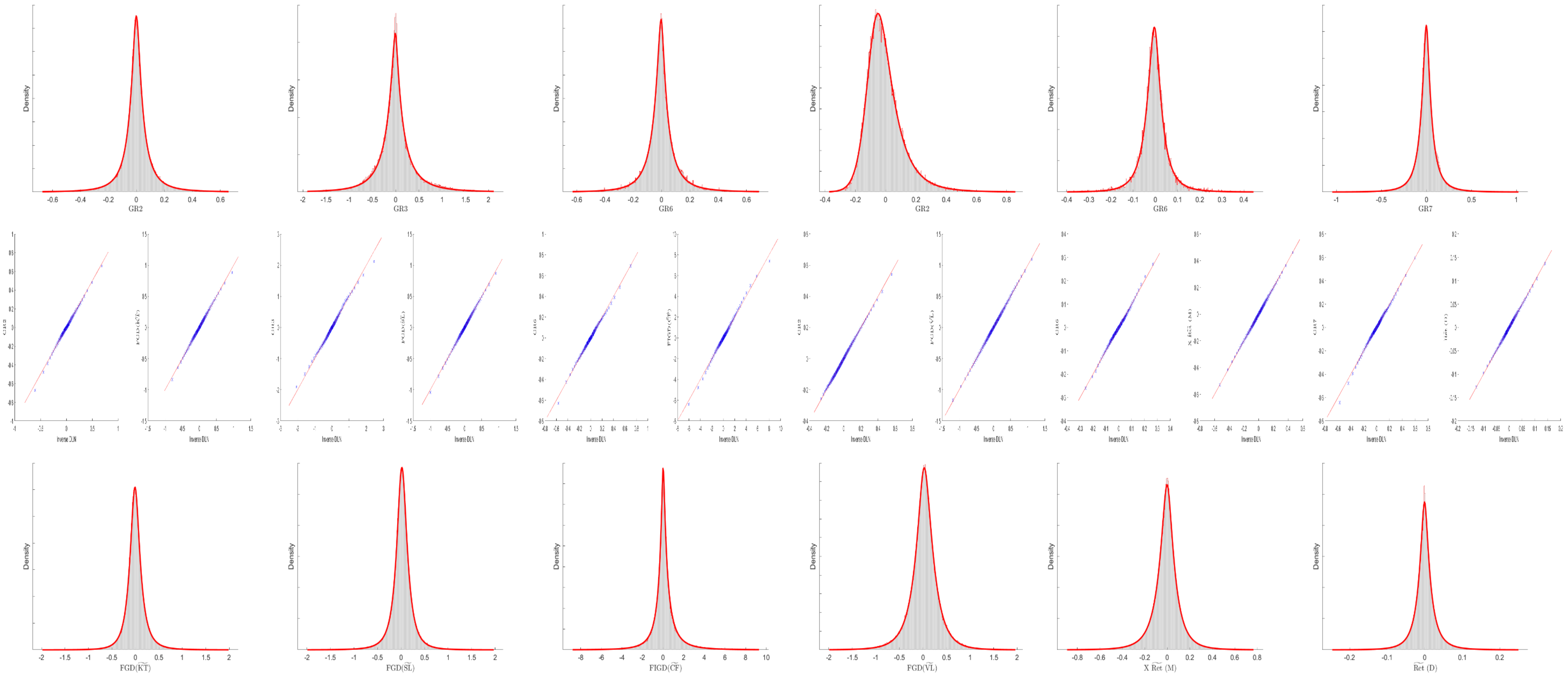
Growth and Differences of Log-Normals

Robert Parham

UVA McINTIRE

What do these all have in common?



Grey: data dist., red line: fitted diff-log-N, middle: q-q plots; Why, and how does it help us write models? (prod func)

Normal

What's so normal about **The Normal**?

$$Y = \lim_{M\to\infty} \frac{1}{M} \sum_{i=1}^{M} X_i \sim \mathcal{N}$$

$$\mathcal{N} + \mathcal{N} \sim \mathcal{N}$$

$$\mathcal{N} - \mathcal{N} \sim \mathcal{N}$$

The "brown" of nature (maximum entropy dist.) e.g. AR(1); Closed under addition and subtraction

Log-Normal

What if we **multiply** rather than add?

$$Y = \lim_{M \to \infty} \left(\prod_{i=1}^{M} X_i \right)^{\frac{1}{M}} \sim \log\text{-}\mathcal{N}$$

$$\log\text{-}\mathcal{N} \times \log\text{-}\mathcal{N} \sim \log\text{-}\mathcal{N} \qquad \log\text{-}\mathcal{N} + \log\text{-}\mathcal{N} \sim\sim \log\text{-}\mathcal{N}$$

$$\log\text{-}\mathcal{N} \div \log\text{-}\mathcal{N} \sim \log\text{-}\mathcal{N} \qquad \log\text{-}\mathcal{N} - \log\text{-}\mathcal{N} \sim \,?$$

Growth: percents don't add, percents multiply; Gibrat (1931), Roy(1950); Closure; What is the distribution of logN-logN?

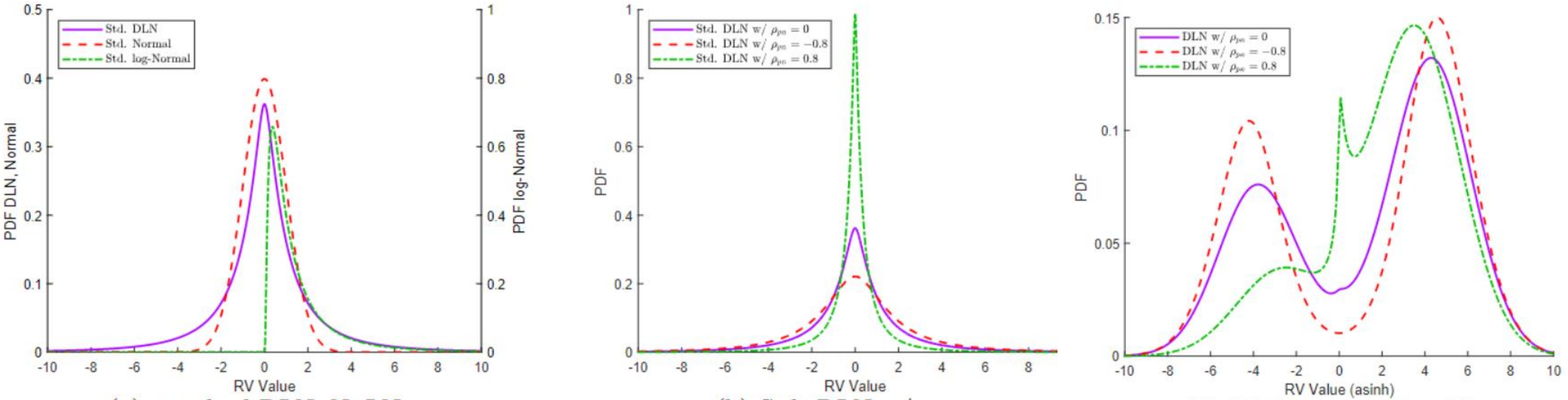
A word from our sponsor*

“There are only two ways to make money: increase sales and decrease costs.”
-- Fred DeLuca (founder of Subway)

* Not really sponsoring.

Difference of Log-Normals

$$W = Y_p - Y_n = \exp(X_p) - \exp(X_n) \quad \text{with} \quad \boldsymbol{X} = (X_p, X_n)^T \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

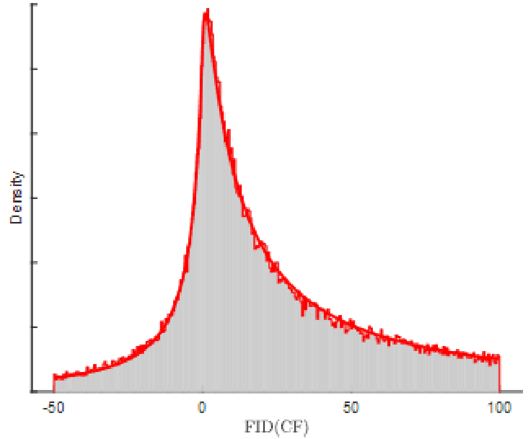


(a) standard DLN, N, LN (b) Std. DLN w/ corrs (d) DLN w/ corrs (asinh)

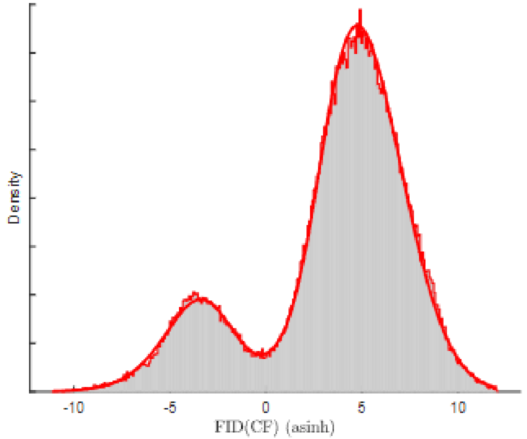
Every marginal benefit has a marginal cost
and they're both multiplicative

Not used anywhere in the sciences, not characterized, just sitting there in plain sight

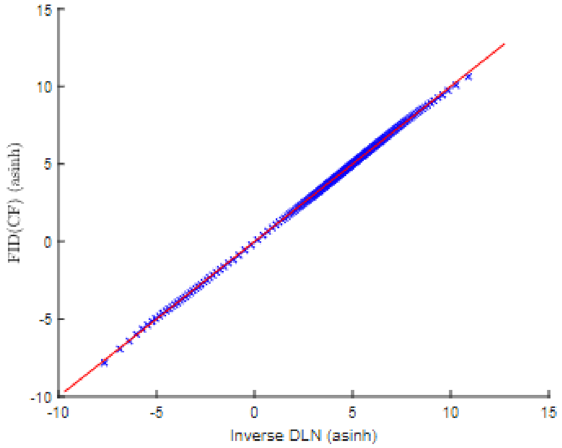
Example: Firm cashflows



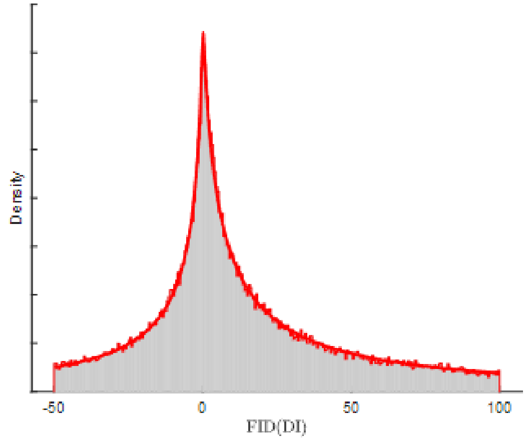
(a) cashflows w/ DLN



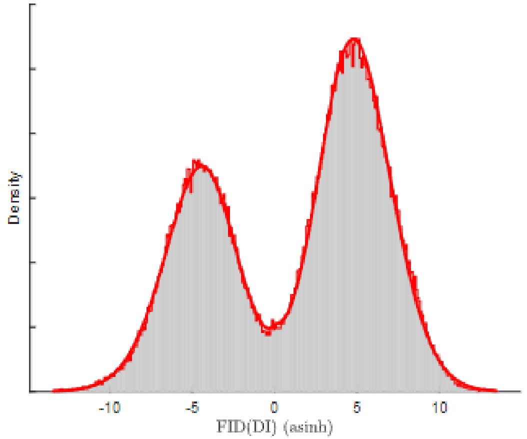
(b) cashflows w/ DLN



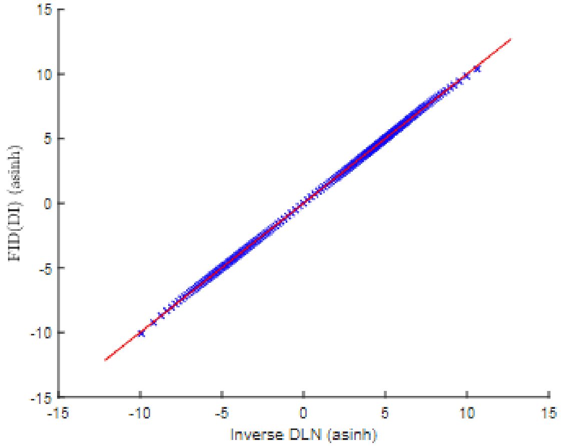
(c) q-q cashflows vs. DLN



(d) dispen. w/ DLN



(e) dispen. w/ DLN



(f) q-q dispen. vs. DLN

The fundamental sources and uses equation of the firm: SALES – EXPENSES = INCOME = DISPENSATIONS + INVESTMENT; “double log”

Production function

$$Y_{LL} = \prod_{i=1}^{M} X_i^{\beta_i} = \exp\left(\boldsymbol{x} \cdot \boldsymbol{\beta}\right)$$

$$Y_{DLL} = Y_p - Y_n = \exp\left(\boldsymbol{x} \cdot \boldsymbol{\beta}_p\right) - \exp\left(\boldsymbol{x} \cdot \boldsymbol{\beta}_n\right)$$

$$Y_{DLL} = 2 \cdot \exp\left(\lambda\right) \cdot \sinh\left(\tau\right)$$

$$\lambda = \boldsymbol{x} \cdot \frac{\beta_p + \beta_n}{2} = \log\left(\sqrt{Y_p \cdot Y_n}\right) = \log(\sqrt{\text{Sales} \cdot \text{Expenses}})$$

$$\tau = \boldsymbol{x} \cdot \frac{\beta_p - \beta_n}{2} = \log\left(\sqrt{Y_p / Y_n}\right) = \log(\sqrt{\text{Sales}/\text{Expenses}})$$

Cartesian to Polar in Hyperbolic Geometry; Lamb = scale, Tau = efficiency. Income mostly depends on Lamb, sign of income solely on Tau. No previous mention of Tau. SL=exp(L+T)

Production function cool stuff

$$Y_p = \exp\left(\lambda + \tau\right)$$

$$Y_n = \exp\left(\lambda - \tau\right)$$

$$\frac{dY_t/dt}{|Y_t|} = \mathrm{sgn}(\tau_t) \cdot \left[\frac{d\lambda_t}{dt} + \frac{d\tau_t}{dt} \cdot \frac{1}{\tanh\left(\tau_t\right)}\right] \approx \mathrm{sgn}(\tau_t) \cdot \left[(\lambda_{t+1} - \lambda_t) + \frac{\tau_{t+1} - \tau_t}{\tau_t}\right]$$

$$\frac{\partial Y_{DLL}}{\partial X_i} \Big/ \frac{Y_{DLL}}{X_i} = \frac{\beta_{p,i} \cdot Y_p - \beta_{n,i} \cdot Y_n}{Y_{DLL}} = \frac{\beta_{p,i} \cdot Y_p - \beta_{n,i} \cdot Y_n}{Y_p - Y_n}$$

Lam and Tau are intuitive (in logs); Growth is heavy-tailed when Tau ~ 0; Elasticities are no longer constant but rather “value weighted”

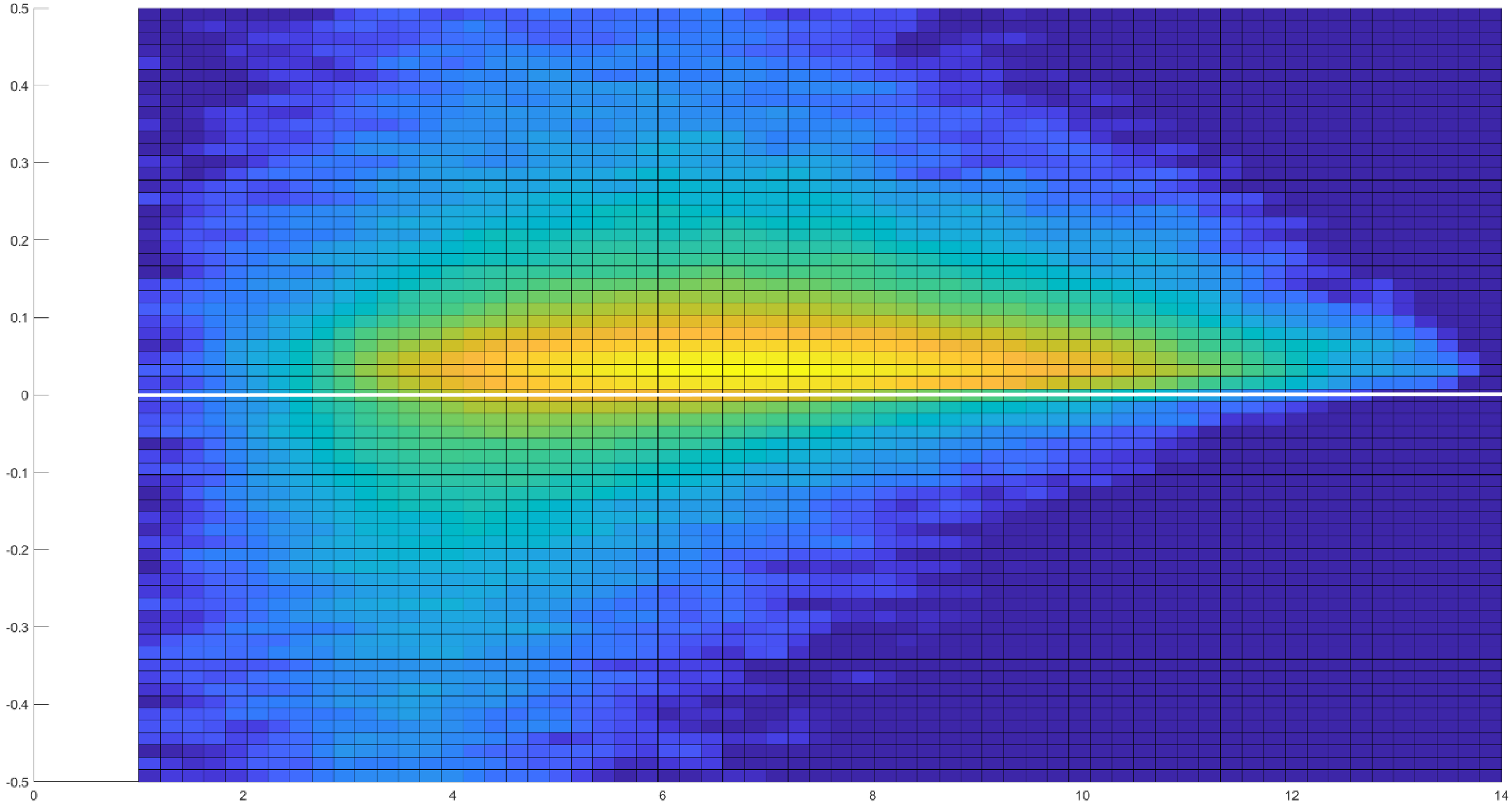
Summary

Gibrat: size is approx. log-Normal
Here: growth is approx. difference-of-log-Normals

Predicted by a sensible production function and confirmed in data
Example prod. func. for firms, cities, and populations in paper

Riddle: A firm has $100 in sales and $90 in expenses.
What is **income growth** if both sales and expenses increase by 10%?
What if sales increase by 10% but expenses decrease by 10%?
What if sales decrease by 10% and expenses increase by 10%?

Sunrise



(a) Firm scale and efficiency heat-map

Lam-Tau heatmap for all yearly firm obs 1970-2019