A. Type 1 Extreme Value Shocks, Choice Probabilities, and Expected utility

This section justifies and derives the expression for utility in (**??**), choice probabilities, expected utility, etc. The idea is that we can recast the planner as choosing cutoff values for the preference shock as the planner directly choosing the migration rates. I do everything below in a simplified form with an agent picking between locations j and j'. And to simplify the notation, I'm not carrying around all the states.

First, the ordinal way to think of the planner's problem is that the planner is choosing $J-1$ cut-off values $\nu_j^j(s)$ for each state (s and current location j). The $J-1$ is because for J locations, one is redundant. In my presentation below, it is only a two location situation, so there is only one cutoff value.

The cutoff value then works in the following way where: $\nu^j + \nu^j(s) > \nu^{j'}$, then you move to j otherwise go to j'. Another, perhaps clearer way to write this is $\nu^j(s) \geq \nu^{j'} - \nu^j$. So, for example, if your preference for location j is very large, this inequality is satisfied and you go to j. Otherwise, if this inequality is not satisfied (you have a very large $\nu^{j'}$ shock, then go to j'.

Now we can go back an forth between the cut-off values described above and migration rates. Given some joint density over the preference shocks ϕ, the probability that people go to location j is:

$$\mu_{j,j}(s) = \texttt{Prob}\{\nu^j + \nu_j^j(s) \geq \nu^{j'}\} = \int_{\infty}^{\infty} \left[\int^{\nu^j + \nu_j^j(s)} \phi(\nu^j, \nu^{j'}) d\nu^{j'} \right] d\nu^j \tag{1}$$

where the internal part of the bracket stays: "fix a ν^j, then how many guys have $\nu^{j'}$ below $\nu^j + \nu_j^j(s)$" or in words "for the given ν^j, how many want to go to j." Then there is the outer part says "add up all those guys for each ν^j." I do this many times, always get confused, so say it again. With the Type 1 extreme value distribution, we can evaluate the integral in (1) and then compute the migration rates as a function of the cut-off values and expected utility as a function of the migration rates as in (**??**). Below we do this in several steps.

1. Train's book is here https://eml.berkeley.edu/books/choice2.html which is a very good resource for working with the Type 1 extreme value distribution. Most of the discussion below simply fills in the holes from aspects of the book with a more explicit derivation.

2. The Type 1 extreme value pdf is:

$$\phi(\nu) = \sigma^{-1} \exp(-\sigma^{-1}\nu) \exp(-\exp(-\sigma^{-1}\nu)) \tag{2}$$

$$\Phi(\nu) = \exp(-\exp(-\sigma^{-1}\nu)) \tag{3}$$

and then let's insert this into (1) and realizing that the inner term of (1) is the cdf Φ evaluated at the cut-off point of $\nu^j + \nu_j^j(s)$.

$$\texttt{Prob}\{\nu^j + \nu_j^j(s) \geq \nu^{j'}\} = \int_{-\infty}^{\infty} \Phi(\nu^j + \nu_j^j(s))\phi(\nu) d\nu^s \tag{4}$$

$$\texttt{Prob}\{\nu^j + \nu_j^j(s) \geq \nu^{j'}\} = \int_{-\infty}^{\infty} \exp(-\exp(-\sigma^{-1}(\nu^j + \nu_j^j(s))))\sigma^{-1} \exp(-\sigma^{-1}\nu^j) \exp(-\exp(-\sigma^{-1}\nu^j)) d\nu^j \tag{5}$$

Now if you collect terms on the $\exp$s you get something like

$$\texttt{Prob}\{\nu^j + \nu^j_j(s) \geq \nu^{j'}\} = \int_{-\infty}^{\infty} \sigma^{-1} \exp\big[-\sigma^{-1}\nu^j - \exp(-\sigma^{-1}\nu^j)(1+\exp(-\sigma^{-1}\nu^j_j(s)))\big] d\nu^j \tag{6}$$

This is tedious, but almost done. First, using antiderivative calculator will take care of this.[1] A more subtle approach is to note that this is essentially the same as the pdf—which we know integrates up to one—except for the constant term $(1+\exp(-\sigma^{-1}\nu^j(s)))$. Either which way, when you compute this you have that:

$$\mu_{j,j}(s) = \texttt{Prob}\{\nu^j + \nu^j_j(s) \geq \nu^{j'}\} = \frac{1}{1+\exp(-\sigma^{-1}\nu^j_j(s))} \tag{7}$$

which is the standard Type 1 extreme value share formula. First note that if the cut-off $\nu^j_j(s)$ becomes arbitrarily large, then this value converges to one, so everyone stays in location j. Second, notice if dispersion parameter σ becomes arbitrarily large, the share converges to a pure lottery, half go one way half go another way. Finally the probability of going to location j' is

$$\mu_{j',j}(s) = \frac{\exp(-\sigma^{-1}\nu^j_j(s))}{1+\exp(-\sigma^{-1}\nu^j_j(s))} \tag{8}$$

3. Next we need to compute expected utility associated with the migration pattern above. Note this is not a simple calculation because one needs to take into account that only the highst relative (or max) values migrate to a destination. So to compute expected utility, we first need to construct the following density:

$$\texttt{Prob}\{\nu^j \mid \nu^j + \nu^j_j(s) \geq \nu^{j'}\} \tag{9}$$

which is the probability of ν^j, conditional on $\nu^j + \nu^j_j(s)$ being greater than the random variable $\nu^{j'}$. Now we can construct this density in the following way, using Bayes rule we have that

$$\texttt{Prob}\{\nu^j \mid \nu^j + \nu^j_j(s) \geq \nu^{j'}\} \;=\; \frac{\texttt{Prob}\{\nu^j \;,\; \nu^j + \nu^j_j(s) \geq \nu^{j'}\}}{\texttt{Prob}\{\nu^j + \nu^j_j(s) \geq \nu^{j'}\}} \tag{10}$$

and one important thing to notice is that we have already computed the value in the denominator as it's the same as the stay in location j probability in (7). Then the object in the top is closely related to the way we computed the earlier density. This is:

$$\begin{aligned}\texttt{Prob}\{\nu^j \mid \nu^j + \nu^j_j(s) \geq \nu^{j'}\} &= \int_{-\infty}^{\nu} \Big[\int^{\nu^j+\nu^j_j(s)} \phi(\nu^j, \nu^{j'}) d\nu^{j'}\Big] d\nu^j &(11)\\ &= \int_{-\infty}^{\nu} \sigma^{-1} \exp\big[-\sigma^{-1}\nu^j - \exp(-\sigma^{-1}\nu^j)(1+\exp(-\sigma^{-1}\nu^j_j(s)))\big] d\nu^j &(12)\end{aligned}$$

where (to repeat my self) the first line says, fix a ν^j and add up all the guys that go to j for that given value. Then add up all the ν^j **up to the point of** ν. This is the joint density of νs that are movers and values up that

[1] The way to see the form is to note it takes $\alpha\exp(-\alpha x + \exp(-\alpha x)\beta)$ for which the antiderivative is $\frac{\exp(-\beta\exp(-\alpha x))}{\beta}$.

point. Now notice this is the same form of the integral as before, so this density is expressed as:

$$\texttt{Prob}\{\nu^j < \nu \ , \ \nu^j + \nu^j_j(s) \geq \nu^{j'}\} = \frac{\exp[-(1+\exp(-\sigma^{-1}\nu^j_j(s)))\exp(-\sigma^{-1}\nu)]}{1+\exp(-\sigma^{-1}\nu^j_j(s))}. \tag{13}$$

Then combining this probability with probability of being a stayer gives the following.

$$\texttt{Prob}\{\nu^j < \nu \mid \nu^j + \nu^j_j(s) \geq \nu^{j'}\} = \exp[-(1+\exp(-\sigma^{-1}\nu^j_j(s)))\exp(-\sigma^{-1}\nu)] \tag{14}$$

$$= \exp[-(\mu_{j,j}(s))^{-1}\exp(-\sigma^{-1}\nu)] \tag{15}$$

where the last line substitutes in the "ACR" like insight that the centering parameter relates to the share of migrants. Now the expected value of ν^j conditional on staying in j is

$$\texttt{E}(\nu^j | \nu^j + \nu^j_j(s) \geq \nu^{j'}) = \int_{-\infty}^{\infty} \nu^j \ d\exp[-(\mu_{j,j}(s))^{-1}\exp(-\sigma^{-1}\nu^j)]. \tag{16}$$

Now to compute this integral we need to do several more things. First, compute the pdf. Second, compute the integral via a change in variables and then make connection with Euler's constant. First the CDF takes the general form

$$\exp[-\beta\exp(-\alpha x)] \tag{17}$$

$$\tag{18}$$

and then the associated PDF takes the form

$$\beta\alpha\exp[-\beta\exp(-\alpha x) - \alpha x] \tag{19}$$

and the integral that we want to compute is

$$\beta\alpha\int \exp[-\beta\exp(-\alpha x) - \alpha x] x dx \tag{20}$$

Now we want to connect this integral with Euler's constant which is

$$-\int \exp(-y)\ln y dy = \gamma \tag{21}$$

4. https://www.randomservices.org/random/special/ExtremeValue.html helps. As does this one with integrals of functions of exponentiation
https://en.wikipedia.org/wiki/List_of_integrals_of_exponential_functions

5. So first notice that we can express the integral in (20) as

$$\beta\alpha\int \exp[-\beta\exp(-\alpha x)]\exp[-\alpha x] x dx \tag{22}$$

by just using rules of exponents. Next here is the proposed change of variables where

$$y = \beta \exp(-\alpha x) \tag{23}$$

$$\frac{y}{\beta} = \exp(-\alpha x) \tag{24}$$

$$x = \frac{-1}{\alpha} \ln\left(\frac{y}{\beta}\right) \tag{25}$$

$$\frac{dx}{dy} = \frac{-1}{\alpha}\frac{1}{y} \tag{26}$$

And notice that the domain of integration for this is now $[\infty, 0]$ (yes, the infinity is at the bottom), but we have to flip giving rise to a minus sign. Now inserting this change of variable into (22) we have

$$-\beta\alpha \int_0^{\infty} \exp(-y)\left(\frac{y}{\beta}\right)\left(\frac{-1}{\alpha}\ln\left(\frac{y}{\beta}\right)\right)\left(\frac{-1}{\alpha}\frac{1}{y}\right) dy \tag{27}$$

and then notice how the terms cancel leaving

$$-\int \exp(-y) \ln\left(\frac{y}{\beta}\right)\left(\frac{1}{\alpha}\right) dy. \tag{28}$$

Almost there. The final step is to expand out the stuff in the $\ln$ function giving.

$$-\frac{1}{\alpha}\int \exp(-y)\ln y \ dy + \frac{\ln\beta}{\alpha}\int \exp(-y) \ dy \tag{29}$$

Then given the domain of integration and the form of these integrals, the first integral is Euler's constant, the second integral just sums up to one giving us:

$$\frac{\gamma}{\alpha} + \frac{\ln\beta}{\alpha} \tag{30}$$

Now let's get some intuition behind this formula. Recall that $\alpha = \sigma^{-1}$ and that $\beta = \frac{1}{\mu_{j,j}(s)}$. So first notice if no one chooses j', everyone is choosing j, then β equals one and then the formula implies that the expected value of the preference draw those going to j is $\sigma\gamma$ which is the standard formula for the mean of a Type 1 extreme value. Good. Then notice as $\mu_{j,j}(s)$ decreases, less people are staying in j and β becomes larger. Why is it increasing? Remember this is the mean of the preference shock in j, so as out migration rate increases (less $\mu_{j,j}(s)$, higher $\mu_{j',j}(s)$, than those remaining in j are more selected and hence the expected value of the preference shock is larger. Even better.

6. To summarize the main result, the expected values **in a location** j conditional on a migration rate are

$$\mathtt{E}(\nu^j | \nu^j + \nu^j_j(s) \geq \nu^{j'}) = \sigma\gamma + \sigma \ln\left(\frac{1}{\mu_{j,k}(s)}\right) \tag{31}$$

And the **expected value across all locations**, conditional on the migration rates (which is a function of the cutoff value $\nu^j(s)$) is

$$E[\ \nu \mid \mu_{j'j}(s)] = \sigma\gamma + -\sigma \sum_{j'} \mu_{j'}(s) \log\left(\mu_{j'j}(s)\right) \tag{32}$$

which takes the familiar log sum formulation that arises in the Type 1 extreme value models. And most importantly, this is cast all in terms of the migration rates, not cutoff values.