Heterogeneous Agent Trade

Michael E. Waugh
Federal Reserve Bank of Minneapolis and NBER

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ABSTRACT

This paper develops a model of heterogenous agents and international trade. Heterogenous agents are modeled as in the standard incomplete markets tradition with household's facing incomplete insurance against idiosyncratic productivity and taste shocks. Trade in goods follows the Armington tradition but is derived from the "bottom up" with micro-level heterogeneity shaping the aggregate pattern of trade. In the efficient allocation, I recover standard results regarding gravity and the gains from trade. In the decentralized allocation, the pattern of trade is distorted, the aggregate trade elasticity is non-constant and the benefits from globalization are distributed unequally. I use model to explore two issues: the ability of trade policy to improve outcomes and how financial globalization complements trade in goods.

Email: michael.e.waugh@gmail.com. The views expressed herein are those of the author and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System. This project was developed with research support from the National Science Foundation (NSF Award number 1948800).

1. The Model

The model is setup in a simple and transparent way. Trade is in the Armigton tradition with each country producing a nationally differentiated variety. Households follow the standard incomplete markets tradition. The key departure is that I lean into household heterogeneity and have households make a discrete choice over the varieties they consume. Aggregate trade flows between countries are, thus, given by the explicit aggregation of households and the choices they make.

1.1. Trade and Production

There are M locations which I will call a country. Each country produces a differentiated product as in the Armington tradition and these differentiated products. In country i, competitive firms have the following production technology to produce variety i:

$$q_i = A_i N_i, \tag{1}$$

where N_i are the efficiency units of labor supplied by households in country i.

Trade faces several obstacles. There are iceberg trade costs d_{ji} for a good to go from supplier i to buyer j. Cross-border trade faces policy obstacles, i.e. tariffs τ_{ji}. The notation here is such that τ_{ji} which is the ad-valorem tariff rate that country j imposes on the commodity that county i produces.

Profit maximization of the competitive goods producers in location i results in the wage per efficiency unit reflecting the value of the marginal product of labor

$$w_i = p_i A_i. \tag{2}$$

Given iceberg trade costs and tariffs, the unit cost for country j to purchase a good from location i is

$$p_{ji} = \frac{d_{ji}(1+\tau_{ji})w_i}{A_i}. \tag{3}$$

1.2. Households

There is a mass of L_i households in each location i. Households are immobile across countries. They are infinite lived and have time-sparable preferences over non-durable consumption varieties:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(\{c_{jt}\}_M), \tag{4}$$

where the notation $\{c_{jt}\}_M$ means that the household has preferences over all j varieties supplied by M countries in the world. My focus is on the situation where households each period receive additive Type 1 extreme value shocks $\epsilon(j)$ with dispersion parameter σ_ϵ and then households make a discrete choice about which variety j to each period and a continuous choice about how much. More specifically, the utility associated with the choice of variety j and leisure is

$$u(c(j)_t) = \frac{c_{jt}^{1-\gamma}}{1-\gamma} + \epsilon(j)_t. \tag{5}$$

where consumption mapped into utils with a standard CRRA function, $\epsilon(j)$ is the taste shock.

A household's efficiency units are stochastic and they evolve according to a discrete state Markov chain. Mathematically, z is a households efficiently units and $\mathcal{P}(z, z')$ describes the probability of a household with state z efficiency units transiting to state z'.

Households can save and borrow in a non-state contingent asset a. The units of the asset are chosen to be the numeraire and pays out with gross interest rate R. I discuss this more in depth below, but the determination of R_i is either exogenously given or the rate that clears the bond market (local or global). An country specific, exogenous debt limit ϕ_i constrains borrowing so:

$$a_{t+1} \geq -\phi_i. \tag{6}$$

All these pieces come together in the household's budget constraint, conditional on choosing variety j to consume, and focusing on a stationary setting where prices and transfers are constant:

$$a_{t+1} + p_{ij}c_{jt} \leq R_i a_t + w_i z_t + T_{i,\tau}. \tag{7}$$

The value of asset purchases and consumption expenditures must be less than or equal to asset payments, labor earnings, and transfers arising from trade policy (the $T_{i,\tau}$).

1.3. Recursive Formulation of the Household Problem

The state variables of a individual household are it's asset holdings and efficiency units. The aggregate states (and outcomes in other countries) only matter through prices and transfers and thus I summarize the aggregate state in country i as $S_i = (\{w_i\}_M, T_i, R_i)$ which is the collection of the wage per efficiency units, transfers, and interest rates. The wage vector $\{w_i\}_M$ is sufficient here since they determine prices in (3) and, thus, consumers can make the appropriate choice of commodities.

The value function of a household in country i is

$$v_i(a, z; S_i) = \max_{ij} \left\{ v_{ij}(a, z; S_i) \right\} \tag{8}$$

which is the maximum across the discrete choices of different national varieties. The value function conditional on a choice of variety is

$$v_{ij}(a, z; S_i) = \max_{a'} \left\{ u(c(j)) + \beta \, \mathbb{E}[v_i(a', z'; S_i')] \right\} \tag{9}$$

$$\text{subject to } (7) \text{ and } (6)$$

where households choose asset holdings and the level of consumption is residually determined through the budget constraint. The continuation value function is the expectation over (8) where the expectation is taken with respect to z' and taste shocks in the future. What this last point means is that households understand that their may be situations where, e.g., they really desire the high priced imported good and, hence, save accordingly.

As is well known, the Type 1 extreme value shocks give rise to the following choice probabilities for each differentiated good. So

$$\pi_{ij}(a, z; S_i) = \exp\left(\frac{v_{ij}(a, z; S_i)}{\sigma_\epsilon}\right) \Big/ \sum_{j'} \exp\left(\frac{v_{ij'}(a, z; S_i)}{\sigma_\epsilon}\right) \tag{10}$$

which is the probability that a household with assets a and efficiency units z chooses country variety j. And then the expectation of (8) with respect to the taste shocks takes the familiar log-sum form

$$\mathbb{E}_\epsilon v_i(a, z; S_i) = \sigma_\epsilon \log\left\{ \sum_{j'} \exp\left(\frac{v_{ij}(a, z; S_i)}{\sigma_\epsilon}\right) \right\} \tag{11}$$

Associated with the problem (8) are an asset policy function $g_{ij}(a, z; S_i)$ which prescribes asset holdings given a state and variety choice, and then define $c_{ij}(a, z; S_i)$ as the consumption function which prescribes consumption given states and variety choice.

1.4. Aggregation

Define the probability distribution of households across individual states $\lambda_i(a, z; S_i)$ which is the probability measure of households with asset levels a individual shock z in country i. This distribution evolves according to

$$\lambda_i(a', z', S_i') = \sum_z \sum_j \sum_{a:a'=g_{ij}(a,z;S_i)} \pi_{ij}(a, z; S_i)\mathcal{P}(z, z')\lambda_i(a, z; S_i). \tag{12}$$

Aggregate Labor Supply. Aggregate efficiency units are

$$N_i = L_i \sum_z \sum_a z\lambda_i(a, z; S_i) \tag{13}$$

where the inner most term reflects efficiency units multiplied by the measure of households with that state. This is all multiplied by L_i which is the mass of households in country i.

Asset Holdings. The aggregate quantity of asset holdings simply sums across the distribution conditional on choosing

$$\mathrm{A}_i' = L_i \sum_z \sum_j \sum_a g_{ij}(a, z; S_i)\pi_{ij}(a, z; S_i)\lambda_i(a, z; S_i). \tag{14}$$

National Income and Consumption Starting from the production side of our economy, the value of aggregate production must equal aggregate payments to labor so

$$p_iY_i = p_iA_iN_i = L_i \sum_z \sum_a w_i \; z \; \lambda_i(a, z; S_i) \tag{15}$$

where the last term sums over wage payments for each household type. Then by summing over individual consumers' budget constraint and substituting in (15), we arrive at the aggregated budget constraint:

$$p_iY_i = \widetilde{P_iC_i} + \Big[- R_i\mathrm{A_i} + \mathrm{A_i'}\Big] - T_{i,\tau}, \tag{16}$$

where national income equals the value of aggregate consumption $\widetilde{P_iC_i}$, the country's the net asset position, all net of transfers. Here the value of aggregate consumption is

$$\widetilde{P_iC_i} = L_i \sum_z \sum_j \sum_a p_{ij}c_{ij}(a, z; S_i)\pi_{ij}(a, z; S_i)\lambda_i(a, z; S_i) \tag{17}$$

where one can see a bug and a feature of this model. Here there is an "index number problem" in the sense that there is not an ideal price index for which one can decompose aggregate values in to a price and quantity component. This is in contrast to, e.g., a model where households consume a CES bundle of goods.

Trade Flows It's first worth walking though imports for a given set of states. So for households with states a and z we have

$$M_{ij}(a, z; S_i) = p_{ij} c_{ij}(a, z; S_i). \tag{18}$$

Then aggregate imports from country i to country j sums over this weighted by the mass of households choosing variety j and who in those states so

$$M_{ij} = L_i \sum_z \sum_a M_{ij}(a, z; S_i) \pi_{ij}(a, z; S_i) \lambda_i(a, z; S_i). \tag{19}$$

The same can be done for a countries exports. Again, focusing on exports to a location given a set of states we have

$$X_{ji}(a, z; S_j) = p_{ji} c_{ji}(a, z; S_j) \tag{20}$$

Then aggregate exports from country i to country j

$$X_{ji} = L_j \sum_z \sum_a X_{ji}(a, z; S_j) \pi_{ji}(a, z; S_j) \lambda_j(a, z; S_j) \tag{21}$$

1.5. Market Clearing and the Decentralized Equilibrium

Given the definitions above, I discuss the market clearing conditions than an equilibrium must respect.

The Goods Market. From here we can equate the value of production of commodity i in country i with global demand for country i's commodity:

$$p_i Y_i = \sum_j^M X_{ji}, \tag{22}$$

where the left hand side is production and the right hand side is world demand for the commodity from (21).

The Bond Market. The second market clearing condition is in the bond market. Two cases are considered "financial globalization" in which there is a global bond market with one real interest rate R. In this case the market clearing condition is

$$\sum_{i}^{M} \mathrm{A}_{\mathrm{i}}' = 0 \tag{23}$$

which says that net asset demand must equal zero across all countries. The second case considerer is "financial autarky" in which there is a local bond market that facilitates within country risk-sharing, but not globally. In this case, there is an interest rate is R_i for each country and the associated market clearing condition is

$$\mathrm{A}_{\mathrm{i}}' = 0 \ \ \forall i \tag{24}$$

Below is a formal definition of a Stationary Equilibrium when the aggregate state S_i is constant and not changing.

A Stationary Equilibrium. A Stationary Equilibrium are asset policy functions and commodity choice probabilities $\{ \ g_{ij}(a,z), \pi_{ij}(a,z) \ \}_i$, probability distributions $\lambda_i(a,z)$, and positive real numbers $\{w, p, R\}_i$ such that

i Prices (w, p) satisfy (2, 3);

ii The policy functions and choice probabilities solve the household's optimization problem in (**??**);

iv The probability distribution $\lambda_i(a,z)$ induced by the policy functions, choice probabilities, and primitives satisfies (12) and is a stationary distribution;

v Goods market clears:

$$p_i Y_i - \sum_{j}^{M} X_{ji} = 0, \ \ \forall i \tag{25}$$

v Bond market clears with either Financial Globalization with $R_i = R$ and

$$\sum_{i}^{M} \mathrm{A}_{\mathrm{i}}' = 0. \tag{26}$$

Or Financial Autarky where

$$\mathrm{A}_{\mathrm{i}}' = 0, \ \ \forall i \tag{27}$$

1.6. The Centralized Equilibrium

Before discussing some properties of the model, the Centralized (Efficient) Equilibrium is an important point of comparison. To characterize this equilibrium, I take a stand on the Social Welfare function and focus on a Utilitarian planner that places equal weight on all households in the global economy. Social welfare is:

$$\mathcal{W} = \sum_{t=0}^{\infty} \sum_z \sum_i \sum_j \beta^t \left\{ u(c_{ij}(z,t)) + \mathrm{E}[\ \epsilon \mid \mu_{ij}(z,t)] \right\} \mu_{ij}(z,t) L_i \lambda_i(z,t). \tag{28}$$

Working from the inside out, the inner most term is utility from consumption $c_{ij}(z,t)$ conditional on shock, country source and destination plus the expected value of the preference shocks conditional on shock-specific choice probabilities. The inner most term is weighted according to the measure of households in i sourcing from j $\mu_{ij}(z,t)$ times the total measure of households $L_i\lambda_i(z,t)$ in country i with shock z at date t. Then this is done at all dates t and discounted at the same rate β that households discount utility.

The Planner maximizes (28) subject to several constraints. The first is the law of motion describing how the measure of households evolves across states

$$\lambda_i(z', t+1) = \sum_z \mathcal{P}(z, z') \lambda_i(z,t). \tag{29}$$

Given a distribution of households, the effective labor units in each country are

$$N_{i,t} = L_i \sum_z z \lambda_i(z,t), \tag{30}$$

and then aggregate production of the final good is

$$Y_{it} = A_i N_{i,t}. \tag{31}$$

Combining the amount of resources available in (31) with the consumption and sourcing decisions yields the following resource constraint:

$$Y_{it} \ \geq \sum_j \sum_z d_{ji} c_{ji}(z,t) \mu_{ji}(z,t) L_j \lambda_j(z,t) \tag{32}$$

which says that production must be greater than or equal to world consumption of variety i inclusive of trade costs d_{ji}.

Given the social welfare function in 28 and the constraints discussed above, the **Centralized**

Planner's Problem is the following:

$$\mathcal{W}^* = \max_{c_{i,j}(z,t),\ \mu_{i,j}(z,t)} \sum_{t=0}^{\infty} \sum_{z} \sum_{i} \sum_{j} \beta^t \Big\{ u(c_{ij}(z,t)) + \mathrm{E}[\ \epsilon \mid \mu_{i,j}(z,t)] \Big\} \mu_{i,j}(z,t) L_i \lambda_i(z,t)$$

$$\text{subject to } (29)\ (31) \text{ and } (32) \text{ and an inital condition } \lambda_i(z,0). \tag{33}$$

A Stationary Centralized Planner Allocation. A Stationary Centralized Planner Allocation are time invariant policy functions $\{\ c_{i,j}(z),\ \mu_{i,j}(z)\ \}$, a probability distribution $\lambda_i(z)$, and positive real numbers N_i for each country i where:

i The policy functions solve the Centralized Planner's Problem in (33);

ii The probability distribution $\lambda_i(z)$ associated with (29) is a stationary distribution;

iii Effective labor units satisfy (30).

1.7. Special Cases / Running Examples

There are two special cases that I will refer back to repeatedly in the text. One is what I will call the CES case in which I mean that the household has access to a constant elasticity aggregator over the different varieties. The second case is the "hand-to-mouth" case in which households have no access to borrowing or lending.

CES case. This case is relatively familiar and standard. Here there is an aggregator over national varieties of the CES class, so

$$c = \left\{ \sum_{j}^{M} c_j^{\frac{\theta-1}{\theta}} \right\}^{\frac{\theta}{\theta-1}}, \tag{34}$$

where θ controls the elasticity of substitution across products. Then each household has the following demand curve for a region's variety:

$$c_{ij}(a, z; S_i) = \left(\frac{p_{ij}}{P_i}\right)^{-\theta} c_i(a, z; S_i). \tag{35}$$

which is given the total amount of consumption $c_i(a, z; S_i)$ a household chooses given their states and P_i is the CES price index:

$$P_i = \left\{ \sum_{j}^{M} p_{ij}^{1-\theta} \right\}^{\frac{1}{1-\theta}}. \tag{36}$$

what is unique about this setting is that household expenditure shares on different goods are independent of their state. So the expenditure share of a household in location i with state a and z is

$$\frac{p_{ij} c_{ij}(a, z; S_i)}{P_i c_i(a, z; S_i)} = \left(\frac{p_{ij}}{P_i}\right)^{1-\theta}, \tag{37}$$

which depends only on prices that all households face. What is happening here is that the CES aggregator is homothetic in total consumption. So while households are choosing different levels of total consumption to solve their income-fluctuations problem, how that demand is divided up is always the same. From here, one can aggregate and arrive at a "gravity-like" import demand system with households with states a and z importing

$$M_{ij}(a, z; S_i) = p_{ij} \left(\frac{p_{ij}}{P_i}\right)^{-\theta} c_i(a, z; S_i). \tag{38}$$

Then aggregate imports from country i to country j sums over this weighted by the mass of households in those states which gives

$$M_{ij} = \left(\frac{p_{ij}}{P_i}\right)^{1-\theta} \times P_i C_i \tag{39}$$

where the last term follows by noting that the sum of $c_i(a, z; S_i)$ across the distribution is aggregate consumption and then this is put in value terms by multiplying and dividing by the CES price index.

Hand-to-Mouth, No Labor Supply,Log Households. This case focuses on the situation described—so there does not exist a risk free asset to smooth consumption, nor can households adjust their labor supply. The value function of a household contemplating the purchase of national variety j is:

$$v_{ij}(a, z; S_i) = u\left(\frac{w_i z}{p_{ij}}\right) + \epsilon(j) + \beta\, \mathbb{E}[v_i(z'; S_i')] \tag{40}$$

and then the choice probability becomes

$$\pi_{ij}(z; S_i) = \exp\left(\frac{v_{ij}(z; S_i)}{\sigma_\epsilon}\right) \Big/ \sum_{j'} \exp\left(\frac{v_{ij'}(z; S_i)}{\sigma_\epsilon}\right) \tag{41}$$

$$\pi_{ij}(z; S_i) = \exp\left(\frac{u\left(\frac{w_i z}{p_{ij}}\right))}{\sigma_\epsilon}\right) \Big/ \sum_{j'} \exp\left(\frac{u\left(\frac{w_i z}{p_{ij}}\right)}{\sigma_\epsilon}\right) \tag{42}$$

where the last line follows from the properties of the $\exp$ function and that the continuation value function is exactly the same independent of the good chosen. Two more steps. Then with $\log$ this expression collapses to

$$\pi_{ij}(S_i) = \frac{p_{ij}^{-1/\sigma_\epsilon}}{\sum_{j'} p_{ij'}^{-1/\sigma_\epsilon}} \tag{43}$$

where here we recover the well known "ces-logit-isomorphisim" between the logit demand system and the CES aggregator in 34.

Appendix

A. Appendix: Static Logit Trade Model

This case focuses on the most basic model of household heterogeneity: static, log preferences, logit shocks and is a key point of contact. The value function of a household contemplating the purchase of national variety j is:

$$v_{ij}(z) = \log\left(\frac{w_i z}{p_{ij}}\right) + \epsilon(j) \tag{44}$$

and then the choice probability becomes

$$\pi_{ij}(z) = \exp\left(\frac{v_{ij}(z)}{\sigma_\epsilon}\right) \Bigg/ \sum_{j'} \exp\left(\frac{v_{ij'}(z)}{\sigma_\epsilon}\right) \tag{45}$$

$$\pi_{ij}(z) = \exp\left(\frac{\log\left(\frac{w_i z}{p_{ij}}\right)}{\sigma_\epsilon}\right) \Bigg/ \sum_{j'} \exp\left(\frac{\log\left(\frac{w_i z}{p_{ij}}\right)}{\sigma_\epsilon}\right) \tag{46}$$

with the $\log$ specification this expression collapses to

$$\pi_{ij} = \frac{p_{ij}^{\frac{-1}{\sigma_\epsilon}}}{\sum_{j'} p_{ij'}^{\frac{-1}{\sigma_\epsilon}}} \tag{47}$$

as wages an efficiency units cancel. To arrive at an expression for aggregate imports, note that individual imports are

$$M_{ij}(z) = p_{ij}\frac{w_i z}{p_{ij}}. \tag{48}$$

then aggregate imports from country i to country j sums over this weighted by the mass of households in those states which gives

$$M_{ij} = \sum_z p_{ij} \frac{w_i z}{p_{ij}} \pi_{ij} \lambda(z) \tag{49}$$

$$M_{ij} = \pi_{ij} \sum_z w_i z \lambda(z) \tag{50}$$

$$M_{ij} = \frac{p_{ij}^{\frac{-1}{\sigma_\epsilon}}}{\sum_{j'} p_{ij'}^{\frac{-1}{\sigma_\epsilon}}} \times \widetilde{P_i C_i} \tag{51}$$

where the last term follows from imposing the aggregated budget constraint with aggregate labor income equalling aggregate expenditure. Then to get towards something that looks like Eaton and Kortum (2002) substitute in the unit shipping costs from (3) and set $-\theta = -1/\sigma_\epsilon$ yielding:

$$M_{ij} = \frac{A_j^\theta (w_j d_{ij})^{-\theta}}{\sum_{j'} A_j^\theta (w_j d_{ij})^{-\theta}} \times \widetilde{P_i C_i} \tag{52}$$

and it becomes immediate that the trade share (imports divided by consumption) is π_{ij}.

Given this representation of trade flows, I'm going to derive the "trade elasticity" and then welfare. I'll define the trade elasticity as the following:

$$\frac{\partial \pi_{ij}/\pi_{ii}}{\partial d_{ij}} \times \frac{d_{ij}}{\pi_{ij}/\pi_{ii}} \tag{53}$$

or the percent change in trade relative to domestic trade due to a change in trade costs which is our labeling of the elasticity in Simonovska and Waugh (2014). In this setup the "trade elasticity" is:

$$\frac{\partial \pi_{ij}/\pi_{ii}}{\partial d_{ij}} \times \frac{d_{ij}}{\pi_{ij}/\pi_{ii}} = -\theta \tag{54}$$

so it simply reflects the dispersion parameter σ_ϵ.

I can also derive the Arkolakis et al. (2012) sufficient statistic welfare result. Expected utility prior to the realization of the tastes shocks is:

$$\mathcal{W} = \sigma_\epsilon \log \left\{ \sum_{j'} \exp\left(\frac{v_{ij'}(z)}{\sigma_\epsilon} \right) \right\} \tag{55}$$

which again with log preferences can be written as:

$$\mathcal{W} = \sigma_\epsilon \log w_i^{\frac{1}{\sigma_\epsilon}} \ + \sigma_\epsilon \log \left\{ \sum_{j'} p_{ij'}^{\frac{-1}{\sigma_\epsilon}} \right\} \tag{56}$$

I'm always free to make one normalization on prices, so a convenient one is to set $w_i = 1.0$. This then implies from 3 that $p_{ii} = 1/A_i$ and then the countries home share is

$$\pi_{ii} = \frac{A_i^{\frac{1}{\sigma_\epsilon}}}{\sum_{j'} p_{ij'}^{\frac{-1}{\sigma_\epsilon}}} \tag{57}$$

or

$$\pi_{ii}^{-1} A_i^{\frac{1}{\sigma_\epsilon}} \ = \sum_{j'} p_{ij'}^{\frac{-1}{\sigma_\epsilon}} \tag{58}$$

And then substituting into the welfare expression gives

$$\mathcal{W} = \log A_i - \sigma_\epsilon \log \pi_{ii} \tag{59}$$

which is, essentially the Arkolakis et al. (2012) expression. The only substantive difference is that the expression above are in units of utils. To put this in consumption units, take the exponent and then swap into the θ notation and one has:

$$\tilde{\mathcal{W}} = A_i \pi_{ii}^{\frac{-1}{\theta}} \tag{60}$$

which is exactly the Arkolakis et al. (2012) expression. The following proposition summarizes the result:

Proposition 1 (Static Heterogeneity: Gravity, and Welfare) *In the static model with log preferences and logit preference shocks where $\theta = 1/\sigma_\epsilon$, the decentralized allocation is characterized a Gravity equation for country i's imports from country j:*

$$M_{ij} = \frac{A_j^\theta (w_j d_{ij})^{-\theta}}{\sum_{j'} A_j^\theta (w_j d_{ij})^{-\theta}} \times \widetilde{P_i C_i} \tag{61}$$

where the trade elasticity defined as

$$\frac{\partial (M_{ij}/M_{ii})}{\partial d_{ij}} \times \frac{d_{ij}}{(M_{ij}/M_{ii})} = -\theta, \tag{62}$$

and is determined by the dispersion of the preferences shocks. Finally the welfare gains from trade in *consumption units are summarized by*

$$\tilde{\mathcal{W}} = A_i \pi_{ii}^{\frac{-1}{\theta}} \tag{63}$$

where π_{ii} a country i's share of consumption expenditure on goods from itself.

B. Appendix: Dynamic Log-Logit Trade Model

This section explores the following problem:

$$v_i(a, z; S_i) = \max_{ij} \left\{ \ v_{ij}(a, z; S_i) \ \right\} \tag{64}$$

which is the maximum across the discrete choices of different national varieties. The value function conditional on a choice of variety is

$$v_{ij}(a, z; S_i) = \max_{a'} \left\{ \log(c_{ij}) + \beta \, \mathbb{E}[v_i(a', z'; S_i')] \right\} \tag{65}$$

$$\text{subject to } (7) \text{ and } (6)$$

where the only restriction now is that preferences are restricted to be $\log$. A couple of observations is that the choice specific value function can be written as

$$v_{ij}(a, z; S_i) = \max_{a'} \left\{ \log\left(\frac{Ra + wz - a'}{p_{ij}}\right) + \beta \, \mathbb{E}[v_i(a', z'; S_i')] \right\} \tag{66}$$

which is then

$$v_{ij}(a, z; S_i) = \max_{a'} \left\{ \log(Ra + wz - a') + \beta \, \mathbb{E}[v_i(a', z'; S_i')] \right\} - \log p_{ij} \tag{67}$$

which then leads to the observation that the optimal a' conditional on a choice j is **independent** of the price of the consumption good. So what is going on is if you consume an expensive or cheap good, then consumption simply scales up or down so that assets next period are exactly the same. This was verified on the computer. Another approach is to conjecture this policy function and plug it into the euler equation above and show that it respects it.

Then the choice probability becomes

$$\pi_{ij}(a,z;S_i) = \exp\left(\frac{v_{ij}(a,z;S_i)}{\sigma_\epsilon}\right) \Big/ \sum_{j'} \exp\left(\frac{v_{ij'}(a,z;S_i)}{\sigma_\epsilon}\right) \tag{68}$$

$$\pi_{ij} = \exp\left(\frac{-\log p_{ij}}{\sigma_\epsilon}\right) \Big/ \sum_{j'} \exp\left(\frac{-\log p_{ij}}{\sigma_\epsilon}\right) \tag{69}$$

$$\pi_{ij} = \frac{p_{ij}^{\frac{-1}{\sigma_\epsilon}}}{\sum_{j'} p_{ij'}^{\frac{-1}{\sigma_\epsilon}}} \tag{70}$$

where the second line follows from (67) which is exactly the same as discussed above in the static model.

C. Appendix: The Generalized Trade Elasticity

Again, I' the trade elasticity defined as

$$\frac{\partial(M_{ij}/M_{ii})}{\partial d_{ij}} \times \frac{d_{ij}}{(M_{ij}/M_{ii})} = \left(\frac{M'_{ij}M_{ii} - M_{ij}M'_{ii}}{M_{ii}^2}\right)\left(\frac{M_{ii}}{M_{ij}}\right) d_{ij} \tag{71}$$

$$= \left(\frac{M'_{ij}}{M_{ij}} - \frac{M'_{ii}}{M_{ii}}\right) d_{ij} \tag{72}$$

which is the difference in elasticities between how trade between i and j change relative to home trade. So let's compute how trade changes

$$\frac{\partial M_{ij}}{\partial d_{ij}} = \sum_{a,z}\left\{\frac{\partial p_{ij}c_{ij}(a,z)}{\partial d_{ij}}\pi_{ij}(a,z)\lambda(a,z) + \frac{\partial \pi_{ij}(a,z)}{\partial d_{ij}} p_{ij}c_{ij}(a,z)\lambda(a,z) + \frac{\partial \lambda(a,z)}{\partial d_{ij}} p_{ij}c_{ij}(a,z)\pi_{ij}(a,z)\right\} \tag{73}$$

and then from here divide the inside of the brackets by $p_{ij}c_{ij}(a,z)\pi_{ij}(a,z)\lambda(a,z)$ and multiply the outside which gives

$$\frac{\partial M_{ij}}{\partial d_{ij}} = \sum_{a,z}\left\{\frac{\partial p_{ij}c_{ij}(a,z)/p_{ij}c_{ij}(a,z)}{\partial d_{ij}} + \frac{\partial \pi_{ij}(a,z)/\pi_{ij}(a,z)}{\partial d_{ij}} + \frac{\partial \lambda(a,z)/\lambda(a,z)}{\partial d_{ij}}\right\} p_{ij}c_{ij}(a,z)\pi_{ij}(a,z)\lambda(a,z) \tag{74}$$

then define the following "weight" which is the share of goods that those with states a, z account for in total expenditures from j giving

$$\omega_{ij}(a,z) = \frac{p_{ij}c_{ij}(a,z)\pi_{ij}(a,z)\lambda(a,z)}{M_{ij}} \tag{75}$$

then from this definition we have

$$\frac{\partial M_{ij}/M_{ij}}{\partial d_{ij}/d_{ij}} = \sum_{a,z}\left\{\frac{\partial p_{ij}c_{ij}(a,z)/p_{ij}c_{ij}(a,z)}{\partial d_{ij}/d_{ij}} + \frac{\partial \pi_{ij}(a,z)/\pi_{ij}(a,z)}{\partial d_{ij}/d_{ij}} + \frac{\partial \lambda(a,z)/\lambda(a,z)}{\partial d_{ij}/d_{ij}}\right\}\omega_{ij}(a,z) \tag{76}$$

which I'll rearrange and label the following way

$$\frac{\partial M_{ij}/M_{ij}}{\partial d_{ij}/d_{ij}} = \sum_{a,z}\underbrace{\left\{\frac{\partial p_{ij}c_{ij}(a,z)/p_{ij}c_{ij}(a,z)}{\partial d_{ij}/d_{ij}}\right\}}_{\theta_{ij}(a,z)^I}\omega_{ij}(a,z) + \sum_{a,z}\underbrace{\left\{\frac{\partial \pi_{ij}(a,z)/\pi_{ij}(a,z)}{\partial d_{ij}/d_{ij}}\right\}}_{\theta_{ij}(a,z)^E}\omega_{ij}(a,z) \tag{77}$$

$$+ \underbrace{\left\{\sum_{a,z}\frac{\partial \lambda(a,z)/\lambda(a,z)}{\partial d_{ij}/d_{ij}}\right\}}_{\theta_{ij}(a,z)^D}\omega_{ij}(a,z) \tag{78}$$

Or in other words **IED**

$$\frac{\partial M_{ij}/M_{ij}}{\partial d_{ij}/d_{ij}} = \sum_{a,z}\left\{\theta_{ij}(a,z)^I + \theta_{ij}(a,z)^E + \theta_{ij}(a,z)^D\right\}\omega_{ij}(a,z) \tag{79}$$

where the aggregate elasticity of imports from j is a weighted average of elasticities at the household level. And household level elasticities are decomposed into the following components

- **I:** The household level intensive margin trade elasticity $\theta(a,z)^I_{ij}$
- **E:** The household level extensive margin trade elasticity $\theta(a,z)^E_{ij}$
- **D:** The distribution elasticity $\theta(a,z)^D_{ij}$, i.e. how the mass of agents changes across household states a, z.

Then from here one can see how things collapse or are generalized depending upon the case. First, let's look at each elasticity state by state. Also note that these are partial equilibrium elasticities, they don't take into account how wages or interest rates change in response to clear the goods or asset market.

Intensive Margin Trade Elasticity This can be determined by just working off the budget constraint so

$$\underbrace{\frac{\partial p_{ij}c_{ij}(a,z)/p_{ij}c_{ij}(a,z)}{\partial d_{ij}/d_{ij}}}_{\theta_{ij}(a,z)^I} = -\frac{\partial a(a,z)'}{\partial d_{ij}/d_{ij}} \times \frac{1}{p_{ij}c_{ij}(a,z)} \tag{80}$$

where the key issue determining the intensive margin elasticity is how the household trades off assets for a change in trade costs. Just to be clear how this works: a reduction in trade costs lowers prices and relaxes the budget constraint, so how the incremental amount of resources available to spend is dived between assets and consumption determines the intensive margin trade elasticity. Here are two special cases:

- **Static Discrete Choice**: Here there is no asset choice and thus the intensive margin trade elasticity is

$$\theta_{ij}(z)^I = 0 \tag{81}$$

  which means that any change in prices via a change in trade costs maps one for one into a change in consumption and hence $p_{ij}c_{ij}(z)$ is constant.

- **Dynamic Log + Discrete Choice:** This cases is interesting because assets are being held, but as argued above in (67) asset holdings are independent of the variety choice j. Hence they are independent of the change in trade costs d_{ij} and we have

$$\theta_{ij}(a,z)^I = 0 \tag{82}$$

  so what is happening here is that with log preferences is that the price elasticity of consumption is minus one. A one percent decrease in prices leads to a one percent increase in consumption and when viewed through the budget constraint, this implies that assets are not changing with respect to prices. This result has a second implication that the distribution elasticity $\theta(a,z)^D_{ij}$ is unchanging even though assets are being held. That is in the log preference case there is not a reshuffling of households across states in the stationary distribution.

Extensive Margin Trade Elasticity This where the Type 1 extreme value functional form matters. First, I'm going to define the denominator term of the choice probability as the following:

$$\Phi_i(a,z) = \sum_{j'} \exp\left(\frac{v_{ij'}(a,z)}{\sigma_\epsilon}\right) \tag{83}$$

Then the elasticity of the choice probability with respect to a change in trade costs is

$$\underbrace{\left\{\frac{\partial \pi_{ij}(a,z)/\pi_{ij}(a,z)}{\partial d_{ij}/d_{ij}}\right\}}_{\theta_{ij}(a,z)^E} = \frac{1}{\sigma}\frac{\partial V_{ij}(a,z)}{\partial d_{ij}/d_{ij}} - \frac{\partial \Phi_i(a,z)/\Phi_i(a,z)}{\partial d_{ij}/d_{ij}} \tag{84}$$

where the key issue here is how a change in trade costs changes the choice specific value function. In other words, how much more valuable does choice j become when it's cheaper to import variety j. And this is always compared to the change overall change in the value of options across variety, i.e., how Φ_i changes. In general, this is a complicated object as it is state dependent and is forward looking in with the value function showing up, not the period utility function. With that said, there are a couple of illustrative cases.

- **Static Logit**: Here there is no asset choice and forward looking component

  $$\theta_{ij}(z)^E = \frac{u'(c_{ij}(z))}{\sigma}\frac{\partial c_{ij}(z)}{\partial d_{ij}/d_{ij}} - \frac{\partial \Phi_i(z)/\Phi_i(z)}{\partial d_{ij}/d_{ij}} \tag{85}$$

  $$= \frac{u'(c_{ij}(z))c_{ij}(z)}{\sigma} - \frac{\partial \Phi_i(z)/\Phi_i(z)}{\partial d_{ij}/d_{ij}} \tag{86}$$

  and focusing on the first part says that the extensive margin trade elasticity is related to the marginal utility of consumption multiplied by the level of consumption (does this have a meaning?). Even in the static case, there is a sense in which the trade elasticity would depend upon income via the state z.

  In the case of log preferences, these issues partially disappear with the extensive margin elasticity becoming

  $$= \frac{1}{\sigma} - \frac{\partial \Phi_i/\Phi_i}{\partial d_{ij}/d_{ij}} \tag{87}$$

  and it's this result in that would deliver a constant, aggregate trade elasticity the second term cancels (shown below).

- **Dynamic Log-Logit**: Interestingly, this case collapses to something that is similar to the Static Log-Logit. The insight is the because the asset policy function does not depend upon the variety choice, then the continuation value function is common across choices and, thus, it cancels in the numerator and denominator of the choice probabilities. Thus the extensive margin trade elasticity becomes

$$\theta_{ij}(a,z)^E = \frac{1}{\sigma} - \frac{\partial \hat{\Phi}_i / \hat{\Phi}_i}{\partial d_{ij}/d_{ij}} \tag{88}$$

where the $\hat{\Phi}$ is the value of the denominator after continuation value functions and period utility functions are canaled out.

D. Appendix: Endogenous Grid Method

First, I'm going to derive the Euler equation for this model. I'll abstract from the situation in which the HH is at the borrowing constraint.

Focus on the within a variety choice component, the households value function can be written as:

$$v_{ij}(a,z) = \max_{a'} u\left(\frac{R_i a + w_i z - a'}{p_{ij}}\right) + \beta \mathrm{E} v(a', z') \tag{89}$$

then the first order condition associated with this problem is:

$$\frac{u'(c_{ij}(a,z))}{p_{ij}} = \beta \mathrm{E} \frac{\partial v(a', z')}{\partial a'} \tag{90}$$

which is saying that, conditional on a variety choice the left hand side is the loss in consumption units which is $1/p_{ij}$ evaluated at the marginal utility of consumption and then this is set equal to the marginal gain from saving a bit more which is how the value function changes with respect to asset holdings. Now we can arrive at the $\frac{\partial v(a',z')}{\partial a'}$ in the following way, so start from the log-sum expression for the expected value function

$$\mathbb{E}_\epsilon v(a', z') = \sigma_\epsilon \log \left\{ \sum_{j'} \exp\left(\frac{v_{ij}(a', z')}{\sigma_\epsilon}\right) \right\} \tag{91}$$

and then differentiate this with respect to asset holdings which gives:

$$\frac{\partial \mathbb{E}_\epsilon v(a', z')}{\partial a'} = \left(\frac{\sigma_\epsilon}{\sum_{j'} \exp\left(\frac{v_{ij}(a',z')}{\sigma_\epsilon}\right)}\right) \left[\sum_{j'} \exp\left(\frac{v_{ij}(a', z')}{\sigma_\epsilon}\right) \frac{1}{\sigma_\epsilon} \frac{\partial v_{ij}(a', z')}{\partial a'}\right] \tag{92}$$

Then if you look at this carefully and notices how the choice probabilities from (10) are embedded in here, we have:

$$\frac{\partial \mathbb{E}_\epsilon v(a', z')}{\partial a'} = \sum_{j'} \pi_{ij}(a', z) \frac{\partial v_{ij}(a', z')}{\partial a'} \tag{93}$$

and then we can just apply the Envelop theorem to the value functions associated with the discrete choices across the options:

$$\frac{\partial \mathbb{E}_\epsilon v(a', z')}{a'} = \sum_{j'} \pi_{ij}(a', z') \frac{u'(c_{ij}(a', z'))R_i}{p_{ij}} \tag{94}$$

So then putting everything together we have:

$$\frac{u'(c_{ij}(a, z))}{p_{ij}} = \beta R_i \mathrm{E}_{z'} \left[\sum_{j'} \pi_{ij}(a', z') \frac{u'(c_{ij}(a', z'))}{p_{ij}} \right] \tag{95}$$

where this has a very natural form: you set the marginal utility of consumption today equal to the marginal utility of consumption tomorrow adjusted by the return on delaying consumption, and the expected value of the marginal utility of consumption which reflects how the uncertainty over both ones' preference over different varieties and shocks to efficiency units. Taking into account the borrowing constraint then gives the generalized Euler equation from which the endogenous grid method will exploit:

$$\frac{u'(c_{ij}(a, z))}{p_{ij}} = \max \left\{ \beta R_i \mathrm{E}_{z'} \left[\sum_{j'} \pi_{ij}(a', z') \frac{u'(c_{ij}(a', z'))}{p_{ij}} \right] \ , \ u\left(\frac{R_i a + w_i - \phi_i}{p_{ij}} \right) \right\} \tag{96}$$

4.1. EGM-Discrete Choice Algorithm

Here is a proposed approach. This focuses on just the consumer side in one country i.

0. Set up an asset grid as usual. Then guess (i) a consumption function $g_{c,ij}(a, z)$ for each a, z, and product choice j and (ii) choice specific value function $v_{ij}(a, z)$.

1. Compute the choice probabilities from (10) for each (a, z) combination, given the guessed value functions.

1. Given the consumption function and choice probabilities compute the RHS of (96) first.

2. Then invert to find the new updated consumption choice so

$$c_{ij}(\tilde{a}, z) = u^{'-1}\left\{p_{ij} \max\left\{\beta R_i \mathrm{E}_{z'}\left[\sum_{j'} \pi_{ij}(a', z')\frac{u'(c_{ij}(a', z'))}{p_{ij}}\right] \ , \ u\left(\frac{R_i a + w_i - \phi_i}{p_{ij}}\right)\right\}\right\} \tag{97}$$

where $u^{'-1}$ is the inverse function of the marginal utility of consumption.

Side note: One of the interesting things about this equation is that the direct j component on the RHS that only affects the consumption choice is through the price. Can this be exploited? We also know the choice probabilities need to sum to one, so is there a way to map the consumption choice into the choice probabilities? Also, can interpolation be done once how p scales things...

3. The key issue in this method is that we have found $c_{ij}(\tilde{a}, z)$ where the consumption function is associated with some asset level that is not necessarily on the grid. The solution is to (i) use the budget constraint and infer $\tilde{a}$ given that a' was chosen above (that's where we started), z, and $c_{ij}(\tilde{a}, z)$. Now we have a map from $\tilde{a}$ to a' for which one can use interpolation to infer the a' chosen given a where a is on the grid.

- Do steps **2.** and **3.** for each j variety choice. This then makes the function $g_{a,ij}(a, z)$ mapping each state and j choice (today) into a', z' states and then from the budget constraint we have an associated consumption function $g_{c,ij}(a, z)$

4. Compute the $\mathrm{E}\left[v(g_{a,ij}(a, z), z')\right]$. This is performed in the `make_Tv_upwind!` function. It fixes a country j, then works through shocks and asset states today and from the policy function $g_{a,ij}(a, z)$ figures out the asset choice tomorrow. Then the $\mathrm{E}\left[v(g_{a,ij}(a, z), z')\right]$ is (11) over the different variety choices tomorrow (this is the integration over ϵ) multiplied by the probability of z' occurring (this is the integration over z).

5. Given **4.** update the value function using the bellman equation evaluated at the optimal policies:

$$Tv_{ij}(a, z) = u(g_{c,ij}(a, z)) + \beta \mathrm{E}\left[v(g_{a,ij}(a, z), z')\right] \tag{98}$$

6. Compare old and new policy functions, old and new value functions, and then update accordingly.

E. Appendix: The Planning Problem

The Lagrangian associated with the Centralized Planning problem is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{z} \sum_{i} \sum_{j} \beta^t \left\{ u(c_{ij}(z,t)) + \mathrm{E}[\, \epsilon \mid \mu_{ij}(z,t)] \right\} \mu_{ij}(z,t) L_i \lambda_i(z,t), \tag{99}$$

$$+ \sum_{t=0}^{\infty} \sum_{i} \chi_i(t) \left\{ Y_{it} \ - \ \sum_{j} \sum_{z} d_{ji} c_{ji}(z,t) \mu_{ji}(z,t) L_j \lambda_j(z,t) \right\}$$

$$+ \sum_{t=0}^{\infty} \sum_{i} \chi_{2i}(t) \left\{ 1 - \sum_{j} \mu_{ij}(z,t) \right\} L_i \lambda_i(z,t)$$

Then associated with this problem are two first order conditions. The first one is consumption in i with respect to variety j so:

$$\frac{\partial \mathcal{L}}{\partial c_{ij}(z,t)} = \beta^t u'(c_{ij}(z,t)) \mu_{ij}(z,t) L_i \lambda_i(z,t) - \chi_j(t) d_{ij} \mu_{ij}(z,t) L_i \lambda_i(z,t) = 0 \tag{100}$$

$$\Rightarrow \quad \beta^t u'(c_{ij}(z,t)) = \chi_j(t) d_{ij}, \tag{101}$$

which says that the marginal utility of consumption of (i consuming variety j) should equal j's multiplier adjusted by the trade cost d_{ij}. Then notice that the right hand size is independent of z, thus within variety choice we have the result that $u'(c_{ij}(z,t)) = u'(c_{ij}(z',t)) = u'(\bar{c}_{ij}(t))$ for all z, z' combinations. Then the ratio of marginal utility across variety j is:

$$\frac{u'(\bar{c}_{ij}(t))}{u'(\bar{c}_{ij'}(t))} = \frac{\chi_j(t) d_{ij}}{\chi_{j'}(t) d_{ij'}} \tag{102}$$

which is starting to look like a gravity type regression where the χ's are related to the price index in the source countries, adjusted by the iceberg trade costs (e.g. equation (5) in SW for example). To be more concrete, compare i, j purchases from i, i purchases and with log utility one has

$$\frac{\bar{c}_{ij}}{\bar{c}_{ii}} = \frac{\chi_i(t)}{\chi_j(t)} d_{ij}^{-1} \tag{103}$$

which is quite close to a gravity equation where relative consumption equals a source and destination effect adjusted by the trade costs. Key differences are that this is in quantities, not expenditure; this does not reflect the extensive margin; and related to this last point is that the elasticity is γ.

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