# Introduction to Gravitational Waves (pmp -- v1.0, Jan. 2021)

## Line Elements in Flat Spacetime

Building on Hartle (Sec. 2.5), the *line element* of three-dimensional space in Cartesian coordinates is

$$dS^2 = dx^2 + dy^2 + dz^2\,. \tag{1}$$

This quantity represents a (very small) distance between two points – call them A and B – in space. Integrating the line element along a particular path from point A to point B would give the distance between A and B along that path. If a different frame of reference (with a different coordinate system) is used to describe the locations of A and B and the path between them, the result for the distance between A and B will still be the same. That distance is known as an *invariant*, a quantity that is the same in all reference frames.

In relativity, time is also a coordinate that must be included in the line element, but it is treated slightly differently than the spatial coordinates. Once again, the "distance" between two points in spacetime must be the same in different reference frames, but now time dilation and length contraction must be taken into account. It turns out that the line element of flat spacetime in Cartesian coordinates is

$$ds^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2\,, \tag{2}$$

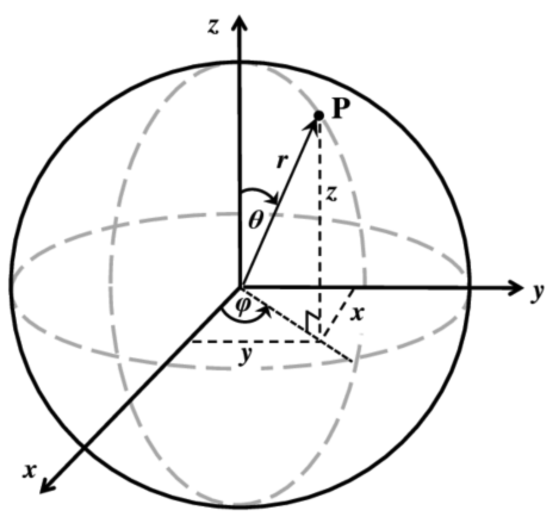
where $c$ is the speed of light. Its presence in the time term is essentially a unit conversion so that all the terms have the same units. In relativity however, we often work in units such that $c = 1$, so the line element simplifies to

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2\,. \tag{3}$$

### Changing Coordinate Systems ►

The line element (3) can be written in other coordinate systems, such as spherical polar coordinates, in which

$$\begin{aligned} x &= r\sin\theta\cos\phi\,, && \text{(4a)}\\ y &= r\sin\theta\sin\phi\,, && \text{(4b)}\\ z &= r\cos\theta\,. && \text{(4c)}\end{aligned}$$




To find the new form of the line element, the differential form of each of these terms must first be computed. For example,

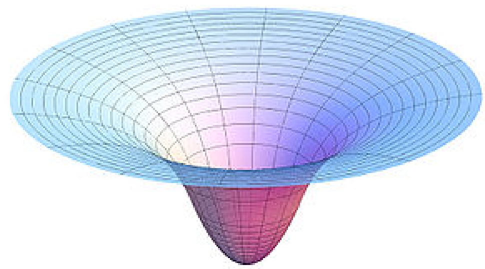
$$\begin{aligned} dx &= (dr)\sin\theta\cos\phi + r(\cos\theta\ d\theta)\cos\phi \\ &\quad + r\sin\theta(-\sin\phi\ d\phi)\,, \end{aligned} \tag{5}$$

where we have applied the product rule for differentiation. If the differentials for $dx$, $dy$, and $dz$ in terms of $dr$, $d\theta$, and $d\phi$ are plugged into the line element (3), the result simplifies to

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\,. \tag{6}$$

It is important to note that this line element (6), while expressed in non-Cartesian coordinates, still describes the same *flat* (or *Minkowski*) spacetime as Eq. (3) does. In this context, "flat" means that two objects on a parallel paths will stay the same distance apart forever. This contrasts with a *curved* spacetime, in which parallel paths would *not* stay the same distance apart indefinitely (more on this below). In practice, you can think of flat spacetime as being empty of anything with a gravitational field, such as a star or planet.

## Gravitational Wave Basics



In the Newtonian description, gravity is a force that attracts objects. General relativity says that gravity is the *curvature* of spacetime itself, and it is that curvature which causes objects to move towards each other. A common way to envision this is to imagine that flat spacetime is a flat rubber sheet. When a massive object is placed on the sheet, it causes the rubber to stretch and deform into a shape like that shown at right. In other words, this image is a representation of the curvature of spacetime in the vicinity of a star or planet. Attempting to a straight line on such a surface will result in a line that is actually curved in some way, and parallel lines will not always remain same distance apart.

In the case of gravitational waves, we will be assuming that spacetime is flat except for small perturbations like ripples on the surface of a pond. These ripples of spacetime curvature are the gravitational waves.

**Notation & Summation Convention ▶**

In general relativity, we often have very complicated expressions in which one line could fill an entire page (or more!). As a result, relativists have come up with a more compact notation in which our four spacetime coordinates can be written as one: $x^\alpha$, where $\alpha$ (or any other Greek letter) is an index indicating which coordinate we want (it does *not* indicate an exponent!). This index takes the values (0, 1, 2, 3), where 0 indicates the time coordinate and the others indicate the spatial coordinates. For example, the Cartesian coordinates would be

$$x^0 = t\,, \qquad x^1 = x\,, \qquad x^2 = y\,, \qquad x^3 = z\,, \tag{7}$$

and the spherical polar coordinates would be

$$x^0 = t\,, \qquad x^1 = r\,, \qquad x^2 = \theta\,, \qquad x^3 = \phi\,. \tag{8}$$

With this notation, the general line element can be written in the form

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta\,, \tag{9}$$

where $g_{\alpha\beta}$ is a symmetric, position-dependent matrix called the *metric*. For example, the metric for flat spacetime in spherical polar coordinates is

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}\,. \tag{10}$$

Notice that the value of (some of) the terms in $g_{\alpha\beta}$ does depend on the value of both $r$ and $\theta$, so this metric is, in fact, position-dependent.

Equation (9) also makes use of the *summation convention*. The summation convention says that, when an index is repeated on the same side of an equation, there is an implied sum over that index. In Eq. (9), therefore, there are sums over both $\alpha$ and $\beta$. To illustrate this more explicitly, we can write the expression out more fully:

$$\begin{aligned} ds^2 &= g_{\alpha\beta}dx^\alpha dx^\beta = \sum_\alpha \sum_\beta g_{\alpha\beta}dx^\alpha dx^\beta \\ &= g_{00}dx^0dx^0 + g_{01}dx^0dx^1 + g_{02}dx^0dx^2 + g_{03}dx^0dx^3 \\ &\quad +g_{10}dx^1dx^0 + g_{11}dx^1dx^1 + g_{12}dx^1dx^2 + g_{13}dx^1dx^3 \\ &\quad +g_{20}dx^2dx^0 + g_{21}dx^2dx^1 + g_{22}dx^2dx^2 + g_{23}dx^2dx^3 \\ &\quad +g_{30}dx^3dx^0 + g_{31}dx^3dx^1 + g_{32}dx^3dx^2 + g_{33}dx^3dx^3 \,. \end{aligned} \tag{11}$$

Using the flat-spacetime spherical-polar metric of Eq. (10) as an example, we can eliminate all the zero terms, then plug in the non-zero terms and the coordinate names given by Eq. (8) to get

$$\begin{aligned} ds^2 &= g_{00}dx^0dx^0 + g_{11}dx^1dx^1 + g_{22}dx^2dx^2 + g_{33}dx^3dx^3 \\ &= (-1)dt^2 + (1)dr^2 + (r^2)d\theta^2 + (r^2\sin^2\theta)d\phi^2 \,, \end{aligned} \tag{12}$$

which is the same as Eq. (6), as it should be.

**Gravitational Wave Metric ▶** ..............................................................
The metric for flat spacetime in Cartesian coordinates has a special name and symbol. It is called the Minkowski metric, and it is represented by the Greek letter "eta,"

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{13}$$

[Notice that, if you plug this into Eq. (9) for $g_{\alpha\beta}$ and compute the sums, you will get back Eq. (3).] As mentioned earlier, gravitational waves are perturbations of the curvature of (otherwise flat) spacetime. We can express this mathematically by writing the gravitational-wave metric as the flat spacetime (Minkowski) metric plus a perturbation,

$$g_{\alpha\beta}(x) = \eta_{\alpha\beta} + h_{\alpha\beta}(x) \,. \tag{14}$$

The *metric perturbation*, $h_{\alpha\beta}$, represents the gravitational wave. Both $g_{\alpha\beta}$ and $h_{\alpha\beta}$ are written with the explicit argument $(x)$ to emphasize the position dependence of those terms.

**Example ▶** ..............................................................................
A simple example of a plane gravitational wave, propagating in the $z$-direction, is

$$h_{\alpha\beta}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} f(t-z) \,, \tag{15}$$

where $f(t-z)$ is any function of $t-z$, as long as the amplitude of $f$ is small (because it is a perturbation). The function $f$ could be a Gaussian wave packet, for example, or a sinusoidal wave. The corresponding line element for this metric perturbation is

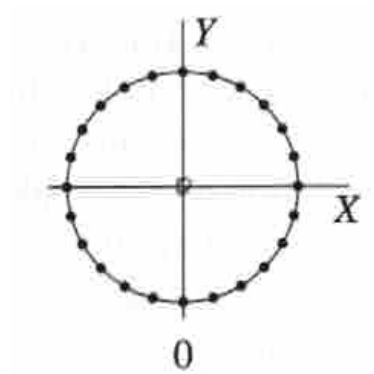
$$ds^2 = -dt^2 + [1 + f(t-z)]dx^2 + [1 - f(t-z)]dy^2 + dz^2 \, . \tag{16}$$

This represents a wave of spacetime curvature propagating in the positive $z$ direction. [Notice that the function $f(t-z)$ is a solution to the one-dimensional wave equation,

$$\frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \, , \tag{17}$$

for a plane wave traveling in the $z$-direction. The solution to this equation is often expressed as $\psi(z,t) = f(z - vt)$, where $f$ is any function. An alternate form (differing only by a minus sign) is $f(vt - z)$, which is the same form as the function $f$ in Eq. (15) or (16) with speed $v = c = 1$. So the given metric perturbation does have the expected form for a traveling plane wave, traveling in the $+z$ direction with speed 1.]

The only non-zero terms in the metric perturbation given by Eq. (15) are the $xx$ and $yy$ terms; this means that there is no perturbation in the $z$ direction, which is the direction of travel. In other words, this gravitational wave is *transverse*. In fact, all gravitational waves are transverse, like electromagnetic waves are.




To try to understand the effect of a gravitational wave on spacetime, let's first assume that the wave is sinusoidal, so $f$ has the form $f(z,t) = A\sin(t-z)$. Now, imagine a circular ring of *test masses*, small particles that are free to move in response to the wave. The ring is placed in the $xy$ plane, so that it is perpendicular to the direction of the wave's travel.

Before the wave reaches the ring (when spacetime is flat), the distance from the origin to any given test mass can be calculated by integrating the flat spacetime line element over the (straight-line) path from the origin to the test mass of interest. As an example, consider the path along the $x$-axis to the test mass sitting on the $+x$ axis; call its position $L_*$. The distance from the origin to this point is given by

$$L = \int_0^{L_*} ds = \int_0^{L_*} \left[-dt^2 + dx^2 + dy^2 + dz^2\right]^{1/2}_{y=z=t=0} = \int_0^{L_*} dx = L_* \, , \tag{18}$$

where all the terms, except $dx$, in the integrand were eliminated by the path condition $y = z = t = 0$.

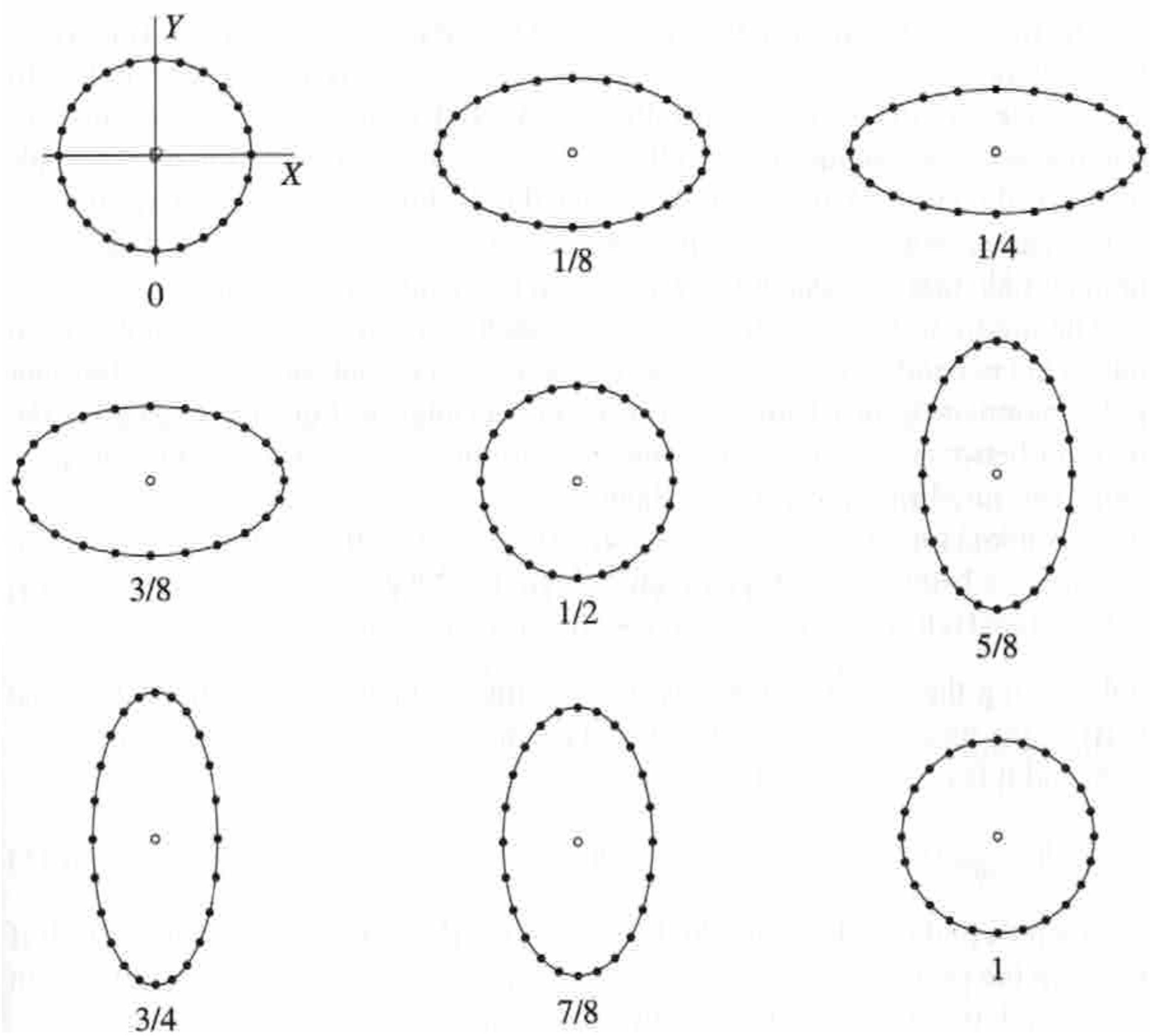
Repeating this calculation when the gravitational wave is present gives

$$\begin{aligned} L_x(t) &= \int_0^{L_*} \left\{-dt^2 + [1 + f(t-z)]dx^2 + [1 - f(t-z)]dy^2 + dz^2\right\}^{1/2}_{y=z=t=0} \\ &= \int_0^{L_*} [1 + f(t-0)]^{1/2} dx \\ &\approx \int_0^{L_*} [1 + \frac{1}{2} f(t)]dx \\ &= L_* \left[1 + \frac{1}{2} f(t)\right] \, , \end{aligned} \tag{19}$$

where we have used the approximation $(1+x)^{1/2} \approx 1 + \frac{1}{2}x$ for $x << 1$ (remembering that the amplitude of $f$ is small). Given our assumption that $f$ is sinusoidal, this expression tells us that the distance of this particular test mass from the origin will vary sinusoidally about the original distance $L_*$, increasing during the first quarter period, returning to the original distance by the time half a period has passed, continuing to decrease for another quarter period, and then returning again to the original distance after one full period. Repeating this calculation for the test mass on the $+y$ axis gives

$$L_y(t) = L_* \left[1 - \frac{1}{2} f(t)\right] , \tag{20}$$

which indicates that when the distance in the $x$ direction is increasing, the distance in the $y$ direction is decreasing, and vice versa. Similar calculations show that the circular ring will be distorted by elongating into a horizontal ellipse and then a vertical one, as shown below (the number below each image represents the fraction of a period).



**Generalization ►** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
The metric perturbation (15) is not the most general. It's actually one of two independent *polarizations* of gravitational waves. Because of the way it would deform a ring of test masses, this polarization is known as the +, or "plus," polarization. The other polarization is similar, but it is rotated by 45°, so that the ring deformations generate diagonal ellipses instead vertical and horizontal ones. Consequently, it is called the ×, or "cross," polarization. The metric perturbation in this case is

$$h_{\alpha\beta}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} f(t-z)\,. \tag{21}$$

These two polarizations are linearly independent, and any gravitational wave can be expressed as a linear combination of them (much like we can write any vector in terms of a linear combination of the unit vectors along each axis). As a result, the most general form of a gravitational wave propagating the the $+z$ direction is of the form

$$\boxed{h_{\alpha\beta}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_+(t-z) & f_\times(t-z) & 0 \\ 0 & f_\times(t-z) & -f_+(t-z) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}\,, \tag{22}$$

where $f_+(t-z)$ and $f_\times(t-z)$ are two different functions.

**Connecting to Wahlquist ►** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
Wahlquist derives the gravitational wave perturbation functions $f_+$ and $f_\times$ for binary star systems; the result is given in his Eqs. (30) and (31), where they are labeled $h_+$ and $h_\times$. His expressions for these functions are given in terms of the polar angle $\theta$ and other quantities defined in his Eqs. (32).

The image below may be helpful in understanding the angles that appear in Wahlquist's expressions. For example, the *inclination angle*, $i$, is marked, and the *line of nodes* is the red dotted line.