Chaotic Motion In a Damped Driven Pohl/Simple Pendulum Hybrid System: Theory and Methods

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Abstract

Here is a few sentence summary of your report. One way to proceed is to write one sentence about each of the following: the motivation, methods, analysis, results, and interpretation.

As usual, this skeleton is an example but only one of many possible structures – do what you think makes the most compelling structure for the goals you are trying to communicate.

1 Theory

I will begin by considering the system in Figure 1. The goal is to determine an equation of motion to describe this system, and show that it can produce a deterministic chaotic system. To derive this equation of motion, the Principle of Least Action will be used. First, define the following variables to describe this system:

- θ: The angular displacement of the arrow, as measured from vertical. Note this completely describes the position of the pendulum.
- I: The moment of inertia of the Pohl Pendulum hoop.
- k: The spring constant of the circular spring.
- m: The mass of the mass attached to the bottom stoke of the Pohl pendulum. This will create a Simple/Pohl hybrid pendulum.
- r: The distance from the center of the Pohl pendulum to the mass m.
- ω_0: Driving frequency of the motor.
- Δx: Radius of driving piston. As the piston arm is much greater than Δx, his will also be considered the maximum horizontal displacement of the driving shaft.
- x_1: Distance from the center of the Pohl Pendulum to where the driving shaft attaches to the top of the spring
- x_2: Distance from the center of the Pohl Pendulum to where the driving arm attaches to the driving shaft.

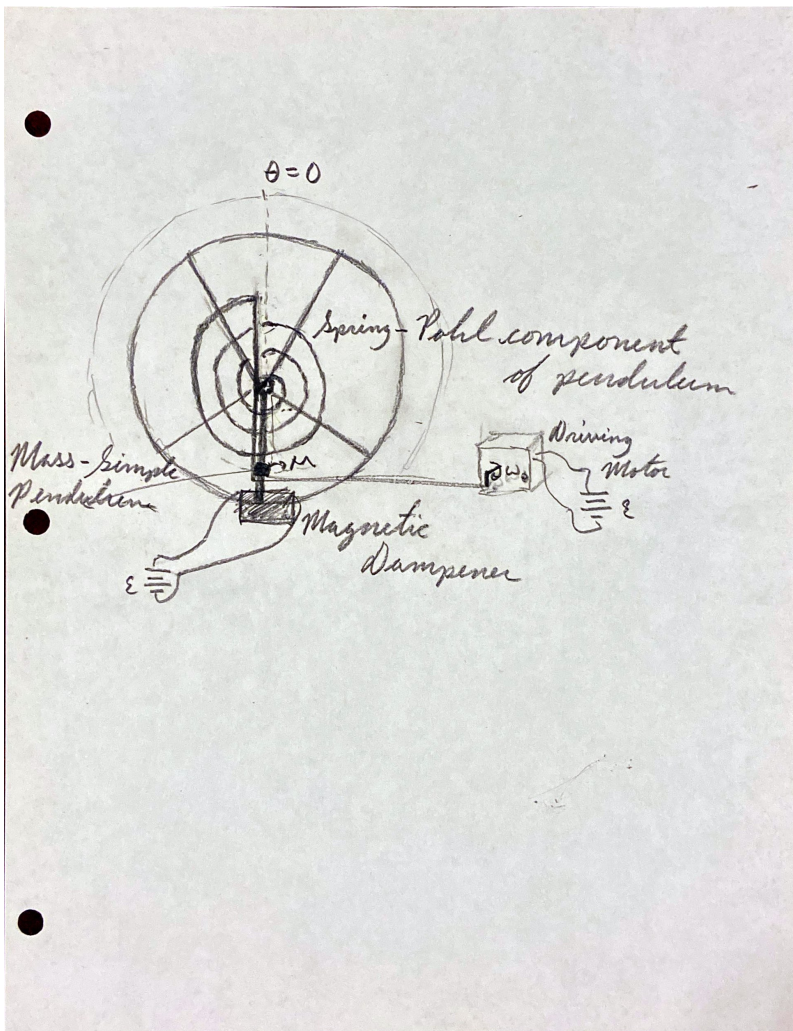

Figure 1: Schematic of system considered in this experiment

- ϕ: Angular displacement of driving shaft from vertical.
- θ': Angular difference between the position of the shaft and the position of the pendulum, defined as $\theta' = |\theta - \phi|$.

Now, consider the kinetic energy of the pendulum. There are two components–the kinetic energy of the Pohl Pendulum and the kinetic energy of the mass attached to the Pohl Pendulum. Therefore:

$$T = \frac{1}{2}(I + rm)\dot{\theta}^2$$

Next, consider the potential energy of the system. Define the zero of potential to be when $\theta = 0$, and assume that, as the Pohl Pendulum is essentially symmetric, that the only gravitational potential energy is due to the mass m. Additionally, assume that the circular spring obeys a hook-like law, $U_s = \frac{1}{2}k\theta'^2$. Note here that the spring potential is due to the angular difference between the top of the spring and where the spring attaches to the axle–namely, θ'. It follows that the potential energy of the system is given by:

$$U = mgr(1 - \cos\theta) + \frac{1}{2}k(\theta')^2 = mgr(1 - \cos\theta) + \frac{1}{2}k(\theta - \phi)^2$$

The Lagrangian is given by:

$$L = T - U = \frac{1}{2}(I + rm)\dot{\theta}^2 - mgr(1 - \cos\theta) - \frac{1}{2}k(\theta - \phi)^2$$

Applying the Principle of Least Action, it follows that:

$$\begin{aligned}
&\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \\
&\frac{\partial}{\partial \theta}[\frac{1}{2}(I + rm)\dot{\theta}^2 - mgr(1 - \cos\theta) - \frac{1}{2}k(\theta - \phi)^2] - \\
&\quad \frac{d}{dt}\frac{\partial}{\partial \dot{\theta}}[\frac{1}{2}(I + rm)\dot{\theta}^2 - mgr(1 - \cos\theta) - \frac{1}{2}k(\theta - \phi)^2] = 0 \\
&-k(\theta - \phi) - mgr\sin\theta - \frac{d}{dt}[(I + rm)\dot{\theta}] = 0, \text{ Implying that:} \\
&\quad \ddot{\theta} = -\frac{k(\theta - \phi) + mgr\sin\theta}{I + rm}
\end{aligned}$$

Notice that ϕ–the angle between the driving shaft and vertical–is a function of time. Therefore, we must determine an expression for it. Let $x(t) = \Delta x * \cos\omega_0 t$ be the horizontal displacement of the shaft as a function of time. Then, $\tan\phi(t) = \frac{x(t)}{x_2}$. Finally, we take the small angle approximation $\tan\phi \approx \phi$ to find that:

$$\phi \approx \frac{x(t)}{x_2} = \frac{\Delta x \cos\omega_0 t}{x_2}$$

Making this substitution into the equation of motion determined above, it follows that:

$$\ddot{\theta} = -\frac{k\theta + mgr\sin\theta}{I + rm} + \frac{k\Delta x}{x_2(I + rm)}\cos\omega_0 t$$

Finally, we add the unknown dampening term $\gamma *$ giving the full equation of motion:

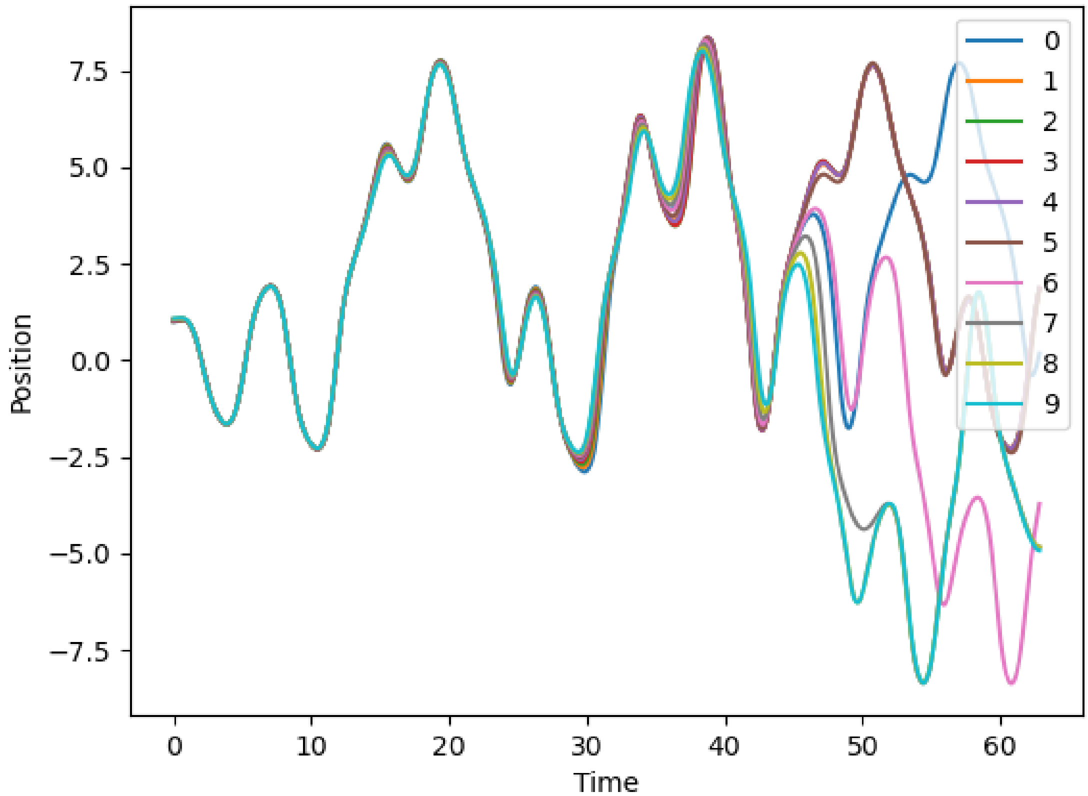

Figure 2: Graph showing chaos with specific, unit less initial conditions. The initial starting position is $\theta = 2$ for the line 0, and each other line has an initial position of $2 + 0.02i$, where i is the number in the legend.

$$\ddot{\theta} = -\frac{k\theta + mgr\sin\theta}{I + rm} + \frac{k\Delta x}{x_2(I + rm)}\cos\omega_0 t - \beta\dot{\theta}$$

Recalling that the goal of this section is simply to show that it is possible to obtain chaotic motion in this system, we paramaterize this equation in terms of constants which can be determined experimentally as:

$$\ddot{\theta} = -C * \theta + A\sin\theta + \gamma * \cos\omega_0 t - \beta\dot{\theta}$$

Under specific conditions, we do find Chaos in this system, as shown in the Figure 2 solving this differential equation near a single starting position. The goal of this experiment will be to figure out these parameters in terms of the variables in the derived equations, and then tweak these parameters to produce chaos

2 Methods

Methods: Unfortunatly running out of time for this. Will quickly go over essential details, and try to elaborate more tomorrow:

- Measure dampening parameter. Use Pohl Pendulum, and collect data for various voltages; fit a curve to determine dampening as a function of voltage.
- Determine moment of inertia of Pohl Hoop by measuring mass, inner diameter, and outer diameter.
- Determine dampening by using just the Pohl Pendulum–which is a simple harmonic oscillator if it is not driven–and determine how much effect the dampening has at different voltages.
- Choose a driving frequency by measuring period of motor at different voltages.
- Choose mass and radius to put mass on stoke.
- Attempt to calculate parameters to produce chaos in this system; attempt to make the same measurement under these conditions multiple times. If the system is chaotic, these measurements should diverge.