Plan for the day

- Show how to frame continuous time dynamic discrete choice models in a way that parallels discrete time dynamic discrete choice models.
- Show that estimation via conditional choice probabilities (CCP's) is in some ways easier in continuous time than in discrete time.
- Provide a structural link to reduced form hazards.

Basic idea

- Standard continuous time models have hazard rates that dictate when events occur.
- These hazard rates are functions of observed characteristics, $\lambda(x)$.
- We're going to decompose this hazard rate into two parts:
  1. A move arrival, λ,
  2. And a probability of making a decision, $\sigma(x)$.
- The hazard rate is then $\lambda\sigma(x)$.
- Framing of the value functions will then be similar to Hotz and Miller (1993) and Arcidiacono and Miller (2011)

Single Agent Framework

- Time is continuous, $t \in [0, \infty)$
- Finite state space, $x \in \mathcal{X}$
- Two competing Poisson processes can change the state:
  1. q_{kl} is the rate of moving from state k to state l
  2. λ is the rate of move opportunities for the agent
- When the agent has the right to move, chooses j from $\{1, \cdots, J\}$.
- Instantaneous payoff for taking action j in state k is $\psi_{jk} + \epsilon_j$
- Flow payoff for being in state k is u_k. Over a period of length τ, the payoff is:

$$\int_0^\tau e^{-\rho t} u_k dt$$

From discrete time to continuous time

- $V(k)$ gives the present value of lifetime utility given initial state k from behaving optimally in the future.
- Let $l(j,k)$ give the state conditional on making choice j in state k
- Can be expressed as:

$$V_k = u_k\Delta t + \frac{\sum_{l\neq k} q_{kl}\Delta t}{1+\rho\Delta t} V_l + \frac{\lambda\Delta t}{1+\rho\Delta t} E\max_j\left\{\psi_{jk} + \epsilon_j + V_{l(j,k)}\right\} + \frac{1-\sum_{l\neq k} q_{kl}\Delta t - \lambda\Delta t}{1+\rho\Delta t} V_k$$

- Moving all the V_k terms on the right hand side to the left hand side yields changes the lefthand side to:

$$\frac{\rho\Delta t + \sum_{l\neq k} q_{kl}\Delta t + \lambda\Delta t}{1+\rho\Delta t} V_k$$

Continuous time value function

- Divide both sides by Δt
- Taking Δt to zero and solving for V_k yields:

$$V_k = \frac{u_k + \sum_{l \neq k} q_{kl} V_l + \lambda E \max_j \left\{ \psi_{jk} + \epsilon_j + V_{l(j,k)} \right\}}{\rho + \lambda + \sum_{l \neq k} q_{kl}}$$

- Given u_k, q_{kl}, ψ_{jk}, λ, and the distribution of the ϵ's, could solve for the value functions using a fixed point algorithm.

Choice Probabilities

- Optimal decision rule, $\delta(k,\epsilon)$, solves:

$$\delta(k,\epsilon) = \arg\max_j \psi_{jk} + \epsilon_j + V_{l(j,k)}$$

- Choice probabilities integrate out over the ϵ's

$$\sigma_{jk} = Pr[\delta(k,\epsilon) = j|k]$$

- In the case where ϵ is Type I extreme value, the choice probabilities are:

$$\sigma_{jk} = \frac{\exp\left(\psi_{jk} + V_{l(j,k)}\right)}{\sum_{j'} \exp\left(\psi_{j'k} + V_{l(j',k)}\right)}$$

Eliminating the *Emax* term 1

Applying Lemma 1 of Arcidiacono and Miller (2011), we can express the *Emax* term as a function of an instantaneous payoff, a value function, and a function of conditional choice probabilities:

Proposition (2)

There is a function $\Gamma^2(j', \sigma_{\cdot k})$ *such that, for all* $j' \in \mathcal{A}$

$$E \max_j \left\{ \psi_{jk} + \varepsilon_j + V_{l(j,k)} \right\} = V_{l(j',k)} + \psi_{j'k} + \Gamma^2(j', \sigma_{\cdot k})$$

Eliminating the *Emax* term 2

When there is an option to do nothing ($j = 0$ and no instantaneous costs), we can express V_k as:

$$\begin{aligned} V_k &= \frac{u_k + \sum_{l \neq k} q_{kl} V_l + \lambda \Gamma^2(0, \sigma_{.k}) + \lambda V_k}{\rho + \lambda + \sum_{l \neq k} q_{kl}} \\ &= \frac{u_k + \sum_{l \neq k} q_{kl} V_l + \lambda \Gamma^2(0, \sigma_{.k})}{\rho + \sum_{l \neq k} q_{kl}} \end{aligned}$$

Towards eliminating remaining rhs value functions

Now want to link the remaining right-hand-side value functions to the value function on the left hand side.
Applying Proposition 1 of Hotz and Miller (1993), we can again link value functions using conditional choice probabilities:

Proposition (1)

There is a function $\Gamma^1(j, j', \sigma_{\cdot k})$ such that, for all $\{j, j'\} \in \mathcal{A}$ such that:

$$V_{l(j,k)} = V_{l(j',k)} + \psi_{j'k} - \psi_{jk} + \Gamma^1(j, j', \sigma_{\cdot k})$$

Eliminating remaining value functions

By successively linking value functions to other value functions, we can sometimes find a chain such that the remaining value functions on the right hand side can be expressed in terms of V_k and conditional choice probabilities.

Definition

A state k^* is *attainable from state k* if there exists a sequence of actions from k that result in state k^*.

Proposition (3)

If for all l such that $q_{kl} > 0$ there exists a k^* that is attainable and is also attainable from k, then there exists a function $\Gamma^3(k, \psi, Q,)$ such that:

$$\rho V_k = u_k + \Gamma^3(k, \psi, Q) \tag{1}$$

Linking Functions with Type 1 extreme value errors

- The forms of these functions are often very simple.
- In the multinomial logit case, we have:

$$\frac{\sigma_{jk}}{\sigma_{j'k}} = \frac{\exp\left[\psi_{jk} + V_{l(j,k)}\right]}{\exp\left[\psi_{j'k} + V_{l(j',k)}\right]}$$

implying:

$$V_{l(j,k)} = V_{l(j',k)} + \psi_{j'k} - \psi_{jk} + \ln(\sigma_{jk}) - \ln(\sigma_{j'k})$$

- Further, the *Emax* function is just:

$$E \max_j \left\{\psi_{jk} + \varepsilon_j + V_{l(j,k)}\right\} = V_{l(j',k)} + \psi_{j'k} - \ln(\sigma_{j'k})$$

Example: Slow Adjustments

- Suppose when nature moves the state can increase between 1 and 10 units.
- When the agent moves, can decrease the state by 1 or do nothing.
- Cost of decreasing the state is c
- Value function is then:

$$V_k = \frac{u_k + \sum_{l=k+1}^{k+10} q_{kl} V_l + \lambda E \max\left\{c + \epsilon_{-1} + V_{k-1}, \epsilon_0 + V_k\right\}}{\rho + \lambda + \sum_{l=k+1}^{k+10} q_{kl}}$$

- Goal is then to make substitutions with CCP's such that all the value functions on the right hand side are eliminated.

Eliminating Value Functions

- Suppose we have logit errors. The expectation on the right hand side then has a closed form solution:

$$V_k = \frac{u_k + \sum_{l=k+1}^{k+10} q_{kl} V_l + \lambda \ln\left[\exp(c + V_{k-1}) + \exp(V_k)\right] + \lambda\gamma}{\rho + \lambda + \sum_{l=k+1} q_{kl}}$$

- Note that:

$$\sigma_{0k} = \frac{\exp(V_k)}{\exp(c + V_{k-1}) + \exp(V_k)}$$

implying:

$$\begin{aligned} V_k &= \frac{u_k + \sum_{l=k+1}^{k+10} q_{kl} V_l - \lambda \ln(\sigma_{0k}) + \lambda V_k + \lambda\gamma}{\rho + \lambda + \sum_{l=k+1}^{k+10} q_{kl}} \\ &= \frac{u_k + \sum_{l=k+1}^{k+10} q_{kl} V_l - \lambda \ln(\sigma_{0k}) + \lambda\gamma}{\rho + \sum_{l=k+1}^{k+10} q_{kl}} \end{aligned}$$

Eliminating Value Functions 2

- Now want to express V_{k+1} through V_{k+10} as functions of V_k plus the adjustment cost, c.
- For $l \in \{k+1, \cdots, k+10\}$, we have:

$$\begin{aligned} V_l &= V_{l-1} + c + \ln(\sigma_{0l}) - \ln(\sigma_{-1l}) \\ &= V_k + (l-k)c + \sum_{l'=k+1}^{l} \left[\ln(\sigma_{0l'}) - \ln(\sigma_{-1l'})\right] \end{aligned}$$

- Substituting in for V_l and rearranging yields:

$$\begin{aligned} \rho V_k &= u_k - \lambda \ln\left(\sigma_{0k}\right) + \lambda\gamma \\ &\quad + \sum_{l=k+1}^{k+10} q_{kl} \left((l-k)c + \sum_{l'=k+1}^{l} \left[\ln(\sigma_{0l'}) - \ln(\sigma_{-1l'})\right] \right) \end{aligned}$$

Choice Probabilities

- The choice probabilities are then simple functions of the flow payoffs, instantaneous payoffs, and transition probabilities:

$$\sigma_{-1k} = \frac{\exp(V_{k-1} + c)}{\exp(V_k) + \exp(V_{k-1} + c)}$$

- Note that if the probability of moving up a set number of states doesn't depend on k, many of the terms in V_k will cancel out with terms in V_{k-1} (choice probabilities only depend on differences in payoffs).

Example: Single Agent Renewal Model

- Consider a bus replacement problem where k denotes mileage.
- Mileage increases 1 at rate q_k.
- Choices are to replace the engine (-1) which resets the mileage or do nothing (0). The cost to replacing the engine is c.
- The value function can then be expressed as:

$$V_k = \frac{u_k - \lambda \ln(\sigma_{0k}) + \lambda\gamma + q_k V_{k+1}}{\rho + q_k}$$

CCP Framework

- We can express V_{k+1} and V_k as:

$$\begin{aligned} V_{k+1} &= V_0 + c + \ln(\sigma_{0k+1}) - \ln(\sigma_{-1k+1}) \\ V_k &= V_0 + c + \ln(\sigma_{0k}) - \ln(\sigma_{-1k}) \end{aligned}$$

implying that:

$$V_{k+1} = V_k + \ln(\sigma_{0k+1}) - \ln(\sigma_{-1k+1}) - \ln(\sigma_{0k}) + \ln(\sigma_{-1k})$$

- Substituting this expression for V_{k+1} and solving for V_k yields:

$$\begin{aligned} \rho V_k = \; & u_k - \lambda \ln\left(\sigma_{0k}\right) + \lambda\gamma \\ & + q_k \left[\ln(\sigma_{0k+1}) - \ln(\sigma_{-1k+1}) - \ln(\sigma_{0k}) + \ln(\sigma_{-1k}))\right] \end{aligned}$$

Forming the likelihood

- Denote the probability of doing nothing in state k by σ_{0k}.
- With Poisson arrivals, the probability of the next state change occurring before t is:

$$G(t;q,\lambda,k) = 1 - \exp\left[-t\left(\sum_{l\neq k} q_{kl} + \lambda\sum_{j\neq 0}\sigma_{jk}\right)\right]$$

- Differentiating with respect to t gives the density of the next state change at t:

$$g(t;q,\lambda,k) = \left(\sum_{l\neq k} q_{kl} + \lambda\sum_{j\neq 0}\sigma_{jk}\right)\exp\left[-t\left(\sum_{l\neq k} q_{kl} + \sum_{j\neq 0}\sigma_{jk}\right)\right]$$

- Conditional on a state change occurring, the probability it is because the agent took action j is:

$$\frac{\lambda\sigma_{jk}}{\sum_{l\neq k} q_{kl} + \lambda\sum_{j\neq 0}\sigma_{jk}}$$

Forming the likelihood 2

- The joint likelihood of the next stage change occurring at t and being the result of action j is then:

$$g(t, j; q, \lambda, k) = \lambda \sigma_{jk} \exp \left[-t \left(\sum_{l \neq q} q_{kl} + \lambda \sum_{j \neq 0} \sigma_{jk} \right) \right]$$

- The likelihood of the data is then the product of the densities for each event, taking into account the fact that nothing occurred between the last event and the end of the period.

Simple case

- When data are observed in continuous time, including when the individual chose do nothing, then the q's and the λ's can be estimated in a first stage.
- For λ, simply count the number of decisions and divide by the sample period.
- For q_{kl}, count the number of times the state went from k to l and then divide this by the amount of time the state was in k.
- For the structural payoffs, treat it just like discrete time: estimate a logit or multinomial logit where now we're plugging in for the value function itself.

Time Aggregation

- **Problem**: What if the data is recorded quarterly (or yearly)?
  - Issue: we don't see actual sequence of moves
- **Solution**: Use transition matrix $P(t)$, which captures all that could happen over period t

Transition Matrix

- Let $P_{kl}(t) = \Pr(X_{t+s} = l \mid X_s = k)$ and $P(t) = (P_{kl}(t))$.
- $P(t)$ can be written in terms of the matrix exponential

$$P(t) = \mathrm{e}^{tQ} = \sum_{j=0}^{\infty} \frac{(tQ)^j}{j!}.$$

- There are many known algorithms for computing $P(t)$.
- Additional information can also be incorporated via simulation.

A Simulation Procedure

- We can decompose the Markov jump process into two components:
  - A state-independent Poisson process dictating when moves occur
  - A Markov chain with the probabilities of moving from one state to another conditional on a move opportunity
- The part of the Markov chain associated with the choices are just the standard discrete time likelihoods (the choice probabilities)
- We then integrate out over the number of moves in a period which is easy to do when the number of moves doesn't depend on the state.

Preview of search presentation

- We can relate these ideas to job search where there are unobserved preferences/signing bonuses/switching costs associated with job offers
- Extremely simplified version:
  1. Job offers arrive at rate λ_1 when employed, λ_2 when unemployed
  2. Exogenous seperation rate δ
  3. Discrete wages with f_w the probability of receiving w given an offer
  4. Pay a cost $c + \epsilon$ to switch jobs, with ϵ distributed logistic

Value function of employed

The value function associated with being employed at wage w:

$$\begin{aligned} V_w &= \frac{u(w) + \delta V_0 + \lambda_1 \sum_{w'} \ln\left(\exp(V_w) + \exp(V_{w'} + c)\right) f_{w'}}{\rho + \delta + \lambda_1} \\ &= \frac{u(w) + \delta V_0 - \lambda_1 \sum_{w'} \ln\left(1 - p_{ww'}\right) f_{w'}}{\rho + \delta} \end{aligned}$$

The probability of changing jobs conditional on an offer is:

$$p_{ww'} = \frac{\exp(V_{w'} - V_w + c)}{\exp(V_{w'} - V_w + c) + 1}$$

Value function of unemployed

The value function associated with being unemployed is:

$$V_0 = \frac{b + \lambda_0 \sum_{w'} \ln\left(\exp(V_0) + \exp(V_{w'})\right) f_{w'}}{\rho + \lambda_0}$$
$$= \frac{u(w) - \lambda_0 \sum_{w'} \ln\left(1 - p_{0w'}\right) f_{w'}}{\rho}$$

(unless unemployment benefits are seen, can't distinguish from

a switching cost)

The probability of accepting a job out of unemployment given an offer is:

$$p_{0w'} = \frac{\exp(V_{w'} - V_0)}{\exp(V_{w'} - V_0) + 1}$$

What is seen in the data

1. $h_{ww'}$, the hazard rate of moving from a job with wage w to a job with wage w';
2. $h_w(t)$, the hazard rate out of unemployment at time t to a job that pays w (assumed to be continuously differentiable);
3. δ, the hazard rate to unemployment.

Wage offers

- Note that the value functions of two jobs that pay the same are equal and the switching cost does not depend on w
- For same wage job changes, we then have $p_{ww} = p_{w'w'}$
- The hazard from a job that pays w to another that pays w can be written as $h_{ww} = \lambda p_{ww} f_w$
- We then have $h_{ww}/h_{w'w'} = f_w/f_{w'}$, implying

$$f_w = \frac{h_{ww}}{\sum_{w'} h_{w'w'}}$$