Chaotic Motion In a Damped Driven Pohl/Simple Pendulum Hybrid System: Theory and Methods

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1 Theory

I will begin by considering the system in Figure 1. The goal is to determine an equation of motion to describe this system, and show that it can produce a deterministic chaotic system. To derive this equation of motion, the Principle of Least Action will be used. First, define the following variables to describe this system:

- θ: The angular displacement of the arrow, as measured from vertical. Note this completely describes the position of the pendulum.
- I: The moment of inertia of the Pohl Pendulum hoop.
- k: The spring constant of the circular spring.
- m: The mass of the mass attached to the bottom stoke of the Pohl pendulum. This will create a Simple/Pohl hybrid pendulum.
- r: The distance from the center of the Pohl pendulum to the mass m.
- ω_0: Driving frequency of the motor.
- Δx: Radius of driving piston. As the piston arm is much greater than Δx, his will also be considered the maximum horizontal displacement of the driving shaft.
- x_1: Distance from the center of the Pohl Pendulum to where the driving shaft attaches to the top of the spring
- x_2: Distance from the center of the Pohl Pendulum to where the driving arm attaches to the driving shaft.
- ϕ: Angular displacement of driving shaft from vertical.
- θ': Angular difference between the position of the shaft and the position of the pendulum, defined as $\theta' = |\theta - \phi|$.
- F_D: The dampening for acting on the pendulum due to eddy currents
- r_P: Average radius of Pohl Pendulum hoop.

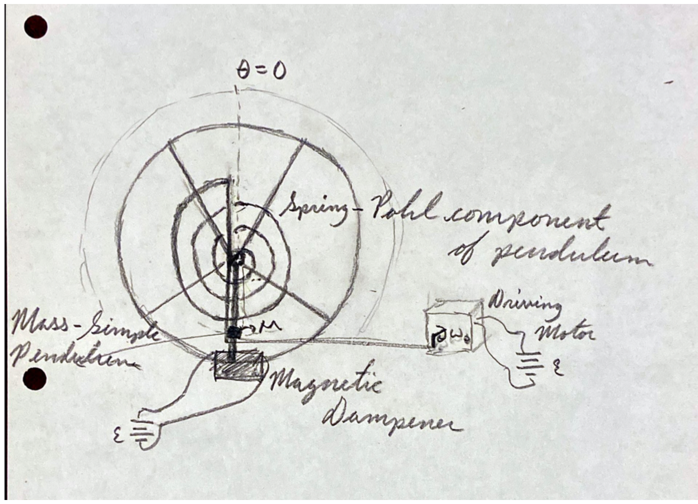

Figure 1: Schematic of system considered in this experiment

We would like to first address the dampening force acting on the pendulum. This dampening is due to eddy currents forming in the Pohl Pendulum as it passes by the electromagnet dampener. Using theory developed in Chapter 14 of Andrew Zangwill' Modern Electrodynamics, we find the this dampening force is roughly given by:

$$F_D \approx V\sigma B^2 v$$

Where V is the volume of the Pohl Pendulum in the magnetic field, B is the strength of the magnetic field, σ is the conductivity of the metal, and $v = r_p\dot{\theta}$ is the speed of the pendulum as it passes through the magnetic field. Notice that due to the symmetry of the Pohl Pendulum, the same volume is always in the magnetic field; additionally, the conductivity of the Pohl Pendulum is some constant. While the magnetic field is a function of the voltage applied to the electromagnetic dampener, it is constant for constant voltage. Therefore, the dampening force is linear in $\dot{\theta}$, and can be considered $F_D \approx \beta(V)\dot{\theta}$, where β is some yet to be determine function of the Voltage applied to the electromagnet.

Now, consider the kinetic energy of the pendulum. There are two components–the kinetic energy of the Pohl Pendulum and the kinetic energy of the mass attached to the Pohl Pendulum. Therefore:

$$T = \frac{1}{2}(I + mr^2)\dot{\theta}^2$$

Next, consider the potential energy of the system. Define the zero of potential to be when $\theta = 0$, and assume that, as the Pohl Pendulum is essentially symmetric, that the only gravitational potential energy is due to the mass m. Additionally, assume that the circular spring obeys a hook-like law, $U_s = \frac{1}{2}k\theta'^2$. Note here that the spring potential is due to the angular difference between the top of the spring and where the spring attaches to the axle–namely, θ'. It follows that the potential energy of the system is given by:

$$U = mgr(1 - \cos\theta) + \frac{1}{2}k(\theta')^2 = mgr(1 - \cos\theta) + \frac{1}{2}k(\theta - \phi)^2$$

The Lagrangian is given by:

$$L = T - U = \frac{1}{2}(I + mr^2)\dot{\theta}^2 - mgr(1 - \cos\theta) - \frac{1}{2}k(\theta - \phi)^2$$

Applying the Principle of Least Action, it follows that:

$$
\begin{aligned}
&\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \\
&\frac{\partial}{\partial \theta}[\frac{1}{2}(I + mr^2)\dot{\theta}^2 - mgr(1 - \cos\theta) - \frac{1}{2}k(\theta - \phi)^2] - \\
&\quad \frac{d}{dt}\frac{\partial}{\partial \dot{\theta}}[\frac{1}{2}(I + mr^2)\dot{\theta}^2 - mgr(1 - \cos\theta) - \frac{1}{2}k(\theta - \phi)^2] = 0 \\
&-k(\theta - \phi) - mgr\sin\theta - \frac{d}{dt}[(I + mr^2)\dot{\theta}] = 0, \text{ Implying that:} \\
&\ddot{\theta} = -\frac{k(\theta - \phi) + mgr\sin\theta}{I + mr^2}
\end{aligned}
$$

Notice that ϕ–the angle between the driving shaft and vertical–is a function of time. Therefore, we must determine an expression for it. Let $x(t) = \Delta x * \cos\omega_0 t$ be the horizontal displacement of the shaft as a function of time. Then, $\tan\phi(t) = \frac{x(t)}{x_2}$. Finally, we take the small angle approximation $\tan\phi \approx \phi$ to find that:

$$\phi \approx \frac{x(t)}{x_2} = \frac{\Delta x \cos\omega_0 t}{x_2}$$

Making this substitution into the equation of motion determined above, it follows that:

$$\ddot{\theta} = -\frac{k\theta + mgr\sin\theta}{I + mr^2} + \frac{k\Delta x}{x_2(I + mr^2)}\cos\omega_0 t$$

Finally, we add the unknown dampening term $\gamma *$ giving the full equation of motion:

$$\ddot{\theta} = -\frac{k\theta + mgr\sin\theta}{I + mr^2} + \frac{k\Delta x}{x_2(I + mr^2)}\cos\omega_0 t - \beta\dot{\theta}$$

Recalling that the goal of this section is simply to show that it is possible to obtain chaotic motion in this system, we paramaterize this equation in terms of constants which can be determined experimentally as:

$$\ddot{\theta} = -C * \theta + A\sin\theta + \gamma * \cos\omega_0 t - \beta\dot{\theta}$$

Under specific conditions, we do find Chaos in this system, as shown in the Figure 2 solving this differential equation near a single starting position. The goal of this experiment will be to figure out these parameters in terms of the variables in the derived equations, and then tweak these parameters to produce chaos

2 Methods

In the Theory section, we identified eight total variables that effect the equation of motion of the Pohl Simple Hybrid Pendulum; some of these can be varied, and some are part of the apparatus given to us and cannot be changed. First, the measurement of the values which cannot be changed will be discussed; then, methods for tweaking those which can be changed to produce chaotic motion will be discussed.

The spring constant k, the moment of inertia of the Pohl Pendulum Wheel I, and driving radius Δx cannot be controlled. Unfortunately, the user manual for the Pohl Pendulum we are using is

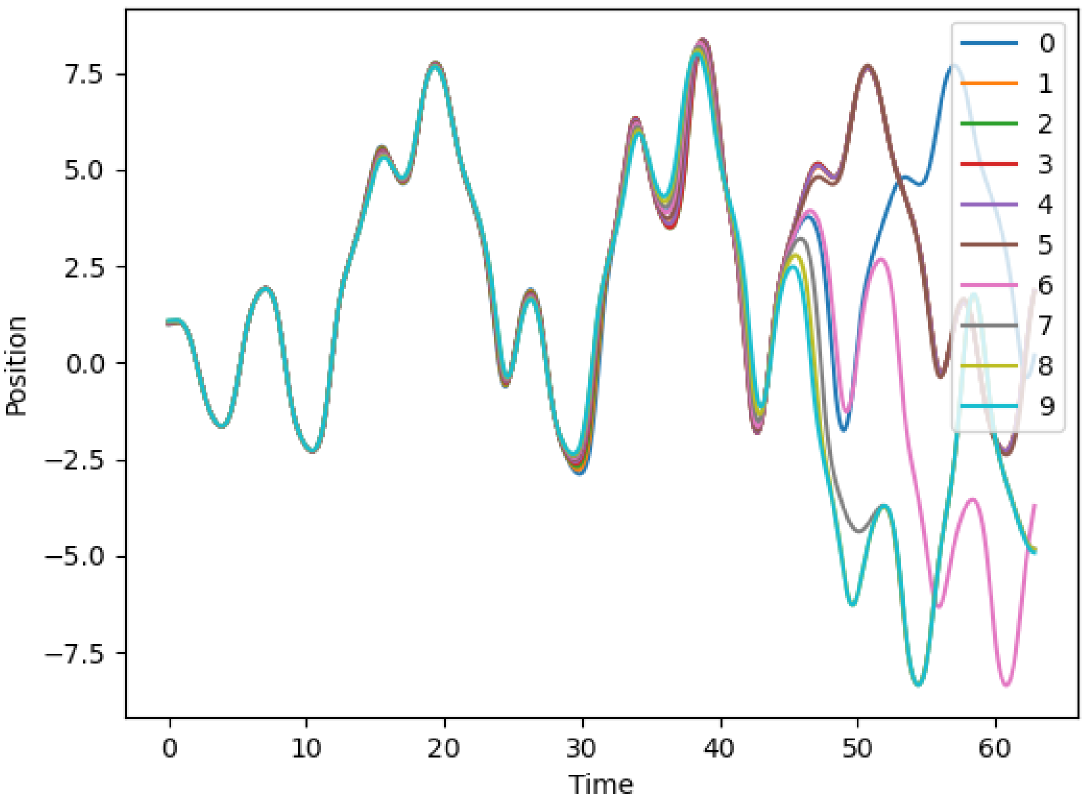

Figure 2: Graph showing chaos with specific, unit less initial conditions. The initial starting position is $\theta = 2$ for the line 0, and each other line has an initial position of $2 + 0.02i$, where i is the number in the legend.

lost and is also likely in German, so we do not have guidance into what these values are. However, they are not too difficult to measure or calculate in terms of other measured values.
The Pohl Pendulum Wheel is assumed to have a constant density; this is a reasonable assumption as it appears to be made out of one material. By splitting it into a hoop of outer radius r_1 and inner radius r_2, and five trapezoidal spokes of from a radius r_3 to r_2 and with an inner base s_2 and outer base s_1, we find the inertia to be given by [1]:

$$I_{net} = \frac{M}{\pi(r_1^2 - r_2^2) + 5(\frac{s_1+s_2}{2})(r_2 - r_3)} * [\frac{\pi}{2}(r_1^4 - r_2^4) + \frac{5}{3}(s_2 + r_3(\frac{s_2 - s_1}{r_2 - r_3}))(r_2^3 - r_3^3) - \frac{5(s_2 - s_1)}{4(r_2 - r_2)}(r_2^4 - r_3^4)]$$

The spatial measurements to be made here are r_1, r_2, r_3 as well as s_1, s_2. These measurements were made with a caliper; the other measurement is the mass M which was made with a scale.
The next parameter to be calculated is the spherical spring constant k. We note that only the Pohl Pendulum is a Simple Harmonic Oscillator described by the equation of motion $\ddot{\theta} = -\frac{k}{I}\theta$. As the moment of inertia can be measured directly, and this simple harmonic oscillator is known to oscillate with angular frequency $\omega = \sqrt{\frac{k}{I}} = \frac{2\pi}{T}$, where T is the period of the oscillator. With this knowledge, it is easy to show that:

$$k = (\frac{2\pi}{T})^2 * I$$

It follows that once the moment of inertia is known, taking data on the period of the Pohl Pendulum–without driving or dampening–is sufficient to quantify k. By taking data for different amplitudes, it can also be confirmed that the spherical spring does in fact obey a hook-like law $U = \frac{1}{2}k\theta'^2$, which was a major assumption of the Theory section; if this is true, the period of the Pohl Pendulum should be independent of the amplitude.
Next, the dampening function $\beta(V)$ must be determined. Again, theory about Simple Harmonic Oscillators–which we expect the Pohl Pendulum to be–will be used. In the case of a under dampened Simple Harmonic Oscillator, starting at some amplitude A and with no initial velocity, we expect that:

$$\theta(t) = e^{-\beta t}(A\cos(\omega_1 t))$$

Where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$. From here, there are two methods that could be used to find β for some voltage; as ω_0 has already been determined, the length of a period in the dampened case could be measured, and the resulting angular frequency ω_1 could be compared with ω_0 as $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$, implying that:

$$\beta = \sqrt{\omega_0^2 - \omega_1^2}$$

Alternatively, the change in amplitude of various periods could be used as well to determine β. Regardless, data will be collected for various voltages V, and a curve will be fit for $\beta(V)$.
Unfortunately running out of time for this. Will quickly go over essential details, and try to elaborate more tomorrow:

- Measure dampening parameter. Use Pohl Pendulum, and collect data for various voltages; fit a curve to determine dampening as a function of voltage.

[1] While it seemed unnecessary to derive this here, check the moment of inertia derivation on this expirements' github page

- Determine moment of inertia of Pohl Hoop by measuring mass, inner diameter, and outer diameter.
- Determine spring constant of spring by placing the Pohl Pendulum into simple harmonic motion with no dampening or driving.
- Determine dampening by using just the Pohl Pendulum–which is a simple harmonic oscillator if it is not driven–and determine how much effect the dampening has at different voltages.
- Choose a driving frequency by measuring period of motor at different voltages.
- Choose mass and where to put the mass on the stoke.
- Attempt to calculate parameters to produce chaos in this system; attempt to make the same measurement under these conditions multiple times. If the system is chaotic, these measurements should diverge.