Heterogeneous Agent Trade

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ABSTRACT ———

This paper develops a model of heterogenous agents and international trade. Heterogenous agents are modeled as in the standard incomplete markets tradition with household's facing incomplete insurance against idiosyncratic productivity and taste shocks. Trade in goods follows the Armington tradition but is derived from the "bottom up" with micro-level heterogeneity shaping the aggregate pattern of trade. In the efficient allocation, I recover standard results regarding gravity and the gains from trade. In the decentralized allocation, the pattern of trade is distorted, the aggregate trade elasticity is non-constant and the benefits from globalization are distributed unequally. I use model to explore two issues: the ability of trade policy to improve outcomes and how financial globalization complements trade in goods.

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1. The Model

The model is setup in a simple and transparent way. Trade is in the Armigton tradition with each country producing a nationally differentiated variety. Households follow the standard incomplete markets tradition. The key departure is that I lean into household heterogeneity and have households make a discrete choice over the varieties they consume. Aggregate trade flows between countries are, thus, given by the explicit aggregation of households and the choices they make.

1.1. Trade and Production

There are M locations which I will call a country. Each country produces a differentiated product as in the Armington tradition and these differentiated products. In country i, competitive firms have the following production technology to produce variety i:

$$q_i = A_i N_i, \tag{1}$$

where N_i are the efficiency units of labor supplied by households in country i.

Trade faces several obstacles. There are iceberg trade costs d_{ji} for a good to go from supplier i to buyer j. Cross-border trade faces policy obstacles, i.e. tariffs τ_{ji}. The notation here is such that τ_{ji} which is the ad-valorem tariff rate that country j imposes on the commodity that county i produces.

Profit maximization of the competitive goods producers in location i results in the wage per efficiency unit reflecting the value of the marginal product of labor

$$w_i = p_i A_i. \tag{2}$$

Given iceberg trade costs and tariffs, the unit cost for country j to purchase a good from location i is

$$p_{ji} = \frac{d_{ji}(1 + \tau_{ji})w_i}{A_i}. \tag{3}$$

1.2. Households

There is a mass of L_i households in each location i. Households are immobile across countries. They are infinite lived and have time-sparable preferences over non-durable consumption va-

rieties:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(\{c_{jt}\}_M), \tag{4}$$

where the notation $\{c_{jt}\}_M$ means that the household has preferences over all j varieties supplied by M countries in the world. My focus is on the situation where households each period receive additive Type 1 extreme value shocks $\epsilon(j)$ with dispersion parameter σ_ϵ and then households make a discrete choice about which variety j to each period and a continuous choice about how much. More specifically, the utility associated with the choice of variety j and leisure is

$$u(c(j)_t) = \frac{c_{jt}^{1-\gamma}}{1-\gamma} + \epsilon(j)_t. \tag{5}$$

where consumption mapped into utils with a standard CRRA function, $\epsilon(j)$ is the taste shock.

A household's efficiency units are stochastic and they evolve according to a discrete state Markov chain. Mathematically, z is a households efficiently units and $\mathcal{P}(z, z')$ describes the probability of a household with state z efficiency units transiting to state z'.

Households can save and borrow in a non-state contingent asset a. The units of the asset are chosen to be the numeraire and pays out with gross interest rate R. I discuss this more in depth below, but the determination of R_i is either exogenously given or the rate that clears the bond market (local or global). An country specific, exogenous debt limit ϕ_i constrains borrowing so:

$$a_{t+1} \geq -\phi_i. \tag{6}$$

All these pieces come together in the household's budget constraint, conditional on choosing variety j to consume, and focusing on a stationary setting where prices and transfers are constant:

$$a_{t+1} + p_{ij}c_{jt} \leq R_i a_t + w_i z_t + T_{i,\tau}. \tag{7}$$

The value of asset purchases and consumption expenditures must be less than or equal to asset payments, labor earnings, and transfers arising from trade policy (the $T_{i,\tau}$).

1.3. Recursive Formulation of the Household Problem

The state variables of a individual household are it's asset holdings and efficiency units. The aggregate states (and outcomes in other countries) only matter through prices and transfers and thus I summarize the aggregate state in country i as $S_i = (\{w_i\}_M, T_i, R_i)$ which is the collection

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of the wage per efficiency units, transfers, and interest rates. The wage vector $\{w_i\}_M$ is sufficient here since they determine prices in (3) and, thus, consumers can make the appropriate choice of commodities.

The value function of a household in country i is

$$v_i(a, z; S_i) = \max_{ij} \left\{ v_{ij}(a, z; S_i) \right\} \tag{8}$$

which is the maximum across the discrete choices of different national varieties. The value function conditional on a choice of variety is

$$v_{ij}(a, z; S_i) = \max_{a'} \left\{ u(c(j)) + \beta \, \mathbb{E}[v_i(a', z'; S_i')] \right\} \tag{9}$$

$$\text{subject to } (7) \text{ and } (6)$$

where households choose asset holdings and the level of consumption is residually determined through the budget constraint. The continuation value function is the expectation over (8) where the expectation is taken with respect to z' and taste shocks in the future. What this last point means is that households understand that their may be situations where, e.g., they really desire the high priced imported good and, hence, save accordingly.

As is well known, the Type 1 extreme value shocks give rise to the following choice probabilities for each differentiated good. So

$$\pi_{ij}(a, z; S_i) = \exp\left(\frac{v_{ij}(a, z; S_i)}{\sigma_\epsilon}\right) \bigg/ \sum_{j'} \exp\left(\frac{v_{ij'}(a, z; S_i)}{\sigma_\epsilon}\right) \tag{10}$$

which is the probability that a household with assets a and efficiency units z chooses country variety j. And then the expectation of (8) with respect to the taste shocks takes the familiar log-sum form

$$\mathbb{E}_\epsilon v_i(a, z; S_i) = \sigma_\epsilon \log\left\{ \sum_{j'} \exp\left(\frac{v_{ij}(a, z; S_i)}{\sigma_\epsilon}\right) \right\} \tag{11}$$

Associated with the problem (8) are an asset policy function $g_{ij}(a, z; S_i)$ which prescribes asset holdings given a state and variety choice, and then define $c_{ij}(a, z; S_i)$ as the consumption function which prescribes consumption given states and variety choice.

1.4. Aggregation

Define the probability distribution of households across individual states $\lambda_i(a, z; S_i)$ which is the probability measure of households with asset levels a individual shock z in country i. This distribution evolves according to

$$\lambda_i(a', z', S_i') = \sum_z \sum_j \sum_{a:a'=g_{ij}(a,z;S_i)} \pi_{ij}(a, z; S_i)\mathcal{P}(z, z')\lambda_i(a, z; S_i). \tag{12}$$

Aggregate Labor Supply. Aggregate efficiency units are

$$N_i = L_i \sum_z \sum_a z\lambda_i(a, z; S_i) \tag{13}$$

where the inner most term reflects efficiency units multiplied by the measure of households with that state. This is all multiplied by L_i which is the mass of households in country i.

Asset Holdings. The aggregate quantity of asset holdings simply sums across the distribution conditional on choosing

$$A_i' = L_i \sum_z \sum_j \sum_a g_{ij}(a, z; S_i)\pi_{ij}(a, z; S_i)\lambda_i(a, z; S_i). \tag{14}$$

National Income and Consumption Starting from the production side of our economy, the value of aggregate production must equal aggregate payments to labor so

$$p_i Y_i = p_i A_i N_i = L_i \sum_z \sum_a w_i \, z \, \lambda_i(a, z; S_i) \tag{15}$$

where the last term sums over wage payments for each household type. Then by summing over individual consumers' budget constraint and substituting in (15), we arrive at the aggregated budget constraint:

$$p_i Y_i = \widetilde{P_i C_i} + \left[- R_i A_i + A_i' \right] - T_{i,\tau}, \tag{16}$$

where national income equals the value of aggregate consumption $\widetilde{P_i C_i}$, the country's the net asset position, all net of transfers. Here the value of aggregate consumption is

$$\widetilde{P_i C_i} = L_i \sum_z \sum_j \sum_a p_{ij}c_{ij}(a, z; S_i)\pi_{ij}(a, z; S_i)\lambda_i(a, z; S_i) \tag{17}$$

where one can see a bug and a feature of this model. Here there is an "index number problem"

in the sense that there is not an ideal price index for which one can decompose aggregate values in to a price and quantity component. This is in contrast to, e.g., a model where households consume a CES bundle of goods.

Trade Flows It's first worth walking though imports for a given set of states. So for households with states a and z we have

$$M_{ij}(a, z; S_i) = p_{ij}c_{ij}(a, z; S_i). \tag{18}$$

Then aggregate imports from country i to country j sums over this weighted by the mass of households choosing variety j and who in those states so

$$M_{ij} = L_i \sum_z \sum_a M_{ij}(a, z; S_i)\pi_{ij}(a, z; S_i)\lambda_i(a, z; S_i). \tag{19}$$

The same can be done for a countries exports. Again, focusing on exports to a location given a set of states we have

$$X_{ji}(a, z; S_j) = p_{ji}c_{ji}(a, z; S_j) \tag{20}$$

Then aggregate exports from country i to country j

$$X_{ji} = L_j \sum_z \sum_a X_{ji}(a, z; S_j)\pi_{ji}(a, z; S_j)\lambda_j(a, z; S_j) \tag{21}$$

1.5. Market Clearing and the Decentralized Equilibrium

Given the definitions above, I discuss the market clearing conditions than an equilibrium must respect.

The Goods Market. From here we can equate the value of production of commodity i in country i with global demand for country i's commodity:

$$p_i Y_i = \sum_j^M X_{ji}, \tag{22}$$

where the left hand side is production and the right hand side is world demand for the commodity from (21).

The Bond Market. The second market clearing condition is in the bond market. Two cases are considered "financial globalization" in which there is a global bond market with one real

interest rate R. In this case the market clearing condition is

$$\sum_{i}^{M} A'_i = 0 \tag{23}$$

which says that net asset demand must equal zero across all countries. The second case consider-erer is "financial autarky" in which there is a local bond market that facilitates within country risk-sharing, but not globally. In this case, there is an interest rate is R_i for each country and the associated market clearing condition is

$$A'_i = 0 \ \ \forall i \tag{24}$$

Below is a formal definition of a Stationary Equilibrium when the aggregate state S_i is constant and not changing.

A Stationary Equilibrium. A Stationary Equilibrium are asset policy functions and commodity choice probabilities $\{\ g_{ij}(a, z), \pi_{ij}(a, z)\ \}_i$, probability distributions $\lambda_i(a, z)$, and positive real numbers $\{w, p, R\}_i$ such that

   i Prices (w, p) satisfy (2, 3);

  ii The policy functions and choice probabilities solve the household's optimization problem in (**??**);

  iv The probability distribution $\lambda_i(a, z)$ induced by the policy functions, choice probabilities, and primitives satisfies (12) and is a stationary distribution;

  v Goods market clears:

$$p_i Y_i - \sum_{j}^{M} X_{ji} = 0, \ \ \forall i \tag{25}$$

  v Bond market clears with either Financial Globalization with $R_i = R$ and

$$\sum_{i}^{M} A'_i = 0. \tag{26}$$

   Or Financial Autarky where

$$A'_i = 0, \ \ \forall i \tag{27}$$

1.6. The Centralized Equilibrium

Before discussing some properties of the model, the Centralized (Efficient) Equilibrium is an important point of comparison. To characterize this equilibrium, I take a stand on the Social Welfare function and focus on a Utilitarian planner that places equal weight on all households in the global economy. Social welfare is:

$$\mathcal{W} = \sum_{t=0}^{\infty} \sum_z \sum_i \sum_j \beta^t \left\{ u(c_{ij}(z,t)) + \mathrm{E}[\,\epsilon \mid \mu_{ij}(z,t)] \right\} \mu_{ij}(z,t) L_i \lambda_i(z,t). \tag{28}$$

Working from the inside out, the inner most term is utility from consumption $c_{ij}(z,t)$ conditional on shock, country source and destination plus the expected value of the preference shocks conditional on shock-specific choice probabilities. The inner most term is weighted according to the measure of households in i sourcing from j $\mu_{ij}(z,t)$ times the total measure of households $L_i \lambda_i(z,t)$ in country i with shock z at date t. Then this is done at all dates t and discounted at the same rate β that households discount utility.

The Planner maximizes (28) subject to several constraints. The first is the law of motion describing how the measure of households evolves across states

$$\lambda_i(z',t+1) = \sum_z \mathcal{P}(z,z') \lambda_i(z,t). \tag{29}$$

Given a distribution of households, the effective labor units in each country are

$$N_{i,t} = L_i \sum_z z \lambda_i(z,t), \tag{30}$$

and then aggregate production of the final good is

$$Y_{it} = A_i N_{i,t}. \tag{31}$$

Combining the amount of resources available in (31) with the consumption and sourcing decisions yields the following resource constraint:

$$Y_{it} \geq \sum_j \sum_z d_{ji} c_{ji}(z,t) \mu_{ji}(z,t) L_j \lambda_j(z,t) \tag{32}$$

which says that production must be greater than or equal to world consumption of variety i inclusive of trade costs d_{ji}.

Given the social welfare function in 28 and the constraints discussed above, the **Centralized**

Planner's Problem is the following:

$$\mathcal{W}^* = \max_{c_{i,j}(z,t),\, \mu_{i,j}(z,t)} \sum_{t=0}^{\infty} \sum_{z} \sum_{i} \sum_{j} \beta^t \left\{ u(c_{ij}(z,t)) + \mathrm{E}[\,\epsilon \mid \mu_{i,j}(z,t)] \right\} \mu_{i,j}(z,t) L_i \lambda_i(z,t)$$

subject to (29) (31) and (32) and an inital condition $\lambda_i(z,0)$. (33)

A Stationary Centralized Planner Allocation. A Stationary Centralized Planner Allocation are time invariant policy functions $\{\, c_{i,j}(z),\ \mu_{i,j}(z)\, \}$, a probability distribution $\lambda_i(z)$, and positive real numbers N_i for each country i where:

i The policy functions solve the Centralized Planner's Problem in (33);

ii The probability distribution $\lambda_i(z)$ associated with (29) is a stationary distribution;

iii Effective labor units satisfy (30).

1.7. Special Cases / Running Examples

There are two special cases that I will refer back to repeatedly in the text. One is what I will call the CES case in which I mean that the household has access to a constant elasticity aggregator over the different varieties. The second case is the "hand-to-mouth" case in which households have no access to borrowing or lending.

CES case. This case is relatively familiar and standard. Here there is an aggregator over national varieties of the CES class, so

$$c = \left\{ \sum_j^M c_j^{\frac{\theta-1}{\theta}} \right\}^{\frac{\theta}{\theta-1}}, \tag{34}$$

where θ controls the elasticity of substitution across products. Then each household has the following demand curve for a region's variety:

$$c_{ij}(a, z; S_i) = \left(\frac{p_{ij}}{P_i} \right)^{-\theta} c_i(a, z; S_i). \tag{35}$$

which is given the total amount of consumption $c_i(a, z; S_i)$ a household chooses given their states and P_i is the CES price index:

$$P_i = \left\{ \sum_j^M p_{ij}^{1-\theta} \right\}^{\frac{1}{1-\theta}}. \tag{36}$$

what is unique about this setting is that household expenditure shares on different goods are independent of their state. So the expenditure share of a household in location i with state a and z is

$$\frac{p_{ij} c_{ij}(a, z; S_i)}{P_i c_i(a, z; S_i)} = \left(\frac{p_{ij}}{P_i} \right)^{1-\theta}, \tag{37}$$

which depends only on prices that all households face. What is happening here is that the CES aggregator is homothetic in total consumption. So while households are choosing different levels of total consumption to solve their income-fluctuations problem, how that demand is divided up is always the same. From here, one can aggregate and arrive at a "gravity-like" import demand system with households with states a and z importing

$$M_{ij}(a, z; S_i) = p_{ij} \left(\frac{p_{ij}}{P_i} \right)^{-\theta} c_i(a, z; S_i). \tag{38}$$

Then aggregate imports from country i to country j sums over this weighted by the mass of

households in those states which gives

$$M_{ij} = \left(\frac{p_{ij}}{P_i}\right)^{1-\theta} \times P_i C_i \tag{39}$$

where the last term follows by noting that the sum of $c_i(a, z; S_i)$ across the distribution is aggregate consumption and then this is put in value terms by multiplying and dividing by the CES price index.

Hand-to-Mouth, No Labor Supply,Log Households. This case focuses on the situation described—so there does not exist a risk free asset to smooth consumption, nor can households adjust their labor supply. The value function of a household contemplating the purchase of national variety j is:

$$v_{ij}(a, z; S_i) = u\left(\frac{w_i z}{p_{ij}}\right) + \epsilon(j) + \beta\, \mathbb{E}[v_i(z'; S_i')] \tag{40}$$

and then the choice probability becomes

$$\pi_{ij}(z; S_i) = \exp\left(\frac{v_{ij}(z; S_i)}{\sigma_\epsilon}\right) \Bigg/ \sum_{j'} \exp\left(\frac{v_{ij'}(z; S_i)}{\sigma_\epsilon}\right) \tag{41}$$

$$\pi_{ij}(z; S_i) = \exp\left(\frac{u\left(\frac{w_i z}{p_{ij}}\right))}{\sigma_\epsilon}\right) \Bigg/ \sum_{j'} \exp\left(\frac{u\left(\frac{w_i z}{p_{ij}}\right)}{\sigma_\epsilon}\right) \tag{42}$$

where the last line follows from the properties of the $\exp$ function and that the continuation value function is exactly the same independent of the good chosen. Two more steps. Then with $\log$ this expression collapses to

$$\pi_{ij}(S_i) = \frac{p_{ij}^{-1/\sigma_\epsilon}}{\sum_{j'} p_{ij'}^{-1/\sigma_\epsilon}} \tag{43}$$

where here we recover the well known "ces-logit-isomorphisim" between the logit demand system and the CES aggregator in 34.

Appendix

A. Appendix: The Generalized Trade Elasticity

My definition of the trade elasticity is the partial equilibrium response of imports from j relative to domestic consumption due to a permanent change in trade costs between steady states. By partial equilibrium, I mean that wages and interest rates as fixed at their initial equilibrium values. By permanent, I work this out under the heuristic of a "steady-state to steady-state comparison" where I factor in how the stationary distribution changes due to changes in behavior. This definition is consistent with the broad terminology used in the trade literature, e.g. Arkolakis et al. (2012) and Simonovska and Waugh (2014).

Mathematically, the trade elasticity equals the difference between the elasticities for how trade between i and j change minus how home trade changes:

$$\frac{\partial(M_{ij}/M_{ii})}{\partial d_{ij}} \times \frac{d_{ij}}{(M_{ij}/M_{ii})} = \frac{\partial(M_{ij}/M_{ij})}{\partial d_{ij}/d_{ij}} - \frac{\partial(M_{ij}/M_{ij})}{\partial d_{ij}/d_{ij}}. \tag{44}$$

The change in imports between i and j with respect to a change in trade costs is:

$$\frac{\partial M_{ij}}{\partial d_{ij}} = \sum_{a,z} \left\{ \frac{\partial p_{ij}c_{ij}(a,z)}{\partial d_{ij}} \pi_{ij}(a,z)\lambda(a,z) + \frac{\partial \pi_{ij}(a,z)}{\partial d_{ij}} p_{ij}c_{ij}(a,z)\lambda(a,z) + \frac{\partial \lambda(a,z)}{\partial d_{ij}} p_{ij}c_{ij}(a,z)\pi_{ij}(a,z) \right\}. \tag{45}$$

Then divide the stuff on the inside of the brackets by household level imports $p_{ij}c_{ij}(a,z)\pi_{ij}(a,z)\lambda(a,z)$ and multiply on the outside giving,

$$\frac{\partial M_{ij}}{\partial d_{ij}} = \sum_{a,z} \left\{ \frac{\partial p_{ij}c_{ij}(a,z)/p_{ij}c_{ij}(a,z)}{\partial d_{ij}} + \frac{\partial \pi_{ij}(a,z)/\pi_{ij}(a,z)}{\partial d_{ij}} + \frac{\partial \lambda(a,z)/\lambda(a,z)}{\partial d_{ij}} \right\} p_{ij}c_{ij}(a,z)\pi_{ij}(a,z)\lambda(a,z). \tag{46}$$

Now define the following "weight" which is the share of goods that those with states a, z account for in total expenditures from j as

$$\omega_{ij}(a,z) = \frac{p_{ij}c_{ij}(a,z)\pi_{ij}(a,z)\lambda(a,z)}{M_{ij}}. \tag{47}$$

where the sum of $\omega_{ij}(a,z)$ equals total imports M_{ij}. This gives a nice expression for the import

elasticity

$$\frac{\partial M_{ij}/M_{ij}}{\partial d_{ij}/d_{ij}} = \sum_{a,z} \left\{ \underbrace{\frac{\partial p_{ij}c_{ij}(a,z)/p_{ij}c_{ij}(a,z)}{\partial d_{ij}/d_{ij}}}_{\theta_{ij}(a,z)^I} + \underbrace{\frac{\partial \pi_{ij}(a,z)/\pi_{ij}(a,z)}{\partial d_{ij}/d_{ij}}}_{\theta_{ij}(a,z)^E} + \underbrace{\frac{\partial \lambda(a,z)/\lambda(a,z)}{\partial d_{ij}/d_{ij}}}_{\theta_{ij}(a,z)^D} \right\} \omega_{ij}(a,z)$$

(48)

or more succinctly as

$$\frac{\partial M_{ij}/M_{ij}}{\partial d_{ij}/d_{ij}} = \sum_{a,z} \left\{ \theta_{ij}(a,z)^I + \theta_{ij}(a,z)^E + \theta_{ij}(a,z)^D \right\} \omega_{ij}(a,z)$$

(49)

where the elasticity of aggregate imports into i from j is a weighted average of micro-level elasticities at the household level. And household level elasticities are decomposed into **(i)** an intensive margin trade elasticity $\theta_{ij}(a,z)^I$ which reflects how much more a household buys as trade costs decline; **(ii)** the extensive margin trade elasticity $\theta(a,z)_{ij}^E$ which reflects how households switch their variety choice as trade costs decline; and **(iii)** the distribution elasticity $\theta(a,z)_{ij}^D$ so how the mass of agents change across household states a, z.

Given the expression in (44), Proposition 1 follows:

Proposition 1 (The Generalized Trade Elasticity) *The trade elasticity between country i and country j is:*

$$\theta_{ij} = \sum_{a,z} \left\{ \theta_{ij}(a,z)^I + \theta_{ij}(a,z)^E + \theta_{ij}(a,z)^D \right\} \omega_{ij}(a,z) - \left\{ \theta_{ii}(a,z)^I + \theta_{ii}(a,z)^E + \theta_{ii}(a,z)^D \right\} \omega_{ii}(a,z)$$

which is the difference in expenditure weighted micro-level elasticities. The micro-level elasticities for households with states a, z are decomposed into an intensive, extensive, and distributional elasticity

$$\theta_{ij}(a,z)^I = \frac{\partial p_{ij}c_{ij}(a,z)/p_{ij}c_{ij}(a,z)}{\partial d_{ij}/d_{ij}}, \quad \theta_{ij}(a,z)^E = \frac{\partial \pi_{ij}(a,z)/\pi_{ij}(a,z)}{\partial d_{ij}/d_{ij}}, \quad \theta_{ij}(a,z)^D = \frac{\partial \lambda(a,z)/\lambda(a,z)}{\partial d_{ij}/d_{ij}}$$

and the expenditure weights are defined as

$$\omega_{ij}(a,z) = \frac{p_{ij}c_{ij}(a,z)\pi_{ij}(a,z)\lambda(a,z)}{M_{ij}}$$

Proposition 1 is just derived off of the aggregation of imports at the micro level—no market clearing, functional forms, etc. However, by using the households budget constraint and the Type 1 extreme value distributional assumption more can be said about the intensive and extensive margin trade elasticity and illustrate how the household's consumption-savings problem matters.

The Micro Intensive Margin Trade Elasticity. By working off the budget constraint one can

express the intensive margin elasticity as

$$\underbrace{\frac{\partial p_{ij}c_{ij}(a,z)/p_{ij}c_{ij}(a,z)}{\partial d_{ij}/d_{ij}}}_{\theta_{ij}(a,z)^I} = -\frac{\partial g_{ij}(a,z)}{\partial p_{ij}/p_{ij}}\frac{\partial p_{ij}/p_{ij}}{\partial d_{ij}/d_{ij}} \times \frac{1}{p_{ij}c_{ij}(a,z)}, \tag{50}$$

$$= -\frac{\partial g_{ij}(a,z)}{\partial p_{ij}/p_{ij}} \times \frac{1}{p_{ij}c_{ij}(a,z)}. \tag{51}$$

where recall that $g_{ij}(a,z)$ is the policy function mapping states into asset holdings next period a' and the last line follows from $\frac{\partial p_{ij}/p_{ij}}{\partial d_{ij}/d_{ij}} = 1$. What this expression means is that the intensive margin elasticity is about a households consumption-savings behavior, i.e., how a household adjusts assets given a change in the prices of goods. For example, the way this works is that a reduction in trade costs lowers prices and relaxes the budget constraint, and then the intensive margin elasticity depends on how the incremental amount of resources available to spend is dived between assets and consumption.

A couple of notes:

- Is there a way to connect this with the marginal propensity to consume (or save)?

- I'm doing everything as if it's discrete. Will need to change notation above.

The Micro Extensive Margin Trade Elasticity. The Type 1 extreme value distributional assumption allows for a characterization of the choice probability and, hence, how the choice probabilities respond to changes in trade costs. As a first step, define the denominator of the choice probability as:

$$\Phi_i(a,z) = \sum_{j'} \exp\left(\frac{v_{ij'}(a,z)}{\sigma_\epsilon}\right). \tag{52}$$

The elasticity of the choice probability with respect to a change in trade costs is

$$\underbrace{\frac{\partial \pi_{ij}(a,z)/\pi_{ij}(a,z)}{\partial d_{ij}/d_{ij}}}_{\theta_{ij}(a,z)^E} = \frac{1}{\sigma_\epsilon}\frac{\partial v_{ij}(a,z)}{\partial d_{ij}/d_{ij}} - \frac{\partial \Phi_i(a,z)/\Phi_i(a,z)}{\partial d_{ij}/d_{ij}}. \tag{53}$$

The key issue here is the first term: which reflects how the choice specific value function changes with trade costs. In other words, how much more valuable does choice j become when it's cheaper to import variety j. The second term reflects the fact that the change choice j depends upon a comparison relative to the overall change in the value of options across variety, i.e., how Φ_i changes.

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In general, the first term in (53) is a complicated object: is state dependent and forward looking in with the value function showing up. More subtle it is the idea that the household's income fluctuations problem imposes a cross-equation restriction on between the intensive margin and extensive margin given how assets and consumption are responding to changes in prices via changes in trade costs.

B. Appendix: The Gains from Trade

This section works through the gains from a permanent change in trade costs, across steady states. So the idea here is, again, heuristic in the sense that I'm thinking a situation where the change is small and there is an immediate jump to the new steady state.

To walk through this I'm going to first express everything relative to the home country i. So the observation is that the expected value function over the taste shocks is

$$v_i(a, z) = \sigma_\epsilon \log \left\{ \sum_{j'} \exp\left(\frac{v_{ij}(a, z)}{\sigma_\epsilon} \right) \right\} \tag{54}$$

and then I'm going to make the observation that I can substitute out the sum part with the exp of the home value function relative to the micro-level "home choice"

$$v_i(a, z) = \sigma_\epsilon \log \left\{ \frac{\exp\left(\frac{v_{ii}(a,z)}{\sigma_\epsilon} \right)}{\pi_{ii}(a, z)} \right\} \tag{55}$$

and thus the value function can be represented as

$$v_i(a, z) = -\sigma_\epsilon \log \pi_{ii}(a, z) + u(c_{ii}(a, z)) + \beta \mathbb{E} v_i(g_{ii}(a, z), z) \tag{56}$$

where the expectation on the right-hand side is with respect to z and ϵ. The key innovation here is that now everything is written with respect to the home choice. Now the strategy is to differentiate (56) with respect to a percent change in trade costs and use the recursive structure to iterate forward and construct the overall change.

Furthermore, in all of this, I set w_i to be the numerarie and $A_i = 1$ so p_{ii} equals one and needs not be carried around.

One more detail, to facilitate interpretation, it will be useful to compute the Euler equation associated with asset holdings when the borrowing constraint does not bind. This euler equation is:

$$-u'(c_{ii}(a, z)) = \beta \mathbb{E} \left\{ -\sigma_\epsilon \frac{\partial \pi_{ii}(a', z')/\pi_{ii}(a', z')}{\partial a'} + u'(c_{ii}(a', z'))R \right\}, \tag{57}$$

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which says that the agent should be indifferent between the marginal utility of consumption forgone to hold some more assets and two components (i) the benefit from how a change in assets changes in their variety choice and (ii) the direct benefit of the returns on the assets evaluated at the marginal utility of consumption. Something that is confusing me is how is this consistent with Euler Equation below for the endogenous grid method.

Let's do this!

$$\frac{\partial v_i(a,z)}{\partial d_{ij}/d_{ij}} = -\sigma_\epsilon \theta_{ii}^E(a,z) + u'(c_{ii}(a,z))\frac{\partial R}{\partial d_{ij}/d_{ij}}a - u'(c_{ii}(a,z))\frac{\partial g_{ii}(a,z)}{\partial d_{ij}/d_{ij}} + \beta E \frac{\partial v_i(g_{ii}(a,z),z')}{\partial d_{ij}/d_{ij}}$$

(58)

Side note: I think I'm conflating total derivatives and partial derivatives... so the notation is a bit messed up. Then the derivative of the continuation value function is:

$$\frac{\partial v_i(g(a,z),z')}{\partial d_{ij}/d_{ij}} = \underbrace{\left[-\sigma_\epsilon \frac{\partial \pi_{ii}(a',z')/\pi_{ii}(a',z')}{\partial a'} + u'(c_{ii}(a',z'))R\right]\frac{\partial g_{ii}(a,z)}{\partial d_{ij}/d_{ij}}}_{\frac{\partial v_i(g_{ii}(a,z),z')}{\partial a}} +$$

(59)

$$- \sigma_\epsilon \theta_{ij}^E(a',z') + u'(c_{ii}(a',z'))\frac{\partial R}{\partial d_{ij}/d_{ij}}a' - u'(c_{ii}(a',z'))\frac{\partial g_{ii}(a',z')}{\partial d_{ij}/d_{ij}} + \beta E \frac{\partial v_i(g_{ii}(a',z'),z'')}{\partial d_{ij}/d_{ij}}$$

And now combine and collect terms so

$$\frac{\partial v_i(a,z)}{\partial d_{ij}/d_{ij}} = - \sigma_\epsilon \theta_{ij}^E(a,z)$$

(60)

$$+ \underbrace{u'(c_{ii}(a,z))\frac{\partial R}{\partial d_{ij}/d_{ij}}a}_{\gamma_{ii}(a,z)}$$

(61)

$$+ \underbrace{\left\{-u'(c_{ii}(a,z)) + \beta \mathbb{E}\left[-\sigma_\epsilon \frac{\partial \pi_{ii}(a',z')/\pi_{ii}(a',z')}{\partial a'} + u'(c_{ii}(a',z'))R\right]\right\}\frac{\partial g_{ii}(a,z)}{\partial d_{ij}/d_{ij}}}_{\delta_{ii}(a,z)}$$

(62)

$$+ \beta \mathbb{E}\left\{-\sigma_\epsilon \theta_{ij}^E(a',z') + u'(c_{ii}(a',z'))\frac{\partial R}{\partial d_{ij}/d_{ij}}a' \dots\right.$$

(63)

Let me walk through the interpretation of each term:

- $-\sigma_\epsilon \theta_{ij}^E(a,z)$ is the standard gains from trade term reflecting the idea that gains work through changes in expenditures across different varieties. With static logit, this is exactly the ACR formula.

- $u'(c_{ii}(a,z))\frac{\partial R}{\partial d_{ij}/d_{ij}}a$ is what I would call how trade facilitates more risk sharing. I think as trade costs decline the amount of resources available for risk sharing expands and this changes the interest rate. Which way does it go?

- The third term is interesting because it is the stuff in the outer brackets is the Euler equation for assets. The idea here is that if the household is unconstrained, then it is zero as there is no gain through changes in asset behavior. Asset holdings are already chosen so that margins are equated, thus, on the margin any benefit of lower trade costs on asset behavior is zero. Essentially an application of the Envelope Theorem.

  Now in this economy, it may not be zeroed our because of borrowing constrained households, thus this term is positive (?) this means that the change in trade costs are relaxing the constraints on some households and get's them back on their euler equation.

  Finally, notice how the outside brackets is multiplied by the change in the asset policy function. What this picks up is that if the household is still constrained, then assets can't change and overall second term is zero. So the only people that benefit and contribute to social welfare are those on the margin between constrained and not-constrained.

- The final term is about this continuing on.

so the total gains from trade are

$$\frac{\partial v_i(a,z)}{\partial d_{ij}/d_{ij}} = \mathbb{E}\sum_{t=0}\beta^t\left\{-\sigma_\epsilon\theta_{ii}^E(a_t,z_t)+\gamma_{ii}(a_t,z_t)+\delta_{ii}(a_t,z_t)\right\} \tag{64}$$

To compute how social welfare changes, I'll focus on a utilitarian social welfare function so:

$$W_i = \sum_{a,z}v_i(a,z)\lambda(a,z) \tag{65}$$

Then the total change in total welfare is

$$\frac{\partial W_i}{\partial d_{ij}/d_{ij}} = \sum_{a,z}\left\{\frac{\partial v_i(a,z)}{\partial d_{ij}/d_{ij}}+v_i(a,z)\theta_{ij}(a,z)^D\right\}\lambda(a,z) \tag{66}$$

where the last term in the brackets is the distribution elasticity and that follows from multiplying it by λ/λ converting it to a elasticity and then pulling out the remaining λ. This can also be converted to an elasticity by noting

$$\frac{\partial W_i/W_i}{\partial d_{ij}/d_{ij}} = \sum_{a,z}\left\{\frac{\partial v_i(a,z)/v_i(a,z)}{\partial d_{ij}/d_{ij}}+\theta_{ij}(a,z)^D\right\}\frac{\lambda(a,z)v_i(a,z)}{W_i} \tag{67}$$

this is clear. It's not ideal since the value function is linear in terms and does not give rise to super clean representation.

What does this pick up. The first component in social welfare reflects changes in households in individual states, i.e. (64). And this is about gains from substitution, gains from risk sharing, and the relaxation of borrowing constraints. The second component reflects how the distribution changes and if—at the old v's—the distribution gets better or worse.

Proposition 2 (The Welfare Gains from Trade) *The welfare gains from trade are given by*

$$\frac{\partial W_i / W_i}{\partial d_{ij} / d_{ij}} = \sum_{a,z} \left\{ \frac{\partial v_i(a,z)/v_i(a,z)}{\partial d_{ij}/d_{ij}} + \theta_{ij}(a,z)^D \right\} \frac{\lambda(a,z)v_i(a,z)}{W_i} \tag{68}$$

which reflects the change in household level gains and how the distribution of households across states changes. Household level gains are given by

$$\frac{\partial v_i(a,z)}{\partial d_{ij}/d_{ij}} = \mathbb{E} \sum_{t=0}^{\infty} \beta^t \left\{ -\sigma_\epsilon \theta_{ij}^E(a_t, z_t) + \gamma_{ij}(a_t, z_t) + \delta_{ij}(a_t, z_t) \right\}$$

where each term represents:

- *Gains from substitution:* $-\sigma_\epsilon \theta_{ii}^E(a_t, z_t)$ *with* θ_{ii}^E *is the elasticity of the home choice probability defined in Proposition 1.*

- *Gains from risk sharing:* $\gamma_{ij}(a,z) = u'(c_{ii}(a,z)) \frac{\partial R}{\partial d_{ij}/d_{ij}} a$

- *Gains from relaxing borrowing constraints:*

$$\delta_{ii}(a,z) = \left\{ -u'(c_{ii}(a,z)) + \beta\mathbb{E}\left[-\sigma_\epsilon \frac{\partial \pi_{ii}(a',z')/\pi_{ii}(a',z')}{\partial a'} + u'(c_{ii}(a',z'))R \right] \right\} \frac{\partial g_{ii}(a,z)}{\partial d_{ij}/d_{ij}}$$

C. Appendix: Examples

3.1. Static, Log-Logit, Arkolakis et al. (2012)

3.2. Dynamic, Log Preferences

D. Appendix: Endogenous Grid Method

First, I'm going to derive the Euler equation for this model. I'll abstract from the situation in which the HH is at the borrowing constraint.

Focus on the within a variety choice component, the households value function can be written as:

$$v_{ij}(a, z) = \max_{a'} u \left(\frac{R_i a + w_i z - a'}{p_{ij}} \right) + \beta \mathbb{E} v(a', z') \tag{69}$$

then the first order condition associated with this problem is:

$$\frac{u'(c_{ij}(a, z))}{p_{ij}} = \beta \mathbb{E} \frac{\partial v(a', z')}{\partial a'} \tag{70}$$

which is saying that, conditional on a variety choice the left hand side is the loss in consumption units which is $1/p_{ij}$ evaluated at the marginal utility of consumption and then this is set equal to the marginal gain from saving a bit more which is how the value function changes with respect to asset holdings. Now we can arrive at the $\frac{\partial v(a', z')}{\partial a'}$ in the following way, so start from the log-sum expression for the expected value function

$$\mathbb{E}_\epsilon v(a', z') = \sigma_\epsilon \log \left\{ \sum_{j'} \exp \left(\frac{v_{ij}(a', z')}{\sigma_\epsilon} \right) \right\} \tag{71}$$

and then differentiate this with respect to asset holdings which gives:

$$\frac{\partial \mathbb{E}_\epsilon v(a', z')}{\partial a'} = \left(\frac{\sigma_\epsilon}{\sum_{j'} \exp \left(\frac{v_{ij}(a', z')}{\sigma_\epsilon} \right)} \right) \left[\sum_{j'} \exp \left(\frac{v_{ij}(a', z')}{\sigma_\epsilon} \right) \frac{1}{\sigma_\epsilon} \frac{\partial v_{ij}(a', z')}{\partial a'} \right] \tag{72}$$

Then if you look at this carefully and notices how the choice probabilities from (10) are embedded in here, we have:

$$\frac{\partial \mathbb{E}_\epsilon v(a', z')}{\partial a'} = \sum_{j'} \pi_{ij}(a', z) \frac{\partial v_{ij}(a', z')}{\partial a'} \tag{73}$$

and then we can just apply the Envelop theorem to the value functions associated with the discrete choices across the options:

$$\frac{\partial \mathbb{E}_\epsilon v(a', z')}{\partial a'} = \sum_{j'} \pi_{ij}(a', z') \frac{u'(c_{ij}(a', z')) R_i}{p_{ij}} \tag{74}$$

So then putting everything together we have:

$$\frac{u'(c_{ij}(a, z))}{p_{ij}} = \beta R_i \mathbb{E}_{z'} \left[\sum_{j'} \pi_{ij}(a', z') \frac{u'(c_{ij}(a', z'))}{p_{ij}} \right] \tag{75}$$

where this has a very natural form: you set the marginal utility of consumption today equal to the marginal utility of consumption tomorrow adjusted by the return on delaying consumption, and the expected value of the marginal utility of consumption which reflects how the uncertainty over both ones' preference over different varieties and shocks to efficiency units. Taking into account the borrowing constraint then gives the generalized Euler equation from which the endogenous grid method will exploit:

$$\frac{u'(c_{ij}(a, z))}{p_{ij}} = \max \left\{ \beta R_i \mathbb{E}_{z'} \left[\sum_{j'} \pi_{ij}(a', z') \frac{u'(c_{ij}(a', z'))}{p_{ij}} \right] , \ u \left(\frac{R_i a + w_i - \phi_i}{p_{ij}} \right) \right\} \tag{76}$$

4.1. EGM-Discrete Choice Algorithm

Here is a proposed approach. This focuses on just the consumer side in one country i.

0. Set up an asset grid as usual. Then guess (i) a consumption function $g_{c,ij}(a, z)$ for each a, z, and product choice j and (ii) choice specific value function $v_{ij}(a, z)$.

1. Compute the choice probabilities from (10) for each (a, z) combination, given the guessed value functions.

1. Given the consumption function and choice probabilities compute the RHS of (76) first.

2. Then invert to find the new updated consumption choice so

$$c_{ij}(\tilde{a}, z) = u'^{-1} \left\{ p_{ij} \max \left\{ \beta R_i \mathbb{E}_{z'} \left[\sum_{j'} \pi_{ij}(a', z') \frac{u'(c_{ij}(a', z'))}{p_{ij}} \right] , \ u \left(\frac{R_i a + w_i - \phi_i}{p_{ij}} \right) \right\} \right\} \tag{77}$$

where u'^{-1} is the inverse function of the marginal utility of consumption.

Side note: One of the interesting things about this equation is that the direct j component on the RHS that only affects the consumption choice is through the price. Can this be exploited? We also know the choice probabilities need to sum to one, so is there a way to map the consumption choice into the choice probabilities? Also, can interpolation be done once how p scales things...

3. The key issue in this method is that we have found $c_{ij}(\tilde{a}, z)$ where the consumption function is associated with some asset level that is not necessarily on the grid. The solution is to (i) use the budget constraint and infer $\tilde{a}$ given that a' was chosen above (that's where we started), z, and $c_{ij}(\tilde{a}, z)$. Now we have a map from $\tilde{a}$ to a' for which one can use interpolation to infer the a' chosen given a where a is on the grid.

- Do steps **2.** and **3.** for each j variety choice. This then makes the function $g_{a,ij}(a, z)$ mapping each state and j choice (today) into a', z' states and then from the budget constraint we have an associated consumption function $g_{c,ij}(a, z)$

4. Compute the $\mathrm{E}\left[v(g_{a,ij}(a, z), z')\right]$. This is performed in the `make_Tv_upwind!` function. It fixes a country j, then works through shocks and asset states today and from the policy function $g_{a,ij}(a, z)$ figures out the asset choice tomorrow. Then the $\mathrm{E}\left[v(g_{a,ij}(a, z), z')\right]$ is (11) over the different variety choices tomorrow (this is the integration over ϵ) multiplied by the probability of z' occurring (this is the integration over z).

5. Given **4.** update the value function using the bellman equation evaluated at the optimal policies:

$$Tv_{ij}(a, z) = u(g_{c,ij}(a, z)) + \beta \mathrm{E}\left[v(g_{a,ij}(a, z), z')\right] \tag{78}$$

6. Compare old and new policy functions, old and new value functions, and then update accordingly.

E. Appendix: The Planning Problem

The Lagrangian associated with the Centralized Planning problem is:

$$\mathcal{L} = \sum_{t=0}^{\infty}\sum_{z}\sum_{i}\sum_{j} \beta^t \left\{ u(c_{ij}(z, t)) + \mathrm{E}[\,\epsilon \mid \mu_{ij}(z, t)]\right\}\mu_{ij}(z, t)L_i\lambda_i(z, t), \tag{79}$$

$$+ \sum_{t=0}^{\infty}\sum_{i} \chi_i(t)\left\{ Y_{it} - \sum_{j}\sum_{z} d_{ji}c_{ji}(z, t)\mu_{ji}(z, t)L_j\lambda_j(z, t)\right\}$$

$$+ \sum_{t=0}^{\infty}\sum_{i} \chi_{2i}(t)\left\{ 1 - \sum_{j} \mu_{ij}(z, t)\right\}L_i\lambda_i(z, t)$$

Then associated with this problem are two first order conditions. The first one is consumption in i with respect to variety j so:

$$\frac{\partial \mathcal{L}}{\partial c_{ij}(z,t)} = \beta^t u'(c_{ij}(z,t))\mu_{ij}(z,t)L_i\lambda_i(z,t) - \chi_j(t)d_{ij}\mu_{ij}(z,t)L_i\lambda_i(z,t) = 0 \tag{80}$$

$$\Rightarrow \quad \beta^t u'(c_{ij}(z,t)) = \chi_j(t)d_{ij}, \tag{81}$$

which says that the marginal utility of consumption of (i consuming variety j) should equal j's multiplier adjusted by the trade cost d_{ij}. Then notice that the right hand size is independent of z, thus within variety choice we have the result that $u'(c_{ij}(z,t)) = u'(c_{ij}(z',t)) = u'(\bar{c}_{ij}(t))$ for all z, z' combinations. Then the ratio of marginal utility across variety j is:

$$\frac{u'(\bar{c}_{ij}(t))}{u'(\bar{c}_{ij'}(t))} = \frac{\chi_j(t)d_{ij}}{\chi_{j'}(t)d_{ij'}} \tag{82}$$

which is starting to look like a gravity type regression where the χ's are related to the price index in the source countries, adjusted by the iceberg trade costs (e.g. equation (5) in SW for example). To be more concrete, compare i, j purchases from i, i purchases and with log utility one has

$$\frac{\bar{c}_{ij}}{\bar{c}_{ii}} = \frac{\chi_i(t)}{\chi_j(t)}d_{ij}^{-1} \tag{83}$$

which is quite close to a gravity equation where relative consumption equals a source and destination effect adjusted by the trade costs. Key differences are that this is in quantities, not expenditure; this does not reflect the extensive margin; and related to this last point is that the elasticity is γ.

References

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