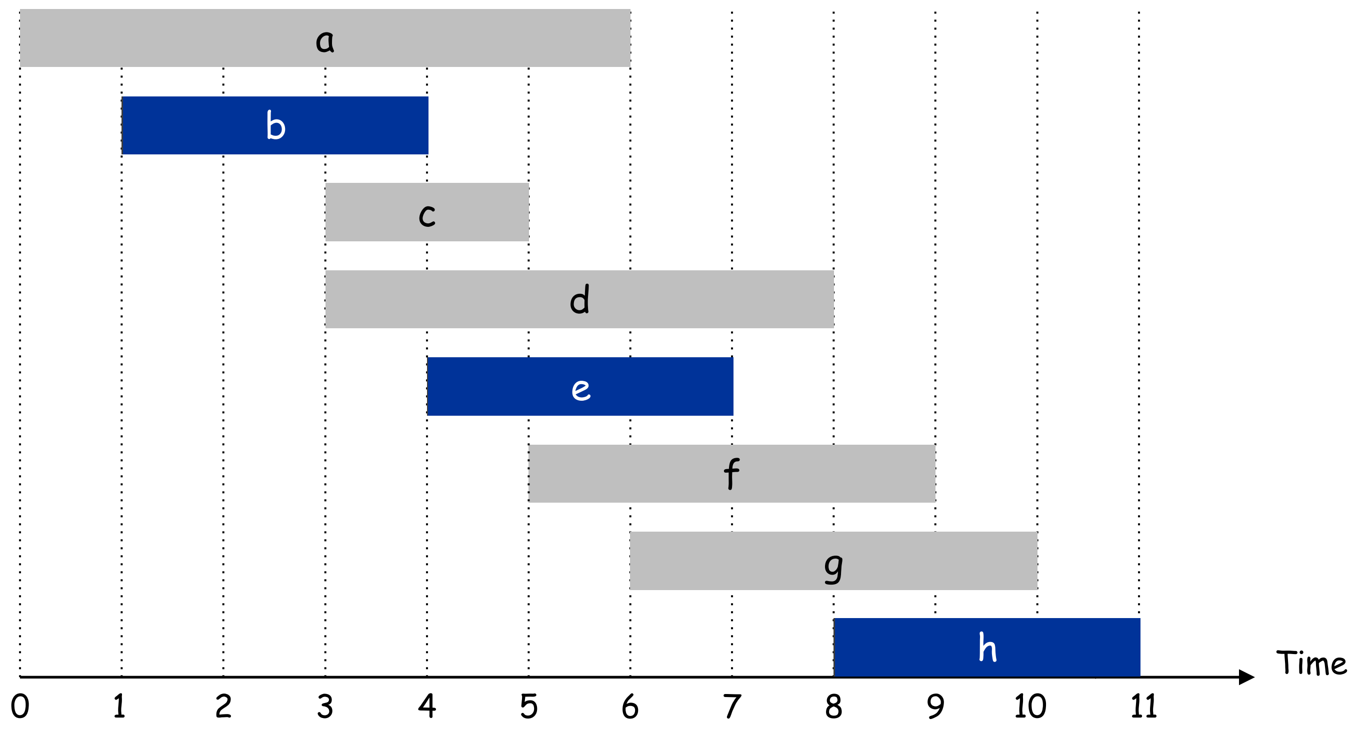
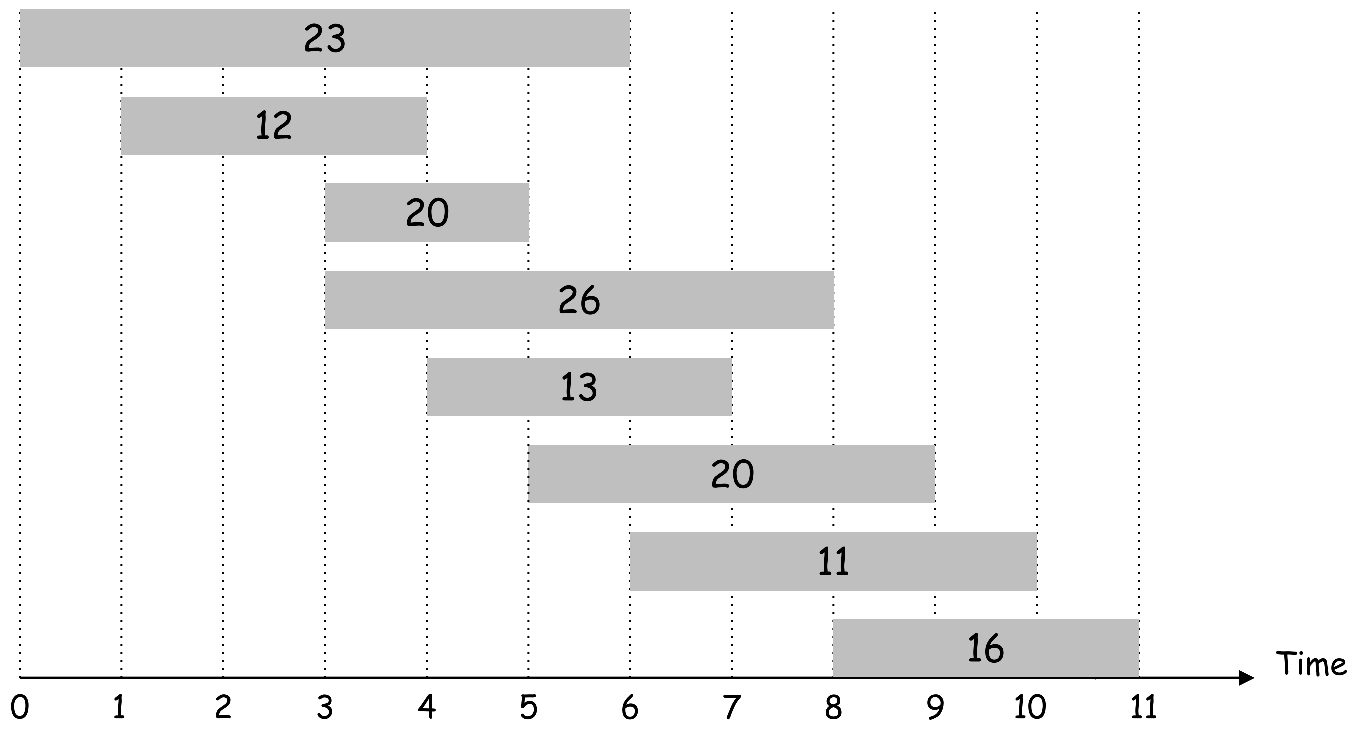


Weighted Interval Scheduling

- Input: Set of jobs with start times, finish times, and weights.
- Goal: Find maximum weight subset of mutually compatible jobs.



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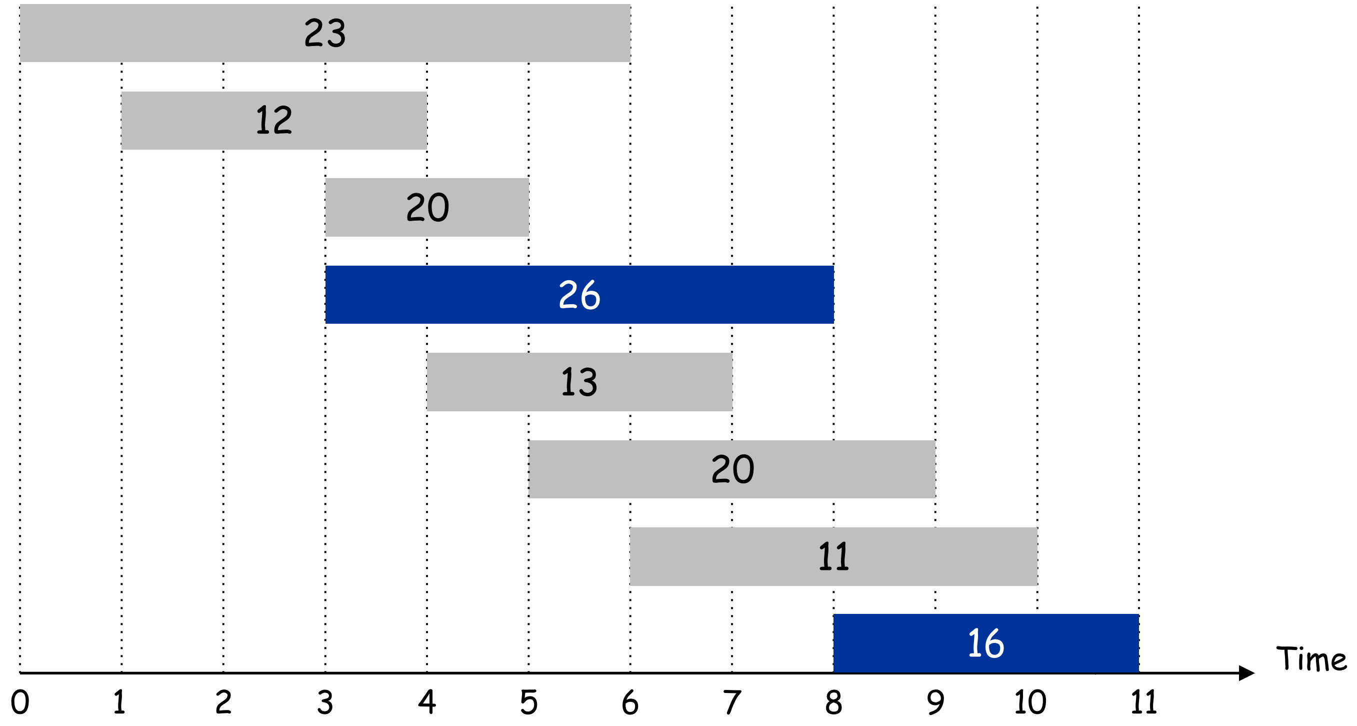
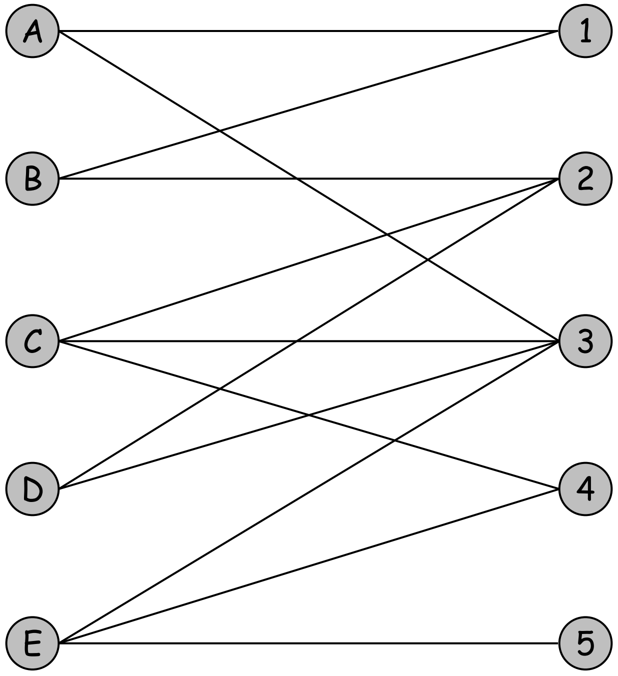


Bipartite Matching

- Input: Bipartite graph.
- Goal: Find maximum cardinality matching.



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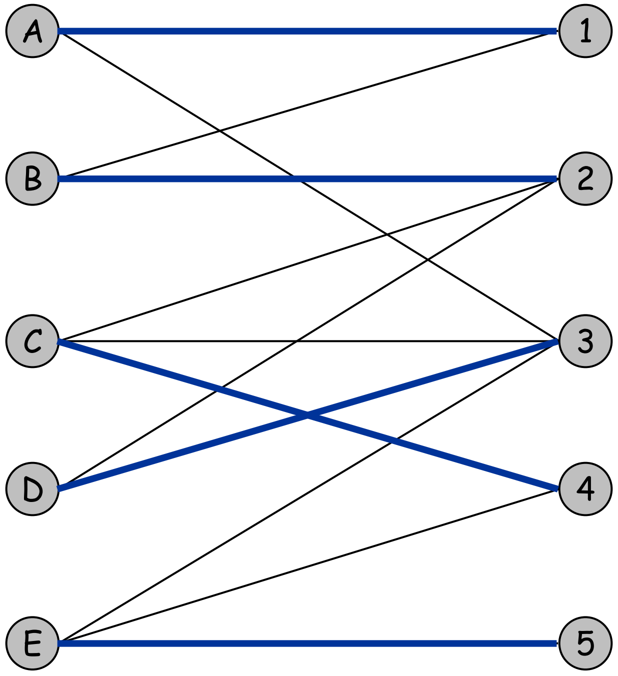
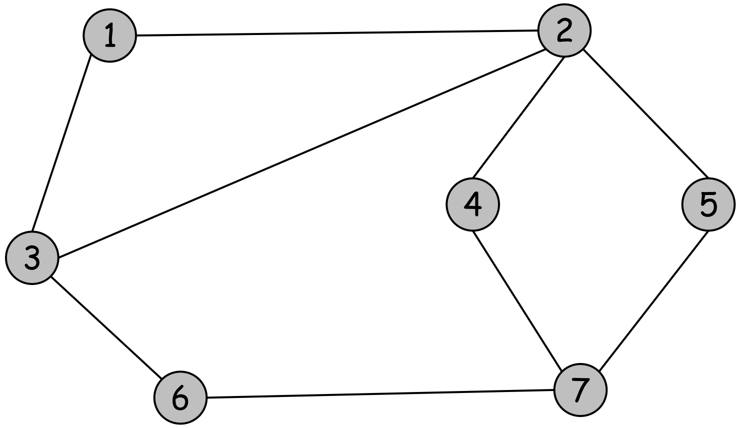


Independent Set

- Input: Graph.
- Goal: Find maximum cardinality independent set.

subset of nodes such that no two joined by an edge



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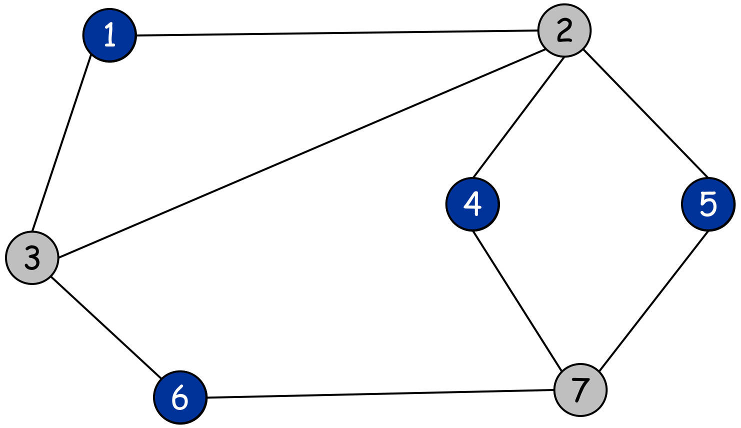
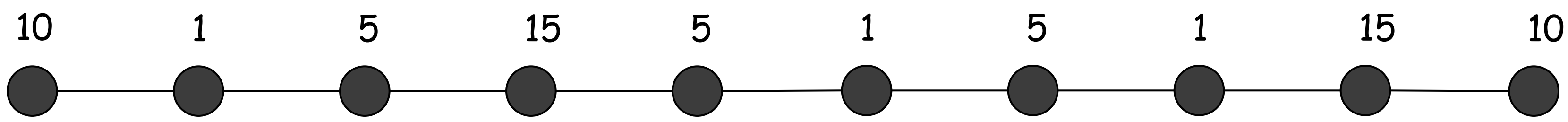


Competitive Facility Location

- Input: Graph with weight on each node.
- Game: Two competing players alternate in selecting nodes. Not allowed to select a node if any of its neighbors have been selected.

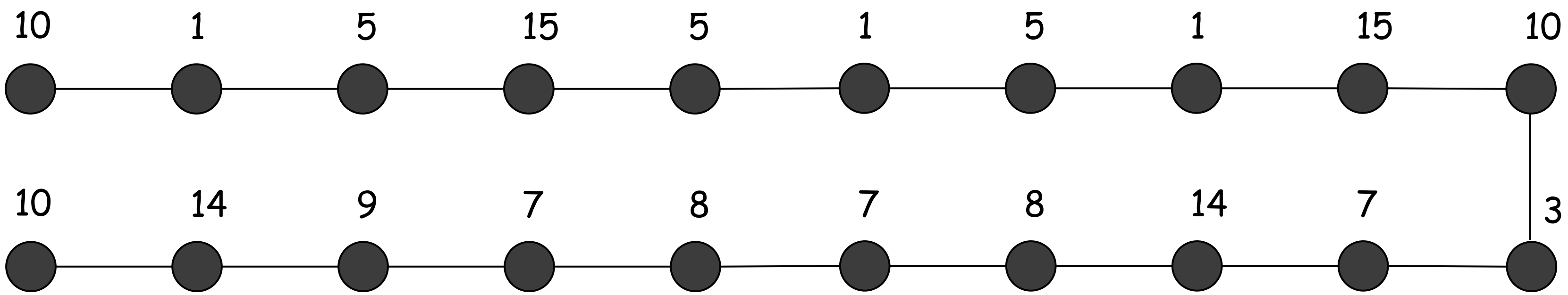
- Goal: Select a **maximum weight** subset of nodes.



Second player can guarantee 20, but not 25.

Competitive Facility Location

- Input: Graph with weight on each node.
- Game: Two competing players alternate in selecting nodes. Not allowed to select a node if any of its neighbors have been selected.



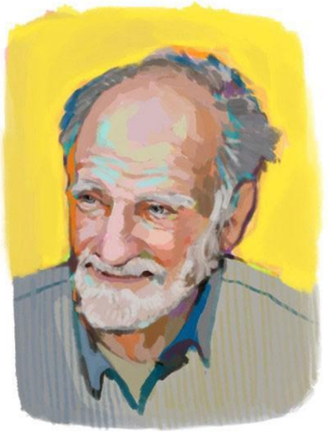
Outline

- Five Representative Problems
- **Stable Matching**

Stable Matching Problem

- In 1962, two mathematical economists asked a question:
  - “Could one design a college admissions process, or a job recruiting process, that was *self-enforcing*?”

David Gale
PROFESSOR, UC BERKELEY



Lloyd Shapley
PROFESSOR EMERITUS, UCLA

A *self-enforcing* process is one in which all participants, acting in their own self-interest, have no incentive to deviate from the assigned outcome. In other words, the system’s stability arises naturally because no pair of agents can both improve their situation by breaking the rules and matching differently.

Stable Matching Problem

- **Goal**: Given a set of preferences among companies and data science students (applicants), design a *self-enforcing* internship recruitment process.
  - Each applicant lists companies in order of preference from best to worst.
  - Each company lists applicants in order of preference from best to worst.

	1st (favorite)	2nd	3rd (least favorite)
Andrea	Meta	Google	Apple
Michela	Google	Meta	Apple
Luca	Meta	Google	Apple

Applicants' Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Meta	Michela	Andrea	Luca
Google	Andrea	Michela	Luca
Apple	Andrea	Michela	Luca

Companies' Preference Profile

Stable Matching Problem

- **Unstable pair**: applicant x and company y are **unstable** if:
  - x prefers y to its assigned company.
  - y prefers x to one of its admitted interns.

	1st (favorite)	2nd	3rd (least favorite)
Andrea	Meta	Google	Apple
Michela	Google	Meta	Apple
Luca	Meta	Google	Apple

Applicants' Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Meta	Michela	Andrea	Luca
Google	Andrea	Michela	Luca
Apple	Andrea	Michela	Luca

Companies' Preference Profile

Stable Matching Problem

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  - Each applicant lists companies in order of preference from best to worst.
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- **Unstable pair**: applicant x and company y are **unstable** if:
  - x prefers y to its assigned company.
  - y prefers x to one of its admitted interns.

- **Stable assignment**. Assignment with no unstable pairs.
  - Natural and desirable condition.
  - Individual self-interest will prevent any applicant/company deal from being made.

- **Perfect matching**: everyone is matched monogamously.

Stable Matching Problem

- **Stability**: no incentive for some pair of participants to undermine assignment by joint action.
  - In matching M, an unmatched pair a-c is unstable if applicant a and company c prefer each other to their current matching.
  - Unstable pair a-c could each improve by eloping.

- **Stable matching**: perfect matching with no unstable pairs.

Stable Matching Problem

- Stable matching problem. Given the preference lists of n participants of gender 1 and n participants of gender 2, find a stable matching if one exists.

- Let us find a stable matching:

	1st	2nd	3rd
Andrea	Meta	Google	Apple
Michela	Google	Meta	Apple
Luca	Meta	Google	Apple

Applicants' Preference Profile

	1st	2nd	3rd
Meta	Michela	Andrea	Luca
Google	Andrea	Michela	Luca
Apple	Andrea	Michela	Luca

Companies' Preference Profile

Stable Matching Problem

- Q. Is the matching highlighted below perfect?

	1st	2nd	3rd
Andrea	Meta	Google	Apple
Michela	Google	Meta	Apple
Luca	Meta	Google	Apple

Applicants' Preference Profile

	1st	2nd	3rd
Meta	Michela	Andrea	Luca
Google	Andrea	Michela	Luca
Apple	Andrea	Michela	Luca

Companies' Preference Profile

Stable Matching Problem

- Q. Is the matching highlighted below perfect?
- A. Yes. Each applicant appears in at most one pair of matching and each student appears in at most one pair of matching.

	1st	2nd	3rd
Andrea	Meta	Google	Apple
Michela	Google	Meta	Apple
Luca	Meta	Google	Apple

Applicants' Preference Profile

	1st	2nd	3rd
Meta	Michela	Andrea	Luca
Google	Andrea	Michela	Luca
Apple	Andrea	Michela	Luca

Companies' Preference Profile

Stable Matching Problem

- Q. Is the matching highlighted below perfect?
- A. Yes. Each applicant appears in at most one pair of matching and each student appears in at most one pair of matching.
- Q. Is the matching highlighted stable?

	1^{st}	2^{nd}	3^{rd}
Andrea	Meta	Google	Apple
Michela	Google	Meta	Apple
Luca	Meta	Google	Apple

Applicants' Preference Profile

	1^{st}	2^{nd}	3^{rd}
Meta	Michela	Andrea	Luca
Google	Andrea	Michela	Luca
Apple	Andrea	Michela	Luca

Companies' Preference Profile

Stable Matching Problem

- Q. Is the matching highlighted below perfect?
- A. Yes. Each applicant appears in at most one pair of matching and each student appears in at most one pair of matching.
- Q. Is the matching highlighted stable?
- A. No. Andrea and Google will connect for their self-interest.

	1st	2nd	3rd
Andrea	Meta	Google	Apple
Michela	Google	Meta	Apple
Luca	Meta	Google	Apple

Applicants' Preference Profile

	1st	2nd	3rd
Meta	Michela	Andrea	Luca
Google	Andrea	Michela	Luca
Apple	Andrea	Michela	Luca

Companies' Preference Profile

Key point. An unstable pair c-a could each improve by joint action

Stable Matching Problem

- Q. Is the matching highlighted stable?

	1st	2nd	3rd
Andrea	Meta	Google	Apple
Michela	Google	Meta	Apple
Luca	Meta	Google	Apple

Applicants' Preference Profile

	1st	2nd	3rd
Meta	Michela	Andrea	Luca
Google	Andrea	Michela	Luca
Apple	Michela	Andrea	Luca

Companies' Preference Profile

Stable Matching Problem

- Q. Is the matching highlighted stable?
- A. Yes.

favorite ↓ … least favorite ↓

	1st	2nd	3rd
Andrea	Meta	Google	Apple
Michela	Google	Meta	Apple
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Applicants' Preference Profile

favorite ↓ … least favorite ↓

	1st	2nd	3rd
Meta	Michela	Andrea	Luca
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Apple	Michela	Andrea	Luca

Companies' Preference Profile

Stable Matching Problem

- Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching (original name: stable marriage).

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

- *Does it return a perfect matching?*
- *Does it return a stable matching?*

Stable Matching Problem

- Analyzing the Algorithm
  - Observation 1. Once a woman is matched, she never becomes unmatched; she only "trades up.“ (**1.1**)
  - Observation 2. Men propose to women in decreasing order of preference. (**1.2**)

- Analyzing the Algorithm. **Proof of Correctness: Termination**
- Claim. Algorithm terminates after at most n^2 iterations of while loop.
- Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals. **▪(1.3)**

- Analyzing the Algorithm. **Proof of Correctness: Perfection**
- The only way to exit the *While loop* is for there to be no free man.
- If m is free at some point in the execution of the algorithm, then there is a woman to whom he has not yet proposed. **(1.4)**

Stable Matching Problem

- Analyzing the Algorithm. **Proof of Correctness: Stability**

- Claim. The set S returned by Gale-Shapley algorithm is a stable matching. No unstable pairs. **(1.6)**
- Pf. (by contradiction)
  - Suppose A-Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching S*.

  - <u>Case 1:</u> Z never proposed to A.
    ⇒ Z prefers his GS partner to A.
    ⇒ A-Z is stable.
  - <u>Case 2:</u> Z proposed to A.
    ⇒ A rejected Z (right away or later)
    ⇒ A prefers her GS partner to Z.
    ⇒ A-Z is stable.

  - In either case A-Z is stable, a contradiction. ▪

Stable Matching Problem

- Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

- An instance with two stable matchings.
  - A-X, B-Y, C-Z.
  - A-Y, B-X, C-Z.

	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Stable Matching Problem

- Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

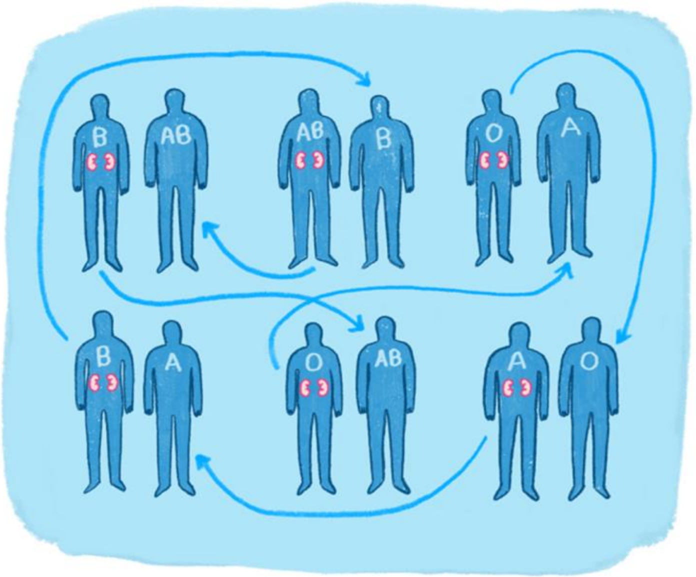
- Def. Man m is a **valid partner** of woman w if there exists some stable matching in which they are matched.
  - A-X, B-Y, C-Z.
  - A-Y, B-X, C-Z.

Both X and Y are valid partners for A.
Both X and Y are valid partners for B.
Z is the only valid partner for C.

- Man-optimal assignment. Each man receives best valid partner.
- Claim. All executions of GS yield man-optimal assignment, which is a stable matching! **(1.7)**

Application of Stable Matching

How Two Matchmakers Won a Nobel Prize!



Blood types: A, B, AB and O

Helping transplant patients find a match

Try the algorithm yourself!

Application of Stable Matching

- Pairing students to public schools
- 8^{th} graders rank top 5 high schools
- High schools rank students and specify limit
- Goal. Match 75k students to 426 high schools

How Game Theory Helped Improve New York City's High School Application Process

By TRACY TULLIS DEC. 5, 2014

Tuesday was the deadline for eighth graders in New York City to submit applications to secure a spot at one of 426 public high schools. After months of school tours and tests, auditions and interviews, 75,000 students have entrusted their choices to a computer program that will arrange their school assignments for the coming year. The weeks of research and deliberation will be reduced to a fraction of a second of mathematical calculation: In just a couple of hours, all the sorting for the Class of 2019 will be finished.

Chapter Summary

- Stable matching problem. Given preference profiles of n men and n women, find a stable matching.
- Gale-Shapley algorithm. Finds a stable matching in at most n^2 iterations.
- We analyzed the algorithm, and we have seen examples of some solutions.
- Other representative problems. (Weighted) Interval Scheduling, Bipartite Matching, Independent Set, Competitive Facility Location.