IDENTIFICATION & ESTIMATION OF DYNAMIC DISCRETE CHOICE GAMES

Econometric Society Summer School
in Dynamic Structural Econometrics

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OUTLINE

1. Introduction

DYNAMIC GAMES: INTRODUCTION

- In oligopoly industries, firms compete in **decisions** that:
  - have returns in the future (forward-looking)
  - involve substantial uncertainty
  - have important effects on competitors' profits

- Some examples are:
  - Investment in R&D, innovation
  - Investment in capacity, physical capital
  - Product design / quality
  - Market entry / exit
  - Pricing ...

DYNAMIC GAMES: INTRODUCTION [2]

- Measuring and understanding the **dynamic strategic interactions** between firms decisions is important to understand the forces behind the evolution of an industry or to evaluate policies.

- Investment costs, uncertainty, and competition effects play an important role in these decisions.

- Estimation of these parameters is necessary to answer some empirical questions.

- Empirical dynamic games provide a framework to estimate these parameters and perform policy analysis.

EXAMPLES OF EMPIRICAL APPLICATIONS

- **Competition in R&D and product innovation**
  - Intel & AMD: Goettler and Gordon (JPE, 2011)
  - Incumbents & new entrants (hard drives): Igami (JPE, 2017).

- **Regulation and industry dynamics**
  - Environmental regulations, entry-exit and capacity in cement industry: Ryan (ECMA, 2012)
  - Land use regulation in the hotel industry: Suzuki (IER; 2013)
  - Subsidies to entry in small medical markets: Dunne et al. (RAND, 2013).

EXAMPLES OF EMPIRICAL APPLICATIONS [2]

- **Product Design, Preemption, and Cannibalization**
  - Choice of format of radio stations: Sweeting (ECMA, 2013)
  - Hub-and-spoke networks and entry deterrrence in the airline industry: Aguirregabiria and Ho (JoE, 2012)
  - Cannibalization and preemption strategies in fast-food industry: Igami and Yang (QE, 2016).

- **Demand uncertainty, Time to build, and Investment**
  - Concrete industry: Collard-Wexler (ECMA, 2013)
  - Shipping industry: Kalouptsidi (AER, 2014)

- **Dynamic price competition**
  - Price adjustment costs: Kano (IJIO, 2013)
  - Frictions (adjustment costs) both in demand and supply: Mysliwski, Sanches, Silva & Srisuma (WP, 2020)

EXAMPLES OF EMPIRICAL APPLICATIONS [3]

- **Dynamic effects of mergers**
  - Dynamic response after airline mergers: Benkard, Bodoh-Creed, and Lazarev (WP, 2010)
  - Endogenous mergers: Jeziorski (RAND, 2014).

- **Exploitation of a common natural resource**
  - Fishing: Huang and Smith (AER, 2014).

- **Dynamic Search & Matching**
  - NYC Taxi industry: Buchholz (AER, 2022)
  - World trade and transoceanic shipping industry: Brancaccio, Kalouptsidi, and Papageorgiou (ECMA, 2020).

2. Structure of Dynamic Games

BASIC STRUCTURE

- Time is discrete and indexed by t.
- The game is played by N firms that we index by i.
- The action is taken to maximize the expected and discounted flow of profits in the market,

$$E_t \left(\sum_{s=0}^{\infty} \delta_i^s \; \pi_{it+s} \right)$$

$\delta_i \in (0, 1)$ is the discount factor, and π_{it} is firm i's profit at period t.

- Every period t, firms make two decisions: a static, and a dynamic.
- For instance: firms compete in prices (static competition), and make investments to improve the quality of their products (dynamic decision).

DECISIONS, STATES, and PROFITS

- We represent firm $i's$ investment/dynamic decision by a_{it}. It can be continuous, discrete, or mixed.

- Current profit π_{it} depends on the firms's own action a_{it}, other firms' actions, $\boldsymbol{a}_{-it} = \{a_{jt} : j \neq i\}$, and a vector of state variables $\boldsymbol{x}_t$.

$$\pi_{it} = \pi_i \left(a_{it}, \boldsymbol{a}_{-it}, \boldsymbol{x}_t \right)$$

- We should interpret $\pi_i \left(a_{it}, \boldsymbol{a}_{-it}, \boldsymbol{x}_t \right)$ as an **"indirect" profit function** that comes from the static equilbrium of the model: e.g., Bertrand equilibrium in prices, Cournot equilibrium in quantities.

- $\boldsymbol{x}_t$ includes:
  - Endogenous state variables that depend on the firms' investment decisions at previous periods, e.g., capital stocks.
  - Exogenous state variables affecting costs and consumer.

EXAMPLE: DYNAMIC COMPETITION IN PRODUCT QUALITY

- Each firm has a **differentiated product**. Consumer demand depends on products' qualities (k_{it}) and prices (p_{it}).
- State $\boldsymbol{x}_t$ consists of product qualities $\boldsymbol{k}_t = (k_{1t}, k_{2t}, ..., k_{Nt})$, and exogenous variables affecting demand and marginal costs ($\boldsymbol{z}_{it}$).
- Given $\boldsymbol{x}_t$, firms' compete in prices a la Bertrand, and this determines **Bertrand equilibrium** variable profits for each firm: $r_i(\boldsymbol{x}_t)$.
- The total profit, π_{it}, consists on $r_i(\boldsymbol{x}_t)$ minus the cost of investing in quality improvement: $IC_i(a_{it}, k_{it})$:

$$\pi_{it} = r_i(\boldsymbol{x}_t) - IC_i(a_{it}, k_{it})$$

- Quality stock evolves endogeneously according to the transition rule:

$$k_{i,t+1} = k_{it} + a_{it}$$

EXAMPLE: DYN. COMPETITION IN PROD. QUALITY (2)

- More specifically, we can consider the following structure for r_{it}:

$$r_{it} = (p_{it} - mc_i(k_{it}, z_{it}))\ q_{it}$$

  - p_{it} and q_{it} are price and quantity for firm i.
  - $mc_i(k_{it}, z_{it})$ is the per-unit or marginal cost.
  - z_{it} is a vector of exogenous product characteristics.

- The demand system could have a simple Logit structure:

$$q_{it} = \frac{\exp\{z_{it}\beta_z + \beta_k\ k_{it} - \alpha\ p_{it}\}}{1 + \sum_{j=1}^{N} \exp\{z_{jt}\beta_z + \beta_k\ k_{jt} - \alpha\ p_{jt}\}}$$

- Bertrand equilibrium implies the **"indirect" variable profit function**:

$$r_i(\mathbf{k}_t, \mathbf{z}_t) = (p_i^*[\mathbf{k}_t, \mathbf{z}_t] - mc_i[k_{it}, z_{it}])\ \ q_i^*[\mathbf{k}_t, \mathbf{z}_t]$$

EXAMPLE: DYN. COMPETITION IN PROD. QUALITY (3)

- A firm faces the following trade-off when choosing investment a_{it}.
  - Investment in quality improvement is costly.
  - Higher quality increases equilibrium quantity $q_i^*[\boldsymbol{k}_t, \boldsymbol{z}_t]$ and price $p_i^*[\boldsymbol{k}_t, \boldsymbol{z}_t]$ and therefore, future profits.
- **Decreasing returns**. Under standard conditions:

$$\frac{\partial^2 \pi_{it}}{\partial k_{it} \partial k_{it}} < 0$$

Best-response investment declines with installed quality.

- **Strategic Complementarity**. Under standard conditions, for $j \neq i$,

$$\frac{\partial^2 \pi_{it}}{\partial k_{it} \partial k_{jt}} > 0$$

Best-response investment increases with competitors installed quality.

Timing of the model: Time-to-Build or Not

- The previous example incorporates an assumption of **time-to-build**.
- **Time-to-build**: The investment decision at period t, a_{it}, takes one period to affect the quality stock. Therefore, variable profit depends on $\boldsymbol{k}_t$ but not on a_{it} or $\boldsymbol{a}_{-it}$:

$$\pi_i\left(a_{it}, \boldsymbol{x}_t\right) = r_i(\boldsymbol{k}_t, z_t) - IC_i\left(a_{it}, k_{it}\right)$$

- We still have a (dynamic) game, as future profits depend on $\boldsymbol{a}_{-it}$.
- **Without Time-to-build:** We can consider a version of the model where investment has an instantaneous effect on quality: demand, marginal costs, and variable profits depend on $\boldsymbol{k}_t + \boldsymbol{a}_t$ instead of $\boldsymbol{k}_t$:

$$\pi_{it} = r_i(\boldsymbol{k}_t + \boldsymbol{a}_t, z_t) - IC_i\left(a_{it}, k_{it}\right) = \pi_i\left(a_{it}, \boldsymbol{a}_{-it}, \boldsymbol{x}_t\right)$$

WHY IS THIS DECISION PROBLEM DYNAMIC?

- We say that a decision problem decision if the decisions at previous periods have a **causal effect** on the optimal decision and profit at the current period.

- Using the notation in this model, the model is dynamic iff k_{it} has an effect on today's profit which is different to the effect of a_{it}.

- Suppose that there is NO time-to-build and demand, marginal cost, and investment cost depends on k_{it} only through the total quality $k_{it} + a_{it}$. Then, the problem is not dynamic because the optimal choice of quality $k_{it} + a_{it}$ is not affected by previous choices in k_{it}.

- The existence of dynamics requires some **cost of "reversing" past decisions**.

EVOLUTION OF THE STATE VARIABLES

- **Exogenous common knowledge state variables:** follow an exogenous Markov process with transition probability function $f_z(\mathbf{z}_{t+1}|\mathbf{z}_t)$.

- **Endogenous state variables:** The form of the transition rule depends on the application:
  - Market entry: $k_{it} = a_{it-1}$, such that $k_{i,t+1} = a_{it}$
  - Investment without depreciation: $k_{i,t+1} = k_{it} + a_{it}$.
  - Investment - deterministic depreciation: $k_{i,t+1} = \lambda(k_{it} + a_{it})$
  - Investment - stochastic depreciation: $k_{i,t+1} = k_{it} + a_{it} - \xi_{i,t+1}$

- In a compact way, we use $f_x(\mathbf{x}_{t+1}|\mathbf{a}_t, \mathbf{x}_t)$ to represent the transition probability function of all the state variables.

3. Markov Perfect Equilibrium

SINGLE-AGENT DYNAMIC DECISION MODEL

- Before we consider the game version of the model, it is convenient to look at the single-agent version.

- We can interpret this single-agent model as the one faced by a **monopolist**, or a firm in a **perfectly competitive market**.

- Vector of state variables for firm i is $\mathbf{x}_{it} = (k_{it}, \mathbf{z}_{it})$.

- Let $V_i(\mathbf{x}_{it})$ be the **value function** of the DP problem of firm i. This value function is the unique solution to the Bellman equation:

$$V_i(\mathbf{x}_{it}) = \max_{a_{it}} \left\{ \pi_i(a_{it}, \mathbf{x}_{it}) + \delta_i \int V_i(\mathbf{x}_{i,t+1}) \ f_x(\mathbf{x}_{i,t+1} | a_{it}, \mathbf{x}_{it}) \ d\mathbf{x}_{i,t+1} \right\}$$

- The **optimal decision rule**, $a_{it} = \alpha_i(\mathbf{x}_{it})$ is the argmax of the expression within brackets $\{.\}$.

- Bellman's operator is a **contraction** such that functions $V_i(.)$ and $\alpha_i(.)$ are unique and can be obtained using fixed-point iterations.

OPTIMAL DECISION RULE IN THE DYNAMIC GAME

- Now, vector of state variables affecting the profit of firm i:

$$\mathbf{x}_t = (k_{it}, \mathbf{z}_{it}, k_{jt}, \mathbf{z}_{jt} \text{ for any firm } j \neq i)$$

- Let's first consider a "naive" approach to the dynamic game.

- Let $V_i(\mathbf{x}_t)$ be the **value function** of the DP problem of firm i. This value function is the unique solution to the Bellman equation:

$$V_i(\mathbf{x}_t) = \max_{a_{it}} \left\{ \pi_i(a_{it}, \mathbf{x}_t) + \delta_i \int V_i(\mathbf{x}_{t+1}) \ f_{x,i}(\mathbf{x}_{t+1} | a_{it}, \mathbf{x}_t) \ d\mathbf{x}_{t+1} \right\}$$

- And the **optimal decision rule**, $a_{it} = \alpha_i(\mathbf{x}_{it})$ is the argmax of the expression within brackets $\{.\}$.

OPTIMAL DECISION RULE IN THE DYNAMIC GAME (2)

- The problem of this "naive" approach is that it is treating the transition rule/probability $f_{x,i}(\mathbf{x}_{t+1}|a_{it}, \mathbf{x}_t)$ as if it were a primitive (exogenous structural function) of the model, but it is not.

- For firms j different to i, the value of $k_{j,t+1}$ depends on firm j's optimal investment decision.

- Therefore, the transition rule/probability $f_{x,i}(\mathbf{x}_{t+1}|a_{it}, \mathbf{x}_t)$ contains implicitly **beliefs that firm i has about the investment behavior of other firms in the market**.

OPTIMAL DECISION RULE IN THE DYNAMIC GAME (3)

- Suppose that firm i believes that the other firms in the market behave – now and in the future – according to the strategy function:

$$\alpha_j(\mathcal{I}_t) \text{ for any } j \neq i$$

where $\mathcal{I}_t$ is a particular specification of the information used by firms at period t. More specifically:

$$\mathcal{I}_t = \left(\mathbf{x}_t,\ \boldsymbol{a}_{t-1},\ \mathbf{x}_{t-1}, ...,\ \boldsymbol{a}_{t-p},\ \mathbf{x}_{t-p}\right)$$

- Given these beliefs, firm i has the following transition probability for the state variables:

$$f^{\boldsymbol{\alpha}}_{\mathcal{I},i}(\mathcal{I}_{t+1}|a_{it}, \mathcal{I}_t) = f_x(\mathbf{x}_{t+1}|a_{it}, \boldsymbol{\alpha}_{-i}(\mathcal{I}_t), \mathbf{x}_t)$$

Note that $f_x(\mathbf{x}_{t+1}|a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t)$ is a primitive of the model.

BEST RESPONSE & NASH EQUILIBRIUM

- Given beliefs $\boldsymbol{\alpha}_{-i}$ and the corresponding $f^{\boldsymbol{\alpha}}_{\mathcal{I},i}(\mathcal{I}_{t+1}|a_{it},\mathcal{I}_t)$, we can define the best response of firm i as the solution to the single-agent DP problem defined by this Bellman equation:

$$V_i^{\boldsymbol{\alpha}}(\mathcal{I}_t) = \max_{a_{it}} \left\{ \pi_i(a_{it}, \mathbf{x}_t) + \delta_i \int V_i^{\boldsymbol{\alpha}}(\mathcal{I}_{t+1}) \; f^{\boldsymbol{\alpha}}_{\mathcal{I},i}(\mathcal{I}_{t+1}|a_{it},\mathcal{I}_t) \right\}$$

- Let $BR_i(\boldsymbol{\alpha}_{-i})$ be the optimal strategy function that solves this DP problem. It is a best response to the beliefs $\boldsymbol{\alpha}_{-i}$.

- A **Nash Equilibrium** of this dynamic game consists of an N-tuple of strategy functions $\{\alpha_i(\mathcal{I}_t) : i = 1, 2, ..., N\}$ such that, for every firm i:

$$\alpha_i = BR_i(\boldsymbol{\alpha}_{-i})$$

That is:

1. Every firm behaves according to its best response strategy.
2. Beliefs are rational, i.e., the actual firms' strategies in equilibrium.

MARKOV PERFECT EQUILIBRIUM

- The previous definition of Nash Equilibrium depends on the choice of the information set $\mathcal{I}_t$. We have as many types of NE as possible selections of $\mathcal{I}_t$.

- Most dynamic IO models assume Markov Perfect Equilibrium (MPE), (Maskin & Tirole, ECMA 1988; Ericson & Pakes, REStud 1995).

- This solution concept corresponds to NE when **players' strategies are functions of only payoff-relevant state variables**, $\mathcal{I}_t = \mathbf{x}_t$.

- **Why this restriction?**:
  - **Rationality (Maskin & Tirole):** if other players use this type of strategies, a player cannot make higher payoff by conditioning its behavior on non-payoff relevant information (e.g., lagged values of the state variables)

  - **Dimensionality:** It is convenient because it reduces the dimensionality of the state space.

MARKOV PERFECT EQUILIBRIUM – DEFINITION

- Let $\boldsymbol{\alpha} = \{\alpha_i(\mathbf{x}_t) : i = 1, 2, ..., N\}$ be a set of strategy functions.

- A MPE is an N-tuple of strategy functions $\boldsymbol{\alpha}$ such that every firm is maximizing its value given the strategies of the other players.

- For given strategies of the other firms, the decision problem of a firm is a single-agent dynamic programming (DP) problem.

MARKOV PERFECT EQUILIBRIUM: Best Response DP

- Let $V_i^{\boldsymbol{\alpha}}(\mathbf{x}_t)$ be the value function of the DP problem that describes the best response of firm i to the strategies of the other firms in $\boldsymbol{\alpha}$.

- This value function is the unique solution to the Bellman equation:

$$V_i^{\boldsymbol{\alpha}}(\mathbf{x}_t) = \max_{a_{it}} \left\{ \pi_i^{\boldsymbol{\alpha}}(a_{it}, \mathbf{x}_t) + \delta_i \int V_i^{\boldsymbol{\alpha}}(\mathbf{x}_{t+1}) \; f_{x,i}^{\boldsymbol{\alpha}}(\mathbf{x}_{t+1} | a_{it}, \mathbf{x}_t) \; d\mathbf{x}_{t+1} \right\}$$

- with (here, I consider there is no time-to-build):

$$\pi_i^{\boldsymbol{\alpha}}(a_{it}, \mathbf{x}_t) \quad = \quad \pi_i \left(a_{it}, \; \boldsymbol{\alpha}_{-i}(\mathbf{x}_t), \; \mathbf{x}_t \right)$$

- and:

$$f_{x,i}^{\boldsymbol{\alpha}}(\mathbf{x}_{t+1} | a_{it}, \mathbf{x}_t) \quad = \quad f_x(\mathbf{x}_{t+1} \mid a_{it}, \; \boldsymbol{\alpha}_{-i}(\mathbf{x}_t), \; \mathbf{x}_t)$$

MPE — EXISTENCE

- Doraszelski & Satterhwaite (RAND, 2010) show that **existence** of a MPE in pure strategies **is not guaranteed** in this model **when the choice set for a_{it} is discrete**.

- A possible approach to guarantee existence is to allow for mixed strategies. However, computing a MPE in mixed strategies poses important computational challenges.

- To establish existence, Doraszelski & Satterhwaite (RAND, 2010) propose **incorporating private information state variables**.

- This incomplete information version of Ericson-Pakes model has been the one adopted in most **empirical applications**.
  - The main reason is that – as we illustrate below – i.i.d. private information shocks are very **convenient type of unobservables from an econometric point of view**.

4. Dynamic Games with Incomplete Information

PRIVATE INFORMATION SHOCKS

- State variables in $\mathbf{x}_t$ are known to all the firms in the market at period t (common knowledge).

- In addition, a firm's investment cost function $IC_i(.)$ depends on a vector of state variables ε_{it} with two properties:

  1. ε_{it} is **private information of firm i**. It is unknown to the other firms.

  2. ε_{it} is **i.i.d. over time and independent across firms** with CDF G_i that has full support on $\mathbb{R}^{|A|}$.

- Strategy functions are now $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$.

- MPE has the same definition as above but with strategies $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$.

CONDITIONAL CHOICE PROBABILITIES

- It is very convenient to represent a firm's strategy using **Conditional Choice Probability (CCP) function**. For any value $(a, \mathbf{x})$:

$$P_i(a|\mathbf{x}) \equiv \Pr\left(\alpha_i(\mathbf{x}_t, \varepsilon_{it}) = a \mid \mathbf{x}_t = \mathbf{x}\right)$$

- Since function P_i results from integrating function α_i over the continuous variables in ε_{it}, P_i is a lower dimensional object than α_i.

- In discrete choice games with $\varepsilon_{it}(a_{it})$ entering additively in the profit function, there is a **one-to-one relationship** between best-response strategy functions $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$ and its CCP function $P_i(.|\mathbf{x}_t)$.

- It is obvious that given $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$ there is a unique $P_i(.|\mathbf{x}_t)$.

- The inverse relationship – given $P_i(.|\mathbf{x}_t)$ there is a unique best response function $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$ – is a corollary of **Hotz-Miller inversion Theorem**.

HOTZ & MILLER INVERSION THEOREM

- Consider a discrete choice game, $a_{it} \in \mathcal{A} = \{0, 1, ..., J\}$, where ε_{it}'s enter **additively in the payoff function**:

$$\pi_i(a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t, \varepsilon_{it}) = \bar{\pi}_i(a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t) + \varepsilon_{it}(a_{it})$$

where $[\varepsilon_{it}(0), \varepsilon_{it}(1), ..., \varepsilon_{it}(J)] \in \mathbb{R}^{J+1}$ are private information shocks.

- This additivity, together with ε_{it} i.i.d. over time, imply that a firm's best response function has the following form:

$$\alpha_i(\mathbf{x}_t, \varepsilon_{it}) = \arg\max_{a_{it} \in \mathcal{A}} [v_i^{\boldsymbol{\alpha}}(a_{it}, \mathbf{x}_t) + \varepsilon_{it}(a_{it})]$$

where $v_i^{\boldsymbol{\alpha}}(a_{it}, \mathbf{x}_t)$ is the **conditional-choice value function**.

- For any choice alternative $a \in \mathcal{A}$:

$$P_i(a|\mathbf{x}) = Pr \left(\begin{array}{cc} v_i^{\boldsymbol{\alpha}}(a, \mathbf{x}_t) - v_i^{\boldsymbol{\alpha}}(0, \mathbf{x}_t) + \varepsilon_{it}(a) - \varepsilon_{it}(0) & \geq \\ v_i^{\boldsymbol{\alpha}}(j, \mathbf{x}_t) - v_i^{\boldsymbol{\alpha}}(0, \mathbf{x}_t) + \varepsilon_{it}(j) - \varepsilon_{it}(0) & \text{for any } j \neq a \end{array} \right)$$

HOTZ & MILLER INVERSION THEOREM [2]

- For a given value of $\mathbf{x}_t$, equation

$$P_i(a|\mathbf{x}) = Pr\left(\begin{array}{ll} v_i^{\alpha}(a,\mathbf{x}_t) - v_i^{\alpha}(0,\mathbf{x}_t) + \varepsilon_{it}(a) - \varepsilon_{it}(0) & \geq \\ v_i^{\alpha}(j,\mathbf{x}_t) - v_i^{\alpha}(0,\mathbf{x}_t) + \varepsilon_{it}(j) - \varepsilon_{it}(0) & \text{for any } j \neq a \end{array}\right)$$

  defines a mapping from the space of the J "free" value differences $\mathbf{v}_i^{\alpha}(\mathbf{x}_t) \equiv [v_i^{\alpha}(a,\mathbf{x}_t) - v_i^{\alpha}(0,\mathbf{x}_t) : a = 1, ..., J]$ into the space of the J "free" choice probabilities $\mathbf{P}_i(\mathbf{x}_t) \equiv [P_i(a|\mathbf{x}) : a = 1, ..., J]$

$$\mathbf{P}_i(\mathbf{x}_t) = \Lambda_i\left(\mathbf{v}_i^{\alpha}(\mathbf{x}_t)\right)$$

- Hotz-Miller inversion Theorem establishes that mapping $\Lambda_i(.)$ is invertible everywhere.

COROLLARY OF HOTZ & MILLER INVERSION THEOREM

- An implication or Corollary of Hotz-Miller inversion Theorem is that there is a **one-to-one relationship between a strategy function $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$ and a vector of CCPs $\mathbf{P}_i(\mathbf{x}_t)$**.

- Given $\alpha_i(.)$, we integrate over ε_{it}, to obtain the CCPs $\mathbf{P}_i(\mathbf{x}_t)$.

- Given CCPs $\mathbf{P}_i(\mathbf{x}_t)$, we apply Hotz-Miller inversion to obtain vector $\mathbf{v}_i^{\boldsymbol{\alpha}}(\mathbf{x}_t)$, and then there is a unique strategy function $\alpha_i(.)$ defined as:

$$\alpha_i(\mathbf{x}_t, \varepsilon_{it}) = \arg\max_{a_{it}} [v_i^{\boldsymbol{\alpha}}(a_{it}, \mathbf{x}_t) + \varepsilon_{it}(a_{it})]$$

- Therefore, we can describe strategies (and equilibrium) using vectors of CCPs $\mathbf{P}_i(\mathbf{x}_t)$.

MPE as FIXED POINT OF a MAPPING IN CCPs

- Given strategy functions described by CCP functions $\mathbf{P}$, we can define **expected profit** $\pi_i^{\mathbf{P}}$ and **expected transition** $f_i^{\mathbf{P}}$ as:

$$\pi_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) = \sum_{a_{-it}} \left[\prod_{j \neq i} P_j\left(a_{jt} \mid \mathbf{x}_t\right) \right] \pi_i\left(a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t\right)$$

$$f_i^{\mathbf{P}}(\mathbf{x}_{t+1} | a_{it}, \mathbf{x}_t) = \sum_{a_{-it}} \left[\prod_{j \neq i} P_j\left(a_{jt} \mid \mathbf{x}_t\right) \right] f_x(\mathbf{x}_{t+1} | a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t)$$

- We also define **expected conditional-choice values**:

$$v_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) \equiv \pi_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) + \delta \int V_i^{\mathbf{P}}(\mathbf{x}_{t+1})\ f_i^{\mathbf{P}}(\mathbf{x}_{t+1} | a_{it}, \mathbf{x}_t)\ d\mathbf{x}_{t+1}$$

- with:

$$V_i^{\mathbf{P}}(\mathbf{x}_t) = \int \max_{a_{it}} \left\{ v_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) + \varepsilon_{it}(a_{it}) \right\} dG_i(\varepsilon_{it})$$

MPE as FIXED POINT OF a MAPPING IN CCPs [2]

- A MPE is a vector of CCPs, $\mathbf{P} \equiv \{P_i(a_i|\mathbf{x}) : \text{for any } (i, a_i, \mathbf{x})\}$, such that, for any $(i, a, \mathbf{x})$:

$$P_i(a_i|\mathbf{x}) = \Pr\left(a_i = \arg\max_{a'} \left\{v_i^{\mathbf{P}}(a', \mathbf{x}) + \varepsilon_i(a')\right\} \mid \mathbf{x}\right)$$

- This system of equations defines a Fixed Point mapping from the space of CCPs $\mathbf{P}$ into itself:

$$\mathbf{P} = \Psi(\mathbf{P})$$

- Mapping $\Psi(.)$ is continuous. Therefore, **by Brower's Fixed Point Theorem an equilibrium exists**.

- In general, this model has multiple equilibria.

MPE IN TERMS OF CCPs: AN EXAMPLE

- Suppose that vector ε_{it}'s are iid Extreme Value Type I.

- Then, a MPE is a vector $\mathbf{P} \equiv \{P_i(a|\mathbf{x}) : \text{for any } (i,a,\mathbf{x})\}$, such that:

$$P_i(a|\mathbf{x}) = \frac{\exp\left\{v_i^{\mathbf{P}}(a,\mathbf{x})\right\}}{\sum_{a'} \exp\left\{v_i^{\mathbf{P}}(a',\mathbf{x})\right\}}$$

- where

$$v_i^{\mathbf{P}}(a_{it},\mathbf{x}_t) \equiv \pi_i^{\mathbf{P}}(a_{it},\mathbf{x}_t) + \delta \sum_{\mathbf{x}_{t+1}} V_i^{\mathbf{P}}(\mathbf{x}_{t+1})\ f_i^{\mathbf{P}}(\mathbf{x}_{t+1}|a_{it},\mathbf{x}_t)$$

- and $V_i^{\mathbf{P}}$ is the unique solution to the Bellman equation:

$$V_i^{\mathbf{P}}(\mathbf{x}_t) = \ln\left(\sum_{a_i} \exp\left\{\pi_i^{\mathbf{P}}(a_{it},\mathbf{x}_t) + \delta \sum_{\mathbf{x}_{t+1}} V_i^{\mathbf{P}}(\mathbf{x}_{t+1})\ f_i^{\mathbf{P}}(\mathbf{x}_{t+1}|a_{it},\mathbf{x}_t)\right\}\right)$$

5. Solution Methods

EQUILIBRIUM MAPPING IN VECTOR FORM

- Suppose that $\mathbf{x}_t$ is discrete: $\mathbf{x}_t \in \{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^{|X|}\}$.

- The primitives of the model are:

1. **Vectors of payoffs**: $\mathbf{\Pi}_i(a_i, a_{-i})$ with dimension $|X| \times 1$, for every value of (a_i, a_{-i}).

2. **Matrices of transition probabilities**: $\mathbf{F}_x(a_i, a_{-i})$ with dimension $|X| \times |X|$, for every value of (a_i, a_{-i}).

3. **Discount factor**: δ.

4. **Distribution of private information shocks**: $G(\varepsilon_i(a_i) : a_i \in \mathcal{A})$.

Equilibrium Mapping in Vector Form [2]

- Let $\mathbf{P}_i(a_i)$ be a vector of CCPs with dimension $|X| \times 1$ and with the probs. that firm i chooses a_i for every state $\mathbf{x}$.

- Let $\mathbf{P}_i \equiv \{\mathbf{P}_i(a_i) : \text{for every } a_i \in A\}$.

- We can define the $|X| \times 1$ **vectors of expected payoffs**:

$$\mathbf{\Pi}_i^{\mathbf{P}_{-i}}(a_i) \equiv \sum_{a_{-i}} \left[\prod_{j \neq i} \mathbf{P}_j(a_j) \right] * \mathbf{\Pi}_i(a_i, a_{-i})$$

- And the $|X| \times |X|$ **matrices of expected transition probabilities**:

$$\mathbf{F}_i^{\mathbf{P}_{-i}}(a_i) \equiv \sum_{a_{-i}} \left[\prod_{j \neq i} \mathbf{P}_j(a_j) \right] * \mathbf{F}_x(a_i, a_{-i})$$

- where $*$ represents the "element-by-element" or Hadamard product.

Equilibrium Mapping in Vector Form [3]

- A MPE is a vector $\mathbf{P} \equiv \{\mathbf{P}_i : i \in I\}$ such that:

$$\mathbf{P}_i = \Psi_i(\mathbf{P}_{-i}) \;\; for\ every\ i \in I$$

where $\Psi_i(.)$ is i's **best response mapping** that is the composition of:

$$\Psi_i = \Lambda_i \circ \Gamma_i$$

- $\mathbf{V}_i^{\mathbf{P}_{-i}} = \Gamma_i(\mathbf{P}_{-i})$ gives the vector of values that solves Bellman's equation for firm i given $\mathbf{P}_{-i}$: (for Logit case):

$$\mathbf{V}_i^{\mathbf{P}_{-i}} = \ln\left(\sum_{a_i} \exp\left\{\mathbf{\Pi}_i^{\mathbf{P}_{-i}}(a_i) + \delta\ \mathbf{F}_i^{\mathbf{P}_{-i}}(a_i)\ \mathbf{V}_i^{\mathbf{P}_{-i}}\right\}\right)$$

- $\mathbf{P}_i = \Lambda_i(\mathbf{V}_i^{\mathbf{P}_{-i}})$ gives optimal CCPs given $\mathbf{V}_i^{\mathbf{P}_{-i}}$: (for Logit case):

$$\mathbf{P}_i(a_i) = \Lambda_i(a_i, \mathbf{V}_i^{\mathbf{P}_{-i}}) = \frac{\exp\left\{\mathbf{\Pi}_i^{\mathbf{P}_{-i}}(a_i) + \delta\ \mathbf{F}_i^{\mathbf{P}_{-i}}(a_i)\ \mathbf{V}_i^{\mathbf{P}_{-i}}\right\}}{\sum_{a'} \exp\left\{\mathbf{\Pi}_i^{\mathbf{P}_{-i}}(a') + \delta\ \mathbf{F}_i^{\mathbf{P}_{-i}}(a')\ \mathbf{V}_i^{\mathbf{P}_{-i}}\right\}}$$

Methods / Algorithms to Compute a MPE

- We study three algorithms that have been used to compute MPE in this class of models.

1. Fixed point iterations in the best response mapping Ψ.
2. Newton's method.
3. Spectral residual method(s)

- Method [1] does not guarantee convergence. [2] does, but it is impractical in most applications. [3] has advantages relative to [1] and [2].

Fixed Point Iterations

- Let $\mathbf{P}^0 \equiv \{\mathbf{P}_i^0 : \text{for any } i\}$ be arbitrary vector of CCPs.

- At iteration n, for any player i:

$$\mathbf{P}_i^n = \Psi_i \left(\mathbf{P}_{-i}^{n-1} \right)$$

- We check for convergence:

$$\begin{cases} \text{if } \left\| \mathbf{P}^n - \mathbf{P}^{n-1} \right\| \leq \kappa & \text{then} \quad \mathbf{P}^n \text{ is a MPE} \\ \text{if } \left\| \mathbf{P}^n - \mathbf{P}^{n-1} \right\| > \kappa & \text{then} \quad \text{Proceed to iteration } n+1 \end{cases}$$

where κ is a small positive constant, e.g., $\kappa = 10^{-6}$.

- **Convergence is NOT guaranteed**. This is a serious limitation.

Newton's Method

- Define the function $f(\mathbf{P}) \equiv \mathbf{P} - \Psi(\mathbf{P})$.
- Finding a fixed point of Ψ is equivalent to finding a zero (root) of f.
- We can use Newton's method to find a root of f.
- At iteration n: ($\nabla f(\mathbf{P})$ is the Jacobian matrix)

$$\mathbf{P}^n = \mathbf{P}^{n-1} + \left[\nabla f(\mathbf{P}^{n-1})\right]^{-1} f(\mathbf{P}^{n-1})$$

- We check for convergence: $\left\|\mathbf{P}^n - \mathbf{P}^{n-1}\right\| \leq \kappa$
- **Convergence is guaranteed** (to one of the multiple equilibria).

Newton's Method [2]

- The main computational cost of a Newton's iteration comes from the computation of Jacobian matrix $\nabla f(\mathbf{P})$.

- There is not a closed-form expression for the derivatives in this matrix. And in this class of models, this matrix is not sparse.

- This matrix is of dimension $N|\mathcal{A}||\mathcal{X}| \times N|\mathcal{A}||\mathcal{X}|$, and the computation of one single element in this matrix involves solving many single-agent dynamic programming problems, each of them with a complexity $O(|\mathcal{X}|^3)$.

- In summary, Newton's method is not practical in most empirical applications, in which $|\mathcal{X}|$ is greater than 10^5.

Spectral Residual Method

- It is a general method for solving high-dimension systems of nonlinear equations, $f(\mathbf{P}) = 0$.

- It has two very attractive features:

1. It is derivative free, and the cost of one iteration is equivalent to evaluation $f(\mathbf{P})$ – the same cost as one fixed point iteration.

2. It converges to a solution under mild regularity conditions – similar good convergence properties to Newton's.

Spectral Residual Method [2]

- Spectral methods propose the following updating rule/iteration:

$$\mathbf{P}_{n+1} = \mathbf{P}_n - \alpha_n \, f\left(\mathbf{P}_n\right)$$

where α_n is the spectral steplength, which is a scalar.

- Different updating rules have been proposed in the literature. Barzilai and Borwein (1988) is commonly used:

$$\alpha_n = \frac{[\mathbf{P}_n - \mathbf{P}_{n-1}]'[f(\mathbf{P}_n) - f(\mathbf{P}_{n-1})]}{[f(\mathbf{P}_n) - f(\mathbf{P}_{n-1})]'[f(\mathbf{P}_n) - f(\mathbf{P}_{n-1})]}$$

- The intuition for the convergence of the Spectral Residual method is that the updating of α_n can guarantee the right direction to convergence.

6. Datasets in Applications

Type of Data in most Empirical Applications

- Panel data of M geographic markets, over T periods, and N firms.

$$Data = \{\boldsymbol{a}_{mt},\ \mathbf{x}_{mt} : m = 1, 2, ..., M;\ t = 1, 2, ..., T\}$$

- **Example 1**: Major airlines in US ($N = 10$), in the markets/routes defined by all the pairs of top-50 US airports ($M = 1,275$), over $T = 20$ quarters (5 years).

- **Example 2**: Supermarket chains in Ontario ($N = 6$), in the geographic markets defined by census tracts ($M > 1k$), over $T = 24$ months.

Type of Data in most Empirical Applications [2]

- This data structure applies to industries characterized by **many geographic markets**, where a separate (dynamic) game is played in each market: e.g., retail industries, services, airline markets, procurement auctions, ...
- However, there are many manufacturing industries where **competition is more global**: a single national or even international market: e.g., microchips.
- For these "global" industries, applications rely on sample variability that comes from a **combination of modest N, M, and T**.
- Some other industries are characterized by a **large number of heterogeneous firms** (large N), e.g., NYC taxis.

7. IDENTIFICATION

Our identification problem

- The **primitives of the model** are:

$$\{\pi_i(.),\ \delta_i,\ F_x(.):\ i \in \mathcal{I}\}$$

- Empirical applications assume that these primitives are known to the researcher up to a **vector of structural parameters $\boldsymbol{\theta}$**.

- The **identification problem** consists in using the data and the restrictions of the model to:
  - uniquely determine the value of $\boldsymbol{\theta}$ **(point identification)**
  - or to obtain bounds on $\boldsymbol{\theta}$ **(partial / set identification)**.

- This is a **revealed preference identification approach**: under the assumption that firms' are maximizing profits, their actions reveal information about the structure of their profit functions.

MAIN IDENTIFICATION CHALLENGES IN DYNAMIC GAMES

1. **Dynamics vs. Unobserved Market Heterogeneity**. How much of the time persistence in decisions comes from "true dynamics" and how much from serially correlated unobservables?

2. **Strategic Interactions vs. Common Unobservables**. How much of the correlation between firms' actions comes from "true competition effectss" and how much from unobservables correlated across firms?

3. **Forwar- Looking Firms (δ_i)**. How much of a firm's decision is determined by current profit and how much by expected future profits?

4. **Multiple Equilibria**. Does multiplicity of equlibria create identification problems? Does it create estimation problems?

IDENTIFICATION & MULTIPLE EQUILIBRIA

- Equilibrium uniqueness is neither a necessary nor a sufficient condition for the identification of a model (Jovanovic, 1989).
- Model with structural parameters $\theta \in \Theta$, is a mapping $C(\theta)$ from Θ to the set of probability distributions $P = Pr(a_1, a_2, ..., a_N | x, \theta)$.
- Let P_0 be the true probability distribution of $(a_1, a_2, ..., a_N | x)$ in the population. Of course, P_0 is unique.
- **Multiple equilibria** means that the mapping $C(.)$ is a correspondence.
- **No Identification** means that the inverse mapping $C^{-1}(.)$ evaluated at P_0 is a correspondence.
- In general, $C(.)$ being a function (that is, equilibrium uniqueness) is neither a necessary nor a sufficient condition for $C^{-1}(.)$ being a function (that is, for point identification).

IDENTIFICATION & MULTIPLE EQUILIBRIA – A Simple Example

- Model with 1 parameter (θ), 1 probability (P), and multiple equilibria.

- Equilibrium equation:
$$P = \Phi(c + \theta \ P)$$
where $\Phi(.)$ is the CDF of N(0,1).

- The researcher observes firms' actions $a_{imt} \in \{0, 1\}$ that come from an equilibrium of this game.

- In the Population, $c = -1.8$ (known to the researcher, for simplicity), $\theta = 3.5$, unknown to the researcher.

- For these true parameters, the model has three equilibria: $P_A = 0.054$, $P_B = 0.551$, and $P_C = 0.924$.

IDENT. & MULTIPLE EQUILIBRIA – A Simple Example [2]

- Let P_0 the value of P in the population: $a_{imt} \sim$ i.i.d. $Bernoulli(P_0)$.

- P_0 is identified. $P_0 = \mathbb{E}(a_{imt})$. Given a sample of a_{imt} we can estimate P_0 consistently using: $\hat{P}_0 = \frac{1}{NMT} \sum_{i,m,t} a_{imt}$.

- Suppose that P_0 is an equilibrium of the game (one of the three, P_A, P_B, or P_C) and **not a mixture of the three possible equilibria**.

$$P_0 = \Phi(c + \theta_0 \; P_0)$$

- Since $\Phi(.)$ is invertible, we have that:

$$\theta_0 = \frac{\Phi^{-1}(P_0) - c}{P_0}$$

- This equation establishes the point identification of θ_0).

Basic Assumptions

- Set of assumptions used in many applications in this literature.

ID.1 **No common knowledge unobservables**. The researcher observes $\mathbf{x}_t$. The only unobservables are ε_{it}.

ID.2 **Single equilibrium in the data**. Every observation (i, m, t) in the data comes from the same MPE.

ID.3 **Additive unobservables**. The unobservables ε_{it} enter additively in the payoff function: $\pi_i(\boldsymbol{a}_t, \mathbf{x}_t) + \varepsilon_{it}(a_{it})$.

ID.4 **Known distribution of unobservables**. The distribution of ε_{it} is completely known to the researcher.

ID.5 **Conditional independence**. Conditional on $(\boldsymbol{a}_t, \mathbf{x}_t)$ the distribution of $\mathbf{x}_{t+1}$ does not depend on $\boldsymbol{\varepsilon}_t$.

A Positive Identification Result but Not for Primitives

- Under Assumptions [ID.1] and [ID.2], the vector of equilibrium CCPs in the population, $\mathbf{P}^0$, is identified from the data. For every $(i, a_i, \mathbf{x}$:

$$P_i^0(a_i|\mathbf{x}) = \mathbb{E}\left(1\{a_{imt} = a_i\} \mid \mathbf{x}_{mt} = \mathbf{x}\right)$$

- Given CCPs and under assumptions [ID.3] to [ID.5], **Hotz-Miller Inversion Theorem** implies the identification of **conditional-choice value function** relative to a baseline alternative (say 0):

$$\widetilde{v}_i^{\mathbf{P}}(a_i, \mathbf{x}) \equiv v_i^{\mathbf{P}}(a_i, \mathbf{x}) - v_i^{\mathbf{P}}(0, \mathbf{x})$$

- For instance, when ε is Type I extreme value:

$$\widetilde{v}_i^{\mathbf{P}}(a_i, \mathbf{x}) = \ln P_i^0(a_i|\mathbf{x}) - \ln P_i^0(0|\mathbf{x})$$

A Negative Identification Result (on Primitives)

- Unfortunately, the identification of function $\widetilde{v}_i^{\mathbf{P}}(a_i, \mathbf{x})$ is not sufficient to identify the primitive preference function $(\pi_i(\boldsymbol{a}, \mathbf{x}),\ \delta_i)$.
  See **Rust (1994, Handbook)**, **Magnac & Thesmar (2002, ECMA)**

- There are three identification issues.

P1. **Non innocuous normalizations**. In contrast to static models, normalizing $\pi_i(0, \mathbf{x}) = 0$ has implications on important empirical questions.

P2. **No identification of discount factor**.

P3. **No identification competition effects**. $\widetilde{v}_i^{\mathbf{P}}(a_i, \mathbf{x})$ does not have a_{-it} as an argument, but we are interested in the effect of a_{-it} on π_i.

Additional Assumptions Restrictions

ID.6 **Normalization of payoff of one choice alternative**.
$\pi_i(a_i = 0, \boldsymbol{a}_{-i}, \mathbf{x}) = 0$ for every $(i, \boldsymbol{a}_{-i}, \mathbf{x})$.

Example: If a firm decides not being in the market its profit is zero, regardless she is a potential entrant or an incumbent (i.e., no scrap value of exit costs).

ID.7 **Known discount factor**. δ_i is known to the researcher.

Example: With annual frequency, $\delta_i = 0.95$ for every firm.

ID.8 **Exclusion restriction in profit function**. $\mathbf{x}_t = (\mathbf{x}_t^c), z_{it} : i \in \mathcal{I}$ such that $\pi_i(\boldsymbol{a}_t, \mathbf{x}^c, z_{it})$ does not depend on $z_{jt})$ for $j \neq i$.

Example: In a game of market entry-exit, firm i's profit depends on the current entry decisions of competitors $(\boldsymbol{a}_{-it})$, and on the own incumbency status $(a_{i,t-1})$ but there is not a (direct) effect of the competitors' incumbency status $(\boldsymbol{a}_{-i,t-1})$.

Positive Identification Result

- Under Assumptions [ID.1] to [ID.8], the profit functions $\pi_i(\boldsymbol{a}, \mathbf{x})$ are **nonparametrically identified** from the conditional-choice values $\widetilde{v}_i^{\mathbf{P}}(a_i, \mathbf{x})$.

  Proposition 3 in Pesendorfer & Schmidt-Dengler (REStud, 2008).

- Assumptions [ID.1] to [ID.8] are very common in empirical applications of dynamic games in IO. In most cases, they are combined with parametric restrictions on function π_i.

RELAXING (Sufficient but Not Necessary) IDEN. RESTRICTIONS

1. **Serially correlated unobservables**

   - Time-invariant market heterogeneity: Aguirregabiria & Mira (ECMA, 2007); Kasahara & Shimotsu (ECMA, 2009); Arcidiacono & Miller (ECMA, 2011).

   - Time-variant market heterogeneity: Hu & Shum (JOE, 2012).

2. **Multiple equilibria in the data**
   - Sweeting (RAND, 2009); De Paula & Tang (ECMA, 2012); Aguirregabiria & Mira (QE, 2019)

3. **Relaxing normalization restrictions**
   - Aguirregabiria & Suzuki (QME, 2014); Kalouptsidi, Souza-Rodrigues, & Scott (QE, 2019; REStud, 2024).

RELAXING (Sufficient but Not Necessary) IDEN. RESTRICTIONS

4. **Identification of discount factors**
   - Abbring & Daljord (QE, 2020)

5. **Non-additive unobservables**
   - Kristensen & De Paula (JOE, 2021).

6. **Nonparametric distribution of unobservables**
   - Norest & Tang (REStud, 014); Buchholz, Hu, & Shum (JOE, 2021).

7. **Non-equilibrium beliefs**
   - Aguirregabiria & Magesan (REStud, 2020)

8. ESTIMATION

ESTIMATION – PRELIMINARIES

- Primitives of the model: $\{\pi_i, \ \beta_i, \ f_x, \ G_\varepsilon\}$, can be described in terms of a vector of parameters $\boldsymbol{\theta}$ that is unknown to the researcher.

- It is convenient to distinguish four sub-vectors in $\boldsymbol{\theta}$, $(\boldsymbol{\theta}_\pi, \ \boldsymbol{\theta}_f, \ \boldsymbol{\beta}, \ ,\boldsymbol{\theta}_\varepsilon)$.

- In most empirical applications, the main challenge is in the **estimation of "dynamic parameters" in $\boldsymbol{\theta}_\pi$**:

  - $\boldsymbol{\theta}_f$ can be estimated "outside" of the dynamic decision model.

  - Consumer demand and firms' variable costs – which are part of $\boldsymbol{\theta}_\pi$ – can be estimated "outside" of the dynamic decision model.

  - Most applications assume that $\boldsymbol{\theta}_\varepsilon$ (distribution of ε) and $\boldsymbol{\beta}$ are known.

  - Often, the focus in the estimation of the dynamic game is parameters capturing dynamics, i.e., investment costs, entry/exit costs, fixed costs.

OUTLINE ON ESTIMATION

1. Maximum Likelihood Est. (MLE) of models with **unique equilibrium**
    - Rust's **Nested Fixed Point (NFXP)** algorithm.
2. Maximum Likelihood Est. (MLE) of models with **multiple equilibria**
3. Sequential **CCP methods**
4. **Finite Dependence Property** + CCP methods

8.1. MLE WITH UNIQUE EQUILIBRIUM

MLE: MODELS WITH UNIQUE EQUILIBRIUM

- There exist sufficient conditions implying that a dynamic game has a unique equilibrium for every possible value of the parameters $\boldsymbol{\theta}$.
- An example of sufficient conditions for equilibrium uniqueness are:
  - i. Finite horizon T.
  - ii. Within every period t, firms make decisions sequentially: firm 1 first, firm 2 second, ..., firm N last. These decisions become common knowledge to the firms later in the sequence.
- Let $P_{it}(a_{it} \mid \mathbf{x}_t, \boldsymbol{\theta})$ be the equilibrium CCP function for firm i at period t when the vector of parameters is $\boldsymbol{\theta}$.
- The **full log-likelihood function** is: $\ell(\boldsymbol{\theta}) = \sum_{m=1}^{M} \ell_m(\boldsymbol{\theta})$, where $\ell_m(\boldsymbol{\theta})$ is the contribution of market m:

$$\ell_m(\boldsymbol{\theta}) = \sum_{i=1}^{N} \sum_{t=1}^{T} \log P_{it}(a_{imt} | \mathbf{x}_{mt}, \boldsymbol{\theta}) + \log f_x(\mathbf{x}_{m,t+1} | \boldsymbol{a}_{mt}, \mathbf{x}_{mt}, \boldsymbol{\theta}_f)$$

NESTED FIXED POINT (NFXP) ALGORITHM

- The MLE is: $\widehat{\boldsymbol{\theta}} = argmax_{\boldsymbol{\theta}} \ \ell(\boldsymbol{\theta})$.
- Rust's NFXP algorithm is a method to compute the MLE. It combines BHHH iterations (**outer algorithm**) with equilibrium solution algorithm (**inner algorithm**) for each trial value $\boldsymbol{\theta}$.

1. Start at an initial guess: $\widehat{\boldsymbol{\theta}}_0$.
2. At every **outer iteration** k, apply a BHHH iteration:

$$\widehat{\boldsymbol{\theta}}_{k+1} = \widehat{\boldsymbol{\theta}}_k + \left(\sum_{m=1}^{M} \frac{\partial \ell_m(\widehat{\boldsymbol{\theta}}_k)}{\partial \boldsymbol{\theta}} \frac{\partial \ell_m(\widehat{\boldsymbol{\theta}}_k)}{\partial \boldsymbol{\theta}'} \right)^{-1} \left(\sum_{m=1}^{M} \frac{\partial \ell_m(\widehat{\boldsymbol{\theta}}_k)}{\partial \boldsymbol{\theta}} \right)$$

3. The score vector $\partial \ell_m(\widehat{\boldsymbol{\theta}}_k) / \partial \boldsymbol{\theta}$ depends on $\partial \log P_i(a_{imt} | \mathbf{x}_{mt}, \widehat{\boldsymbol{\theta}}_k) / \partial \boldsymbol{\theta}$. To obtain these derivatives, the **inner algorithm** solves for the equilibrium CCPs given $\widehat{\boldsymbol{\theta}}_k$ using fixed point iterations.
4. Outer BHHH iterations until $||\widehat{\boldsymbol{\theta}}_{k+1} - \widehat{\boldsymbol{\theta}}_k|| <$ small constant

8.2. MLE WITH MULTIPLE EQUILIBRIA

MLE WITH MULTIPLE EQUILIBRIA

- With Multiple Equilibria, $\ell(\boldsymbol{\theta})$ is not a function but a correspondence. The MLE cannot be defined as the $argmax$ of $\ell(\boldsymbol{\theta})$.

- To define the MLE in a model with multiple equilibria, it is convenient to define an *extended* or **Pseudo Likelihood function**.

- For arbitrary values of $\boldsymbol{\theta}$ and firms' CCPs $\mathbf{P}$, define:

$$Q(\boldsymbol{\theta}, \mathbf{P}) = \sum_{m=1}^{M} \sum_{i=1}^{N} \sum_{t=1}^{T} \log \Psi_i(a_{imt} \mid \mathbf{x}_{mt}, \boldsymbol{\theta}, \mathbf{P})$$

where Ψ_i is the *best response probability function*.

MLE WITH MULTIPLE EQUILIBRIA [2]

- The MLE is the pair $(\widehat{\boldsymbol{\theta}}_{MLE}, \widehat{\mathbf{P}}_{MLE})$ that **maximizes Q subject to the constraint that CCPs are equilibrium strategies**:

$$(\widehat{\boldsymbol{\theta}}_{MLE}, \widehat{\mathbf{P}}_{MLE}) = \begin{cases} \arg\max\limits_{(\boldsymbol{\theta},\mathbf{P})} & Q(\boldsymbol{\theta}, \mathbf{P}) \\ & \\ \text{subject to:} & \mathbf{P} = \Psi(\boldsymbol{\theta}, \mathbf{P}) \end{cases}$$

- Or using the Lagrangian function:

$$(\widehat{\boldsymbol{\theta}}_{MLE}, \widehat{\mathbf{P}}_{MLE}, \widehat{\boldsymbol{\lambda}}_{MLE}) = \arg\max_{(\boldsymbol{\theta},\mathbf{P},\boldsymbol{\lambda})} \; Q(\boldsymbol{\theta}, \mathbf{P}) + \boldsymbol{\lambda}' \left[\mathbf{P} - \Psi(\boldsymbol{\theta}, \mathbf{P})\right]$$

- The F.O.C. are the Lagrangian equations:

$$\begin{cases} \widehat{\mathbf{P}}_{MLE} - \Psi(\widehat{\boldsymbol{\theta}}_{MLE}, \widehat{\mathbf{P}}_{MLE}) & = & \mathbf{0} \\ \nabla_{\boldsymbol{\theta}} Q(\widehat{\boldsymbol{\theta}}_{MLE}, \widehat{\mathbf{P}}_{MLE}) - \widehat{\boldsymbol{\lambda}}'_{MLE} \; \nabla_{\boldsymbol{\theta}} \Psi(\widehat{\boldsymbol{\theta}}_{MLE}, \widehat{\mathbf{P}}_{MLE}) & = & \mathbf{0} \\ \nabla_{\mathbf{P}} Q(\widehat{\boldsymbol{\theta}}_{MLE}, \widehat{\mathbf{P}}_{MLE}) - \widehat{\boldsymbol{\lambda}}'_{MLE} \; \nabla_{\mathbf{P}} \Psi(\widehat{\boldsymbol{\theta}}_{MLE}, \widehat{\mathbf{P}}_{MLE}) & = & \mathbf{0} \end{cases}$$

MLE WITH MULTIPLE EQUILIBRIA [3]

- A Newton method can be used to obtain a root of this system of Lagrangian equations.
- A key computational problem is the **very high dimensionality of this system of equations**.
- The most costly part of this algorithm is the **calculation of the Jacobian matrix** $\nabla_{\mathbf{P}}\Psi(\widehat{\theta}, \widehat{\mathbf{P}})$. In dynamic games, in general, this is not a sparse matrix, and can contain billions or trillions of elements.
- The evaluation of the best response mapping $\Psi(\theta, \mathbf{P})$ for a new value of $\mathbf{P}$ requires solving for a valuation operator and solving a system of equations with the same dimension as $\mathbf{P}$.
- Due to serious computational issues, there are no empirical applications of dynamic games with multiple equilibria that compute the MLE, with either the NFXP or MPEC algorithms.

DP & CURSE OF DIMENSIONALITY

- Solving Dynamic Programming (DP) involves the **Computation of Present Values (PV)**:

  i.e., expected and discounted value of the stream of future payoffs.

- The computation of PVs is subject to the so called **Curse of Dimensionality**.

- With discrete state variables, with n state variables, each of dimension S, computing time is $O(S^n)$.

- This is an **exponential time problem**, which in computer science are denoted as **"Intractable problems"** – in comparison to polynomial time problem which are "Tractable".

DP & CURSE OF DIMENSIONALITY: EXAMPLE

- N-players dynamic game. Each player with firm-specific state variables of dimension $|X_i|$. Computing time for solving 1 firm's best response: $T = \gamma \ |X_i|^N$, where γ is a constant measured in units of time.

- Suppose that $|X_i| = 100$ and $\gamma = 10^{-5}$ seconds. Then:
  - With $N = 1$, CPU time $= 0.001$ seconds.
  - With $N = 2$, CPU time $= 0.1$ seconds.
  - With $N = 3$, CPU time $= 10$ seconds.
  - With $N = 4$, CPU time $= 1,000$ seconds $\approx$ 16 minutes.
  - With $N = 5$, CPU time $= 10^5$ seconds $\approx$ 28 hours.
  - With $N = 6$, CPU time $= 10^7$ seconds $\approx$ 115 days

- This can make researchers to use parsimonious models, with **few state variables, crude discretization, & few players**.

8.3. TWO-STEP CCP METHODS

TWO-STEP CCP METHODS

- Methods that avoid solving for firms' best responses or an equilibrium, even once.

- **Hotz & Miller (REStud, 1993)** was a seminal contribution on this class of methods. They show that the conditional choice values can be written as known functions of CCPs, transition probabilities, and $\boldsymbol{\theta}$.

- Suppose that one-period profit is linear-in-parameters:

$$\pi_i(a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t) \; = \; h(a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t)' \, \boldsymbol{\theta}_{\pi,i}$$

where $h(a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t)$ is a vector of known functions to the researcher.

- The conditional-choice value function $v_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t)$ is:

$$v_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) = \mathbb{E}\left(\sum_{j=0}^{\infty} \beta^j \; h(\boldsymbol{a}_{t+j}, \mathbf{x}_{t+j})' \; \boldsymbol{\theta}_{\pi,i} + \varepsilon_{i,t+j}(a_{i,t+j}) \mid a_{it}, \mathbf{x}_t \right)$$

where future actions, $\boldsymbol{a}_{t+j}$, are taken according to equilibrium CCPs.

TWO-STEP CCP METHODS [2]

- We can write:

$$v_i^{\mathbf{P}}(a_{it},\mathbf{x}_t) \;=\; \widetilde{h}_i^{\mathbf{P}}(a_{it},\mathbf{x}_t)\;\boldsymbol{\theta}_{\pi,i} \;+\; \widetilde{e}_i^{\mathbf{P}}(a_{it},\mathbf{x}_t)$$

with:

$$\widetilde{h}_i^{\mathbf{P}}(a_{it},\mathbf{x}_t) \;=\; \mathbb{E}\left(\sum_{j=0}^{\infty}\beta^j\; h(\boldsymbol{a}_{t+j},\mathbf{x}_{t+j}) \mid a_{it},\mathbf{x}_t\right)$$

$$\widetilde{e}_i^{\mathbf{P}}(a_{it},\mathbf{x}_t) \;=\; \mathbb{E}\left(\sum_{j=0}^{\infty}\beta^j\; [\gamma - \ln P_i(a_{i,t+j}|\mathbf{x}_{t+j})] \mid a_{it},\mathbf{x}_t\right)$$

- Given firms' equilibrium CCPs, P, β, and the transition probability of $\mathbf{x}$, we can calculate these present values using, for instance, forward Monte Carlo Simulation.

TWO-STEP CCP METHODS [3]

- Given this representation of conditional choice values, the pseudo likelihood function $Q(\boldsymbol{\theta}, \mathbf{P})$ has practically the same structure as in a static or reduced form discrete choice model.

- Best response probabilities that enter in $Q(\boldsymbol{\theta}, \mathbf{P})$ can be seen as the choice probabilities in a standard random utility model:

$$\Psi_i(a_{imt} = j | \mathbf{x}_{mt}, \boldsymbol{\theta}, \mathbf{P}) = \frac{\exp\{\widetilde{h}_i^{\mathbf{P}}(j, \mathbf{x}_{mt}) \ \boldsymbol{\theta}_i + \widetilde{e}_i^{\mathbf{P}}(j, \mathbf{x}_{mt})\}}{\sum_{k=0}^{J} \exp\{\widetilde{h}_i^{\mathbf{P}}(k, \mathbf{x}_{mt}) \ \boldsymbol{\theta}_i + \widetilde{e}_i^{\mathbf{P}}(k, \mathbf{x}_{mt})\}}$$

- Given $\widetilde{h}_i^{\mathbf{P}}(., \mathbf{x}_{mt})$ and $\widetilde{e}_i^{\mathbf{P}}(., \mathbf{x}_{mt})$ and a parametric specification for the distribution of ε (e.g., logit, probit), the vector of parameters $\boldsymbol{\theta}_i$ can be estimated as in a standard logit or probit model.

TWO-STEP CCP METHODS [3]

- The method proceeds in two steps.
- Let $\widehat{\mathbf{P}}^0$ be a consistent nonparametric estimator of true $\mathbf{P}^0$. The two-step estimator of $\boldsymbol{\theta}$ is defined as:

$$\widehat{\boldsymbol{\theta}}_{2S} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \widehat{\mathbf{P}}^0)$$

- Under standard regularity conditions, this two-step estimator is root-M consistent and asymptotically normal.
- It can be extended to incorporate market unobserved heterogeneity (e.g., Aguirregabiria & Mira (2007); Arcidiacono & Miller (2011)).
- Monte Carlo Simulation can be used to compute present values: Bajari, Benkard, & Levin (2007).
- Limitation: Finite sample bias due to imprecise estimates of CCPs in the first step.

Nested Pseudo Likelihood (NPL)

- Imposes equilibrium restrictions but does NOT require:

  - Repeatedly solving for MPE for each trial value of $\boldsymbol{\theta}$ (as NFXP)
  - Computing $\nabla_{\mathbf{P}}\Psi(\widehat{\boldsymbol{\theta}}, \widehat{\mathbf{P}})$ (as NFXP and MPEC)

- A NPL $(\widehat{\boldsymbol{\theta}}_{NPL}, \widehat{\mathbf{P}}_{NPL})$, that satisfy two conditions:

  (1) given $\widehat{\mathbf{P}}_{NPL}$, we have that: $\widehat{\boldsymbol{\theta}}_{NPL} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \widehat{\mathbf{P}}_{NPL})$

  (2) given $\widehat{\boldsymbol{\theta}}_{NPL}$, we have that: $\widehat{\mathbf{P}}_{NPL} = \Psi(\widehat{\boldsymbol{\theta}}_{NPL}, \widehat{\mathbf{P}}_{NPL})$

- The NPL estimator is consistent and asymptotically normal under the same regularity conditions as the MLE. For dynamic games, the NPL estimator has larger asymptotic variance than the MLE.

Nested Pseudo Likelihood (NPL) [2]

- An algorithm to compute the NPL is the **NPL fixed point algorithm**.
- Starting with an initial $\widehat{\mathbf{P}}_0$, at iteration $k \geq 1$:

  (Step 1) given $\widehat{\mathbf{P}}_{k-1}$, $\widehat{\boldsymbol{\theta}}_k = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \widehat{\mathbf{P}}_{k-1})$;

  (Step 2) given $\widehat{\boldsymbol{\theta}}_k$, $\widehat{\mathbf{P}}_k = \Psi(\widehat{\boldsymbol{\theta}}_k, \widehat{\mathbf{P}}_{k-1})$.
- A natural choice for the initial $\widehat{\mathbf{P}}_0$ is a frequency estimator of CCPs using the data.
- Step 1 is very simple in most applications. It has the same comp. cost as obtaining the MLE in a static single-agent discrete choice model.
- Step 2 is equivalent to solving once a system of linear equations with the same dimension as $\mathbf{P}$.
- A limitation of this fixed point algorithm is that **convergence is not guaranteed**. An alternative algorithm that has been used to compute NPL is a **Spectral Residual algorithm**.