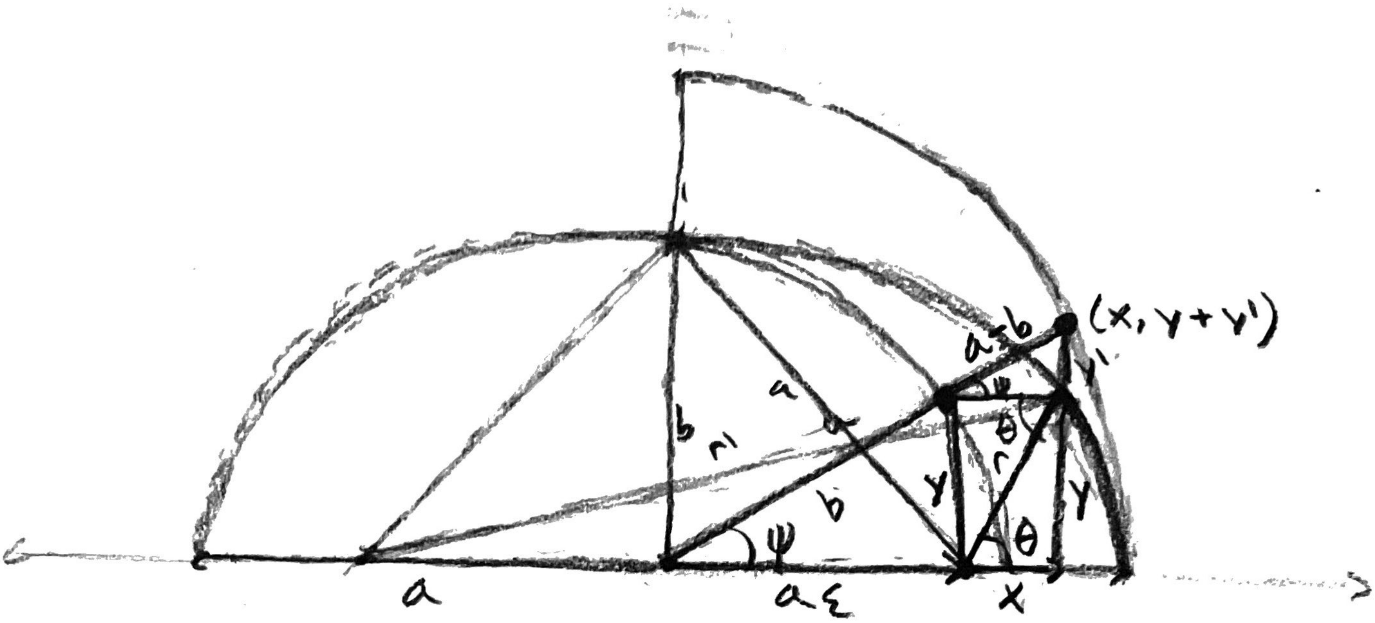
Keppler's Equation



The equation of our ellipse is:

$$\frac{(x+a\varepsilon)^2}{a^2} + \frac{y^2}{b^2} = 1$$

additionally, by geometry, $\cos\psi = \frac{x+a\varepsilon}{a}$;
Therefore:

$$\cos^2\psi + \frac{y^2}{b^2} = 1$$

By the Pythagorean Identity,
$\cos^2\psi + \sin^2\psi = 1$, so $\sin\psi = \frac{y}{b}$.

We then have:

$$y = b\sin\psi \qquad x = a(\cos\psi - \varepsilon)$$
$$= a\sqrt{1-\varepsilon^2}\,\sin\psi$$

As $r^2 = x^2 + y^2$:

$$r^2 = a^2\left((1-\varepsilon^2)\sin^2\psi + \cos^2\psi - 2\varepsilon\cos\psi + \varepsilon^2\right)$$

As $\sin^2\psi = 1 - \cos^2\psi$:

$$r^2 = a^2\left[1 - \cancel{\varepsilon^2} - \cancel{\cos^2\psi} + \varepsilon^2\cos^2\psi + \cancel{\cos^2\psi} - 2\varepsilon\cos\psi + \cancel{\varepsilon^2}\right]$$

$$= a^2\left[1 + \varepsilon^2\cos^2\psi - 2\varepsilon\cos\psi\right]$$

$$= a^2(1 - \varepsilon\cos\psi)$$, and so:

$$r(\psi) = a(1 - \varepsilon\cos\psi)$$

Thus far, our derivation has been purely geometrical. Now, we can use our ~~shape~~ equation $r(\theta)$ to relate θ and ψ. For a body under the an inverse square force, $\theta(r)$ is:

$$\theta(r) = \int \frac{(1/r^2)\,dr}{\sqrt{2\mu\left(E + \frac{k}{r} - \frac{\ell^2}{2\mu r^2}\right)}}$$

$$\theta = \frac{\ell}{\sqrt{2\mu}} \int \frac{(1/r^2)\,dr}{\sqrt{E + \frac{k}{r} - \frac{\ell^2}{2\mu r^2}}}, \text{ letting } u = \frac{1}{r},\ du = -\frac{1}{r^2}dr:$$

$$= -\frac{\ell}{\sqrt{2\mu}} \int \frac{du}{\sqrt{E + ku - a^2u^2}}, \text{ where } a^2 = \frac{\ell^2}{2\mu}$$

$$ku \quad -\frac{k}{2a}$$
$$au \quad a^2u^2 \quad -\frac{k}{2}u$$
$$-\frac{k}{2a} \quad -\frac{k}{2}u \qquad \frac{k^2}{4a^2}$$

as $E + ku - a^2u^2 = E + \frac{k^2}{4a^2} - \left(au - \frac{k}{2a}\right)^2$:

$$= -\frac{\ell}{\sqrt{2\mu}} \int \frac{du}{\sqrt{\left(E + \frac{k^2}{4a^2}\right) - \left(au - \frac{k}{2a}\right)^2}}, \text{ letting } b^2 = E + \frac{k^2}{4a^2},$$

$$\omega = au - \frac{k}{2a},$$

$$d\omega = a\,du:$$

$$= -\frac{\ell}{a\sqrt{2\mu}} \int \frac{d\omega}{\sqrt{b^2 - \omega^2}}, \text{ letting } \omega = b\cos\phi,$$

$$d\omega = -b\sin\phi\, d\phi$$

$$= -\frac{\cancel{b}\ell}{a\sqrt{2\mu}} \int \frac{\cancel{\sin\phi\, d\phi}}{\cancel{b}\cancel{\sqrt{1 - \cos^2\phi}}} = \frac{1}{a\sqrt{2\mu}}\,\phi = \frac{\ell}{a\sqrt{2\mu}} \arccos\left(\frac{\omega}{b}\right)$$

$$= \frac{\cancel{\ell}\cancel{\sqrt{2\mu}}}{\cancel{\ell}\cancel{\sqrt{2\mu}}} \arccos\left(\frac{au - \frac{k}{2a}}{\sqrt{E + \frac{k^2}{4a^2}}}\right) = \arccos\left(\frac{\frac{\ell}{\sqrt{2\mu}\,r} - \frac{k\sqrt{2\mu}}{\ell}}{\sqrt{E + \frac{2k^2\mu}{\ell^2}}}\right)$$

$$= \arccos\left(\frac{\frac{\ell^2}{2\mu r} - 1}{\sqrt{\frac{E\ell^2}{2k^2\mu} + 1}}\right), \text{ otherwise correct}$$

Which is all equivalent to θ, so:

$$\cos(\theta) = \frac{\frac{\ell^2}{2\mu r} - 1}{\sqrt{\frac{E\ell^2}{2k^2\mu} + 1}}, \text{ where } \varepsilon = \sqrt{\frac{E\ell^2}{2k^2\mu} + 1},$$

$$\alpha = \frac{\ell^2}{2\mu}, \text{ so:}$$

$$r = \frac{\alpha}{1 + \varepsilon\cos\theta}$$ (which describes a conic section with eccentricity ε)

$$r + \varepsilon r\cos\theta = \alpha,$$

Because the orbit is an ellipse, we require that $\alpha = a(1-\varepsilon^2)$, so:

$$r + \varepsilon r\cos\theta = a(1-\varepsilon^2)$$

$$\varepsilon r\cos\theta = a(1-\varepsilon^2) - r,$$ adding εr to both sides:

$$\varepsilon r + \varepsilon r\cos\theta = \varepsilon r + a(1-\varepsilon^2) - r$$

$$\varepsilon r(1+\cos\theta) = r(\varepsilon - 1) + a(1-\varepsilon)(1+\varepsilon)$$

$$\varepsilon r(1+\cos\theta) = (1-\varepsilon)[a(1+\varepsilon) - r]$$

Now, we recall that $r = a(1-\varepsilon\cos\psi)$; making this substitution, it follows that:

$$\varepsilon r(1+\cos\theta) = (1-\varepsilon)[a(1+\varepsilon) - a(1-\varepsilon\cos\psi)]$$

$$\cancel{\varepsilon} r(1+\cos\theta) = a(1-\varepsilon)[\cancel{\varepsilon} + \cancel{\varepsilon}\cos\psi]$$

$$r(1+\cos\theta) = a(1-\varepsilon)(1+\cos\psi)$$

Going back to the relation $r + \varepsilon r\cos\theta = a(1-\varepsilon^2)$ we instead subtract εr, giving:

$$\varepsilon r\cos\theta - \varepsilon r = a(1-\varepsilon^2) - r - \varepsilon r$$

$$\varepsilon r(\cos\theta - 1) = a(1-\varepsilon)(1+\varepsilon) - r(1+\varepsilon)$$

$$\varepsilon r(\cos\theta - 1) = (1+\varepsilon)[a(1-\varepsilon) - r]$$

as $r = a(1-\varepsilon\cos\psi)$,

$$\varepsilon r(\cos\theta - 1) = (1+\varepsilon)[a(1-\varepsilon) - a(1-\varepsilon\cos\psi)]$$

$$\varepsilon r(\cos\theta - 1) = a(1+\varepsilon)[1-\varepsilon-1+\varepsilon\cos\psi]$$

$$\cancel{\varepsilon} r(\cos\theta - 1) = a(1+\varepsilon)[-\cancel{\varepsilon} + \cancel{\varepsilon}\cos\psi]$$

$$r(\cos\theta - 1) = a(1+\varepsilon)(\cos\psi - 1)$$

Dividing the result from subtracting εr by that from adding εr gives:

$$\frac{\cos\theta + 1}{\cos\theta + 1} = \frac{1+\varepsilon}{1-\varepsilon} \cdot \frac{1-\cos\psi}{1+\cos\psi}$$

and as $\tan\frac{\theta}{2} = \sqrt{\frac{\cos\theta - 1}{\cos\theta + 1}}$:

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\psi}{2}$$

$$\frac{\theta}{2} = \frac{\psi}{2}\arctan\left(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\right)$$

$$\theta = \psi\arctan\left(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\right)$$

Differentiating $\tan\frac{\theta}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\psi}{2}$ gives:

$$\tfrac{1}{2}\sec^2\tfrac{\theta}{2}\,d\theta = \tfrac{1}{2}\sec^2\tfrac{\psi}{2}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\,d\psi$$

$$d\theta = \frac{\cos^2\frac{\theta}{2}}{\cos^2\frac{\psi}{2}}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\,d\psi$$

as $r(1+\cos\theta) = a(1-\varepsilon)(1+\cos\psi)$, $r = a(1-\varepsilon)\frac{1+\cos\psi}{1+\cos\theta}$; however,
as $\cos^2\frac{\theta}{2} = \frac{1}{2}(1+\cos\theta)$;

$$r = a(1-\varepsilon)\frac{\cos^2\left(\frac{\psi}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right)}$$

Keppler's second law states that equal areas are swept out by the orbit in equal times. Say that the period of orbit is T; the area of the ellipse is $A = \pi a b$. Because area is swept out at a constant rate, this rate must be the area swept out in an orbit in a time T, $\frac{\Delta A}{\Delta t} = \frac{A}{T} = \frac{\pi a b}{T}$; we can equate this with the area integral, $\int dA = \frac{1}{2}\int r^2 d\theta$ (analogous to $\Delta\theta r^2$ for a circle). That is:

$$\frac{\pi a b}{T} t = \frac{1}{2}\int r^2 d\theta$$

As $d\theta = \frac{\cos^2\frac{\theta}{2}}{\cos^2\frac{\psi}{2}}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\,d\psi$, $r = a(1-\varepsilon)\frac{\cos^2\frac{\psi}{2}}{\cos^2\frac{\theta}{2}}$:

$$\frac{\pi a b}{T} t = \frac{1}{2}\int a\sqrt{(1-\varepsilon)(1+\varepsilon)}\, r\, d\psi$$

Taking $r = a(1-\varepsilon\cos\psi)$:

$$\frac{\pi a b}{T} t = \frac{1}{2} a^2 \sqrt{1-\varepsilon^2}\int 1-\varepsilon\cos\psi\, d\psi$$

$$\frac{\pi b}{T} t = \frac{a}{2}\sqrt{1-\varepsilon^2}(\psi - \varepsilon\sin\psi)$$

as $b = a\sqrt{1-\varepsilon^2}$;

$$\frac{\pi \cancel{a\sqrt{1-\varepsilon^2}}}{T}\, t = \frac{\cancel{a}}{2}\cancel{\sqrt{1-\varepsilon^2}}\,(\psi - \varepsilon \sin\psi)$$

$$\frac{2\pi}{T}\, t = \psi - \varepsilon \sin\psi$$

which must somehow be approximated for $\psi(t)$.