Graduate Macro Theory II: A Medium-Scale New Keynesian DSGE Model

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1 Introduction

This set of notes lays out, estimates, and studies a so-called "medium scale" New Keynesian DSGE model. These models are used extensively in central banks to conduct policy analysis. The pioneering medium scale DSGE models are Christiano, Eichenbaum, and Evans (2005, *JPE*) and Smets and Wouters (2007, *AER*).

At its core, the medium scale DSGE model is just a real business cycle model. But it incorporates numerous real and nominal frictions, and features several shocks. Relative to the simple New Keynesian models we have heretofore studied, the medium scale model includes capital accumulation. The features imbedded in the medium scale model I develop in these notes are:

1. Physical capital accumulation
2. Sticky prices
3. Sticky wages
4. Backward indexation of non-updated prices and wages
5. Habit formation in consumption
6. Investment adjustment costs
7. Variable capital utilization
8. A fixed cost of production
9. Monetary policy conducted according to a Taylor rule
10. The following shocks:
    (a) Productivity
    (b) Marginal efficiency of investment

(c) Government spending

(d) Monetary policy

(e) Intertemporal preference shock

(f) Intratemporal preference shock (labor supply shock)

One way to think about this model is that it is a “godfather model” in the sense that it essentially incorporates, all in one model, all of the features we’ve talked about in isolation over the course of the semester. The only features listed above which are new are backward price and wage indexation and a fixed cost of production. Everything else we’ve already seen.

Another feature of medium scale DSGE models is that the parameters of the model are typically estimated, usually via Bayesian maximum likelihood. This is in contrast to conventional calibration which targets moments external to the model (e.g. long run moments) or to simply picking parameters in an ad hoc way so as to be “reasonable.” I will not offer a full treatment of Bayesian estimation (which itself would be an independent course), but will only briefly discuss how it can be done in Dynare. After developing the model, I will focus mostly on properties of the estimated model and will point out some areas where the model as written is deficient.

2 The Model

The model features the following actors: a “labor packer” that combines heterogeneous labor inputs into a homogeneous labor input available to firms for production; households who consume, invest in physical capital, supply labor, make a capital utilization decision, lease capital services to firms, set wages according to the downward-sloping demand for their heterogeneous labor input, and accumulate bonds; a final good firm that bundles heterogeneous outputs of intermediate firms into a final output good; intermediate goods firms who use capital services and labor to produce heterogeneous output goods; a central bank which conducts monetary policy according to a Taylor rule; and a fiscal authority (i.e. government) which chooses some spending exogenously and finances this spending with a mix of lump sum taxes and debt.

A couple of notes on what is not in the model. First, I abstract from money altogether – i.e. the economy is “cashless.” I could bring money in by assuming households get utility from money in an additively separable way and it would not impact the equilibrium of the model. Second, I don’t model any usefulness of government spending. Similarly to money, I could assume that households get utility from government spending in an additively separable way and the equilibrium of the economy would not be affected. Third, I assume that all government finance is via lump sum taxes. This means that the economy exhibits Ricardian equivalence and the mix between lump sum taxes and bonds is indeterminate. I could easily write down the model with distortionary tax finance. Fourth, I ignore the zero lower bound on interest rates. Fully incorporating the non-linear constraint of the zero lower bound in this model would be challenging because the model features many state variables, but it is feasible (and relatively straightforward) to think about the implications of an interest rate peg by augmenting the Taylor rule with news shocks as we have previously discussed.

The subsections below lay out the decision problems and optimality conditions of the different actors in the model, and then discuss equilibrium and aggregation.

2.1 Labor Packer

In the model there are a continuum of households, indexed by by $l \in [0,1]$. They supply differentiated labor input to a "labor packing firm" (or a union, if you like), who then bundles the differentiated labor input into a homogeneous labor input available for production, which I denote by $N_{d,t}$. The bundling technology is:

$$N_{d,t} = \left(\int_0^1 N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w}} dl \right)^{\frac{\epsilon_w}{\epsilon_w - 1}}, \quad \epsilon_w > 1 \tag{1}$$

The parameter ϵ_w measures the elasticity of substitution among different types of labor, and is assumed greater than one so that different types of labor are substitutes. The profit maximization problem of the labor packer is:

$$\max_{N_t(l)} \quad W_t \left(\int_0^1 N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w}} dl \right)^{\frac{\epsilon_w}{\epsilon_w - 1}} - \int_0^1 W_t(l) N_t(l) dl$$

Here W_t is the aggregate nominal wage, while $W_t(l)$ denotes the nominal wage of labor of variety l. The first order condition gives a downward-sloping demand for each variety of labor:

$$N_t(l) = \left(\frac{W_t(l)}{W_t} \right)^{-\epsilon_w} N_{d,t} \tag{2}$$

Demand for labor of variety l depends on aggregate labor demand and the relative wage of variety l labor. Since it depends on the relative wage, this specification is the same in terms of the real wage or the nominal wage ratio.

Define the aggregate wage index as:

$$W_t N_{d,t} = \int_0^1 W_t(l) N_t(l) dl = \int_0^1 W_t(l)^{1-\epsilon_w} W_t^{\epsilon_w} N_{d,t} dl$$

This can be simplified to be:

$$W_t^{1-\epsilon_w} = \int_0^1 W_t(l)^{1-\epsilon_w} dl \tag{3}$$

Define aggregate labor supply as equaling the sum of labor by variety:

$$N_t = \int_0^1 N_t(l) dl \tag{4}$$

Plugging in the demand for each variety of labor, we can write this as:

$$N_t = \int_0^1 \left(\frac{W_t(l)}{W_t}\right)^{-\epsilon_w} N_{d,t} dl$$

Simplifying, we get:

$$N_t = N_{d,t} v_t^w \tag{5}$$

Where:

$$v_t^w = \int_0^1 \left(\frac{W_t(l)}{W_t}\right)^{-\epsilon_w} dl \tag{6}$$

This is a measure of wage dispersion across different varieties of labor, in a way analogous to price dispersion. If it is greater than one, then aggregate labor used in production will be smaller than the aggregate labor supplied by households.

2.2 Households

There are a continuum of households indexed by $l \in [0, 1]$. These households are subject to a Calvo-style wage-setting friction, which means they will have heterogeneous wages and hence heterogeneous labor input and therefore income. We assume that there exist state-contingent securities which insure them against idiosyncratic wage risk. If utility is separable between consumption and labor (which I assume here), then households will be identical along all margins except labor supply and wage, and I will suppress dependence on l with the exception of those two margins.

The household owns the capital stock. It can choose how intensively to utilize that capital stock, and then leases capital services (the product of physical capital and utilization) to firms. The household can also choose how many bonds to accumulate. The household problem can be written:

$$\max_{C_t, N_t(l), W_t(l), u_t, K_{t+1}, B_{t+1}} \quad E_t \sum_{s=0}^{\infty} \beta^s \nu_{t+s} \left\{ \ln\left(C_{t+s} - bC_{t+s-1}\right) - \psi_{t+s} \frac{N_{t+s}(l)^{1+\chi}}{1+\chi} \right\}$$

s.t.

$$P_t C_t + P_t I_t + B_{t+1} \leq W_t(l) N_t(l) + R_t^n K_t u_t + \Pi_t^n - P_t T_t + (1 + i_{t-1}) B_t$$

$$K_{t+1} = Z_t \left[1 - \frac{\kappa}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2\right] I_t + (1 - \delta(u_t)) K_t$$

$$N_t(l) = \left(\frac{W_t(l)}{W_t}\right)^{-\epsilon_w} N_{d,t}$$

$$\delta(u_t) = \delta_0 + \delta_1(u_t - 1) + \frac{\delta_2}{2}(u_t - 1)^2$$

Allow me to explain what each parameter and variable measures. β is the discount factor, b is a measure of internal habit formation, and χ is the inverse Frisch elasticity of labor supply. κ

measures an investment adjustment cost. $\delta(u_t)$ is a function mapping utilization of capital into the depreciation rate, with parameters δ_0, δ_1, and δ_2. ν_t is an exogenous intertemporal preference shock. ψ_t is an exogenous intratemporal preference shock affecting the relative valuation of utility from consumption and disutility from labor supply. P_t is the nominal price of goods. C_t is consumption, I_t investment, $N_t(l)$ labor input, and K_t physical capital. Z_t is an exogenous marginal efficiency of investment shock. R_t^n is a nominal rental rate on capital services, Π_t^n is nominal profit distributed from firms, and T_t is a real lump sum tax/transfer from the government. i_t is the nominal interest rate and B_t is the stock of nominal bonds with which a household enters a period.

Form a Lagrangian for the part of the problem not related to wage-setting:

$$\mathcal{L} = E_t \sum_{s=0}^{\infty} \beta^s \Big\{ \nu_{t+s} \ln \left(C_{t+s} - bC_{t+s-1}\right) + \dots$$

$$\lambda_{t+s}^n \left(W_{t+s}(l)N_{t+s}(l) + R_{t+s}^n K_{t+s} u_{t+s} + \Pi_{t+s}^n - P_{t+s}T_{t+S} + (1+i_{t+s-1})B_{t+S} - P_{t+s}C_{t+S} - P_{t+s}I_{t+s} - B_{t+s+1}\right) +$$

$$+ \mu_{t+s} \left(Z_{t+S} \left[1 - \frac{\kappa}{2} \left(\frac{I_{t+s}}{I_{t+s-1}} - 1 \right)^2 \right] I_{t+s} + (1 - \delta(u_{t+s}))K_{t+s} - K_{t+s+1} \right) \Big\}$$

The FOC are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Leftrightarrow P_t \lambda_t^n = \frac{\nu_t}{C_t - bC_{t-1}} - \beta b E_t \frac{\nu_{t+1}}{C_{t+1} - bC_t} \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial u_t} = 0 \Leftrightarrow \lambda_t^n R_t^n K_t = \mu_t \delta'(u_t) K_t \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = 0 \Leftrightarrow \lambda_t^n = \beta E_t \lambda_{t+1}^n (1 + i_t) \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial I_t} = 0 \Leftrightarrow P_t \lambda_t^n = \mu_t Z_t \left(1 - \frac{\kappa}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right)^2 - \kappa \left(\frac{I_t}{I_{t-1}} - 1 \right) \frac{I_t}{I_{t-1}} \right) + \dots$$

$$\beta E_t \mu_{t+1} Z_{t+1} \kappa \left(\frac{I_{t+1}}{I_t} - 1 \right) \left(\frac{I_{t+1}}{I_t} \right)^2 \tag{10}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \Leftrightarrow \mu_t = \beta E_t \left(\lambda_{t+1}^n R_{t+1}^n u_{t+1} + \mu_{t+1}(1 - \delta(u_{t+1})) \right) \tag{11}$$

To get everything in real terms, define $\lambda_t = P_t \lambda_t^n$. Then we can write these conditions as:

$$\lambda_t = \frac{\nu_t}{C_t - bC_{t-1}} - \beta b E_t \frac{\nu_{t+1}}{C_{t+1} - bC_t} \tag{12}$$

$$\lambda_t R_t = \mu_t \delta'(u_t) \tag{13}$$

$$\lambda_t = \beta E_t (1 + i_t)(1 + \pi_{t+1})^{-1} \tag{14}$$

$$\lambda_t = \mu_t Z_t \left(1 - \frac{\kappa}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2 - \kappa\left(\frac{I_t}{I_{t-1}} - 1\right)\frac{I_t}{I_{t-1}}\right) + \beta E_t \mu_{t+1} Z_{t+1} \kappa \left(\frac{I_{t+1}}{I_t} - 1\right)\left(\frac{I_{t+1}}{I_t}\right)^2 \quad (15)$$

$$\mu_t = \beta E_t \left(\lambda_{t+1} R_{t+1} u_{t+1} + \mu_{t+1}(1 - \delta(u_{t+1}))\right) \quad (16)$$

In these conditions, $R_t = R_t^n / P_t$ and $1 + \pi_{t+1} = \frac{P_{t+1}}{P_t}$.

Next, consider the parts of the Lagrangian related to wage-setting. Each period, a randomly selected fraction of households, $1 - \phi_w$, is given the opportunity to adjust their wage. This means that ϕ_w of households cannot adjust their nominal wage. We assume that non-updating households can index their nominal wage to lagged inflation at rate $\zeta_w \in [0, 1]$. In other words, the period t wage of a household not getting to update is $(1+\pi_{t-1})^{\zeta_w} W_{t-1}(l)$, where W_{t-1} is the wage it charged the period before. Formally, the nominal wage of a household l in period t is:

$$W_t(l) = \begin{cases} W_t^{\#}(l) & \text{if } W_t(l) \text{ chosen optimally} \\ (1+\pi_{t-1})^{\zeta_w} W_{t-1}(l) & \text{otherwise} \end{cases} \quad (17)$$

Now, consider the problem of a household given the opportunity to adjust its wage in period t. With probability θ_w^s, the wage it chooses today will still be relevant in $t + s$. Because of indexing, the wage it gets to charge in each subsequent period if it does not get to adjust will be equal to:

$$\begin{aligned} W_{t+1}(l) &= (1+\pi_t)^{\zeta_w} W_t^{\#}(l) \\ W_{t+2}(l) &= ((1+\pi_{t+1})(1+\pi_t))^{\zeta_w} W_t^{\#}(l) \\ &\vdots \\ W_{t+s}(l) &= \left(\prod_{j=0}^{s-1}(1+\pi_{t+j})\right)^{\zeta_w} W_t^{\#}(l) \end{aligned}$$

Where:

$$\prod_{j=0}^{s-1}(1+\pi_{t+j}) = (1+\pi_t)(1+\pi_{t+1})(1+\pi_{t+2}) \times \cdots \times (1+\pi_{t+s-1})$$

In terms of price levels, this can be written:

$$\prod_{j=0}^{s-1}(1+\pi_{t+j}) = \frac{P_t}{P_{t-1}} \frac{P_{t+1}}{P_t} \frac{P_{t+2}}{P_{t+1}} \times \ldots \frac{P_{t+s-1}}{P_{t+s-1}} = \frac{P_{t+s-1}}{P_{t-1}}$$

In other words, all the price terms except the last and the previous cancel out. Hence, we can express the non-updated wage in period $t + s$ as:

$$W_{t+s}(l) = \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\zeta_w} W_t^{\#}(l)$$

Let's re-create the part of the Lagrangian related to wage-setting for a household given the opportunity to adjust its wage. To do this, plug in the demand for labor to express the problem just in terms of choosing the wage. The household will discount period $t+s$ by $(\beta\phi_w)^s$, reflecting the fact that the probability that a wage chosen in period t is still in effect in $t+s$ is ϕ_w^s. The problem is:

$$\mathcal{L} = E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \Bigg\{ -\nu_{t+s}\psi_{t+s}(1+\chi)^{-1} \left(\frac{\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\zeta_w} W_t^{\#}(l)}{W_{t+s}} \right)^{-\epsilon_w(1+\chi)} N_{d,t+s}^{1+\chi} \\ + \lambda_{t+s}^n \left(\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\zeta_w} W_t^{\#}(l) \right)^{1-\epsilon_w} W_{t+s}^{\epsilon_w} N_{d,t+s}$$

The FOC is:

$$\frac{\partial \mathcal{L}}{\partial W_t^{\#}(l)} = 0 \Leftrightarrow \epsilon_w W_t^{\#}(l)^{-\epsilon_w(1+\chi)-1} E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \nu_{t+s}\psi_{t+s} \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\zeta_w\epsilon_w(1+\chi)} W_{t+s}^{\epsilon_w(1+\chi)} N_{d,t+s}^{1+\chi} \\ + (1-\epsilon_w) W_t^{\#}(l)^{-\epsilon_w} E_t E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \lambda_{t+s}^n \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\zeta_w(1-\epsilon_w)} W_{t+s}^{\epsilon_w} N_{d,t+s} = 0$$

Or simplifying a little bit:

$$(\epsilon_w - 1) W_t^{\#}(l)^{-\epsilon_w} E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \lambda_{t+s}^n \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\zeta_w(1-\epsilon_w)} W_{t+s}^{\epsilon_w} N_{d,t+s} = \\ \epsilon_w W_t^{\#}(l)^{-\epsilon_w(1+\chi)-1} E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \nu_{t+s}\psi_{t+s} \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\zeta_w\epsilon_w(1+\chi)} W_{t+s}^{\epsilon_w(1+\chi)} N_{d,t+s}^{1+\chi}$$

Simplifying further, we have:

$$W_t^{\#1+\epsilon_w\chi} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{H_{1,t}}{H_{2,t}} \tag{18}$$

Here:

$$H_{1,t} = E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \nu_{t+s}\psi_{t+s} \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\zeta_w\epsilon_w(1+\chi)} W_{t+s}^{\epsilon_w(1+\chi)} N_{d,t+s}^{1+\chi} \tag{19}$$

$$H_{2,t} = E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \lambda_{t+s}^n \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\zeta_w(1-\epsilon_w)} W_{t+s}^{\epsilon_w} N_{d,t+s} \tag{20}$$

In writing this I have eliminated dependence on l since nothing on the right hand side depends on l – in other words, all updating households will update to a common reset wage. Now, to make this useable, we have to do some algebraic manipulations. First, let's write everything in real terms. Define $w_t = \frac{W_t}{P_t}$. Write $H_{1,t}$ and $H_{2,t}$ in terms of this. We get:

$$H_{1,t} = E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \nu_{t+s}\psi_{t+s} \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\zeta_w \epsilon_w (1+\chi)} w_{t+s}^{\epsilon_w(1+\chi)} P_{t+s}^{\epsilon_w(1+\chi)} N_{d,t+s}^{1+\chi}$$

$$H_{2,t} = E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \lambda_{t+s}^n \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\zeta_w (1-\epsilon_w)} w_{t+s}^{\epsilon_w} P_{t+s}^{\epsilon_w} N_{d,t+s}$$

Write $H_{2,t}$ in terms of the real value of the multiplier on the flow budget constraint by multiplying and dividing by P_t as:

$$H_{2,t} = E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \lambda_{t+s} \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\zeta_w (1-\epsilon_w)} w_{t+s}^{\epsilon_w} P_{t+s}^{\epsilon_w - 1} N_{d,t+s}$$

Now write $H_{1,t}$ and $H_{2,t}$ recursively:

$$H_{1,t} = \nu_t \psi_t w_t^{\epsilon_w(1+\chi)} P_t^{\epsilon_w(1+\chi)} N_{d,t}^{1+\chi} + \beta\theta_w E_t \left(\frac{P_t}{P_{t-1}}\right)^{-\zeta_w \epsilon_w (1+\chi)} H_{1,t+1}$$

$$H_{2,t} = \lambda_t w_t^{\epsilon_w} P_t^{\epsilon_w - 1} N_{d,t} + \phi_w \beta E_t \left(\frac{P_t}{P_{t-1}}\right)^{\zeta_w (1-\epsilon_w)} H_{2,t+1}$$

In terms of inflation rates:

$$H_{1,t} = \nu_t \psi_t w_t^{\epsilon_w(1+\chi)} P_t^{\epsilon_w(1+\chi)} N_{d,t}^{1+\chi} + \beta\theta_w E_t (1+\pi_t)^{-\zeta_w \epsilon_w (1+\chi)} H_{1,t+1}$$

$$H_{2,t} = \lambda_t w_t^{\epsilon_w} P_t^{\epsilon_w - 1} N_{d,t} + \phi_w \beta E_t (1+\pi_t)^{\zeta_w (1-\epsilon_w)} H_{2,t+1}$$

Now we need to get rid of price levels. Define $h_{1,t} = H_{1,t}/P_t^{\epsilon_w(1+\chi)}$. We get:

$$h_{1,t} = \nu_t \psi_t w_t^{\epsilon_w(1+\chi)} N_{d,t}^{1+\chi} + \beta\theta_w E_t (1+\pi_t)^{-\zeta_w \epsilon_w (1+\chi)} \frac{H_{1,t+1}}{P_t^{\epsilon_w(1+\chi)}}$$

Playing around with the last term:

$$h_{1,t} = \nu_t \psi_t w_t^{\epsilon_w(1+\chi)} N_{d,t}^{1+\chi} + \beta\theta_w E_t (1+\pi_t)^{-\zeta_w \epsilon_w (1+\chi)} \frac{H_{1,t+1}}{P_{t+1}^{\epsilon_w(1+\chi)}} \frac{P_{t+1}^{\epsilon_w(1+\chi)}}{P_t^{\epsilon_w(1+\chi)}}$$

Or:

$$h_{1,t} = \nu_t \psi_t w_t^{\epsilon_w(1+\chi)} N_{d,t}^{1+\chi} + \beta\theta_w (1+\pi_t)^{-\zeta_w \epsilon_w (1+\chi)} E_t (1+\pi_{t+1})^{\epsilon_w(1+\chi)} \widehat{h}_{1,t+1} \tag{21}$$

Now define $h_{2,t} = H_{2,t}/P_t^{\epsilon_w - 1}$. We get:

$$h_{2,t} = \lambda_t w_t^{\epsilon_w} N_{d,t} + \phi_w \beta E_t (1+\pi_t)^{\zeta_w (1-\epsilon_w)} \frac{H_{2,t+1}}{P_t^{\epsilon_w - 1}}$$

Playing with the last term:

$$h_{2,t} = \lambda_t w_t^{\epsilon_w} N_{d,t} + \phi_w \beta E_t \left(1+\pi_t\right)^{\zeta_w(1-\epsilon_w)} \frac{H_{2,t+1}}{P_{t+1}^{\epsilon_w - 1}} \frac{P_{t+1}^{\epsilon_w - 1}}{P_t^{\epsilon_w - 1}}$$

Or:

$$h_{2,t} = \lambda_t w_t^{\epsilon_w} N_{d,t} + \phi_w \beta (1+\pi_t)^{\zeta_w(1-\epsilon_w)} E_t (1+\pi_{t+1})^{\epsilon_w - 1} h_{2,t+1} \tag{22}$$

Hence, the reset wage expression is:

$$W_t^{\# 1+\epsilon_w \chi} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{\left(H_{1,t}/P_t^{\epsilon_w(1+\chi)}\right) P_t^{\epsilon_w(1+\chi)}}{\left(H_{2,t}/P_t^{\epsilon_w - 1}\right) P_t^{\epsilon_w - 1}}$$

Or:

$$W_t^{\# 1+\epsilon_w \chi} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{h_{1,t}}{h_{2,t}} P_t^{1+\epsilon_w \chi}$$

Define $w_t^{\#} = W_t^{\#}/P_t$ as the real reset wage. Then this condition can be written:

$$w_t^{\# 1+\epsilon_w \chi} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{h_{1,t}}{h_{2,t}} \tag{23}$$

Equations (21)-(23) characterize the optimal wage-setting process. For the purposes of calculating steady states, it is computationally easier if we write this condition in terms of relative reset wages on the right hand side. To that end, define $\widehat{h}_{1,t} = h_{1,t}/w_t^{\# \epsilon_w(1+\chi)}$ and $\widehat{h}_{2,t} = h_{2,t}/w_t^{\# \epsilon_w}$. Let's start playing around:

$$\widehat{h}_{1,t} = \nu_t \psi_t \left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w(1+\chi)} N_{d,t}^{1+\chi} + \beta \phi_w \left(1+\pi_t\right)^{-\zeta_w \epsilon_w(1+\chi)} E_t (1+\pi_{t+1})^{\epsilon_w(1+\chi)} \frac{h_{1,t+1}}{w_t^{\# \epsilon_w(1+\chi)}}$$

Playing with the last term:

$$\widehat{h}_{1,t} = \nu_t \psi_t \left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w(1+\chi)} N_{d,t}^{1+\chi} + \beta \phi_w \left(1+\pi_t\right)^{-\zeta_w \epsilon_w(1+\chi)} E_t (1+\pi_{t+1})^{\epsilon_w(1+\chi)} \frac{h_{1,t+1}}{w_{t+1}^{\# \epsilon_w(1+\chi)}} \frac{w_{t+1}^{\# \epsilon_w(1+\chi)}}{w_t^{\# \epsilon_w(1+\chi)}}$$

Or finally:

$$\widehat{h}_{1,t} = \nu_t \psi_t \left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w(1+\chi)} N_{d,t}^{1+\chi} + \beta \phi_w \left(1+\pi_t\right)^{-\zeta_w \epsilon_w(1+\chi)} E_t (1+\pi_{t+1})^{\epsilon_w(1+\chi)} \left(\frac{w_{t+1}^{\#}}{w_t^{\#}}\right)^{\epsilon_w(1+\chi)} h_{1,t+1} \tag{24}$$

Now process similarly for the other auxiliary variable:

$$\widehat{h}_{2,t} = \lambda_t \left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w} N_{d,t} + \phi_w \beta (1+\pi_t)^{\zeta_w(1-\epsilon_w)} E_t (1+\pi_{t+1})^{\epsilon_w - 1} \frac{h_{2,t+1}}{w_t^{\#\epsilon_w}}$$

Playing with the last term:

$$\widehat{h}_{2,t} = \lambda_t \left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w} N_{d,t} + \phi_w \beta (1+\pi_t)^{\zeta_w(1-\epsilon_w)} E_t (1+\pi_{t+1})^{\epsilon_w - 1} \frac{h_{2,t+1}}{w_{t+1}^{\#\epsilon_w}} \frac{w_{t+1}^{\#\epsilon_w}}{w_t^{\#\epsilon_w}}$$

Or finally:

$$\widehat{h}_{2,t} = \lambda_t \left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w} N_{d,t} + \phi_w \beta (1+\pi_t)^{\zeta_w(1-\epsilon_w)} E_t (1+\pi_{t+1})^{\epsilon_w - 1} \left(\frac{w_{t+1}^{\#}}{w_t^{\#}}\right)^{\epsilon_w} \widehat{h}_{2,t+1} \tag{25}$$

Then we can write the wage-setting equation as:

$$w_t^{\#1+\epsilon_w \chi} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{h_{1,t}/w_t^{\#\epsilon_w(1+\chi)}}{h_{2,t}/w_t^{\#\epsilon_w}} \frac{w_t^{\#\epsilon_w(1+\chi)}}{w_t^{\#\epsilon_w}}$$

Or more compactly:

$$w_t^{\#} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{\widehat{h}_{1,t}}{\widehat{h}_{2,t}} \tag{26}$$

Equations (24)-(26), as well as (12)-(16) characterize the household side of the model.

2.3 Final Good Firm

There are a continuum of intermediate goods firms indexed by $j \in [0,1]$ producing differentiated output $Y_t(j)$ at prices $P_t(j)$. These intermediates are bundled into a final output via:

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon_p - 1}{\epsilon_p}} dj\right)^{\frac{\epsilon_p}{\epsilon_p - 1}}, \quad \epsilon_p > 1 \tag{27}$$

$\epsilon_p > 1$ is the elasticity of substitution among goods. The profit maximization problem is:

$$\max_{Y_t(j)} \quad P_t \left(\int_0^1 Y_t(j)^{\frac{\epsilon_p - 1}{\epsilon_p}} dj\right)^{\frac{\epsilon_p}{\epsilon_p - 1}} - \int_0^1 P_t(j) Y_t(j) dj$$

The first order condition is a demand curve for each intermediate:

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t \tag{28}$$

The aggregate price index is given by:

$$P_t^{1-\epsilon_p} = \int_0^1 P_t(j)^{1-\epsilon_p} dj \tag{29}$$

2.4 Intermediate Goods Firms

Intermediate goods production is given by:

$$Y_t(j) = A_t \widehat{K}_t(j)^{\alpha} N_{d,t}(j)^{1-\alpha} - F \tag{30}$$

$F \geq 0$ is a fixed cost. It can be set to ensure that profits are zero in steady state, which rules out entry. It will have an effect on the equilibrium, but we could also write the model without the fixed cost. We require production to be non-negative. $\widehat{K}_t(j)$ is capital services (the product of utilization and physical capital leased from households).

Firms may not be able to adjust their price in a given period, but they will always choose inputs to minimize total cost each period. The cost-minimization problem is:

$$\min_{\widehat{K}_t(j), N_{d,t}} \quad W_t N_{d,t}(j) + R_t^n \widehat{K}_t$$

$$\text{s.t.}$$

$$A_t \widehat{K}_t(j)^{\alpha} N_{d,t}(j)^{1-\alpha} - F \geq \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon_p} Y_t$$

A Lagrangian is:

$$\mathcal{L} = -W_t N_{d,t}(j) - R_t^n \widehat{K}_t(j) + \varphi_t(j) \left(A_t \widehat{K}_t(j)^{\alpha} N_{d,t}(j)^{1-\alpha} - F - \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon_p} Y_t \right)$$

The FOC are:

$$\frac{\partial \mathcal{L}}{\partial N_{d,t}} = 0 \Leftrightarrow W_t = (1-\alpha) \varphi_t(j) A_t \left(\frac{\widehat{K}_t(j)}{N_{d,t}(j)} \right)^{\alpha}$$

$$\frac{\partial \mathcal{L}}{\partial \widehat{K}_t} = 0 \Leftrightarrow R_t^n = (1-\alpha) \varphi_t(j) A_t \left(\frac{\widehat{K}_t(j)}{N_{d,t}(j)} \right)^{\alpha-1}$$

Divide these two conditions:

$$\frac{W_t}{R_t^n} = \frac{1-\alpha}{\alpha} \left(\frac{\widehat{K}_t(j)}{N_{d,t}(j)} \right)$$

This is an important equation. Since firms face the same factor prices, this equation says that they must all have the same capital/labor ratio, which will in turn be equal to the aggregate ratio of factors. Writing the factor price ratio in real terms (by dividing numerator and denominator by P_t), and dropping the j subscripts, we get:

$$\frac{w_t}{R_t} = \frac{1-\alpha}{\alpha} \left(\frac{\widehat{K}_t}{N_{d,t}} \right) \tag{31}$$

But if firms face the same factor prices and hire capital and labor in the same ratio, and all face the same productivity shock, then they have the same marginal cost. Re-arranging the FOC for labor, we get:

$$mc_t = \frac{w_t}{(1-\alpha)A_t\left(\frac{\widehat{K}_t}{N_{d,t}}\right)^{\alpha}} \tag{32}$$

Here I have written real marginal cost as $mc_t = \varphi_t/P_t$ and written it in terms of the real wage. Real marginal cost is the same across firms.

Now, let's consider the price-setting problem of a firm. First, nominal profits are:

$$\Pi_t^n(j) = P_t(j)Y_t(j) - W_t N_{d,t}(j) - R_t^n \widehat{K}_t(j)$$

Now, from the FOC, we have that:

$$W_t N_{d,t}(j) = (1-\alpha)\varphi_t A_t \widehat{K}_{d,t}(j)^{\alpha} N_{d,t}(j)^{1-\alpha}$$

$$R_t^n \widehat{K}_t(j) = \alpha\varphi_t A_t \widehat{K}_{d,t}(j)^{\alpha} N_{d,t}(j)^{1-\alpha}$$

Here I have imposed the result that all firms have the same nominal marginal cost, φ_t. The sum of these is then:

$$W_t N_{d,t}(j) + R_t^n \widehat{K}_t(j) = \varphi_t A_t \widehat{K}_{d,t}(j)^{\alpha} N_{d,t}(j)^{1-\alpha}$$

Now, if you use (30), this can be written as:

$$W_t N_{d,t}(j) + R_t^n \widehat{K}_t(j) = \varphi_t Y_t(j) + \varphi_t F$$

Hence, we can write the profit function as:

$$\Pi_t^n(j) = P_t(j)Y_t(j) - \varphi_t Y_t(j) - \varphi_t F$$

Firms are subject to Calvo price-setting. Each period, there is a fixed probability of $1-\phi_p$ that a firm can adjust its price. If it cannot adjust, it can partially index its price to lagged aggregate inflation at $\zeta_p \in [0,1]$. That is, the price a firm can charge in period t is:

$$P_t(j) = \begin{cases} P_t^{\#}(j) & \text{if } P_t(j) \text{ chosen optimally} \\ (1+\pi_{t-1})^{\zeta_p} P_{t-1}(j) & \text{otherwise} \end{cases} \tag{33}$$

Consider the price charged by a firm who can adjust in period t in subsequent periods under this setup. It would be:

$$P_{t+1}(j) = (1+\pi_t)^{\zeta_p} P_t^{\#}(j)$$
$$P_{t+2}(j) = ((1+\pi_{t+1})(1+\pi_t))^{\zeta_p} P_t^{\#}(j)$$
$$\vdots$$
$$P_{t+s}(j) = \left(\prod_{g=0}^{s-1} (1+\pi_{t+g}) \right)^{\zeta_p} P_t^{\#}(j)$$

As in the wage-setting exercise, we can write this product of gross inflation rates as:

$$P_{t+s}(j) = \left(\frac{P_{t+s-1}}{P_{t-1}} \right)^{\zeta_p} P_t^{\#}(j)$$

In other words, this is the price that a firm who updates in period t will charge in period $t+s$ if it has not been given the opportunity to adjust again by $t+s$.

Given this, consider the profit maximization problem of a firm given the opportunity to adjust its price in period t. This problem is now dynamic because the price it chooses in period t will in expectation be relevant in future periods. Firms will discount profit flows via the nominal stochastic discount factor, equal to $E_t \beta^s \frac{\lambda_{t+s}^n}{\lambda_t^n}$, as well as the probability that a price chosen today is still in effect in the future, ϕ_p^s. Plugging in the demand for goods, the problem is:

$$\max_{P_t^{\#}(j)} \quad E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s \frac{\lambda_{t+s}^n}{\lambda_t^n} \Bigg[\left(\frac{P_{t+s-1}}{P_{t-1}} \right)^{\zeta_p(1-\epsilon_p)} P_t^{\#}(j)^{1-\epsilon_p} P_{t+s}^{\epsilon_p} Y_{t+s} - \varphi_{t+s} \left(\frac{P_{t+s-1}}{P_{t-1}} \right)^{-\zeta_p \epsilon_p} P_t^{\#}(j)^{-\epsilon_p} P_{t+s}^{\epsilon_p} Y_{t+s} - \varphi_{t+s} F \Bigg]$$

The first order condition is:

$$(1-\epsilon_p) P_t^{\#}(j)^{-\epsilon_p} E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s \frac{\lambda_{t+s}^n}{\lambda_t^n} \left(\frac{P_{t+s-1}}{P_{t-1}} \right)^{\zeta_p(1-\epsilon_p)} P_{t+s}^{\epsilon_p} Y_{t+s} + \epsilon_p P_t^{\#}(j)^{-\epsilon_p - 1} E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s \frac{\lambda_{t+s}^n}{\lambda_t^n} \varphi_{t+s} \left(\frac{P_{t+s-1}}{P_{t-1}} \right)^{-\zeta_p \epsilon_p} P_{t+s}^{\epsilon_p} Y_{t+s} = 0$$

Re-arranging, we get:

$$P_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1} \frac{E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s \lambda_{t+s}^n \varphi_{t+s} \left(\frac{P_{t+s-1}}{P_{t-1}} \right)^{-\zeta_p \epsilon_p} P_{t+s}^{\epsilon_p} Y_{t+s}}{E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s \lambda_{t+s}^n \left(\frac{P_{t+s-1}}{P_{t-1}} \right)^{\zeta_p(1-\epsilon_p)} P_{t+s}^{\epsilon_p} Y_{t+s}}$$

In doing this the λ_t^n terms in the denominators cancel out. Let's write the right hand side in terms of real marginal utility, noting that $\lambda_{t+s} = P_{t+s}\lambda_{t+s}^n$. Hence, multiply and divide each term in the sums in both numerator and denominator by P_{t+s}. This will leave terms like $\lambda_{t+s}mc_{t+s}$ in the numerator and $P_{t+s}^{\epsilon_p-1}$ in the denominator:

$$P_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1} \frac{E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s \lambda_{t+s} mc_{t+s} \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\zeta_p \epsilon_p} P_{t+s}^{\epsilon_p} Y_{t+s}}{E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s \lambda_{t+s} \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\zeta_p(1-\epsilon_p)} P_{t+s}^{\epsilon_p-1} Y_{t+s}}$$

Now write the price-setting condition recursively as:

$$P_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1} \frac{X_{1,t}}{X_{2,t}}$$

Where:

$$X_{1,t} = \lambda_t mc_t P_t^{\epsilon_p} Y_t + \phi_p \beta \left(\frac{P_t}{P_{t-1}}\right)^{-\zeta_p \epsilon_p} E_t X_{1,t+1}$$

$$X_{2,t} = \lambda_t P_t^{\epsilon_p - 1} Y_t + \phi_p \beta \left(\frac{P_t}{P_{t-1}}\right)^{\zeta_p(1-\epsilon_p)} E_t X_{2,t+1}$$

Or written in terms of inflation rates:

$$X_{1,t} = \lambda_t mc_t P_t^{\epsilon_p} Y_t + \phi_p \beta (1 + \pi_t)^{-\zeta_p \epsilon_p} E_t X_{1,t+1}$$

$$X_{2,t} = \lambda_t P_t^{\epsilon_p - 1} Y_t + \phi_p \beta (1 + \pi_t)^{\zeta_p(1-\epsilon_p)} E_t X_{2,t+1}$$

Now we need to write these in real terms. Define $x_{1,t} = X_{1,t}/P_t^{\epsilon_p}$. We get:

$$x_{1,t} = \lambda_t mc_t Y_t + \phi_p \beta (1 + \pi_t)^{-\zeta_p \epsilon_p} \frac{E_t X_{1,t+1}}{P_t^{\epsilon_p}}$$

Now play with the last term:

$$x_{1,t} = \lambda_t mc_t Y_t + \phi_p \beta (1 + \pi_t)^{-\zeta_p \epsilon_p} \frac{E_t X_{1,t+1}}{P_{t+1}^{\epsilon_p}} \frac{P_{t+1}^{\epsilon_p}}{P_t^{\epsilon_p}}$$

Or:

$$x_{1,t} = \lambda_t mc_t Y_t + \phi_p \beta (1 + \pi_t)^{-\zeta_p \epsilon_p} E_t (1 + \pi_{t+1})^{\epsilon_p} x_{1,t+1} \tag{34}$$

Now, define $x_{2,t} = X_{2,t}/P_t^{\epsilon_p - 1}$. We get:

$$x_{2,t} = \lambda_t Y_t + \phi_p \beta (1 + \pi_t)^{\zeta_p(1-\epsilon_p)} \frac{E_t X_{2,t+1}}{P_t^{\epsilon_p - 1}}$$

Playing with the last term:

$$x_{2,t} = \lambda_t Y_t + \phi_p \beta \left(1+\pi_t\right)^{\zeta_p(1-\epsilon_p)} \frac{E_t X_{2,t+1}}{P_{t+1}^{\epsilon_p - 1}} \frac{P_{t+1}^{\epsilon_p - 1}}{P_t^{\epsilon_p - 1}}$$

Or:

$$x_{2,t} = \lambda_t Y_t + \phi_p \beta \left(1+\pi_t\right)^{\zeta_p(1-\epsilon_p)} E_t (1+\pi_{t+1})^{\epsilon_p - 1} x_{2,t+1} \tag{35}$$

In terms of the reset price equation, we can write it as:

$$P_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1} \frac{X_{1,t}/P_t^{\epsilon_p}}{X_{2,t}/P_t^{\epsilon_p - 1}} \frac{P_t^{\epsilon_p}}{P_t^{\epsilon_p - 1}}$$

Or:

$$P_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1} \frac{x_{1,t}}{x_{2,t}} P_t$$

Define $1 + \pi_t^{\#} = \frac{P_t^{\#}}{P_{t-1}}$ as reset price inflation. This condition can then be written:

$$1 + \pi_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1} (1+\pi_t) \frac{x_{1,t}}{x_{2,t}} \tag{36}$$

Equations (36) plus (34)-(35) summarizing the optimal price-setting conditions.

2.5 Government

The government consumes an exogenous amount of output, G_t. This follows an AR(1) process in the log:

$$\ln G_t = (1 - \rho_G) \ln G + \rho_G \ln G_{t-1} + s_G \varepsilon_{G,t} \tag{37}$$

It is assumed that $0 \leq \rho_G < 1$. $\varepsilon_{G,t}$ is drawn from a standard normal distribution. s_G is the standard deviation of the shock.

The government raises revenue via lump sum taxes and issues debt:

$$P_t G_t + i_{t-1} D_t \leq P_t T_t + D_{t+1} - D_t \tag{38}$$

D_t is the stock of nominal debt with which the government enters the period. It owes interest i_{t-1} on that. It raises revenue from lump sum taxes, or can issue new debt, $D_{t+1} - D_t$, to finance its nominal expenditure.

Monetary policy is set according to the following partial adjustment Taylor rule:

$$i_t = (1 - \rho_i) i + \rho_i i_{t-1} + (1 - \rho_i) \big[\phi_\pi (\pi_t - \pi) + \phi_y (\ln Y_t - \ln Y_{t-1})\big] + s_i \varepsilon_{i,t} \tag{39}$$

I assume that $0 \leq \rho_i < 1$. i and π are the steady state interest rate and inflation rate, respectively. The coefficients ϕ_π and ϕ_y are positive and are assumed to be such that the economy is in the determinacy region. I write the Taylor rule as responding to output growth (implicitly relative to steady state growth, which is zero in the model) instead of an output gap. There is no closed form expression for the flexible price and wage level of output in the model with capital. I could solve for Y_t^f numerically but writing the Taylor rule in terms of output growth seems roughly consistent with the data as well and avoids this complication.

2.6 Equilibrium and Aggregation

Total profit in the economy is the integral of profit across intermediate goods firms:

$$\Pi_t^n = \int_0^1 \Pi_t^n(j)dj = \int_0^1 \left[P_t(j)Y_t(j) - W_t N_{d,t}(j) - R_t^n \widehat{K}_t(j)\right] dj$$

Now, let's play with this some. Break up the integral on the right hand side:

$$\Pi_t^n = \int_0^1 P_t(j)Y_t(j)dj - W_t \int_0^1 N_{d,t}(j)dj - R_t^n \int_0^1 \widehat{K}_t(j)dj$$

Now, market-clearing requires that $N_{d,t} = \int_0^1 N_{d,t}(j)$ and $u_t K_t = \int_0^1 \widehat{K}_t(j)dj$. Hence:

$$\Pi_t^n = \int_0^1 P_t(j)Y_t(j)dj - W_t N_{d,t} - R_t^n u_t K_t$$

Now, use the demand function for goods:

$$\Pi_t^n = \int_0^1 P_t(j)^{1-\epsilon_p} P_t^{\epsilon_p} Y_t dj - W_t N_{d,t} - R_t^n u_t K_t$$

Or:

$$\Pi_t^n = P_t^{\epsilon_p} Y_t \int_0^1 P_t(j)^{1-\epsilon_p} dj - W_t N_{d,t} - R_t^n u_t K_t$$

But since $P_t^{1-\epsilon_p} = \int_0^1 P_t(j)^{1-\epsilon_p} dj$, this simply becomes:

$$\Pi_t^n = P_t Y_t - W_t N_{d,t} - R_t^n u_t K_t \tag{40}$$

Now, integrate over household budget constraints. We get:

$$P_t C_t + P_t I_t + B_{t+1} = \int_0^w W_t(l) N_t(l) dl + R_t^n K_t u_t + \Pi_t^n - P_t T_t + (1 + i_{t-1}) B_t$$

Let's deal with the integral using the demand curve for labor:

$$P_t C_t + P_t I_t + B_{t+1} = \int_0^w W_t(l)^{1-\epsilon_w} W_t^{\epsilon_w} N_{d,t} dl + R_t^n K_t u_t + \Pi_t^n - P_t T_t + (1 + i_{t-1}) B_t$$

Or:

$$P_tC_t + P_tI_t + B_{t+1} = W_t^{\epsilon_w} N_{d,t} \int_0^w W_t(l)^{1-\epsilon_w} dl + R_t^n K_t u_t + \Pi_t^n - P_tT_t + (1+i_{t-1})B_t$$

But since $W_t^{1-\epsilon_w} = \int_0^1 W_t(l)^{1-\epsilon_w} dl$, this simply becomes:

$$P_tC_t + P_tI_t + B_{t+1} = W_tN_{d,t} + R_t^n K_t u_t + \Pi_t^n - P_tT_t + (1+i_{t-1})B_t$$

Now plug in the expression for profit we just derived above, and we get:

$$P_tC_t + P_tI_t + B_{t+1} = P_tY_t - P_tT_t + (1+i_{t-1})B_t$$

For bond market-clearing, we require that $B_t = D_t$ and $B_{t+1} = D_{t+1}$ (i.e. households hold the government bonds). Solving for P_tT_t from the government's budget constraint, we get:

$$P_tT_t = P_tG_t + (1+i_{t-1})D_t - D_{t+1}$$

Plugging this in to the integrated household budget constraint, we end up with the usual looking aggregate accounting identity:

$$Y_t = C_t + I_t + G_t \tag{41}$$

Now, let's derive an expression for the aggregate production function. Integrate over demand for intermediate goods, equating this to the intermediate goods production function:

$$\int_0^1 \left[A_t \widehat{K}_t(j)^\alpha N_t(j)^{1-\alpha} - F\right] dj = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t dj$$

Distributing the integral, and noting that all firms hire capital and labor in the same ratio (a result from above), we get:

$$A_t \left(\frac{\widehat{K}_t}{N_{d,t}}\right)^\alpha \int_0^1 N_{d,t}(j)dj - F = Y_t \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} dj$$

Using the labor market-clearing condition, and defining $v_t^p = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} dj$, we get:

$$A_t \widehat{K}_t^\alpha N_{d,t}^{1-\alpha} - F = Y_t v_t^p \tag{42}$$

Now, what is v_t^p? Here we make use of the Calvo assumption. $1-\phi_p$ of firms update to the same reset price, while ϕ_p can only partially index their price to whatever it is they charged in the last period. Splitting up the integral:

$$\int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} dj = \int_0^{1-\phi} \left(\frac{P_t^\#}{P_t}\right)^{-\epsilon_p} dj + \int_{1-\phi}^1 \left(\frac{(1+\pi_{t-1})^{\zeta_p} P_{t-1}(j)}{P_t}\right)^{-\epsilon_p} dj$$

We can write this as:

$$v_t^p = (1-\phi_p)\left(\frac{P_t^{\#}}{P_t}\right)^{-\epsilon_p} + \int_{1-\phi}^{1}(1+\pi_{t-1})^{-\zeta_p\epsilon_p}\left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon_p}\left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon_p} dj$$

This can be written:

$$v_t^p = (1-\phi_p)\left(\frac{P_t^{\#}}{P_t}\right)^{-\epsilon_p} + (1+\pi_{t-1})^{-\zeta_p\epsilon_p}(1+\pi_t)^{\epsilon_p}\int_{1-\phi}^{1}\left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon_p} dj$$

The first term can be written in terms of inflation rates by dividing numerator and denominator by P_{t-1}. The last term can be written:

$$v_t^p = (1-\phi_p)\left(\frac{1+\pi_t^{\#}}{1+\pi_t}\right)^{-\epsilon_p} + (1+\pi_{t-1})^{-\zeta_p\epsilon_p}(1+\pi_t)^{\epsilon_p}\phi\int_{0}^{1}\left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon_p} dj$$

The last operation follows from the fact that these firms are randomly chosen. But that term is just v_{t-1}^p. Hence:

$$v_t^p = (1-\phi_p)\left(\frac{1+\pi_t^{\#}}{1+\pi_t}\right)^{-\epsilon_p} + (1+\pi_{t-1})^{-\zeta_p\epsilon_p}(1+\pi_t)^{\epsilon_p}\phi v_{t-1}^p \tag{43}$$

What about the aggregate price index? Using features of Calvo pricing again, we have:

$$P_t^{1-\epsilon_p} = \int_0^{1-\phi} P_t^{\#1-\epsilon_p} dj + \int_{1-\phi}^{1}(1+\pi_{t-1})^{\zeta_p(1-\epsilon_p)}P_{t-1}(j)^{1-\epsilon_p} dj$$

This can be written:

$$P_t^{1-\epsilon_p} = (1-\phi_p)P_t^{\#1-\epsilon_p} + (1+\pi_{t-1})^{\zeta_p(1-\epsilon_p)}\phi\int_0^1 P_{t-1}(j)^{1-\epsilon_p}$$

But this last term is just $P_{t-1}^{1-\epsilon_p}$. Hence:

$$P_t^{1-\epsilon_p} = (1-\phi_p)P_t^{\#1-\epsilon_p} + (1+\pi_{t-1})^{\zeta_p(1-\epsilon_p)}\phi_p P_{t-1}^{1-\epsilon_p}$$

Now we need to write this in terms of inflation rates. So divide both sides by $P_{t-1}^{1-\epsilon_p}$. We get:

$$(1+\pi_t)^{1-\epsilon_p} = (1-\phi_p)(1+\pi_t^{\#})^{1-\epsilon_p} + (1+\pi_{t-1})^{\zeta_p(1-\epsilon_p)}\phi_p \tag{44}$$

What about the wage expression? Recall that we have:

$$W_t^{1-\epsilon_w} = \int_0^1 W_t(l)^{1-\epsilon_w} dl$$

Use properties of Calvo wage-setting to break up the right hand side:

$$W_t^{1-\epsilon_w} = \int_0^{1-\phi} W_t^{\#1-\epsilon_w} dl + \int_{1-\phi}^1 (1+\pi_{t-1})^{\zeta_w(1-\epsilon_w)} W_{t-1}(l)^{1-\epsilon_w} dl$$

Simplifying a bit, we have:

$$W_t^{1-\epsilon_w} = (1-\phi_w) W_t^{\#1-\epsilon_w} + (1+\pi_{t-1})^{\zeta_w(1-\epsilon_w)} \phi_w \int_0^1 W_{t-1}(l)^{1-\epsilon_w} dl$$

But the last term is just $W_{t-1}^{1-\epsilon_w}$. So we get:

$$W_t^{1-\epsilon_w} = (1-\phi_w) W_t^{\#1-\epsilon_w} + (1+\pi_{t-1})^{\zeta_w(1-\epsilon_w)} \phi_w W_{t-1}^{1-\epsilon_w}$$

Now, we want to write this in real terms. Divide both sides by $P_t^{1-\epsilon_w}$. We get:

$$w_t^{1-\epsilon_w} = (1-\phi_w) w_t^{\#1-\epsilon_w} + (1+\pi_{t-1})^{\zeta_w(1-\epsilon_w)} \phi_w \left(\frac{W_{t-1}}{P_t}\right)^{1-\epsilon_w}$$

Play with the last term:

$$w_t^{1-\epsilon_w} = (1-\phi_w) w_t^{\#1-\epsilon_w} + (1+\pi_{t-1})^{\zeta_w(1-\epsilon_w)} \phi_w \left(\frac{W_{t-1}}{P_{t-1}}\right)^{1-\epsilon_w} \left(\frac{P_{t-1}}{P_t}\right)^{1-\epsilon_w}$$

Or just:

$$w_t^{1-\epsilon_w} = (1-\phi_w) w_t^{\#1-\epsilon_w} + (1+\pi_{t-1})^{\zeta_w(1-\epsilon_w)} \phi_w (1+\pi_t)^{\epsilon_w - 1} w_{t-1}^{1-\epsilon_w} \tag{45}$$

2.7 Exogenous Processes

It is assumed that the productivity, A_t, marginal efficiency of investment, Z_t, and intertemporal preference shock, ν_t, follow AR(1) processes in the log with non-stochastic means normalized to unity (so zero in logs):

$$\ln A_t = \rho_A \ln A_{t-1} + s_A \varepsilon_{A,t} \tag{46}$$

$$\ln Z_t = \rho_Z \ln Z_{t-1} + s_Z \varepsilon_{Z,t} \tag{47}$$

$$\ln \nu_t = \rho_\nu \ln \nu_{t-1} + s_\nu \varepsilon_{\nu,t} \tag{48}$$

All the AR parameters are assumed to lie between 0 and 1 and the shocks are drawn from standard normal distributions. The intratemporal preference shock follows an AR(1) process in the log with non-stochastic mean of ψ:

$$\ln \psi_t = (1-\rho_\psi) \ln \psi + \rho_\psi \ln \psi_{t-1} + s_\psi \varepsilon_{\psi,t} \tag{49}$$

2.8 Full Set of Equilibrium Conditions

$$\lambda_t = \frac{\nu_t}{C_t - bC_{t-1}} - \beta b E_t \frac{\nu_{t+1}}{C_{t+1} - bC_t} \tag{50}$$

$$\lambda_t R_t = \mu_t \delta'(u_t) \tag{51}$$

$$\lambda_t = \beta E_t (1+i_t)(1+\pi_{t+1})^{-1} \tag{52}$$

$$\lambda_t = \mu_t Z_t \left(1 - \frac{\kappa}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right)^2 - \kappa \left(\frac{I_t}{I_{t-1}} - 1 \right) \frac{I_t}{I_{t-1}} \right) + \beta E_t \mu_{t+1} Z_{t+1} \kappa \left(\frac{I_{t+1}}{I_t} - 1 \right) \left(\frac{I_{t+1}}{I_t} \right)^2 \tag{53}$$

$$\mu_t = \beta E_t \left(\lambda_{t+1} R_{t+1} u_{t+1} + \mu_{t+1}(1 - \delta(u_{t+1})) \right) \tag{54}$$

$$\widehat{h}_{1,t} = \nu_t \psi_t \left(\frac{w_t}{w_t^\#} \right)^{\epsilon_w(1+\chi)} N_{d,t}^{1+\chi} + \phi_w \beta \left(1+\pi_t\right)^{-\zeta_w \epsilon_w (1+\chi)} E_t (1+\pi_{t+1})^{\epsilon_w(1+\chi)} \left(\frac{w_{t+1}^\#}{w_t^\#} \right)^{\epsilon_w(1+\chi)} \widehat{h}_{1,t+1} \tag{55}$$

$$\widehat{h}_{2,t} = \lambda_t \left(\frac{w_t}{w_t^\#} \right)^{\epsilon_w} N_{d,t} + \phi_w \beta (1+\pi_t)^{\zeta_w(1-\epsilon_w)} E_t (1+\pi_{t+1})^{\epsilon_w - 1} \left(\frac{w_{t+1}^\#}{w_t^\#} \right)^{\epsilon_w} \widehat{h}_{2,t+1} \tag{56}$$

$$w_t^\# = \frac{\epsilon_w}{\epsilon_w - 1} \frac{\widehat{h}_{1,t}}{\widehat{h}_{2,t}} \tag{57}$$

$$\frac{w_t}{R_t} = \frac{1-\alpha}{\alpha} \left(\frac{\widehat{K}_t}{N_{d,t}} \right) \tag{58}$$

$$mc_t = \frac{w_t}{(1-\alpha) A_t \left(\frac{\widehat{K}_t}{N_{d,t}} \right)^\alpha} \tag{59}$$

$$x_{1,t} = \lambda_t mc_t Y_t + \phi_p \beta \left(1+\pi_t\right)^{-\zeta_p \epsilon_p} E_t (1+\pi_{t+1})^{\epsilon_p} x_{1,t+1} \tag{60}$$

$$x_{2,t} = \lambda_t Y_t + \phi_p \beta \left(1+\pi_t\right)^{\zeta_p(1-\epsilon_p)} E_t (1+\pi_{t+1})^{\epsilon_p - 1} x_{2,t+1} \tag{61}$$

$$1 + \pi_t^\# = \frac{\epsilon_p}{\epsilon_p - 1} (1+\pi_t) \frac{x_{1,t}}{x_{2,t}} \tag{62}$$

$$Y_t = C_t + I_t + G_t \tag{63}$$

$$K_{t+1} = Z_t \left[1 - \frac{\kappa}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right)^2 \right] I_t + (1 - \delta(u_t) K_t \tag{64}$$

$$A_t \widehat{K}_t^\alpha N_{d,t}^{1-\alpha} - F = Y_t v_t^p \tag{65}$$

$$\widehat{K}_t = u_t K_t \tag{66}$$

$$v_t^p = (1-\phi_p) \left(\frac{1+\pi_t^\#}{1+\pi_t} \right)^{-\epsilon_p} + (1+\pi_{t-1})^{-\zeta_p \epsilon_p} (1+\pi_t)^{\epsilon_p} \phi_p v_{t-1}^p \tag{67}$$

$$(1+\pi_t)^{1-\epsilon_p} = (1-\phi_p)(1+\pi_t^\#)^{1-\epsilon_p} + (1+\pi_{t-1})^{\zeta_p(1-\epsilon_p)} \phi_p \tag{68}$$

$$w_t^{1-\epsilon_w} = (1-\phi_w) w_t^{\#1-\epsilon_w} + (1+\pi_{t-1})^{\zeta_w(1-\epsilon_w)} \phi_w (1+\pi_t)^{\epsilon_w - 1} w_{t-1}^{1-\epsilon_w} \tag{69}$$

$$i_t = (1-\rho_i) i + \rho_i i_{t-1} + (1-\rho_i) \left[\phi_\pi (\pi_t - \pi) + \phi_y (\ln Y_t - \ln Y_{t-1}) \right] + s_i \varepsilon_{i,t} \tag{70}$$

$$\ln G_t = (1-\rho_G) \ln G + \rho_G \ln G_{t-1} + s_G \varepsilon_{G,t} \tag{71}$$

$$\ln A_t = \rho_A \ln A_{t-1} + s_A \varepsilon_{A,t} \tag{72}$$

$$\ln Z_t = \rho_Z \ln Z_{t-1} + s_Z \varepsilon_{Z,t} \tag{73}$$

$$\ln \nu_t = \rho_\nu \ln \nu_{t-1} + s_\nu \varepsilon_{\nu,t} \tag{74}$$

$$\ln \psi_t = (1-\rho_\psi)\ln \psi + \rho_\psi \ln \psi_{t-1} + s_\psi \varepsilon_{\psi,t} \tag{75}$$

This is 26 equations in 26 variables:
$\left\{\lambda_t, \mu_t, C_t, i_t, \pi_t, R_t, u_t, Z_t, I_t, \nu_t, \psi_t, w_t, w_t^{\#}, \widehat{h}_{1,t}, \widehat{h}_{2,t}, N_{d,t}, \widehat{K}_t, K_t, mc_t, \pi_t^{\#}, x_{1,t}, x_{2,t}, Y_t, G_t, A_t, v_t^p\right\}$.

The parameters of the model are:
$\left\{\beta, b, \alpha, \delta_0, \delta_1, \delta_2, \pi, \kappa, \epsilon_w, \chi, \theta_w, \zeta_w, \alpha, \zeta_p, \phi_p, \epsilon_p, F, \rho_i, \phi_\pi, \phi_y, s_i, \rho_A, \rho_Z, \rho_G, \rho_\nu, \rho_\psi, s_A, s_Z, s_\nu, s_G, s_\psi, G, \psi\right\}$.
In writing this I treat the level of trend inflation as an exogenous parameter. I do not necessarily want to assume zero trend inflation.

2.9 The Non-Stochastic Steady State

Let's solve for the non-stochastic steady state. In doing so, denote variables with a time subscript as non-stochastic steady state values.

From the Euler equation, we get the steady state nominal interest rate as:

$$i = \frac{1}{\beta}(1+\pi) - 1 \tag{76}$$

Now, go to the FOC for investment, evaluated in steady state. Note that $Z = 1$. Since the investment growth terms will drop out, this reduces to:

$$\lambda = \mu \tag{77}$$

In other words, the multipliers are equal in steady state (equivalently, q is 1 in steady state, since q is the ratio of μ_t to λ_t). We want to impose that $u = 1$ in the steady state. Recall that the depreciation function is $\delta(u_t) = \delta_0 + \delta_1(u_t - 1) + \frac{\delta_2}{2}(u_t - 1)^2$. If $u = 1$, then steady state depreciation is just δ_0. Now, go to the Euler equation for capital, taking note of these features:

$$\mu = \beta\left(\lambda R + \mu(1-\delta_0)\right)$$

Hence, we get:

$$R = \frac{1}{\beta} - (1-\delta_0) \tag{78}$$

Now, since $\delta'(u_t) = \delta_1 + \delta_2(u_t - 1)$, if $u = 1$, then $\delta'(1) = \delta_1$. From the FOC for utilization, this provides the following restriction on the value of δ_1 to be consistent with $u = 1$:

$$\delta_1 = \frac{1}{\beta} - (1-\delta_0) \tag{79}$$

Now, go to the evolution of inflation condition to solve for steady state reset price inflation in terms of the exogenous steady state inflation rate:

$$1+\pi^{\#} = \left[\frac{(1+\pi)^{1-\epsilon_p} - \phi_p(1+\pi)^{\zeta_p(1-\epsilon_p)}}{1-\phi_p}\right]^{\frac{1}{1-\epsilon_p}} \tag{80}$$

Note that, only if $\pi = 0$ or $\zeta_p = 1$, will we have $\pi^{\#} = \pi$. Once we have this, we can also get an expression for steady state price dispersion:

$$v^p = \frac{(1-\phi_p)\left(\frac{1+\pi^{\#}}{1+\pi}\right)^{-\epsilon_p}}{1-\phi_p(1+\pi)^{\epsilon_p(1-\zeta_p)}} \tag{81}$$

Note that only if $\pi = 0$ or $\zeta_p = 1$ will $v^p = 1$.

Now solve for steady state expressions of x_1 and x_2:

$$x_1 = \frac{\lambda mc Y}{1-\phi_p\beta(1+\pi)^{\epsilon_p(1-\zeta_p)}}$$

$$x_2 = \frac{\lambda Y}{1-\phi_p\beta(1+\pi)^{(\epsilon_p-1)(\zeta_p-1)}}$$

The ratio of these two expressions is thus:

$$\frac{x_1}{x_2} = mc\frac{1-\phi_p\beta(1+\pi)^{(\epsilon_p-1)(\zeta_p-1)}}{1-\phi_p\beta(1+\pi)^{\epsilon_p(1-\zeta_p)}}$$

If $\pi = 0$, or $\zeta_p = 1$, this will just be mc. Otherwise the inflation terms will still be floating around. We can now use this, in conjunction with the reset inflation condition (for which we now know the steady state value), to solve for steady state marginal cost:

$$mc = \frac{\epsilon_p - 1}{\epsilon_p}\frac{1+\pi^{\#}}{1+\pi}\frac{1-\phi_p\beta(1+\pi)^{\epsilon_p(1-\zeta_p)}}{1-\phi_p\beta(1+\pi)^{(\epsilon_p-1)(\zeta_p-1)}} \tag{82}$$

Now that we know stead state marginal cost, we can solve for the steady state capital labor ratio by noting that:

$$R = \alpha mc\left(\frac{K}{N_d}\right)^{\alpha-1}$$

Note that with $u = 1$, there is no distinction between steady state capital and steady state capital services. Solving for the capital-labor ratio, we have:

$$\frac{K}{N_d} = \left(\frac{\alpha mc}{R}\right)^{\frac{1}{1-\alpha}} \tag{83}$$

But once we know this, we can solve for the steady state real wage as:

$$w = (1-\alpha)mc\left(\frac{K}{N_d}\right)^{\alpha} \tag{84}$$

But now we can solve for the steady state reset wage from the wage evolution equation:

$$w^{\#} = \left[\frac{w^{1-\epsilon_w} \left(1 - \phi_w (1+\pi)^{(\epsilon_w - 1)(1-\zeta_w)}\right)}{1 - \phi_w} \right]^{\frac{1}{1-\epsilon_w}} \tag{85}$$

Note that, only if $\pi = 0$ or $\zeta_w = 1$ will $w^{\#} = w$. Now, solve for steady state $\widehat{h}_1$ and $\widehat{h}_2$ in terms of things which are now known:

$$\widehat{h}_1 = \frac{\psi \left(\frac{w}{w^{\#}}\right)^{\epsilon_w (1+\chi)} N_d^{1+\chi}}{1 - \phi_w \beta (1+\pi)^{\epsilon_w (1+\chi)(1-\zeta_w)}}$$

$$\widehat{h}_2 = \frac{\lambda \left(\frac{w}{w^{\#}}\right)^{\epsilon_w} N_d}{1 - \phi_w \beta (1+\pi)^{(\epsilon_w - 1)(1-\zeta_w)}}$$

The ratio of these is:

$$\frac{\widehat{h}_1}{\widehat{h}_2} = \psi \lambda^{-1} N_d^{\chi} \left(\frac{w}{w^{\#}}\right)^{\epsilon_w \chi} \frac{1 - \phi_w \beta (1+\pi)^{(\epsilon_w - 1)(1-\zeta_w)}}{1 - \phi_w \beta (1+\pi)^{\epsilon_w (1+\chi)(1-\zeta_w)}}$$

From the reset wage condition, this must equal:

$$w^{\#} = \frac{\epsilon_w}{\epsilon_w - 1} \psi \lambda^{-1} N_d^{\chi} \left(\frac{w}{w^{\#}}\right)^{\epsilon_w \chi} \frac{1 - \phi_w \beta (1+\pi)^{(\epsilon_w - 1)(1-\zeta_w)}}{1 - \phi_w \beta (1+\pi)^{\epsilon_w (1+\chi)(1-\zeta_w)}}$$

Now, to make sense of this expression, suppose for a moment that $\pi = 0$, so that $w = w^{\#}$. This condition would reduce to:

$$\psi N_d^{\chi} = \lambda \frac{\epsilon_w - 1}{\epsilon_w} w$$

Aside from the term involving ϵ_w, this looks like a standard labor supply condition. With monopolistic competition, the steady state wage would be a markup, $\frac{\epsilon_w - 1}{\epsilon_w}$, over the marginal rate of substitution between labor and consumption, $\psi N_d^{\chi} \lambda^{-1}$.

For the more general case, we can use this to isolate N_d^{χ}:

$$N_d^{\chi} = \frac{\epsilon_w - 1}{\epsilon_w} \frac{1}{\psi} \lambda w^{\#} \left(\frac{w}{w^{\#}}\right)^{-\epsilon_w \chi} \frac{1 - \phi_w \beta (1+\pi)^{\epsilon_w (1+\chi)(1-\zeta_w)}}{1 - \phi_w \beta (1+\pi)^{(\epsilon_w - 1)(1-\zeta_w)}}$$

What is λ? From the FOC, we get:

$$\lambda = \frac{1}{C} \frac{1 - \beta b}{1 - b}$$

Now, plug this in to the condition above for labor:

$$N_d^{\chi} = \frac{\epsilon_w - 1}{\epsilon_w} \frac{1}{\psi} \frac{1}{C} \frac{1 - \beta b}{1 - b} w^{\#} \left(\frac{w}{w^{\#}}\right)^{-\epsilon_w \chi} \frac{1 - \phi_w \beta (1+\pi)^{\epsilon_w (1+\chi)(1-\zeta_w)}}{1 - \phi_w \beta (1+\pi)^{(\epsilon_w - 1)(1-\zeta_w)}}$$

Multiply both sides by N_d:

$$N_d^{1+\chi} = \frac{\epsilon_w - 1}{\epsilon_w}\frac{1}{\psi}\frac{N_d}{C}\frac{1-\beta b}{1-b}w^{\#}\left(\frac{w}{w^{\#}}\right)^{-\epsilon_w\chi}\frac{1-\phi_w\beta(1+\pi)^{\epsilon_w(1+\chi)(1-\zeta_w)}}{1-\phi_w\beta(1+\pi)^{(\epsilon_w-1)(1-\zeta_w)}}$$

Now why is this useful? Let's go to the aggregate resource constraint. Suppose that $G = \omega Y$, where ω is the steady state share of government spending in output. The accounting identity can be written:

$$(1-\omega)Y = C + I$$

From the capital accumulation equation, we know that:

$$I = \delta_0 K$$

So we have:

$$(1-\omega)Y = C + (1-\delta_0)K$$

Now, divide both sides by N_d:

$$(1-\omega)\frac{Y}{N_d} = \frac{C}{N_d} + \delta_0\frac{K}{N_d}$$

We already know what $\frac{K}{N_d}$ is. If we know what $\frac{Y}{N_d}$ was, we could solve for $\frac{C}{N_d}$, which we could then use to solve for N_d itself from the above expression. From above, we can write this as:

$$\frac{Y}{N}v^p = \left(\frac{K}{N_d}\right)^{\alpha} - \frac{F}{N}$$

If we were working in a model without a fixed cost (i.e. $F = 0$), then we'd be done at this point. But what if there is a fixed cost? Recall that total aggregate profit, in real terms, is given by (here I'm evaluating in steady state):

$$\Pi = Y - wN_d - RK$$

Suppose that we want to pick F such that $\Pi = 0$. This means that:

$$\frac{Y}{N_d} = w + R\frac{K}{N_d}$$

But this means that we can solve for $\frac{C}{N_d}$ in terms of things we now know:

$$\frac{C}{N_d} = (1-\omega)\left[w + R\frac{K}{N_d}\right] - \delta_0\frac{K}{N_d}$$

But once we know $\frac{C}{N_d}$, we can solve for N_d:

$$N_d = \left[\frac{\epsilon_w - 1}{\epsilon_w} \frac{1}{\psi} \frac{N_d}{C} \frac{1-\beta b}{1-b} w^{\#} \left(\frac{w}{w^{\#}} \right)^{-\epsilon_w \chi} \frac{1 - \phi_w \beta (1+\pi)^{\epsilon_w (1+\chi)(1-\zeta_w)}}{1 - \phi_w \beta (1+\pi)^{(\epsilon_w - 1)(1-\zeta_w)}} \right]^{\frac{1}{1+\chi}} \tag{86}$$

Now once we know this, we can solve for the F which is consistent with all this as:

$$F = N_d \left[\left(\frac{K}{N_d} \right)^{\alpha} - \left(w + R \frac{K}{N_d} \right) v^p \right] \tag{87}$$

Once I have N_d, it is straightforward to recover everything else.

2.10 Why Does the Fixed Cost Matter?

As noted above, you can solve the model either with a fixed cost, $F = 0$, or with one set so that profits are zero in steady state. Zero profit in steady state is nice because it rules out entry, but you can solve the model either way, ignoring entry in the case where there is no fixed cost. But how will the presence of a fixed cost impact the equilibrium of the model?

To see this, it is easiest to think about the model with flexible wages (but sticky prices), without capital, without preference shocks, and without government spending linearized about a zero inflation steady state (i.e. the textbook NK model). The production function, labor supply condition, labor demand, and aggregate resource conditions are:

$$Y_t v_t^p = A_t N_t - F \tag{88}$$

$$\theta N_t^{\chi} = C_t^{-\sigma} w_t \tag{89}$$

$$mc_t = \frac{w_t}{A_t} \tag{90}$$

$$Y_t = C_t \tag{91}$$

The linearized Phillips Curve (this would actually be the same in the more complicated model) is:

$$\pi_t = \frac{(1-\phi_p)(1-\phi_p \beta)}{\phi} \widetilde{mc}_t + \beta E_t \pi_{t+1} \tag{92}$$

Now, let's log-linearized (88)-(91) to write the Phillips curve in terms of the output gap instead of real marginal cost. Start with the production function:

$$\ln Y_t + \ln v_t^p = \ln \left[A_t N_t - F \right]$$

$$\widetilde{Y}_t + \widetilde{v}_t^p = \frac{1}{Y v^p} \left[A dN_t + dA_t N \right]$$

If we are linearizing about a zero inflation steady state, then $\widetilde{v}_t^p = 0$ and $v^p = 1$. Furthermore, we have normalized $A = 1$. Then this becomes:

$$\widetilde{Y}_t = \frac{N}{Y}\widetilde{N}_t + \frac{N}{Y}\widetilde{A}_t$$

Hence:

$$\widetilde{N}_t = \frac{Y}{N}\widetilde{Y}_t - \widetilde{A}_t$$

Log-linearizing the labor supply condition, making use of the resource constraint and labor demand condition, we get:

$$\chi\widetilde{N}_t = -\sigma\widetilde{Y}_t + \widetilde{mc}_t + \widetilde{A}_t$$

Or:

$$\chi\frac{Y}{N}\widetilde{Y}_t - \chi\widetilde{A}_t = -\sigma\widetilde{Y}_t + \widetilde{mc}_t + \widetilde{A}_t$$

Or:

$$\left(\sigma + \chi\frac{Y}{N}\right)\widetilde{Y}_t - (1+\chi)\widetilde{A}_t = \widetilde{mc}_t$$

Now, where is this getting us? In the flexible price allocation, the markup is constant, and hence $\widetilde{mc}_t^f = 0$. Hence, we have:

$$\widetilde{Y}_t^f = \frac{1+\chi}{\sigma + \chi\frac{Y}{N}}\widetilde{A}_t$$

Hence:

$$\widetilde{A}_t = \frac{\sigma + \chi\frac{Y}{N}}{1+\chi}\widetilde{Y}_t^f$$

This means we can write real marginal cost as a function of the output gap as:

$$\widetilde{mc}_t = \left(\sigma + \chi\frac{Y}{N}\right)(\widetilde{Y}_t - \widetilde{Y}_t^f) \tag{93}$$

Why is this useful? When we did this before, without a fixed cost, we had that $\frac{Y}{N} = 1$, so that the elasticity of real marginal cost with respect to the output gap was $(\sigma + \chi)$. With a fixed cost, we will have $\frac{Y}{N} = 1 - \frac{F}{N}$. For $F > 0$, we will have $\frac{Y}{N} < 1$. This means that the presence of the fixed cost lowers the elasticity of real marginal cost with respect to the output gap. Plugging in this into the Phillips Curve:

$$\pi_t = \frac{(1-\phi_p)(1-\phi_p)\left(\sigma + \chi\frac{Y}{N}\right)}{\phi_p}\widetilde{X}_t + \beta E_t\pi_{t+1} \tag{94}$$

If we call γ the coefficient on the output gap, the presence of a fixed cost makes this coefficient smaller. This is isomorphic to prices being stickier. We say that the fixed cost is a source of real rigidity because it makes real marginal cost less sensitive to the output gap.

3 Estimating the Model

Rather than simply parameterizing the model according to convention, or calibrating parameters of the model to match moments external to the model, the modern convention within macroeconomics is to estimate the parameters of the model. There are many ways one could consider estimating the parameters of the model. First, one could do moment matching (an extension of the generalized method of moments). In a nutshell, you could pre-specify some moments you'd like to target, and then pick the parameters of the model to match those moments. An alternative would be to estimate the parameters via maximum likelihood. In a nutshell, if the model is linear, you can use the Kalman filter to approximate the likelihood function given observed data (the Kalman filter essentially allows you to infer unobserved state variables from observed data, and therefore to form the likelihood function given observed data). A third alternative is Bayesian maximum likelihood. Effectively, Bayesian maximum likelihood seeks to maximize the likelihood but puts prior distributions on parameters. I think about this (probably incorrectly, as I am not a trained Bayesian) as maximum likelihood with a penalty for the parameters differing from your priors. One needs to be a bit careful, because there is an even bigger divide between Bayesian and classical statistics. In classical inference, there are true parameters and the data are random, and you are trying to infer parameter values from data. In Bayesian statistics, you reverse this. The parameters are themselves random, and the data is "true." I'm not going to focus on this idealogical divide and will (incorrectly) think about Bayesian estimation as providing estimates of "the" parameters of interest in a model. Francisco Ruge-Murcia has a nice survey paper which gives an overview on different methods to estimate DSGE models – "Methods to Estimate Dynamic Stochastic Equilibrium Models," *Journal of Economic Dynamics and Control* (2007).

Dynare will do either classical maximum likelihood or Bayesian maximum likelihood for you. Most papers coming out these days use Bayesian maximum likelihood. One could argue that the use of priors is in a sense hiding the fact that the parameters of the model are not well-identified; I will not get into that here. Rather, I will simply take the medium scale model, estimate the parameters using Bayesian maximum likelihood, discuss some features of the estimated parameters, and then will discuss features of the estimated model.

3.1 Data for Estimation

The starting point for estimating a model is data to which you compare the model. In maximum likelihood (or Bayesian maximum likelihood), you can have as many observable variables as you have shocks. In the model there are six shocks – a productivity shock, the marginal efficiency of investment (or just investment) shock, a government spending shock, a monetary policy shock, and two preference shocks – an intertemporal preference shock (ν_t) and an intratemporal preference shock (or labor supply shock) ψ_t. This means that we can use at most six observable variables. We can include additional variables if we want by including measurement error shocks to some variables (shocks which just affect the measured value of one particular variable, but which otherwise have no effects on the equilibrium of the model).

I'm going to assume that the following variables are observed – the growth rates (log first differences) of output, consumption, investment, labor hours, and the real wage; and the levels of the nominal interest rate and the inflation rate. This is seven observable variables, so I have to include measurement error somewhere. I'm going to assume that the growth rate of the real wage is measured with an iid error:

$$\Delta \ln w_t = \ln w_t - \ln w_t + \varepsilon_{w,t} \tag{95}$$

$\varepsilon_{w,t}$ is just an iid measurement error shock, and has no effect on any other variables.

Why do I assume that growth rates are observable, as opposed to levels, for several of these variables? By looking at growth rates I don't need to (i) worry about matching steady state levels (which just scales the economy) and (ii) don't need to worry about the exact source of trend growth (which I really haven't taken a stand on in writing down a stationary model without trend growth, but where there is clearly trend growth in the actual data). This is also convention – most of the papers using Bayesian maximum likelihood include variables like output in terms of growth rates. Since the nominal interest rate and inflation are stationary in the model, I'm going to measure those variables in levels.

Let me briefly discuss how I measure these variables in the data. I do so in a way meant to make the actual data as close as possible to the model. In the actual data net exports are a component of aggregate expenditure, but they are not in the model. I therefore measure nominal output in the data as the sum of nominal expenditure in the following different categories: non-durable consumption, services consumption, durable consumption, gross private fixed investment, and government gross investment and consumption expenditure. Note that in addition to excluding net exports I'm summing up nominal values of each expenditure category. In the NIPA accounts you can get real series for each component, there are separate price deflators for each expenditure category, and the model features constant relative prices for different categories of expenditure. Given the nominal GDP series, I deflate by the GDP price deflator to put it in real terms. I then take logs and first difference to get a series for real output growth.

I define consumption as the sum of non-durable and services consumption. Durable expenditure is excluded because that's economically more like investment. So to get the consumption series, I sum up nominal expenditure on non-durables and services. To put it in real terms, I deflate by the GDP price deflator (the same deflator I used to get a real GDP series). Again, in the model there is one price deflator, so I don't want to define real consumption using the price deflator for the different consumption series (or, if I did, I'd want to deflate GDP by the consumption price deflator). I then take logs and first difference. I measure investment as the sum of durable consumption expenditure and gross private fixed investment. I deflate by the GDP price deflator and take log first differences to put it in growth rates. I do not assume that government spending is observed, so I don't need to do anything there, but note that I do use the government spending series to construct the GDP series.

I measure hours using total hours worked in the aggregate economy. This series is not on the BLS website, but is downloaded from Valerie Ramey's website, which provides updated data each quarter from the BLS. An alternative hours series is hours worked in the non-farm business sector. I take logs and first difference. I measure the nominal wage as total hourly compensation in the non-farm business sector (available from the BLS as an index). I then take logs and deflate by the log GDP price deflator to put it in real terms. I then first difference to get it in growth rates. Part of the reason why I assume that the wage is observed with measurement error is because of the well-known issues with mapping the observed compensation series to the model concept of the wage. Finally, I measure inflation as the log first difference of the GDP price deflator. The nominal interest rate is the effective Federal Funds rate. This series is available at frequencies higher than quarterly, so I use within-quarter averages to convert to a quarterly frequency. Also, the series as presented is expressed at an annualized rate, so I divide by 400 so that the series corresponds to the model's concept of the maturity of bonds (one quarter).

Table 1 below shows some basic summary statistics for the variables I use in the estimation. I focus on the period 1960q1 through 2007q4 (so I exclude the Great Recession and ensuing zero lower bound period).

Table 1: Moments of the Data

Variable	Std.	AR(1)	Correlation w/ $\Delta \ln Y_t$
$\Delta \ln Y_t$	0.0091	0.3531	1.0000
$\Delta \ln C_t$	0.0047	0.3073	0.5153
$\Delta \ln I_t$	0.0305	0.2509	0.9200
$\Delta \ln N_t$	0.0067	0.3914	0.6175
$\Delta \ln w_t$	0.0060	0.0921	0.1071
π_t	0.0059	0.8969	-0.2403
i_t	0.0082	0.9553	-0.2266

We see that the volatility of output growth is a little less than 1 percent. Investment growth is about 3 times more volatile than output growth. Hours growth and wage growth are both about two-thirds as volatile as output growth. The nominal interest rate is a little more volatile than inflation. All the series are positively autocorrelated. This is true of the growth rate series, which with the exception of the real wage have autocorrelations of about 0.3 (the autocorrelation of real wage growth is about 0.1). The autocorrelations of both inflation and the interest rate are quite high. In terms of cyclicalities (correlations with output growth), we see that consumption, hours, and investment growth are all strongly correlated with output growth. The correlation is strongest for investment and weakest for consumption. In contrast, the inflation rate and the nominal interest rate are both negatively correlated with output growth.

Table 2 shows the same moments, but this time focuses on HP detrended log-levels of the variables. We see a pretty similar picture when focusing on growth rates in terms of relative volatilities. The HP filtered log-levels are much more positively autocorrelated than growth rates. The correlations with output are also higher, and for both inflation and the nominal interest rate, the correlations with HP filtered output are positive (as opposed to negative when looking at correlations with output growth).

Table 2: Moments of the Data, HP Filtered Levels

Variable	Std.	AR(1)	Correlation w/ $\ln Y_t$
$\ln Y_t$	0.0167	0.8714	1.0000
$\ln C_t$	0.0085	0.8647	0.8161
$\ln I_t$	0.0512	0.8387	0.9536
$\ln N_t$	0.0136	0.8992	0.8324
$\ln w_t$	0.0088	0.7971	0.1279
π_t	0.0028	0.5339	0.1494
i_t	0.0039	0.8212	0.2914

3.2 Prior and Posterior Distributions of Parameters

Before estimating parameters, there are a few parameters which I am not going to estimate (but which instead are calibrated). First, I set the discount factor to $\beta = 0.995$. This implies an annualized risk free real interest rate of about two percent, which is close to what comes out of the data. I set $\delta_0 = 0.025$, implying an average depreciation rate of 10 percent. The parameter δ_1 is pinned down so as to normalize the steady state utilization to 1. I set the fixed cost of production, F, to be consistent with zero steady state profit. I specify the non-stochastic value of $\psi = 6$, which is roughly consistent with steady state labor hours in the neighborhood of one-third. I specify the steady state value of the government spending share of output to 0.2, which is consistent with the long run average government spending share in the data.

The remaining parameters are estimated. To estimate these parameters using Bayesian maximum likelihood, I have to specify prior distributions for the parameters of interest. This is always somewhat arbitrary. You have to specify a type of distribution (beta, normal, gamma, inverse gamma) as well as the mean and standard deviation of the distribution. The standard deviation of the distribution is in a sense how strongly you believe the prior mean – the smaller is the prior standard deviation, the bigger penalty you are implicitly assigning to the estimation procedure picking a value of the parameter far from the prior mean. Because priors matter, you want to pick prior means that are reasonable or close to what other people have found. In terms of which distribution to use, the rule of thumb is this. If a parameter is restricted to lie between 0 and 1, use the beta distribution. For the shock standard deviations, use the inverse gamma distribution. For other parameters, use either a gamma or a normal distribution. The chief difference between these two distributions is that the gamma distribution is skewed while the normal is not.

I pick prior distributions that are roughly in line with the existing literature. The prior distributions are show in the middle columns of Table 3 below. After the model block of the Dynare code, I need to first specify the observable variables with this command:

```
varobs dy dc dI infl int dN dw;
```

These variables correspond to output growth, consumption growth, investment growth, inflation, the nominal interest rate, hours growth, and real wage growth. Next, I need to specify the prior distributions for the parameters. I do this with the following code:

```
estimated_params;
b,  0.7, beta_pdf, 0.7, 0.1;
phiw,  0.75, beta_pdf,  0.66, 0.1;
phip, 0.66, beta_pdf,  0.66, 0.1;
zetaw, 0.2, beta_pdf, 0.5, 0.15;
zetap, 0.2, beta_pdf,  0.5, 0.15;
phipi, 1.5, 1.0, 3, normal_pdf, 1.7, 0.3;
phiy, 0.2, 0.0, 3, normal_pdf, 0.4, 0.3;
chi, 2.0, gamma_pdf, 2.0, 0.75;
kappa, 4, gamma_pdf, 4, 0.2;
stderr eA, 0.005, inv_gamma_pdf, 0.01, 0.01;
stderr eZ, 0.02,  inv_gamma_pdf, 0.005, 0.01;
stderr ei, 0.002,  inv_gamma_pdf, 0.0015, 0.01;
stderr ev,  0.01, inv_gamma_pdf, 0.001, 0.01;
stderr eG,  0.01, inv_gamma_pdf, 0.005, 0.01;
stderr epsi,  0.002, inv_gamma_pdf, 0.0015, 0.01;
stderr ew,  0.01, inv_gamma_pdf, 0.01, 0.01;
rhoA, 0.92, beta_pdf, 0.5, 0.2;
rhoZ, 0.94, beta_pdf,  0.5, 0.2;
rhov, 0.9, beta_pdf, 0.5, 0.2;
rhopsi, 0.7, beta_pdf, 0.5, 0.2;
rhoi, 0.8, beta_pdf, 0.5, 0.2;
rhoG, 0.8, beta_pdf, 0.5, 0.2;
Δ2, 0.05, gamma_pdf, 0.05, 0.1;
alpha, 0.33, normal_pdf, 0.30, 0.05;
end;
```

The structure of each line is "parameter name, initial guess, distribution type, prior mean, and prior standard deviation." The initial guess isn't part of the prior, it's just an initial value for the numerical optimizer. The distribution types are self-evident. For the prior means, I pick mean values that are broadly in line with the existing literature, although for the autoregressive terms I pick means that are pretty low. I pick low means for the two kinds of preference shocks, in a sense trying to bias the estimation against picking the preference shock standard deviations from being too big. I use prior standard deviations of 0.01 (1 percent) for the shocks.

Then commands the command that estimates the parameters (it is kind of the replacement for the "stoch_simul" command when you are just solving and simulating a model). My code is the following:

```
estimation(datafile=med_data_new,mh_replic=20000,mh_jscale=0.5,mode_compute=6);
```

The first argument here is a file where you have the data (with the same names as the observable variables declared in the Dynare code). My file is called "med_data_new.m" In other words, I just have a .m file where I copied the data series in from Excel. You can also save the data file as a .mat file. "mh_replic" is the number of Metropolis-Hastings replications used to generate the posterior distribution. "mh_jscale" governs the "acceptance rate" on the Metropolis-Hastings draws. I am not going to focus on what exactly these mean here – you can read about these concepts online or in the Dynare manual. "mode_compute" governs the type of numerical optimizer that Matlab uses in the background. The default (what Dynare runs if you leave this field blank) is 4. For complicated models, the optimization will often fail and you won't get to the Metropolis-Hastings draws. Mode_compute 6 is the slowest of the options but is based on Monte Carlo methods and almost always works. I've also had success with mode_compute 9.

Dynare produces a lot of output and diagnostics when it estimates a model. Recalling that in Bayesian statistics parameters are themselves random, the primary output is a posterior (as opposed to prior) distribution of parameters. Effectively what is going on is you specify prior distributions for the parameters. Then you observe some actual data. Then you update the distributions of parameters (i.e. go to the posterior) to maximize the likelihood. What you get are posterior distributions. Dynare produces lots of different moments from the posterior – the mode, the mean, the median, standard deviations, and percentiles of the distribution. You can see this output for yourself if you run a Dynare estimation. Below I present the mode, mean, and standard deviation of the posterior distributions. With some abuse of terminology and thinking like a classical statistician, I am going to take "the" parameter estimates to be the mean of the posterior and the standard deviation of the posterior distributions to be the standard error of the estimates.

Table 3: Prior and Posterior Distributions for Parameters

			Prior		Posterior		
Parameter	Description	Dist	Mean	Std	Mode	Mean	Std
α	Cap. Share	N	0.30	0.05	0.30	0.30	0.0069
b	Habit	B	0.70	0.10	0.95	0.94	0.0108
ϕ_w	Calvo wage	B	0.66	0.10	0.60	0.60	0.0607
ϕ_p	Calvo price	B	0.66	0.10	0.71	0.72	0.0245
ζ_w	Wage indexation	B	0.50	0.15	0.42	0.38	0.1296
ζ_p	Price indexation	B	0.50	0.15	0.07	0.06	0.0325
χ	Inverse Frisch	G	2.00	0.75	1.96	1.66	0.5549
κ	Inv. adjustment	G	4.00	0.20	4.08	4.05	0.2005
δ_2	Util adjustment	G	0.05	0.10	0.03	0.02	0.0116
ρ_i	TR smoothing	B	0.50	0.20	0.87	0.87	0.0122
ϕ_π	TR inflation	N	1.70	0.30	2.25	2.23	0.1792
ϕ_y	TR output growth	N	0.40	0.30	1.06	1.06	0.1580
ρ_A	AR productivity	B	0.50	0.20	0.99	0.99	0.0067
ρ_Z	AR investment	B	0.50	0.20	0.83	0.83	0.0374
ρ_G	AR government	B	0.50	0.20	0.98	0.98	0.0072
ρ_ν	AR intertemporal pref	B	0.50	0.20	0.33	0.31	0.0760
ρ_ψ	AR intratemporal pref	B	0.50	0.20	0.94	0.96	0.0340
s_A	SD productivity	IG	0.01	0.01	0.0056	0.0056	0.0003
s_Z	SD investment	IG	0.005	0.01	0.0668	0.0647	0.0051
s_G	SD government	IG	0.005	0.01	0.0115	0.0109	0.0013
s_ν	SD intertemporal pref	IG	0.001	0.01	0.0741	0.0758	0.0149
s_ψ	SD intratemporal pref	IG	0.0015	0.01	0.1180	0.0962	0.0337
s_i	SD TR	IG	0.0015	0.01	0.0026	0.0026	0.0002
s_w	Wage measurement error	IG	0.01	0.01	0.0061	0.0061	0.0003

Let's walk through the parameters estimates. First, we find that $\alpha = 0.3$, This is close to the conventional value of one-third. I could have calibrated α instead of estimating it, and it wouldn't have made much difference. You typically find in Bayesian estimations that the model wants α to be pretty small – often times lower than 0.2. The model wants a lot of habit formation, with $b = 0.94$. This is a bit high compared to most estimates, which are typically in the region of 0.6-0.9. The Calvo parameters are $\phi_w = 0.6$ for wages and $\phi_p = 0.72$ for prices. These are fairly standard, with most estimates of these parameters between 0.6 and 0.8. These parameters are all fairly precisely estimated, in the sense of having low standard errors. The price indexation parameter is very close to zero at $\zeta_p = 0.06$. There is a bit more evidence of wage indexation, but it's still smaller than 0.5. This parameter is also relatively poorly estimated, as evidence by the high standard error.

The inverse Frisch elasticity is estimated to be about 1.66 (so Frisch elasticity is between 0.5 and 0.7). The fact that the mode is quite different than the mean (these are off by 0.3) is consistent with the gamma prior, which is skewed. This parameter is not very well identified, which we can see from the high standard error. The investment adjustment cost is estimated to be about 4. The utilization adjustment cost parameter, δ_2, is estimated to be 0.02. That this is close to zero suggests that utilization is a big source of amplification in the model. There is significant smoothing in the Taylor rule ($\rho_i = 0.87$), and strong responses to both inflation ($\phi_\pi = 2.23$) and output growth ($\phi_y = 1.06$).

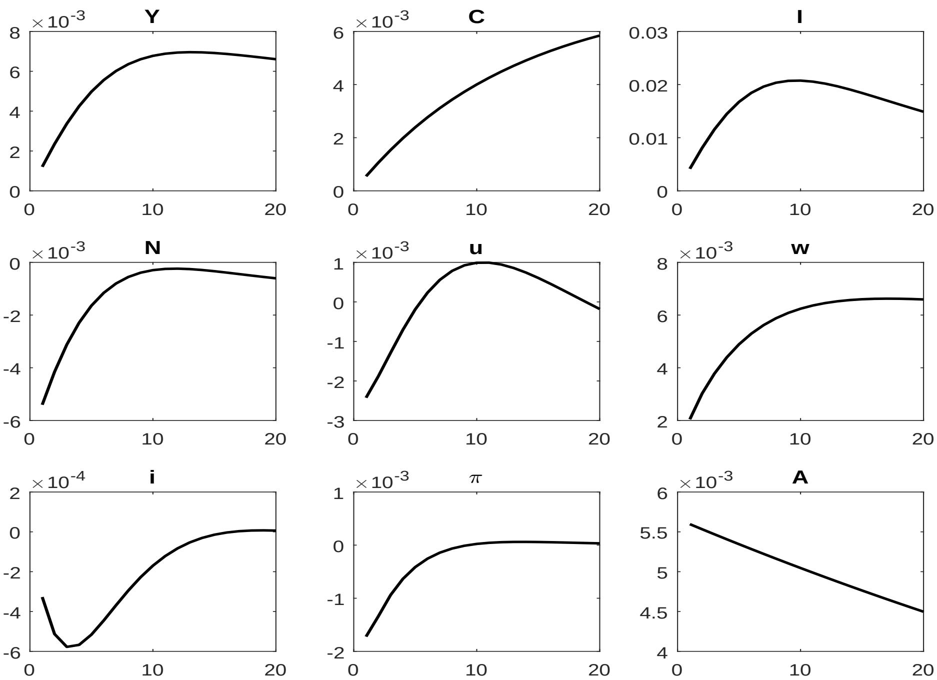
Let's turn next to the estimation of the parameters governing the shock processes. I find a very high AR parameter for the productivity shock ($\rho_A = 0.99$). There is a much smaller AR parameter for the investment shock ($\rho_Z = 0.83$). This is consistent with the estimates in Justiniano, Primiceri, and Tambalotti (2010, JME, and 2011, RED). The government spending shock is very persistent ($\rho_G = 0.98$). The intertemporal preference shock is not very persistent ($\rho_\nu = 0.31$). Recall that this shock will have bigger effects the less persistent it is. The intratemporal preference shock is fairly persistent ($\rho_\psi = 0.96$). The shock standard deviations are such that the productivity shock is fairly small ($s_A = 0.0056$), while the investment shock is pretty large ($s_Z = 0.0668$). This is also consistent with Justiniano, Primiceri, and Tambalotti. The government spending shock is about 1 percent. The two preference shocks are pretty big. The Taylor rule shock is small. The standard deviation of the wage measurement error shock is estimated to be about 0.6 percent. This suggests that a lot of the observed variation in the real wage is attributed to noise.

It is a useful exercise to look at how the posterior distributions compare to the prior. If they're close, it could be that I just picked a really good prior or that the data are not very informative about the underlying parameter of interest.

3.3 Properties of the Estimated Model

Let us now turn to properties of the estimated model. What I'm going to do is this. I'm going to take the posterior means as "the" parameter values. Then I run the model with these parameters. I produce impulse responses to the different shocks, a variance decomposition to see which shocks are most important for which variables, and then I look at moments from the model and compare those to moments from the data.

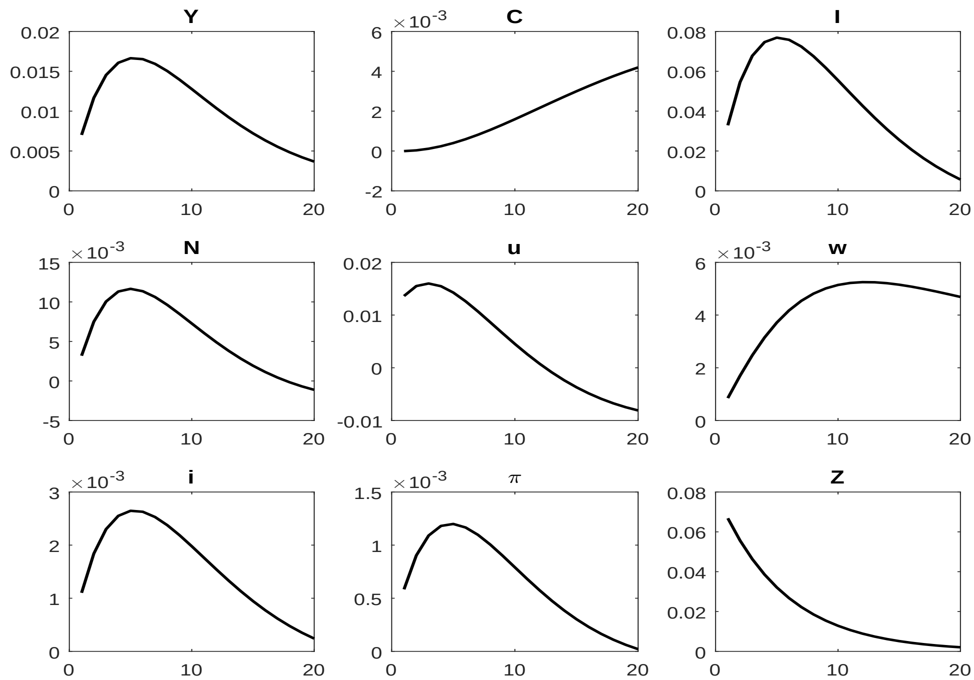
Figure 1: Productivity Shock



Let's start with impulse responses to the productivity shock, which are shown in Figure 1. Output, consumption, and investment all go up on impact, but they significantly undershoot what they would do in a simple RBC model. Hours worked decline. This is consistent with what we observed in the simplest New Keynesian model. But here the hours decline is driven more by real frictions (in particular the investment adjustment cost) as opposed to sticky prices. The response of utilization looks pretty similar to hours. The path of the real wage is similar to output. Inflation falls.

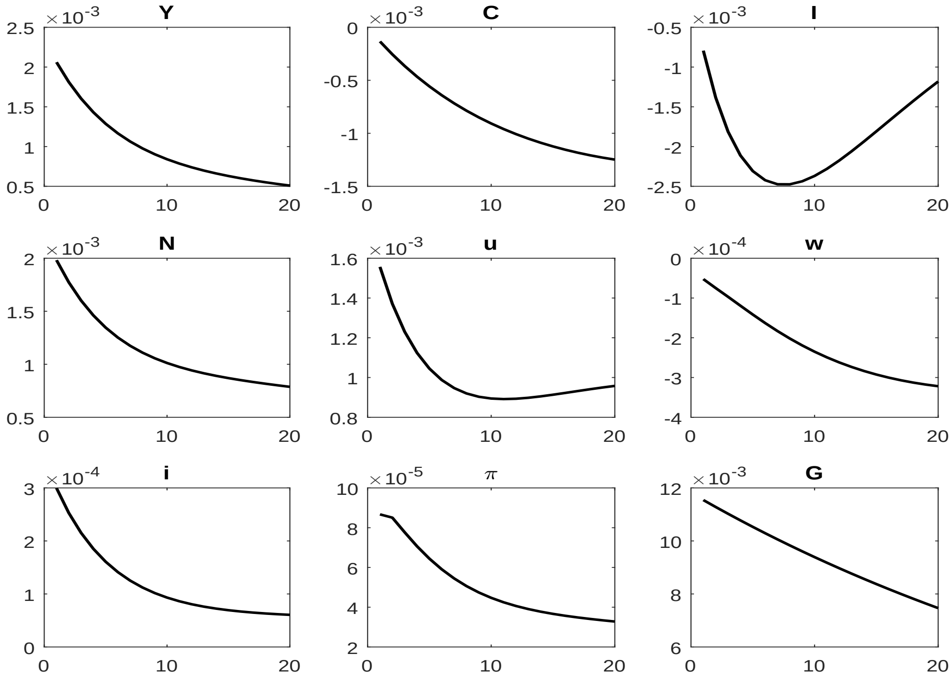
Next, consider the marginal efficiency of investment shock. Output and investment go up and follow hump-shapes. The response of consumption on impact is mildly negative (though not as negative as it would be in a flexible price, frictionless model), but then goes up. Hours and utilization both go up, as does the real wage. Inflation rises, and so too does the nominal interest rate. So the MEI shock looks like a "demand shock" in the sense of raising both prices and output simultaneously.

Figure 2: Marginal Efficiency of Investment Shock



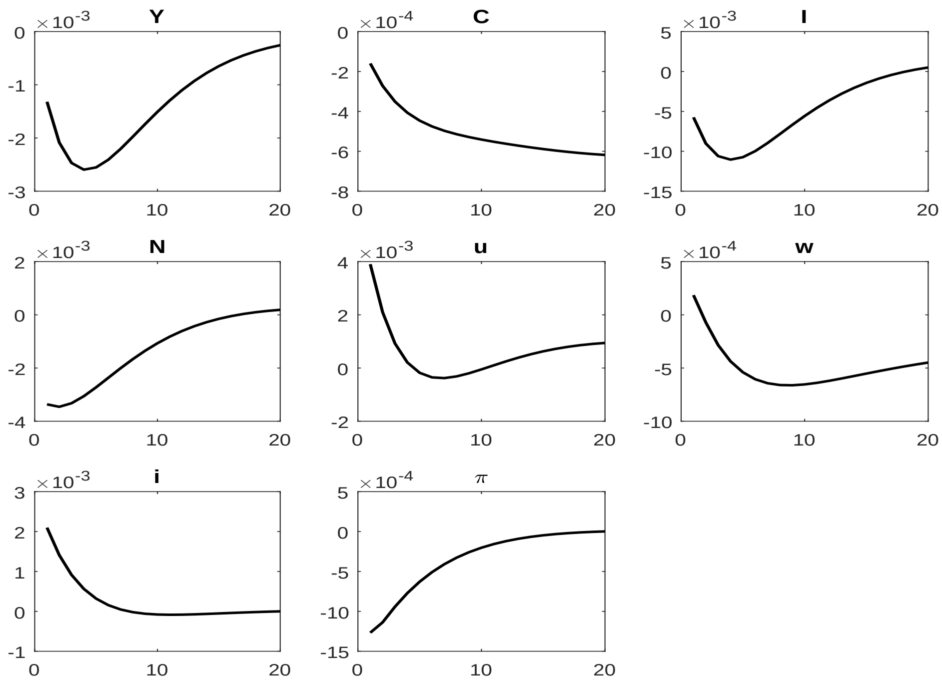
Next, let's look at the impulse responses to a government spending shock. This also raises output and inflation. Hours and utilization go up, while the real wage falls. Consumption and investment both fall. The multiplier comes out to be about 0.9 – so private expenditure is (mildly) crowded out.

Figure 3: Government Spending Shock



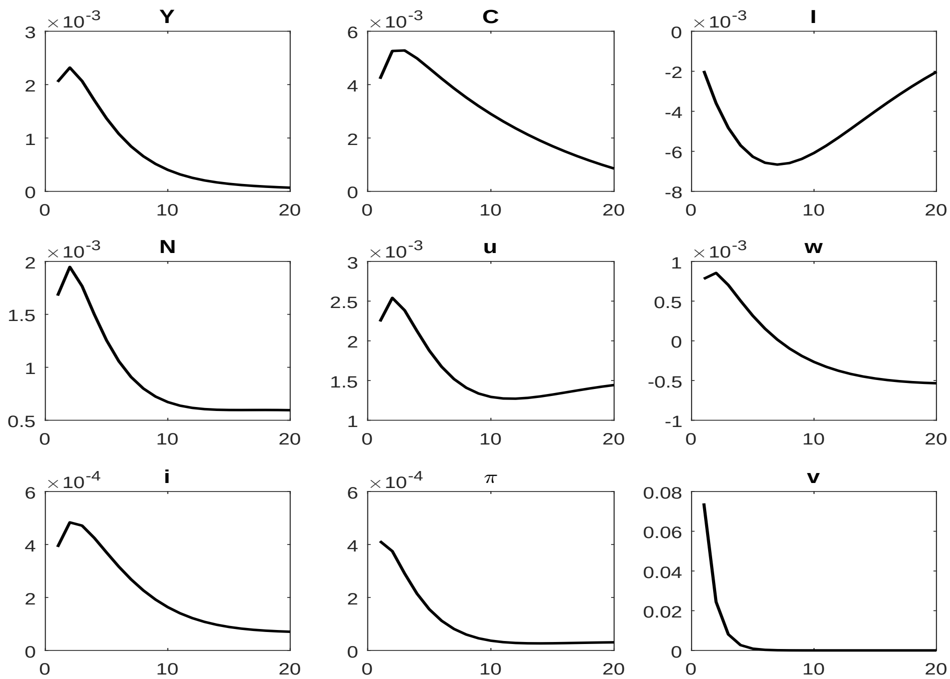
Now, let's focus on the impulse responses to the monetary policy shock. Since this is a positive shock to the interest rate rule, it is a contractionary shock. Output falls on impact and follows a hump-shape before reverting back to trend. Consumption and investment both fall. Interestingly, capital utilization rises on impact, as does the real wage, after which time both fall. Inflation falls.

Figure 4: Monetary Policy Shock



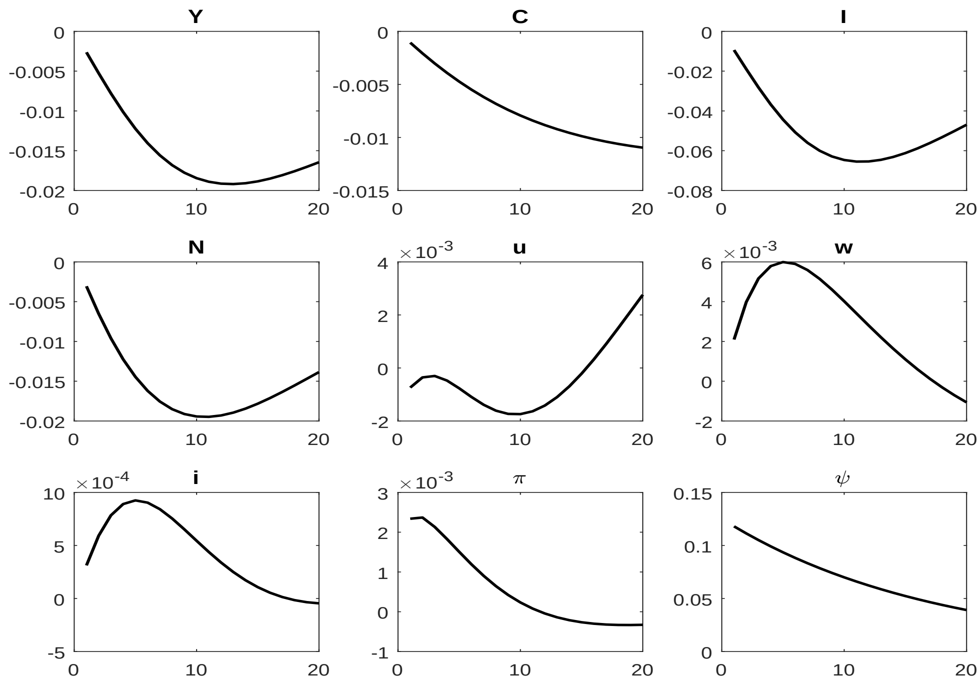
Now, let's look at the intertemporal preference shock. This shock is isomorphic to a decrease in the rate of time preference, making people want to consume more in the present. We naturally therefore observe an increase in consumption. Unlike what we would get in a frictionless real business cycle model, hours and output go up. Investment still falls. Inflation rises.

Figure 5: Intertemporal Preference Shock



Finally, consider the responses to the intratemporal preference shock, or labor supply shock. These are shown in Figure 6. Output, hours, consumption, and investment all fall, and these decreases are quite persistent. Inflation and the real wage, in contrast, rise. These responses are essentially the same to what we would observe to an increase in the tax rate on labor income. Two important features here – inflation and output move in opposite directions (so this, like the productivity shock, is a “supply shock”) and the real wage moves in the opposite direction from output.

Figure 6: Intratemporal Preference Shock



Now that we've seen impulse responses, let's look at the unconditional variance decomposition of the model to get a sense of how important the six different shocks are for the evolution of the different variables. I focus on the same observable variables I used in the estimation. For the purposes of this exercise I "turn off" the measurement error shock in the observed wage. Table 4 shows the variance decomposition. The investment shock is the most important shock for understanding output movements, explaining about 60 percent of the variance of output growth. This is consistent with the analysis in Justiniano, Primiceri, and Tambalotti (2010 and 2011). The next most important shock is the labor supply shock, accounting for about 30 percent of the variance of output growth. The other shocks, including the productivity shock, are unimportant in explaining output fluctuations.

Table 4: Unconditional Variance Decomposition, Estimated Model

Variable	Productivity	Investment	Gov.	TR	Intertemporal Pref.	Intratemporal Pref
$\Delta \ln Y_t$	3.84	59.50	2.79	1.82	3.01	29.04
$\Delta \ln C_t$	7.01	4.64	0.35	0.17	60.86	26.98
$\Delta \ln I_t$	2.41	72.95	0.05	1.89	0.39	22.31
$\Delta \ln N_t$	20.50	30.33	2.50	7.53	1.92	37.22
$\Delta \ln w_t$	27.21	14.25	0.03	0.75	3.12	54.64
π_t	12.70	23.97	0.17	10.97	1.04	51.15
i_t	2.64	75.48	0.58	9.02	1.67	10.61

There are some differences in terms of the contributions of shocks for other variables. First, the intertemporal preference shock explains the bulk of the variance of consumption, but is unimportant for all other variables. Second, the labor supply shock explains much more of the variance of wages (and, to a lesser extent, hours) than the other shocks. Third, the investment shock accounts for much more of the variance of investment than it does for other variables. Fourth, the monetary policy shock explains a much larger share of the variances of inflation and the interest rate than it does for other variables.

Table 5 shows the variance decomposition of HP-filtered variables as opposed to first differences (or straight up levels in the case of the inflation and interest rates). For the most part, the variance decomposition here is very similar to what is presented above.

Table 5: Unconditional Variance Decomposition, Estimated Model, HP Filtered Levels

Variable	Productivity	Investment	Gov.	TR	Intertemporal Pref.	Intratemporal Pref
$\ln Y_t$	3.93	60.62	0.76	1.64	1.24	31.81
$\ln C_t$	6.83	2.90	0.39	0.22	59.93	29.72
$\ln I_t$	2.64	71.08	0.05	1.63	0.43	24.17
$\ln N_t$	5.58	40.03	0.76	3.71	0.84	49.07
$\ln w_t$	20.46	15.61	0.02	0.52	1.54	61.86
π_t	18.43	12.74	0.05	11.47	1.19	56.10
i_t	3.32	59.92	0.56	25.32	2.00	8.88

Finally, let's look at unconditional moments from the model using the means of the posterior distributions for the parameters. In particular, we want to compare these moments to the moments from the actual data. Our estimation procedure is not a method of moments procedure, but it's still a good "eye test" to see if the model produces moments that look like the data. Table 6 shows moments based on first differenced log variables (i.e. the same variables used in the estimation), while Table 7 shows moments from HP filtered data.

Table 6: Moments from the Estimated Model

Variable	Std.	AR(1)	Correlation w/ $\Delta \ln Y_t$
$\Delta \ln Y_t$	0.0126	0.7173	1.0000
$\Delta \ln C_t$	0.0058	0.5403	0.3579
$\Delta \ln I_t$	0.0547	0.7438	0.9472
$\Delta \ln N_t$	0.0128	0.5762	0.7887
$\Delta \ln w_t$	0.0079	0.3016	0.0597
π_t	0.0071	0.8785	-0.3081
i_t	0.0093	0.9498	-0.0621

Table 7: Moments from the Estimated Model, HP Filtered Levels

Variable	Std.	AR(1)	Correlation w/ $\ln Y_t$
$\ln Y_t$	0.0289	0.9361	1.0000
$\ln C_t$	0.0095	0.8823	0.3384
$\ln I_t$	0.1284	0.9388	0.9766
$\ln N_t$	0.0275	0.9225	0.9200
$\ln w_t$	0.0113	0.9363	0.0507
π_t	0.0049	0.7567	0.3438
i_t	0.0048	0.8298	0.5045

Let's focus first on moments on first differenced data and compare those to the model. First, note that the estimated model tends to over-estimate volatilities – comparing the standard deviation entries in Table 6 to 1, we see that they are all bigger in the estimated model. For example, the volatility of output growth in the estimated model is 0.0126, compared to 0.0091 in the data (i.e. higher by a factor of about one-third). This over-estimation of volatility is a fairly standard result in estimation of models like this – see, for example, Justiniano, Primiceri, and Tambalotti (2011, RED). That said, the model does a reasonably good job with relative volatilities – for example, the relative volatility of consumption growth to output growth is 0.46 (0.51 in the data) and it 4.34 for investment growth (3.35 in the data). The estimated model tends to over-estimate the autocorrelations of the growth rate series, but is virtually spot on for the autocorrelations of the interest rate and inflation. Qualitatively, the model does a good job with the correlations with output growth – consumption, investment, and hours growth are strongly positively correlated with output growth (as in the data), real wage growth is close to uncorrelated with output growth (as in the data), and the interest rate and inflation are negatively correlated with output growth (as in the data). The relative ranking of the magnitudes of these correlations are also close matches with the data, though the model produces too small of a correlation between consumption growth and output growth. If you look at the model moments using HP filtered data (Table 7) compared to data moments using HP filtered data (Table 2), you see a fairly similar picture. The model overestimates volatilities, and it's a little bigger in terms of HP filtered data than it is in growth rates. The model does a pretty good job with relative volatilities and a good job with autocorrelations. The model does a reasonable job with correlations with HP filtered output, and in particular matches the fact that these are positive for the interest rate and inflation (whereas they are negative in terms of correlations with output growth). The model still generates too small a correlation of consumption with output.

3.4 Discussion

I would like to discuss briefly some broader features of the model and how they related to some of the recent literature. For this I'm going to focus mostly on the variance decomposition.

First, note that the intratemporal preference shock is an important driver of output and other variables – it explains about 30 percent of the unconditional variance of output, and more than 50 percent of the variance of wages. Smets and Wouters (2007, AER) also find that this shock is very important (indeed, a little more important than I find here). Chari, Kehoe, and McGrattan (2009, AEJ Macro) point out that this is problematic in the sense that the labor supply shock (how I have modeled it here) is isomorphic to a shock to the desired wage markup (how Smets and Wouters model it), which is essentially a shock to ε_w. The policy implications depending on the shock interpretation are quite different – in the wage markup shock interpretation, there is a change in the monopoly power of households in wage-setting, whereas in the labor supply shock households have a change in how they dislike labor. This isomorphism between the two interpretations of the shock makes it difficult to use the model to draw policy conclusions.

Why does the estimated model "like" the labor supply shock? Using the real wage series we did in the data, the real wage is not very procyclical. With the exception of the government spending shock (which itself can't be very important since it induces negative co-movement between consumption and investment with output), the real wage moves in the same direction as output conditional on all but the intratemporal labor supply shock. In response to the labor supply shock, the real wage and output move in opposite directions. This means that without the wage markup shock in the model, the real wage would be quite procyclical in the model, much more so than what we observe in the data. So the model "likes" the wage markup shock because it helps keep the real wage from being too procyclical.

A relevant issue here (and one that we've discussed previously) is whether the wage data from the BLS accurately corresponds to the model concept of the real wage. This relates to the composition bias, which the model as written cannot account for given that it does not have much interesting heterogeneity. If your starting point were that the real wage was in fact a lot more procyclical, the model would rely less heavily on the labor supply shock (and indeed, wage rigidity itself would be less important). This point and others are made in Basu and House (2016, Handbook of Macroeconomics).

Second, in the estimated model the productivity shock is not very important for the behavior of output. This is very different than in a standard real business cycle model, which relies on productivity shocks. In this model, hours worked decline when productivity increases, inducing a negative correlation between hours and output. Since hours and output are strongly positively correlated in the data, the model isn't going to "like" the productivity shock. The features which drive hours down when productivity improves are: price stickiness, investment adjustment costs, and habit formation. The model needs all three of these features to match other features of the data – investment adjustment costs break the connection between the real interest rate and the marginal product of capital, and also induce positive autocorrelations of the growth rates of variables (which is an important feature of the data); price stickiness is necessary to explain the behavior of inflation and to account for monetary non-neutrality; and habit formation in consumption is needed to generate sluggish behavior of consumption. Since the model needs these features, it is going to produce negative co-movement between hours and output conditional on a productivity shock, and is therefore not going to "like" the productivity shock as a main driver of output.

Third, the marginal efficiency of investment shock is estimated to be quite important – accounting for about half of the variance of output growth. There are two potential downsides to this. First, the importance of the investment shock is what makes the model correlation between consumption and output lower than it is in the data – the model is ill-equipped to generate much of a consumption boom in response to the investment shock. Second, what is the structural interpretation of this shock? Justiniano, Primiceri, and Tambalotti (2011, RED) have argued that it is a proxy for a financial shock, and they show that the estimated investment shock closely correlates with bond spreads, long thought to be a measure of financial distress. But to take the model seriously, we need to better micro-found these financial frictions, a task which is currently being undertaken.

Fourth, for all of its bells and whistles, the model still lacks much amplification and propagation. If you look at the variance decomposition table, you see that "own shocks" drive a lot of the variability in individual series. What do I mean by "own shocks"? We can *loosely* think of the intertemporal preference shock as being a shock to consumption, the investment shock as being a shock to investment, and the intratemporal labor supply shock as being a shock to wages/hours. The intertemporal preference shock explains 60 percent of the variance of consumption but virtually none of output – in other words, there is not much spillover in the model from the preference shock to other variables. The investment shock explains 72 percent of the variance of investment but less of other variables. The intratemporal preference shock accounts for more than half of the variance of wages, but less of the variances of other variables. Put differently – the model lacks amplification/propagation from shocks mostly focused on a particular endogenous variable (consumption, investment, wages) into other variables. The estimated model ends up attributing a lot of the variance of individual series to shocks most closely connected to those series.