

A. Type 1 Extreme Value Shocks, Choice Probabilities, and Expected utility

This section justifies and derives the expression for utility in (??), choice probabilities, expected utility, etc. The idea is that we can recast the planner as choosing cutoff values for the preference shock as the planner directly choosing the migration rates. I do everything below in a simplified form with an agent picking between locations j and j' . And to simplify the notation, I'm not carrying around all the states.

First, the ordinal way to think of the planner's problem is that the planner is choosing $J - 1$ cut-off values $\nu_j^j(s)$ for each state (s and current location j). The $J - 1$ is because for J locations, one is redundant. In my presentation below, it is only a two location situation, so there is only one cutoff value.

The cutoff value then works in the following way where: $\nu^j + \nu_j^j(s) > \nu^{j'}$, then you move to j otherwise go to j' . Another, perhaps clearer way to write this is $\nu^j(s) \geq \nu^{j'} - \nu^j$. So, for example, if your preference for location j is very large, this inequality is satisfied and you go to j . Otherwise, if this inequality is not satisfied (you have a very large $\nu^{j'}$ shock, then go to j' .

Now we can go back and forth between the cut-off values described above and migration rates. Given some joint density over the preference shocks ϕ , the probability that people go to location j is:

$$\mu_{j,j}(s) = \text{Prob}\{\nu^j + \nu_j^j(s) \geq \nu^{j'}\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\nu^j + \nu_j^j(s)} \phi(\nu^j, \nu^{j'}) d\nu^{j'} \right] d\nu^j \quad (1)$$

where the internal part of the bracket stays: "fix a ν^j , then how many guys have $\nu^{j'}$ below $\nu^j + \nu_j^j(s)$ " or in words "for the given ν^j , how many want to go to j ." Then there is the outer part says "add up all those guys for each ν^j ." I do this many times, always get confused, so say it again. With the Type 1 extreme value distribution, we can evaluate the integral in (1) and then compute the migration rates as a function of the cut-off values and expected utility as a function of the migration rates as in (??).

Below we do this in several steps.

1. Train's book is here <https://eml.berkeley.edu/books/choice2.html> which is a very good resource for working with the Type 1 extreme value distribution. Most of the discussion below simply fills in the holes from aspects of the book with a more explicit derivation.
2. The Type 1 extreme value pdf is:

$$\phi(\nu) = \sigma^{-1} \exp(-\sigma^{-1}\nu) \exp(-\exp(-\sigma^{-1}\nu)) \quad (2)$$

$$\Phi(\nu) = \exp(-\exp(-\sigma^{-1}\nu)) \quad (3)$$

and then let's insert this into (1) and realizing that the inner term of (1) is the cdf Φ evaluated at the cut-off point of $\nu^j + \nu_j^j(s)$.

$$\text{Prob}\{\nu^j + \nu_j^j(s) \geq \nu^{j'}\} = \int_{-\infty}^{\infty} \Phi(\nu^j + \nu_j^j(s)) \phi(\nu) d\nu^s \quad (4)$$

$$\text{Prob}\{\nu^j + \nu_j^j(s) \geq \nu^{j'}\} = \int_{-\infty}^{\infty} \exp(-\exp(-\sigma^{-1}(\nu^j + \nu_j^j(s)))) \sigma^{-1} \exp(-\sigma^{-1}\nu^j) \exp(-\exp(-\sigma^{-1}\nu^j)) d\nu^j \quad (5)$$

Now if you collect terms on the exponents you get something like

$$\text{Prob}\{\nu^j + \nu_j^j(s) \geq \nu^{j'}\} = \int_{-\infty}^{\infty} \sigma^{-1} \exp[-\sigma^{-1}\nu^j - \exp(-\sigma^{-1}\nu^j)(1 + \exp(-\sigma^{-1}\nu_j^j(s)))] d\nu^j \quad (6)$$

This is tedious, but almost done. First, using antiderivative calculator will take care of this.¹ A more subtle approach is to note that this is essentially the same as the pdf—which we know integrates up to one—except for the constant term $(1 + \exp(-\sigma^{-1}\nu_j^j(s)))$. Either way, when you compute this you have that:

$$\mu_{j,j}(s) = \text{Prob}\{\nu^j + \nu_j^j(s) \geq \nu^{j'}\} = \frac{1}{1 + \exp(-\sigma^{-1}\nu_j^j(s))} \quad (7)$$

which is the standard Type 1 extreme value share formula. First note that if the cut-off $\nu_j^j(s)$ becomes arbitrarily large, then this value converges to one, so everyone stays in location j . Second, notice if dispersion parameter σ becomes arbitrarily large, the share converges to a pure lottery, half go one way half go another way. Finally the probability of going to location j' is

$$\mu_{j',j}(s) = \frac{\exp(-\sigma^{-1}\nu_j^j(s))}{1 + \exp(-\sigma^{-1}\nu_j^j(s))} \quad (8)$$

3. Next we need to compute expected utility associated with the migration pattern above. Note this is not a simple calculation because one needs to take into account that only the highest relative (or max) values migrate to a destination. So to compute expected utility, we first need to construct the following density:

$$\text{Prob}\{\nu^j \mid \nu^j + \nu_j^j(s) \geq \nu^{j'}\} \quad (9)$$

which is the probability of ν^j , conditional on $\nu^j + \nu_j^j(s)$ being greater than the random variable $\nu^{j'}$. Now we can construct this density in the following way, using Bayes rule we have that

$$\text{Prob}\{\nu^j \mid \nu^j + \nu_j^j(s) \geq \nu^{j'}\} = \frac{\text{Prob}\{\nu^j, \nu^j + \nu_j^j(s) \geq \nu^{j'}\}}{\text{Prob}\{\nu^j + \nu_j^j(s) \geq \nu^{j'}\}} \quad (10)$$

and one important thing to notice is that we have already computed the value in the denominator as it's the same as the stay in location j probability in (7). Then the object in the top is closely related to the way we computed the earlier density. This is:

$$\text{Prob}\{\nu^j \mid \nu^j + \nu_j^j(s) \geq \nu^{j'}\} = \int_{-\infty}^{\nu} \left[\int_{\nu^j + \nu_j^j(s)}^{\nu^{j'}} \phi(\nu^j, \nu^{j'}) d\nu^{j'} \right] d\nu^j \quad (11)$$

$$= \int_{-\infty}^{\nu} \sigma^{-1} \exp[-\sigma^{-1}\nu^j - \exp(-\sigma^{-1}\nu^j)(1 + \exp(-\sigma^{-1}\nu_j^j(s)))] d\nu^j \quad (12)$$

where (to repeat myself) the first line says, fix a ν^j and add up all the guys that go to j for that given value. Then add up all the ν^j **up to the point of** ν . This is the joint density of ν s that are movers and values up that

¹The way to see the form is to note it takes $\alpha \exp(-\alpha x + \exp(-\alpha x)\beta)$ for which the antiderivative is $\frac{\exp(-\beta \exp(-\alpha x))}{\beta}$.

point. Now notice this is the same form of the integral as before, so this density is expressed as:

$$\text{Prob}\{\nu^j < \nu, \nu^j + \nu_j^j(s) \geq \nu^{j'}\} = \frac{\exp[-(1 + \exp(-\sigma^{-1}\nu_j^j(s))) \exp(-\sigma^{-1}\nu)]}{1 + \exp(-\sigma^{-1}\nu_j^j(s))}. \quad (13)$$

Then combining this probability with probability of being a stayer gives the following.

$$\text{Prob}\{\nu^j < \nu \mid \nu^j + \nu_j^j(s) \geq \nu^{j'}\} = \exp[-(1 + \exp(-\sigma^{-1}\nu_j^j(s))) \exp(-\sigma^{-1}\nu)] \quad (14)$$

$$= \exp[-(\mu_{j,j}(s))^{-1} \exp(-\sigma^{-1}\nu)] \quad (15)$$

where the last line substitutes in the “ACR” like insight that the centering parameter relates to the share of migrants. Now the expected value of ν^j conditional on staying in j is

$$E(\nu^j \mid \nu^j + \nu_j^j(s) \geq \nu^{j'}) = \int_{-\infty}^{\infty} \nu^j d \exp[-(\mu_{j,j}(s))^{-1} \exp(-\sigma^{-1}\nu^j)]. \quad (16)$$

Now to compute this integral we need to do several more things. First, compute the pdf. Second, compute the integral via a change in variables and then make connection with Euler’s constant. First the CDF takes the general form

$$\exp[-\beta \exp(-\alpha x)] \quad (17)$$

$$(18)$$

and then the associated PDF takes the form

$$\beta \alpha \exp[-\beta \exp(-\alpha x) - \alpha x] \quad (19)$$

and the integral that we want to compute is

$$\beta \alpha \int \exp[-\beta \exp(-\alpha x) - \alpha x] x dx \quad (20)$$

Now we want to connect this integral with Euler’s constant which is

$$- \int \exp(-y) \ln y dy = \gamma \quad (21)$$

4. <https://www.randomservices.org/random/special/ExtremeValue.html> helps. As does this one with integrals of functions of exponentiation

https://en.wikipedia.org/wiki/List_of_integrals_of_exponential_functions

5. So first notice that we can express the integral in (20) as

$$\beta \alpha \int \exp[-\beta \exp(-\alpha x)] \exp[-\alpha x] x dx \quad (22)$$

by just using rules of exponents. Next here is the proposed change of variables where

$$y = \beta \exp(-\alpha x) \quad (23)$$

$$\frac{y}{\beta} = \exp(-\alpha x) \quad (24)$$

$$x = \frac{-1}{\alpha} \ln \left(\frac{y}{\beta} \right) \quad (25)$$

$$\frac{dx}{dy} = \frac{-1}{\alpha} \frac{1}{y} \quad (26)$$

And notice that the domain of integration for this is now $[\infty, 0]$ (yes, the infinity is at the bottom), but we have to flip giving rise to a minus sign. Now inserting this change of variable into (22) we have

$$-\beta\alpha \int_0^\infty \exp(-y) \left(\frac{y}{\beta} \right) \left(\frac{-1}{\alpha} \ln \left(\frac{y}{\beta} \right) \right) \left(\frac{-1}{\alpha} \frac{1}{y} \right) dy \quad (27)$$

and then notice how the terms cancel leaving

$$- \int \exp(-y) \ln \left(\frac{y}{\beta} \right) \left(\frac{1}{\alpha} \right) dy. \quad (28)$$

Almost there. The final step is to expand out the stuff in the \ln function giving.

$$-\frac{1}{\alpha} \int \exp(-y) \ln y \, dy + \frac{\ln \beta}{\alpha} \int \exp(-y) \, dy \quad (29)$$

Then given the domain of integration and the form of these integrals, the first integral is Euler's constant, the second integral just sums up to one giving us:

$$\frac{\gamma}{\alpha} + \frac{\ln \beta}{\alpha} \quad (30)$$

Now let's get some intuition behind this formula. Recall that $\alpha = \sigma^{-1}$ and that $\beta = \frac{1}{\mu_{j,j}(s)}$. So first notice if no one chooses j' , everyone is choosing j , then β equals one and then the formula implies that the expected value of the preference draw those going to j is $\sigma\gamma$ which is the standard formula for the mean of a [Type 1 extreme value](#). Good. Then notice as $\mu_{j,j}(s)$ decreases, less people are staying in j and β becomes larger. Why is it increasing? Remember this is the mean of the preference shock in j , so as out migration rate increases (less $\mu_{j,j}(s)$, higher $\mu_{j',j}(s)$), than those remaining in j are more selected and hence the expected value of the preference shock is larger. Even better.

6. To summarize the main result, the expected values **in a location** j conditional on a migration rate are

$$E(\nu^j | \nu^j + \nu_j^j(s) \geq \nu^{j'}) = \sigma\gamma + \sigma \ln \left(\frac{1}{\mu_{j,k}(s)} \right) \quad (31)$$

And the **expected value across all locations**, conditional on the migration rates (which is a function of the cutoff value $\nu^j(s)$) is

$$E[\nu | \mu_{j',j}(s)] = \sigma\gamma + -\sigma \sum_{j'} \mu_{j'}(s) \log(\mu_{j',j}(s)) \quad (32)$$

which takes the familiar log sum formulation that arises in the Type 1 extreme value models. And most importantly,

this is cast all in terms of the migration rates, not cutoff values.