A CCP Primer

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Introduction

Stations on this whistle-stop journey through CCP land

1. Inversion
2. Identification
3. Estimation
4. Finite Dependence
5. Unobserved Heterogeneity
6. Two Monte Carlo Exercises

. . . switching metaphors (and centuries). . . this is a flyover . . . see the references for this lecture

1. Inversion

Conditional independence assumption

- Let $T \in \{1, 2, \ldots\}$ with $T \leq \infty$ denote the horizon of the optimization problem and $t \in \{1, \ldots, T\}$ denote the time period.
- Each period the individual chooses amongst J mutually exclusive actions.
- Let $d_t \equiv (d_{1t}, \ldots, d_{Jt})$ where $d_{jt} = 1$ if action $j \in \{1, \ldots, J\}$ is taken at time t and $d_{jt} = 0$ if action j is not taken at t.
- $x_t \in \{1, 2, \ldots, X\}$ for some finite X for each t (*observed*).
- $\epsilon_t \equiv (\epsilon_{1t}, \ldots, \epsilon_{Jt})$ where $\epsilon_{jt} \in \mathbb{R}$ for all (j, t) (*unobserved*).
- Assume the data comprises observations on (d_t, x_t).
- The joint mixed density function for the state in period $t+1$ conditional on (x_t, ϵ_t), denoted by $g_{t,x,\epsilon}\left(x_{t+1}, \epsilon_{t+1} \left| x_t, \epsilon_t \right.\right)$, satisfies the *conditional independence assumption*:

$$g_{t,j,x,\epsilon}\left(x_{t+1}, \epsilon_{t+1} \left| x_t, \epsilon_t \right.\right) = g_{t+1}\left(\epsilon_{t+1} \middle| x_{t+1}\right) f_{jt}(x_{t+1} | x_t)$$

where $g_t\left(\epsilon_t \middle| x_t\right)$ is a conditional density for the disturbances, and $f_{jt}(x_{t+1}|x)$ is a transition probability for x conditional on (j, t).

1. Inversion

Social surplus and conditional value functions

- Denote the optimal decision rule at t as $d_t^o(x_t, \epsilon_t)$, with j^{th} element $d_{jt}^o(x_t, \epsilon_t)$, and define the *social surplus function* as:

$$V_t(x_t) \equiv E\left\{ \sum_{\tau=t}^{T} \sum_{j=1}^{J} \beta^{\tau-t-1} d_{j\tau}^o\left(x_\tau, \epsilon_\tau\right)\left(u_{j\tau}(x_\tau) + \epsilon_{j\tau}\right) \right\}$$

- The *conditional value function*, $v_{jt}(x_t)$, is defined as:

$$v_{jt}(x_t) \equiv u_{jt}(x_t) + \beta \sum_{x=1}^{X} V_{t+1}(x) f_{jt}\left(x | x_t\right)$$

- Integrating $d_{jt}^o(x_t, \epsilon)$ over $\epsilon \equiv (\epsilon_1, \ldots, \epsilon_J)$ define the *conditional choice probabilities* CCPs by:

$$p_{jt}(x_t) \equiv E\left[d_{jt}^o\left(x_t, \epsilon\right) | x_t \right] = \int d_{jt}^o\left(x_t, \epsilon\right) g_t\left(\epsilon \,| x_t\right) d\epsilon$$

1. Inversion

Each CCP is a mapping of differences in the conditional value functions

- The starting point for our analysis is to define differences in the conditional value functions with respect to choice J as:

$$\Delta v_{jt}(x) \equiv v_{jt}(x) - v_{Jt}(x)$$

$$\begin{aligned}
\Rightarrow p_{jt}(x) &\equiv \int d_{jt}^{o}(x, \epsilon)\, dG_t(\epsilon \,|x) \\
&= \int I\{\epsilon_k \leq \epsilon_j + \Delta v_{jt}(x) - \Delta v_{kt}(x) \forall\ k \neq j\}\, dG_t(\epsilon \,|x) \\
&= \int_{-\infty}^{\epsilon_j + \Delta v_{jt}(x) - \Delta v_{1t}(x)} \cdots \int_{-\infty}^{\epsilon_j + \Delta v_{jt}(x)} dG_t(\epsilon \,|x) \\
&= \int_{-\infty}^{\infty} G_{jt}\left(\begin{array}{c} \epsilon_j + \Delta v_{jt}(x) - \Delta v_{1t}(x), \ldots \\ \ldots, \epsilon_j, \ldots, \epsilon_j + \Delta v_{jt}(x) \end{array} |x \right) d\epsilon_j
\end{aligned}$$

where $G_{jt}(\epsilon \,|x) \equiv \partial G_t(\epsilon \,|x) / \partial \epsilon_j$.

1. Inversion

CCPs are invertible in conditional value functions (Hotz and Miller, 1993)

- For any vector $J-1$ dimensional vector $\delta \equiv (\delta_1, \ldots, \delta_{J-1})$ define:

$$Q_{jt}(\delta, x) \equiv \int_{-\infty}^{\infty} G_{jt}(\epsilon_j + \delta_j - \delta_1, \ldots, \epsilon_j, \ldots, \epsilon_j + \delta_j \,|x)\, d\epsilon_j$$

- $Q_{jt}(\delta, x)$ is the probability choosing j in a static random utility model (RUM) with payoff $\delta_j + \epsilon_j$ and disturbance distribution $G_t(\epsilon\,|x)$.
- $Q_t(\delta, x) \equiv (Q_{1t}(\delta, x), \ldots Q_{J-1,t}(\delta, x))'$ is invertible in δ.
- This *inversion theorem* implies:

$$\begin{bmatrix} \Delta v_{1t}(x) \\ \vdots \\ \Delta v_{J-1,t}(x) \end{bmatrix} = \begin{bmatrix} Q_{1t}^{-1}[p_t(x), x] \\ \vdots \\ Q_{J-1,t}^{-1}[p_t(x), x] \end{bmatrix}$$

1. Inversion

Interpreting the inversion expression

- $Q_{jt}^{-1}(p, x)$ has an intuitive interpretation:
  - Given x and $p(x)$ the agent is indifferent between the j^{th} and J^{th} choices for values of ϵ'_{jt} and ϵ'_{Jt} satisfying:

$$\begin{aligned} v_{jt}(x) + \epsilon'_{jt} &= v_{Jt}(x) + \epsilon'_{Jt} \\ \Rightarrow \Delta v_{jt}(x) &= \epsilon'_{j} - \epsilon'_{J} \\ &= Q_{jt}^{-1}[p_t(x), x] \end{aligned}$$

  - Thus the value of $Q_{jt}^{-1}[p_t(x), x]$ is the difference between the j^{th} and J^{th} taste shocks that would make the agent indifferent between those two choices.
  - More generally the value of the vector mapping:

$$Q_t^{-1}[p_t(x), x] = \left(Q_{1t}^{-1}[p_t(x), x], \ldots, Q_{J-1t}^{-1}[p_t(x), x]\right)$$

    corresponds to the value of a vector $\epsilon'_t \equiv (\epsilon'_{1t}, \ldots, \epsilon'_{Jt})$ that renders the agent indifferent to all the choices.

1. Inversion

Using the inversion theorem

- The inversion theorem exploits conditional independence to finesse optimization and integration.
- More specifically we use the inversion theorem to:
  1. provide tractable *representations of the conditional value functions*.
  2. analyze *identification* in dynamic discrete choice models.
  3. yield parametric forms for ϵ_t density *generalizing T1EV*.
  4. estimate the model without solving it
  5. generalize the renewal and terminal state properties often used in empirical work to *finite dependence*, by obtaining restrictions on the state variable transitions used to implement CCP estimators.
  6. incorporate *unobserved state variables* within the model.

The conditional value function correction (Arcidiacono and Miller, 2011)

- Define the *conditional value function correction* as:

$$\psi_{jt}(x) \equiv V_t(x) - v_{jt}(x)$$

- For example if $G(\epsilon)$ is a nested logit then:

$$G(\epsilon) \equiv \exp\left[-\left(\sum_{j\in\mathcal{J}} \exp\left[-\epsilon_j/\sigma\right]\right)^{\sigma}\right]$$

  and:

$$\psi_{jt}(p) = \gamma - \sigma \ln(p_{jt}) - (1-\sigma)\ln\left(\sum_{k\in\mathcal{J}} p_{kt}\right)$$

  where $\gamma = 0.577\ldots$ is Euler's constant.

- Assume (T, β, g) is known, and note f is identified (by inspection). Then $\psi_{jt}(x)$ is identified too (off the CCPs).

- We can write:

$$\begin{aligned} v_{jt}(x_t) &= u_{jt}(x_t) + \beta \sum_{x=1}^{X} V_{t+1}(x) f_{jt}(x|x_t) \\ &= u_{jt}(x_t) \\ &+ \sum_{\tau=t+1}^{T} \sum_{k=1}^{J} \sum_{x=1}^{X} \beta^{\tau-t} \left\{ \begin{array}{c} (u_{k\tau}(x) + \psi_k[p_\tau(x)]) \\ \times \omega_{k\tau}(x,j) \kappa_{\tau-1}(x|x_t,j) \end{array} \right\} \end{aligned} \tag{1}$$

where the weights $\omega_{k\tau}(x_\tau, j)$ satisfy:

$$-\infty < \omega_{k\tau}(x_\tau, j) < \infty \text{ and } \sum_{k=1}^{J} \omega_{k\tau}(x_\tau, j) = 1$$

while the $\tau - \mathit{period}$ state transitions $\kappa_\tau(x_{\tau+1}|x_t, j)$ are defined as:

$$\kappa_\tau(x_{\tau+1}|x_t, j) \equiv \left\{ \begin{array}{l} \kappa_t(x_{t+1}|x_t, j) \equiv f_{jt}(x_{t+1}|x_t) \\ \sum_{x_\tau=1}^{X} \sum_{k=1}^{J} \omega_{k\tau}(x_\tau, j) f_{k\tau}(x_{\tau+1}|x_\tau) \kappa_{\tau-1}(x_\tau|x_t, j) \end{array} \right.$$

- Differencing $v_{jt}(x_t)$ and $v_{1t}(x_t)$ using (1) gives:

$$
\begin{aligned}
u_{jt}(x) &= u_{1t}(x) + \psi_{1t}(x) - \psi_{jt}(x) \\
&+ \sum_{\tau=t+1}^{T} \sum_{x_\tau=1}^{X} \beta^{\tau-t} \left\{ \begin{array}{l} [u_{1\tau}(x_\tau) + \psi_{1t}(x_\tau)] \times \\ [\kappa_{\tau-1}(x_\tau|x,1) - \kappa_{\tau-1}(x_\tau|x,j)] \end{array} \right\}
\end{aligned} \tag{2}
$$

- If (T, β, f, g) is known, along with a payoff, say the first, is also known for every state and time, then u is exactly point identified.
- The identification is exact because there are as many $u_{jt}(x)$ terms as there are $p_{jt}(x)$ terms, and the CCPs are a sufficient statistic for the sample (that is given $f_{jt}(x_{t+1}|x_t)$).

- Assume $u_{1t}(x) = 0$ and set:

$$\widehat{p}_{jt}(x) = \sum_{n=1}^{N} \mathbf{1}\{x_{nt} = x, d_{njt} = 1\} \Big/ \sum_{n=1}^{N} \mathbf{1}\{x_{nt} = x\}$$

$$\widehat{f}_{jt}(x' | x) = \frac{\sum_{n=1}^{N} \mathbf{1}\{x_{nt} = x, d_{njt} = 1, x_{n,t+1} = x'\}}{\sum_{n=1}^{N} \mathbf{1}\{x_{nt} = x, d_{njt} = 1\}}$$

$$\widehat{\kappa}_{\tau}(x_{\tau+1} | t, x_t, j) \equiv \begin{cases} \widehat{f}_{jt}(x_{t+1} | x_t) & \tau = t \\ \sum_{x=1}^{X} \widehat{f}_{1\tau}(x_{\tau+1} | x) \kappa_{\tau-1}(x | t, x_t, j) & \tau = t+1, \ldots \end{cases}$$

to obtain $\widehat{\psi}_{jt}(x)$ and hence from (2):

$$\begin{aligned} \widehat{u}_{jt}(x_t) \equiv\ & \widehat{\psi}_{1t}(x_t) - \widehat{\psi}_{jt}(x_t) \\ & + \sum_{\tau=1}^{T-t} \sum_{x=1}^{X} \beta^{\tau-t} \widehat{\psi}_{1,t+\tau}(x) \left[\widehat{\kappa}_{t1,\tau-1}(x | x_t) - \widehat{\kappa}_{tj,\tau-1}(x | x_t) \right] \end{aligned} \qquad (3)$$

- The only way to increase asymptotic efficiency is to place restrictions on $u_{jt}(x)$ through parametric assumptions.

- One approach is to estimate:
  - $\theta^{(2)}$ with LIML off the transitions $f_{jt}(x|x_{nt}, \theta^{(2)})$
  - $\theta_0^{(1)}$ by minimizing the distance between the unrestricted estimates $\widehat{u}_{jt}(x_t)$ given in (3) and its parameterization $u_{jt}(x_t, \theta^{(1)})$:

$$\theta_{MD}^{(1)} = \underset{\theta^{(1)} \in \Theta^{(1)}}{\arg\min} \left[u(x, \theta^{(1)}) - \widehat{u}(x_t) \right]' W \left[u(x, \theta^{(1)}) - \widehat{u}(x_t) \right]$$

    where $u(x, \theta^{(1)})$ and $\widehat{u}(x_t)$ are stacked vectors of $u_{jt}(x_t, \theta^{(1)})$ and $\widehat{u}_{jt}(x_t)$, and W is a weight matrix (MD).

- Note:
  - $\theta_{MD}^{(1)}$ has a closed form if $u(x; \theta_0^{(1)})$ is linear in $\theta_0^{(1)}$.
  - the overidentifying restrictions can be tested.

- Alternatively to implement a QML estimator, first estimate $p_{jt}(x)$, $\theta_0^{(2)}$ and $\kappa_\tau(x|t, x_t, k, \theta_0^{(2)})$ and $\psi_{1t}(x)$ as above, and then:

$$\theta_{QML}^{(1)} \equiv \arg\max_{\theta_1} \sum_{n=1}^{N} \sum_{t=1}^{T} \sum_{j=1}^{J} d_{njt} \left\{ \ln \left[\widehat{p}_{jt}(x_{nt}, \theta^{(1)}, \theta_{LIML}^{(2)}) \right] \right\}$$

where in T1EV applications:

$$\widehat{p}_{jt}(x, \theta^{(1)}, \widehat{h}) = \frac{\exp\left[u_{jt}(x, \theta^{(1)}) + \widehat{h}_{jt}(x) \right]}{\sum_{k=1}^{J} \exp\left[u_{kt}(x, \theta^{(1)}) + \widehat{h}_{kt}(x) \right]}$$

and $\widehat{h}_{kt}(x)$ is a numeric *dynamic correction factor* defined:

$$\widehat{h}_{jt}(x) \equiv \sum_{\tau=t+1}^{T} \sum_{x_\tau=1}^{X} \beta^{\tau-t} \widehat{\psi}_{1\tau}(x_\tau) \kappa_{\tau-1}(x_\tau | t, x, j, \theta_{LIML}^{(2)})$$

- Similarly, to form a MSM estimator first:
  1. Estimate $p_{jt}(x)$, $\theta_0^{(2)}$ and $\kappa_\tau(x|t, x_t, k, \theta_0^{(2)})$ and $\psi_{kt}(x)$ for all $k \in \{1, \ldots, K\}$ as above.
  2. Simulate a lifetime path from x_{nt_n} onwards for each j, using $\widehat{f}$ and $\widehat{p}$. This generates $\widehat{x}_{ns}$ and $\widehat{d}_{ns} \equiv \left(\widehat{d}_{n1s}, \ldots, \widehat{d}_{nJs}\right)$ for all $s \in \{t_n + 1, \ldots, T\}$.
  3. Obtain estimates of:

$$\widehat{E}\left[\epsilon_{jt} \middle| d_{jt}^o = 1, x_t\right] \equiv$$

$$p_{jt}^{-1}(x_t) \int\limits_{\epsilon_t} \prod_{k=1}^{J} \mathbf{1}\left\{ \begin{array}{r} \widehat{\psi}_{jt}(x_t) - \widehat{\psi}_{kt}(x_t) \\ \leq \epsilon_{jt} - \epsilon_{kt} \end{array} \right\} \epsilon_{jt} dG(\epsilon_t)$$

  or simulate it from the selected population $\widehat{\epsilon}_{jt}$.

3. Estimation

The last three steps for an MSM estimator

1. Stitch together a simulated lifetime utility outcome for each n from the j^{th} choice at t_n onwards: $\widehat{v}_{jt_n}\left(x_{nt_n};\theta^{(1)},\widehat{f},\widehat{p}\right)\equiv$

$$u_{jt}(x_{nt_n},\theta^{(1)}) + \sum_{s=t+1}^{T}\sum_{j=1}^{J}\beta^{t-1}1\left\{\widehat{d}_{njs}=1\right\}\left\{\begin{array}{l} u_{js}(\widehat{x}_{ns},\theta^{(1)}) \\ +\widehat{E}\left[\epsilon_{js}\left|\widehat{x}_{ns},\widehat{d}_{njs}=1\right.\right] \end{array}\right\}$$

2. Form the $J-1$ dimensional vector $h_n\left(x_{nt_n};\theta^{(1)},\widehat{f},\widehat{p}\right)$ from:

$$\begin{aligned} h_{nj}\left(x_{nt_n};\theta^{(1)},\widehat{f},\widehat{p}\right) \equiv\ & \widehat{v}_{jt_n}\left(x_{nt_n},\theta^{(1)},\widehat{f},\widehat{p}\right)-\widehat{v}_{Jt_n}\left(x_{nt_n},\theta^{(1)},\widehat{f},\widehat{p}\right) \\ & +\widehat{\psi}_{jt}(x_{nt_n})-\widehat{\psi}_{Jt}(x_{nt_n}) \end{aligned}$$

3. Given a weighting matrix W_S and an instrument vector z_n minimize:

$$N^{-1}\left[\sum_{n=1}^{N}z_n h_n\left(x_{nt_n};\theta^{(1)},\widehat{f},\widehat{p}\right)\right]' W_S\left[\sum_{n=1}^{N}z_n h_n\left(x_{nt_n};\theta^{(1)},\widehat{f},\widehat{p}\right)\right]$$

4. Finite Dependence

Definition (Arcidiacono and Miller, 2019)

- *Finite dependence* is a useful property reducing the number of pre-estimated CCPs.
- The pair of choices $\{i, j\}$ exhibits *ρ-period dependence* at (t, x_t) if there exist a pair of sequences of decision weights:

$$\{\omega_{k\tau}(t, x_\tau, i)\}_{(k,\tau)=(1,t+1)}^{(J,t+\rho)} \text{ and } \{\omega_{k\tau}(t, x_\tau, j)\}_{(k,\tau)=(1,t+1)}^{(J,t+\rho)}$$

  such that for all $x_{t+\rho+1} \in \{1, \ldots, X\}$:

$$\kappa_{t+\rho+1}(x_{t+\rho+1}|t, x_t, i) = \kappa_{t+\rho+1}(x_{t+\rho+1}|t, x_t, j)$$

- From (1), if there is finite dependence at (t, x_t, i, j) then:

$$u_{jt}(x_t) + \psi_j[p_t(x_t)] - u_{it}(x_t) - \psi_i[p_t(x_t)] =$$

$$\sum_{(k,\tau,x_\tau)=(1,t+1,1)}^{(J,t+\rho,X)} \beta^{\tau-t} \left\{ \begin{array}{l} u_{k\tau}(x_\tau) \\ +\psi_k[p_\tau(x_\tau)] \end{array} \right\} \left[\begin{array}{l} \omega_{k\tau}(t, x_\tau, i)\kappa_\tau(x_\tau|t, x_t, i) \\ -\omega_{k\tau}(t, x_\tau, j)\kappa_\tau(x_\tau|t, x_t, j) \end{array} \right] \tag{4}$$

- Terminal choices are widely assumed in structural econometric applications of dynamic optimization problems and games.
- A *terminal choice* ends the evolution of the state variable with an *absorbing state* that is independent of the current state.
- If the first choice denotes a terminal choice, then:

$$f_{1t}(x_{t+1}|x) \equiv f_{1t}(x_{t+1})$$

for all (t, x) and hence:

$$\sum\nolimits_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}) f_{jt}(x_{t+1}|x_t) = f_{1,t+1}(x_{t+2})$$

- Setting $\omega_{k\tau}(t, x, i) = 0$ for all (x, i) and $k \neq 1$, (4) implies:

$$\begin{aligned} & u_{1t}(x_t) + \psi_1[p_t(x_t)] - u_{jt}(x_t) - \psi_j[p_t(x_t)] \\ = \; & \sum\nolimits_{x_{t+1}=1}^{X} \beta \left\{ u_{1,t+1}(x_{t+1}) + \psi_1\left[p_{t+1}(x_{t+1})\right]\right\} f_{jt}(x_{t+1}|x_t) \end{aligned}$$

- Similarly a *renewal choice* yields a probability distribution of the state variable next period that does not depend on the current state.
- If the first choice is a renewal choice, then for all $j \in \{1, \dots, J\}$:

$$\begin{aligned}
\sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}|x_{t+1}) f_{jt}(x_{t+1}|x_t) &= \sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}) f_{jt}(x_{t+1}|x_t) \\
&= f_{1,t+1}(x_{t+2}) \sum_{x_{t+1}=1}^{X} f_{jt}(x_{t+1}|x_t) \\
&= f_{1,t+1}(x_{t+2}) \qquad (5)
\end{aligned}$$

- In this case Equation (4) implies:

$$\begin{aligned}
& u_{1t}(x_t) + \psi_1[p_t(x_t)] - u_{jt}(x_t) - \psi_j[p_t(x_t)] \\
= & \sum_{x=1}^{X} \beta \{u_{1,t+1}(x) + \psi_1 [p_{t+1}(x)]\} [f_{jt}(x|x_t) - f_{1t}(x|x_t)]
\end{aligned}$$

- First, consider the infeasible case where $s \in \{1, \ldots, S\}$ is unobserved but $p(x, s)$ is known.
- Let π_s denote population probability of being in unobserved state s.
- Supposing β is known the ML estimator for this "easier" problem is:

$$\{\hat{\theta}, \hat{\pi}\} = \arg\max_{\theta,\pi} \sum_{n=1}^{N} \ln \left[\sum_{s=1}^{S} \pi_s \prod_{t=1}^{T} l(d_{nt}|x_{nt}, s, p, \theta) \right]$$

where $p \equiv p(x, s)$ is a string of probabilities assigned/estimated for each (x, s) and $l(d_{nt}|x_{nt}, s_n, p, \theta)$ is derived from our representation of the conditional valuation functions and takes the form:

$$\frac{d_{1nt} + d_{2nt} \exp(\theta_1 x_{nt} + \theta_2 s + \beta \ln [p(0, s)] - \beta \ln [p(x_{nt} + 1, s)]}{1 + \exp(\theta_1 x_{nt} + \theta_2 s + \beta \ln [p(0, s)] - \beta \ln [p(x_{nt} + 1, s)])}$$

- Maximizing over the sum of a log of summed products is computationally burdensome.

- The EM algorithm is a computationally attractive alternative to directly maximizing the likelihood.
- Denote by $d_n \equiv (d_{n1}, \ldots, d_{nT})$ and $x_n \equiv (x_{n1}, \ldots, x_{nT})$ the full sequence of choices and mileages observed in the data for bus n.
- At the m^{th} iteration:

$$\begin{aligned} q_{ns}^{(m+1)} &= \Pr\left\{ s \,\middle|\, d_n, x_{n,} \theta^{(m)}, \pi_s^{(m)}, p \right\} \\ &= \frac{\pi_s^{(m)} \prod_{t=1}^{T} l(d_{nt} | x_{nt}, s, p, \theta^{(m)})}{\sum_{s'=1}^{S} \pi_{s'}^{(m)} \prod_{t=1}^{T} l(d_{nt} | x_{nt}, s', p, \theta^{(m)})} \end{aligned}$$

$$\pi_s^{(m+1)} = N^{-1} \sum_{n=1}^{N} q_{ns}^{(m+1)}$$

$$\theta^{(m+1)} = \arg\max_{\theta} \sum_{n=1}^{N} \sum_{s=1}^{S} \sum_{t=1}^{T} q_{ns}^{(m+1)} \ln[l(d_{nt} | x_{nt}, s, p, \theta)]$$

Our algorithm begins by setting initial values for $\theta^{(1)}$, $\pi^{(1)}$, and $p^{(1)}(\cdot)$:

Step 1 Compute $q_{ns}^{(m+1)}$ as:

$$q_{ns}^{(m+1)} = \frac{\pi_s^{(m)} \prod_{t=1}^{T} l\left[d_{nt}|x_{nt}, s, p^{(m)}, \theta^{(m)}\right]}{\sum_{s'=1}^{S} \pi_s^{(m)} \prod_{t=1}^{T} l\left(d_{nt}|x_{nt}, s', p^{(m)}, \theta^{(m)}\right)}$$

Step 2 Compute $\pi_s^{(m+1)}$ according to:

$$\pi_s^{(m+1)} = \frac{\sum_{n=1}^{N} q_{ns}^{(m+1)}}{N}$$

Step 3 Update $p^{(m+1)}(x, s)$ using one of two rules below

Step 4 Obtain $\theta^{(m+1)}$ from:

$$\theta^{(m+1)} = \arg\max_{\theta} \sum_{n=1}^{N} \sum_{s=1}^{S} \sum_{t=1}^{T} q_{ns}^{(m+1)} \ln\left[l\left(d_{nt}|x_{nt}, s_n, p^{(m+1)}, \theta\right)\right.$$

- Take a weighted average of decisions to replace engine, conditional on x, where weights are the conditional probabilities of being in unobserved state s.

Step 3A Update CCP's with:

$$p^{(m+1)}(x,s) = \frac{\sum_{n=1}^{N}\sum_{t=1}^{T} d_{1nt} q_{ns}^{(m+1)} I(x_{nt}=x)}{\sum_{n=1}^{N}\sum_{t=1}^{T} q_{ns}^{(m+1)} I(x_{nt}=x)}$$

- Or in a stationary infinite horizon model use identity from model that likelihood returns CCP of replacing the engine:

Step 3B Update CCP's with:

$$p^{(m+1)}(x_{nt},s_n) = I(d_{nt1}=1|x_{nt},s_n,p^{(m)},\theta^{(m)})$$

6. Two Monte Carlo Exercises

Two types of bus engines (unobserved by econometrician)

- Mr. Zurcher decides whether to replace the existing engine ($d_{1t} = 1$), or keep it for at least one more period ($d_{2t} = 1$), maximizing:

$$E\left\{\sum_{t=1}^{\infty}\beta^{t-1}\left[d_{2t}(\theta_1 x_t + \theta_2 s + \epsilon_{2t}) + d_{1t}\epsilon_{1t}\right]\right\}$$

- The fleet comprises equal numbers of Ford versus GM buses $s \in \{0, 1\}$. Total accumulated mileage is:

$$x_{t+1} = \begin{cases} \Delta_t \text{ if } d_{1t} = 1 \\ x_t + \Delta_t \text{ if } d_{2t} = 1 \end{cases}$$

where $\Delta_t \in \{0, 0.125, \ldots, 24.875, 25\}$ is drawn from:

$$f(\Delta_t | x') = \exp\left[-x'(\Delta_t - 25)\right] - \exp\left[-x'(\Delta_t - 24.875)\right]$$

and $x' \in \{0.25, 0.26, \ldots, 1.25\}$ equi-probable (observed).

- Transitory *iid* choice-specific shocks, ϵ_{jt} are Type 1 Extreme value.

- Let θ_{0t} be an aggregate shock (denoting cost fluctuations say).
- The difference in current payoff from retaining versus replacing the engine is:

$$u_{2t}(x_t, s) - u_{1t}(x_t, s) \equiv \theta_{0t} + \theta_1 \min\{x_t, 25\} + \theta_2 s$$

- From (5):

$$\begin{aligned} & v_{2t}(x_t, x', s) - v_{1t}(x_t, x', s) \\ = \; & \theta_{0t} + \theta_1 \min\{x_t, 25\} + \theta_2 s \\ & + \beta \sum_{\Delta_t \in \Lambda} \left\{ \ln \left[\frac{p_{1t}(0, s)}{p_{1t}(x_{1t} + \Delta_t, s)} \right] \right\} f(\Delta_t | x') \end{aligned}$$

6. Two Monte Carlo Exercises

Table 1 of Arcidiacono and Miller (2011, page 1854)

MONTE CARLO FOR THE OPTIMAL STOPPING PROBLEM[a]

		s Observed			*s* Unobserved		Time Effects	
				Ignoring *s*			*s* Observed	*s* Unobserved
	DGP	FIML	CCP	CCP	FIML	CCP	CCP	CCP
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
θ_0 (intercept)	2	2.0100	1.9911	2.4330	2.0186	2.0280		
		(0.0405)	(0.0399)	(0.0363)	(0.1185)	(0.1374)		
θ_1 (mileage)	−0.15	−0.1488	−0.1441	−0.1339	−0.1504	−0.1484	−0.1440	−0.1514
		(0.0074)	(0.0098)	(0.0102)	(0.0091)	(0.0111)	(0.0121)	(0.0136)
θ_2 (unobs. state)	1	0.9945	0.9726		1.0073	0.9953	0.9683	1.0067
		(0.0611)	(0.0668)		(0.0919)	(0.0985)	(0.0636)	(0.1417)
β (discount factor)	0.9	0.9102	0.9099	0.9115	0.9004	0.8979	0.9172	0.8870
		(0.0411)	(0.0554)	(0.0591)	(0.0473)	(0.0585)	(0.0639)	(0.0752)
Time (minutes)		130.29	0.078	0.033	275.01	6.59	0.079	11.31
		(19.73)	(0.0041)	(0.0020)	(15.23)	(2.52)	(0.0047)	(5.71)

[a] Mean and standard deviations for 50 simulations. For columns 1–6, the observed data consist of 1000 buses for 20 periods. For columns 7 and 8, the intercept (θ_0) is allowed to vary over time and the data consist of 2000 buses for 10 periods. See the text and the Supplemental Material for additional details.

- Entrants pay startup cost to enter market, but not incumbents.
- Startup cost transforms entrant into incumbent next period.
- Declining to compete in any given period is tantamount to exit.
- When a firm exits another firm potentially enters next period.
- There are two sources of dynamics in this model:
  1. An entrant depreciates startup cost over its anticipated lifetime.
  2. Becoming an incumbent reduces the probability of other firms entering the market, and hence increases expected profits.

- Each market has a permanent market characteristic, denoted by x_1, common to each player within the market and constant over time, but differing independently across markets, with equal probabilities on support $\{1, \ldots, 10\}$.
- The number of firm exits in the previous period is also common knowledge to the market, and this variable is indicated by:

$$x_{2t} \equiv \sum_{h=1}^{I} d_{1,t-1}^{(h)}$$

- This variable is a useful predictor for the number of firms that will compete in the current period.
- Intuitively, the more players paying entry costs, the lower the expected number of competitors.

- The unobserved state variable $s_t \in \{1, \ldots, 5\}$ follows a first order Markov chain.
- We assume that the probability of the unobserved variable remaining unchanged in successive periods is fixed at some $\pi \in (0, 1)$, and that if the state does change, any other state is equally likely to occur with probability $(1 - \pi)/4$.
- We generated also price data on each market, denoted by w_t, with the equation:

$$w_t = \alpha_0 + \alpha_1 x + \alpha_2 s_t + \alpha_3 \sum_{h=1}^{I} d_{1t}^{(h)} + \eta_t$$

where η_t is distributed as a standard normal disturbance independently across markets and periods, revealed to each market after the entry and exit decisions are made.

- The flow payoff of an active firm i in period t, net of private information $\epsilon_{2t}^{(i)}$ is modeled as:

$$U_2\left(x_t^{(i)}, s_t^{(i)}, d_t^{(-i)}\right) = \theta_0 + \theta_1 x + \theta_2 s_t + \theta_3 \sum_{h=1}^{I} d_{1t}^{(h)} + \theta_4 d_{1,t-1}^{(i)}$$

- We normalize exit utility as $U_1\left(x_t^{(i)}, s_t^{(i)}, d_t^{(-i)}\right) = 0$
- We assume $\epsilon_{jt}^{(i)}$ is distributed as Type 1 Extreme Value.
- The number of firms in each market in our experiment is 6.
- We simulated data for $3,000$ markets, and set $\beta = 0.9$.
- Starting at an initial date with 6 entrants in the market, we ran the simulations forward for twenty periods.

6. Two Monte Carlo Exercises

Table 2 of Arcidiacono and Miller (2011, page 1862)

	DGP (1)	s_t Observed (2)	Ignore s_t (3)	CCP Model (4)	CCP Data (5)	Two-Stage (6)	No Prices (7)
Profit parameters							
θ_0 (intercept)	0	0.0207	−0.8627	0.0073	0.0126	−0.0251	−0.0086
		(0.0779)	(0.0511)	(0.0812)	(0.0997)	(0.1013)	(0.1083)
θ_1 (obs. state)	0.05	−0.0505	−0.0118	−0.0500	−0.0502	−0.0487	−0.0495
		(0.0028)	(0.0014)	(0.0029)	(0.0041)	(0.0039)	(0.0038)
θ_2 (unobs. state)	0.25	0.2529		0.2502	0.2503	0.2456	0.2477
		(0.0080)		(0.0123)	(0.0148)	(0.0148)	(0.0158)
θ_3 (no. of competitors)	−0.2	−0.2061	0.1081	−0.2019	−0.2029	−0.1926	−0.1971
		(0.0207)	(0.0115)	(0.0218)	(0.0278)	(0.0270)	(0.0294)
θ_4 (entry cost)	−1.5	−1.4992	−1.5715	−1.5014	−1.4992	−1.4995	−1.5007
		(0.0131)	(0.0133)	(0.0116)	(0.0133)	(0.0133)	(0.0139)
Price parameters							
α_0 (intercept)	7	6.9973	6.6571	6.9991	6.9952	6.9946	
		(0.0296)	(0.0281)	(0.0369)	(0.0333)	(0.0335)	
α_1 (obs. state)	−0.1	−0.0998	−0.0754	−0.0995	−0.0996	−0.0996	
		(0.0023)	(0.0025)	(0.0028)	(0.0028)	(0.0028)	
α_2 (unobs. state)	0.3	0.2996		0.2982	0.2993	0.2987	
		(0.0045)		(0.0119)	(0.0117)	(0.0116)	
α_3 (no. of competitors)	−0.4	−0.3995	−0.2211	−0.3994	−0.3989	−0.3984	
		(0.0061)	(0.0051)	(0.0087)	(0.0088)	(0.0089)	
π (persistence of unobs. state)	0.7			0.7002	0.7030	0.7032	0.7007
				(0.0122)	(0.0146)	(0.0146)	(0.0184)
Time (minutes)		0.1354	0.1078	21.54	27.30	15.37	16.92
		(0.0047)	(0.0010)	(1.5278)	(1.9160)	(0.8003)	(1.6467)

[a]Mean and standard deviations for 100 simulations. Observed data consist of 3000 markets for 10 periods with 6 firms in each market. In column 7, the CCP's are updated with the model. See the text and the Supplemental Material for additional details.