1. Mongey & Waugh Model

A. Notation Guide

- c is the competitive good.
- x is the differentiated good.
- e is efficiency units of the household.

B. Households

There is a unit mass of households. They are infinite lived and have time-sparable preferences over consumption of two types of goods:

$$E_0 \sum_{t=0}^{\infty} \beta^t \tilde{u}(c_t, \{x_{j,t}\}_J), \tag{1}$$

where the notation c_t is the consumption of the "competitive good" and $x_{j,t}$ is consumption over the differentiated "market power" good. Only one unit of the market power good is consumed each period and the utility associated with is of the random utility class. Specifically, the utility associated with the choice of variety j is

$$\tilde{u}(c_{j,t}) = u(c_t) + \alpha_j + \epsilon_{j,t}. \tag{2}$$

where the $\epsilon_{j,t}$ are iid random variables across time and households. These shocks are distributed Type 1 Extreme Value with dispersion parameter σ. The α_j term is the "quality" of product j. This is like in section 3.2 of Anderson, De Palma, and Thisse (1992) for Linear Random Utility models, or in a simple form this picks up the way how product characteristics show up in say Berry (1994). Below I discuss some implications of this.

Digressions on this specification. I think the way to think about this specification is more akin to BLP rather than CES. First, consider the static model with a given distribution of assets and then efficiency units e.

$$s_j(a, e) = \exp\left(\frac{u(Ra + we - p_j) + \alpha_j}{\sigma}\right) \Big/ \sum_{j'} \exp\left(\frac{u(Ra + we - p_j) + \alpha_j}{\sigma}\right), \tag{3}$$

And then the aggregate market share is

$$s_j = \int_{A \times E} \exp\left(\frac{u(Ra + we - p_j) + \alpha_j}{\sigma}\right) \Big/ \sum_{j'} \exp\left(\frac{u(Ra + we - p_j) + \alpha_j}{\sigma}\right). \tag{4}$$

Now if one does linear utility, one gets

$$s_j = \exp\left(\frac{-p_j + \alpha_j}{\sigma}\right) \Big/ \sum_{j'} \exp\left(\frac{-p_j + \alpha_j}{\sigma}\right), \tag{5}$$

which is independent of a households wealth and income. A couple of observations here:

- First, this is **not** equivalent to CES. In Anderson et al. (1992) (section 3.7), they ask the question: "can we find a random utility model that can rationalize demand systems of CES?" There answer is yes when the preference structure is like in my trade model, so

$$\tilde{u}(c_j) = \log(c_j) + \epsilon_j. \tag{6}$$

  where the households have log preferences over the physical commodity j and face a continuous choice over the quantity, c_j. **If** we wanted to make the demand system kind of look more CES, this would be a path to go. What about quality here? It would look like quality in the CES world?

- Here (3) the non-linearity introduced by u is delivering features that look like BLP and in particular Berry, Levinsohn, and Pakes (2004) (need to investigate more). I think one way to think about this is that imagine:

$$v(Ra + we - p_j) \approx p_j \times \left\{ \beta_{pj} + \sum_i \beta_{a,i} a^i + \sum_i \beta_{w,i} we^i \right\} \tag{7}$$

  or something like this (what if we did a proper Taylor series). So one is interacting the observed price with observed characteristics of the households and then the β's are coefficients to be estimated (this is literally equations (1) and (2) in Berry et al. (2004)). And then one would get aggregate market shares as

$$s_j = \int_{A \times E} \exp\left(p_j \left\{ \tilde{\beta}_{pj} + \sum_i \tilde{\beta}_{a,i} a^i + \sum_i \tilde{\beta}_{w,i} we^i \right\} + \tilde{\alpha}_j \right) \Big/ \sum_{j'} \exp\left(p_j \left\{ \tilde{\beta}_{pj} + \sum_i \beta_{a,i} a^i + \sum_i \beta_{w,i} we^i \right\} + \tilde{\alpha}_j \right), \tag{8}$$

  where one can subsume the elasticity σ into the coefficients that are estimated. Note all this logic applies in the dynamic case. In this situation then the BLP approach above is simply approximating the value function.

- What is the case when elasticities don't vary? I don't see it in this formulation.

A household's efficiency units are stochastic and they evolve according to a Markov chain, i.e. , e is a household's efficiently units and $\mathcal{P}(e, e')$ describes the probability of a household with state e efficiency units transiting to state e'.

Households can save and borrow in a non-state contingent asset a. One unit of the asset pays out with gross interest rate R next period. I discuss this more in depth below, but the determination of R is with that which clears the bond market (local or global). A exogenous debt limit ϕ constrains borrowing so:

$$a_{t+1} \geq -\phi. \tag{9}$$

All these pieces come together in the household's budget constraint, conditional on choosing variety j to consume, and focusing on a stationary setting where prices are constant:

$$c_{jt} + p_j + a_{t+1} \leq Ra_t + w_i e_t. \tag{10}$$

The value of asset purchases and consumption expenditures must be less than or equal to asset payments, labor earnings.

C. The Household Problem

The state variables of a individual household are its asset holdings and efficiency units. As alluded to above, for now I focus on a stationary setting where aggregates are not changing and, thus, I abstract from carrying the notation associated with them around. The value function of a household after the variety shocks are realized is

$$\tilde{v}(a, e) = \max_{j} \{ \ v_j(a, e) \ \} \tag{11}$$

which is the maximum across the value functions associated with the discrete choices of different national varieties. The value function conditional on a choice of variety is

$$v_j(a, e) = \max_{a', c} \left\{ u(c) + \alpha_j + \epsilon_j + \beta \, \mathbb{E}[v(a', e')] \right\} \tag{12}$$

$$\text{subject to (10) and (9)}$$

where households chooses asset holdings and the level of consumption is residually determined through the budget constraint. Associated with the solution to this problem is a policy function $g_{ij}(a, e)$ which solves (12) and maps current states into asset holdings tomorrow a' contingent upon the variety choice. The continuation value function on the right-hand side of (12) is the expectation over (11) with respect to (i) e' and (ii) the variety taste shocks.

One implication of this last point means is that households understand that their may be situations where they really desire, say, a high priced imported good and, hence, save accordingly.

The Type 1 extreme value distribution on the taste shocks give rise to the following choice probabilities for each differentiated good:

$$s_j(a, e) = \exp\left(\frac{v_j(a, e)}{\sigma}\right) \Bigg/ \sum_{j'} \exp\left(\frac{v_{j'}(a, e)}{\sigma}\right), \tag{13}$$

which is the probability that a household with assets a and efficiency units z chooses country variety j. And then the expectation of (11) with respect to the taste shocks takes the familiar log-sum form

$$v_i(a, e) = \sigma \log \left\{ \sum_{j'} \exp\left(\frac{v_j(a, e)}{\sigma}\right) \right\}. \tag{14}$$

D. Products and Firms

There are J products. A firm has the following production technology to produce variety j:

$$q_j = A_j N_j, \tag{15}$$

where N_j are the efficiency units of labor supplied by households to produce product j.

The deal is this, so think about a firm supplying j as a monopolistic competitor. So he knows he can influence his own price j, but will treat all other prices of his competitors as given. This is what I mean by "monopolistic competition" does this make sense? Then the firms problem is:

$$\Pi_j = \max_{p_j, N_j} \; p_j \; Q_j(p_j) - wN_j \tag{16}$$

where $Q_j(p_j)$ is from the "aggregate demand" that is an aggregate across individual households demands state by state. Then substituting in the production technology one has

$$\Pi_j = \max_{p_j, N_j} \; p_j \; Q_j(p_j) - wQ_j(p_j)/A_j \tag{17}$$

and the first order condition associated with this problem is

$$Q_j(p_j) + p_j \frac{\partial Q_j}{\partial p_j} - \frac{w}{A_j} \frac{\partial Q_j}{\partial p_j} = 0 \tag{18}$$

and this can then be rearranged so that

$$p_j + p_j \left(\underbrace{\frac{\partial Q_j / Q}{\partial p_j / p_j}}_{-\theta_j} \right)^{-1} = \frac{w}{A_j} \tag{19}$$

where θ_j is elasticity of a firm j's quantity with respect to it's price. Then this delivers:

$$p_j = \left(\frac{\theta_j}{1 - \theta_j} \right) \frac{w}{A_j} \tag{20}$$

where the price is a markup over it's marginal cost and notice that (i) the markup is specific to firm j and (ii) presumably not constant in this economy.

E. Elasticities

So the amount of goods sold to households with state z is

$$Q_j(a, e) = s_j(a, e)\lambda(a, e) \tag{21}$$

where s_j is the share of households with states a, e choosing commodity j. Then aggregate demand is

$$Q_j = \int_{A \times E} s_j(a, e)\lambda(a, e) \tag{22}$$

and we want to derive (i) the elasticity of demand with respect to price and (ii) how the elasticity of demand changes with respect to price (the super elasticity).

Step 1. Elasticity of Demand. Differentiating Q with respect to p gives

$$\frac{\partial Q_j}{\partial p_j} = \int_{A \times E} \left\{ \frac{\partial s_j(a, e)}{\partial p_j} \right\} \lambda(a, e) \tag{23}$$

and then we can pull stuff out and divide through by p/Q giving

$$\frac{\partial Q_j/Q_j}{\partial p_j/p_j} == \int_{A\times E} \left\{ \frac{\partial s_j(a,e)/s_j(a,e)}{\partial p_j/p_j} \right\} \omega_j(a,e) \tag{24}$$

$$\text{where} \quad \omega_j(a,e) = \frac{s_j(a,e)\lambda(a,e)}{Q_j} \tag{25}$$

All we did was differentiate stuff, we have nice expression for the elasticity of demand

$$\theta_j = \int_{A\times E} \left\{ \theta_j(a,e)^E \right\} \omega_j(a,e) \tag{26}$$

so the "aggregate" elasticity of demand is simply a weighted average of the extensive margin micro-level elasticities where the weights are the share of consumption that household type consumes. I want to think through a bit more about what economic forces shape the elasticity. To do so, impose the Type 1 extreme value assumption:

$$\theta_j(a,e)^E = \frac{\partial s_j(a,e)/s_j(a,e)}{\partial p_j/p_j} = \frac{1}{\sigma} \times \frac{\partial v_j(a,e)}{\partial p_j/p_j} \tag{27}$$

where I took this partial derivative not takeing into account how the denominator of the choice probability changes, this is where the monopolistic competition bit matters. It's important to think through what this says which is how choice probability changes with (i) a "constant elasticity" term reflecting how much variation in tastes there is times (ii) the marginal value associated with the change in the price. Marginal value here could be the how the value function changes in a BHA economy.

Example #1: The BAH Economy

This is a bit more involved, but promising. Just recap, the budget constraint is

$$c + p_j = we + Ra - a' \quad a' > \phi, \tag{28}$$

and associated with this problem is (i) a policy function $g(a,e,j)$ which prescribes the map from states and variety choices into asset holdings next period and (ii) the measure $\lambda(a,e)$ which describes how many households are in state a, z. Given the elasticity calculation above 27, we just need to compute the elasticity.

Now I'm going to make a couple of observations: The Euler Equation, conditional on choice j is

$$u'(c(a,e,j)) = \max \left\{ \beta \mathbb{E} \frac{\partial v(a',e')}{\partial e'} \frac{\partial g(a,e,j)}{\partial p_j}, \quad u'\left(wz + Ra - \phi - p_j\right) \right\} \tag{29}$$

which says that the household sets the marginal utility of foregone consumption equal to the marginal benefit of holding some extra assets in the future, but if constrained then marginal utility is just at the level it's at. Now we can plug this in

$$\frac{\partial s_j(a,e)/s_j(a,e)}{\partial p_j} = \frac{1}{\sigma} \left\{ -u'(c(a,e,j)) - u'(c(a,e,j)) \frac{\partial g(a,e,j)}{\partial p_j} + \beta \mathbb{E} \frac{\partial v(a',e')}{\partial a'} \frac{\partial g(a,e,j)}{\partial p_j} \right\} \tag{30}$$

So this says the elasticity of the choice probability is $\frac{1}{\sigma}$ times how the value function changes with respect to prices. How the value function changes with respect to prices reflects the direct effect on period utility and then the indirect effect on asset choices. Notice, I am making the observation that firms are small, so the likelihood that they are faced again in the future is small, so no need to take derivative about future. Then I'm going to be explicit and then invoke an Envelope argument. So the elasticity is

$$\frac{\partial s_j(a,e)/s_j(a,e)}{\partial p_j/p_j} = \frac{-u'(c(a,e,j))p_j}{\sigma} + \frac{1}{\sigma}\left\{ -u'(c(a,e,j)) + \beta\mathbb{E}\,\frac{\partial v(a',e')}{\partial a'}\right\}\frac{\partial g(a,e,j)}{\partial p_j/p_j} \tag{31}$$

which says that the elasticity of the choice probability equals (i) how it directly changes hh's valuation of the commodity which is the first term (ii) how it may affect hh's via borrowing constraints. This is the argument for the last term to be zero. If off constraint, it's zero. If at constraint, then $\frac{\partial g(a,e,j)}{\partial p_j/p_j}$ is zero. If **just on constraint**, so on the margin between being constrained or not and thus assets may change, then the inside part has to be zero as the hh is effectively indifferent between holding more assets or not. In other words, marginal switchers from constrained to unconstrained, well are marginal and hence the value from being on the constraint to being off has to be equal. It's like a smooth pasting type condition.

So the extensive margin elasticity is

$$\theta_j(a,e)^E = \frac{\partial s_j(a,e)/s_j(a,e)}{\partial p_j/p_j} = \frac{-u'(c(a,e,j))p_j}{\sigma} \tag{32}$$

and then the aggregate elasticity of demand is

$$\theta_j = -\frac{p_j}{\sigma} \times \left\{ \int_{A\times E} u'(c(a,e,j))\omega_j(a,e) \right\} \tag{33}$$

still feel like my sign is wrong. It's because dQ stuff is defines as $-\theta$.

A couple of thought experiments:

- Just to remind our selves. If σ is small, then this is like goods are becoming perfect substitutes and (holding everything else fixed) the **price elasticity of demand becomes larger**.
- Fix a price p_j and imagine only rich people buy good j. Then with diminishing marginal utility, θ_j becomes small, so the **price elasticity of demand becomes smaller**. This is this idea that rich guys are more inelastic and then face larger markups.
- Fix a price p_j and imagine only poor people buy good j. Then θ_j becomes large and the **price elasticity of demand becomes larger**. This is this idea that poor guys are more elastic and then face smaller markups.
- Imagine u is linear, then this becomes

$$\theta_j = \frac{p_j}{\sigma} \tag{34}$$

- Note that these elasticities look a bit different than what shows up in say explanations of BLP. One cosmetic difference is that they normalize the $\sigma = 1$ as it's always subsumed in the coefficients they estimate. The second difference is their own price elasticities take into account the move in the denominator, so for example in the linear utility bit, that's why one does not see the $1 - s_j$ bit.

Step 2. Super Elasticity of Demand.

This is a bit fuzzy but we want to compute

$$\frac{\partial \theta_j}{\partial p_j} = \int_{A\times E} \left\{ \frac{\partial \theta_j(a,e)^E}{\partial p_j} \omega_j(a,e) + \theta_j(a,e)^E \frac{\partial \omega_j(a,e)}{\partial p_j} \right\} \tag{35}$$

and then convert it to an elasticity which gives

$$\frac{\partial \theta_j/\theta_j}{\partial p_j/p_j} = \int_{A\times E} \left\{ \frac{\partial \theta_j(a,e)^E/\theta_j(a,e)^E}{\partial p_j/p_j} + \frac{\partial \omega_j(a,e)/\omega_j(a,e)}{\partial p_j/p_j} \right\} \frac{\theta_j(a,e)^E \omega_j(a,e)}{\theta_j} \tag{36}$$

and then if I plug in how the extensive margin trade elasticity changes is

$$\frac{\partial \theta_j(a,e)^E/\theta_j(a,e)^E}{\partial p_j/p_j} = \frac{1}{\sigma}\left[u''(c(a,e,j)) \frac{\partial c(a,e,j)}{\partial p_j} + u'(c(a,e,j)) \right] \tag{37}$$

$$= \frac{1}{\sigma}\left[-\gamma(a,e,j) \frac{\partial c(a,e,j)/c(a,e,j)}{\partial p_j/p_j} + 1 \right] \tag{38}$$

where $\gamma(a,e,j)$ is relative risk aversion. So then if you put in the CRRA it becomes

$$= \frac{1}{\sigma}\left[-\gamma \frac{\partial c(a,e,j)/c(a,e,j)}{\partial p_j/p_j} + 1 \right] \tag{39}$$

Then I think the share elasticity is this:

$$\frac{\partial \omega_j(a,e)}{\partial p_j} = Q^{-2}\left[\frac{\partial s_j(a,e)}{\partial p_j} Q_j \lambda(a,e) - s_j \frac{\partial Q_j}{\partial p_j} \lambda(a,e) \right] \tag{40}$$

$$= Q^{-1}\left[\frac{\partial s_j(a,e)}{\partial p_j} \lambda(a,e) - s_j \frac{\partial Q_j/Q_j}{p_j} \lambda(a,e) \right] \tag{41}$$

and now divide through by ω so

$$\frac{Q^{-1}}{Q s_j(a,e)\lambda(a,e)}\left[\frac{\partial s_j(a,e)}{\partial p_j} \lambda(a,e) - s_j \frac{\partial Q_j/Q_j}{\partial p_j} \lambda(a,e) \right] \tag{42}$$

which gives

$$\frac{\partial \omega_j(a,e)}{\partial p_j/p_j} = \frac{\partial s_j(a,e)/s_j(a,e)}{\partial p_j/p_j} - \frac{\partial Q_j/Q_j}{\partial p_j/p_j} \tag{43}$$

which then is

$$\frac{\partial \omega_j(a,e)}{\partial p_j/p_j} = \theta_j(a,e)^E - \theta_j \tag{44}$$

Then the total super elasticity (after rearranging things and pulling stuff out)

$$\frac{\partial \theta_j/\theta_j}{\partial p_j/p_j} = -\theta_j + \frac{1}{\sigma} + \int_{A\times E} \left\{ \theta_j(a,e)^E - \frac{\gamma}{\sigma} \frac{\partial c(a,e,j)/c(a,e,j)}{\partial p_j/p_j} \right\} \frac{\theta_j(a,e)^E \omega_j(a,e)}{\theta_j} \tag{45}$$

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