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The Doppler Response to Gravitational Waves from a Binary Star Source

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The equations are developed for spacecraft Doppler detection of periodic gravitational waves from a single binary star source. Graphical examples are included to indicate the great variety of Doppler signals which can be generated by these systems.

1. INTRODUCTION

The purpose of this paper is to develop the equations for the spacecraft Doppler signals which would be expected from a binary star source of gravitational waves. Supermassive black hole binaries appear to be the most likely candidate source for strong periodic gravitational waves in the very-low-frequency region available to the spacecraft Doppler detection method. The astrophysical arguments for binaries with black hole components are becoming more persuasive [1]. Although there is no evidence as yet for systems sufficiently massive to produce gravitational waves detectable with the current Doppler sensitivity, their existence is considered theoretically plausible [2]. Possibly, their first detection will be achieved by future Doppler gravitational wave experiments.

The analysis of waves from a nonrelativistic binary source will be developed here ab initio starting from the standard quadrupole formula. This may appear quite unnecessary in view of the definitive treatment of this subject published many years ago in the classic papers of Peters and

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Mathews [3]. Their analysis did provide complete results on the gravitational luminosity of a nonrelativistic binary as a function of direction and frequency—everything one would need for a detection technique which responds to the local flux of energy in a gravitational wave. However, the broadband Doppler technique responds to a geometrical projection of the time-dependent metric of the wave rather than to the square of its time derivative (which gives the energy flux), and at each instant the Doppler signal depends on the metric nonlocally; i.e., at three separated space-time events [4]. So, in fact, it is not possible to infer the Doppler response directly from the results found in the papers of Peters and Mathews. An additional purpose here is to present the equations in a form adapted to Doppler observations and suitable for computer calculations of the detailed waveforms and Fourier components.

Section 2 summarizes the pertinent equations for a binary system, the dynamics of which are assumed to be adequately described by Newtonian theory. Section 3 gives the quadrupole formula for the metric of linearized gravitational waves specialized to the binary case, and Section 4 calculates the explicit form of the metric in the standard celestial frame of reference. Finally, Section 5 determines the Doppler response to this gravitational wave metric from an arbitrary binary source. Graphs of a variety of computed waveforms and their spectra are also presented.

2. THE BINARY SOURCE

We consider a distant binary with component masses m_1 and m_2 situated at $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, respectively, relative to an origin, 0, in the vicinity of Earth. The origin is chosen at rest in the center-of-mass (CM) frame of the binary, and the vector to the CM is defined by

$$M\mathbf{R} \equiv m_1\mathbf{r}_1 + m_2\mathbf{r}_2 \tag{1}$$

with $M = m_1 + m_2$ and $\dot{\mathbf{R}} = 0$ (see Fig. 1). The relative separation vector of the binary components is

$$\mathbf{r} = r\hat{r} = \mathbf{r}_2 - \mathbf{r}_1 \tag{2}$$

and satisfies the Newtonian equation of motion

$$\ddot{\mathbf{r}} = -(GM/r^2)\hat{r} \tag{3}$$

The binary is assumed to be astronomically distant from 0, so

$$|\mathbf{R}| \gg |\mathbf{r}| \qquad \text{and} \qquad |\mathbf{R}| \gg |\mathbf{r}_E| \tag{4}$$

if $\mathbf{r}_E$ is the vector from the origin to Earth.

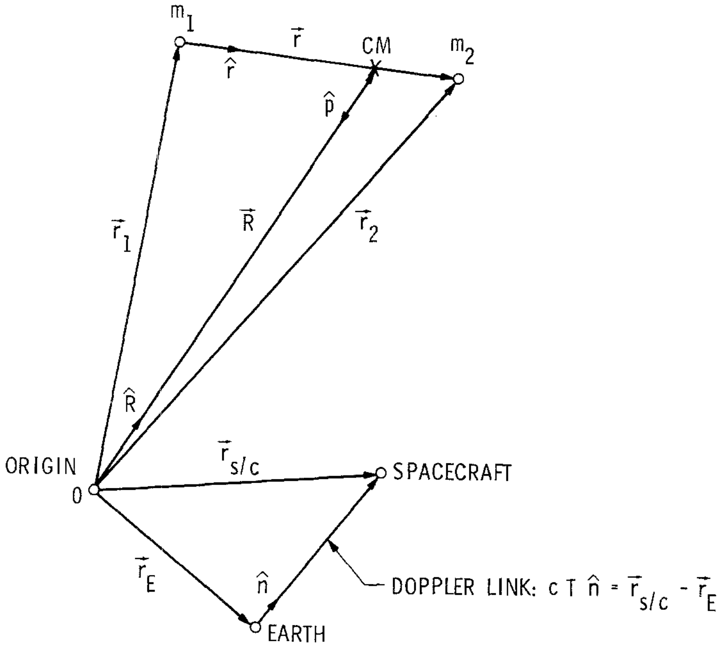

Fig. 1. Diagram of the vectors (not to scale). See text for definitions.

In terms of polar coordinates in the orbital plane of the binary, the usual equation for the relative orbit ellipse is

$$r = \frac{a(1-e^2)}{1+e\cos(\theta-\theta_p)} \tag{5}$$

where a is the semimajor axis, e is the eccentricity, and θ_p is the value of θ at the periapse, $r_p = r(\theta_p) = a(1-e)$. The binary period is given by

$$P = \left(\frac{4\pi^2 a^3}{GM}\right)^{1/2} \tag{6}$$

and the differential equation for the Keplerian motion can be written as

$$\dot{\theta} = (2\pi/P)(1-e^2)^{-3/2}[1+e\cos(\theta-\theta_p)]^2 \tag{7}$$

Adopting θ as the independent variable, the equation of motion can be integrated to give

$$\frac{2\pi}{P}[t(\theta)-t_p] = \arccos\left[\frac{e+\cos(\theta-\theta_p)}{1+e\cos(\theta-\theta_p)}\right] - \frac{e(1-e^2)^{1/2}\sin(\theta-\theta_p)}{1+e\cos(\theta-\theta_p)} \tag{8}$$

where t_p is the integration constant determined so that $t(\theta_p) = t_p + (\text{integer}) \times P$.

3. THE GRAVITATIONAL WAVE METRIC IN LINEARIZED THEORY

In the linearized Einstein theory of gravitational waves, the metric perturbation is given to lowest order by the second-time derivative of the quadrupole moment of the source [5]

$$\underset{\sim}{h}^{\mathrm{TT}}(t) = \frac{2G}{c^4}\frac{1}{R}\ddot{\underset{\sim}{Q}}^{\mathrm{TT}}(t-R) \tag{9}$$

where $R = |\mathbf{R}|$ is the distance to the source, $\underset{\sim}{h}^{\mathrm{TT}}$ is the three-dimensional metric perturbation in transverse-traceless (TT) gauge, and $\underset{\sim}{Q}^{\mathrm{TT}}$ is the TT quadrupole moment of the source, evaluated at the retarded time $t - R$.

The quadrupole moment for a binary system is

$$\underset{\sim}{Q} \equiv \int \mathbf{r}_m \mathbf{r}_m \, dm = m_1 \mathbf{r}_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 \mathbf{r}_2 \tag{10}$$

or expressing this in terms of the CM and relative separation vectors

$$\underset{\sim}{Q} = M\mathbf{R}\mathbf{R} + (m_1 m_2/M)\,\mathbf{r}\mathbf{r} \tag{11}$$

Since $|\mathbf{R}| \gg |\mathbf{r}_E|$, the unit propagation vector, $\hat{p}$, of a wave coming to Earth will be the unit vector $-\hat{R} = -\mathbf{R}/R$, and we may use the projection tensor $\underset{\sim}{I} - \hat{R}\hat{R}$ to obtain the transverse components of $\underset{\sim}{Q}$. Thus

$$\underset{\sim}{Q}^{\mathrm{T}} = (\underset{\sim}{I} - \hat{R}\hat{R}) \cdot \underset{\sim}{Q} \cdot (\underset{\sim}{I} - \hat{R}\hat{R}) = (m_1 m_2/M)\,\mathbf{r}^{\mathrm{T}}\mathbf{r}^{\mathrm{T}} \tag{12}$$

where the first term on the right side of (11) projects to zero, and we have defined the transverse vector

$$\mathbf{r}^{\mathrm{T}} \equiv (\underset{\sim}{I} - \hat{R}\hat{R}) \cdot \mathbf{r} \tag{13}$$

Subtracting the trace then gives $\underset{\sim}{Q}^{\mathrm{TT}}$ as

$$\underset{\sim}{Q}^{\mathrm{TT}} = \underset{\sim}{Q}^{\mathrm{T}} - \tfrac{1}{2}\mathrm{Tr}(\underset{\sim}{Q}^{\mathrm{T}})(\underset{\sim}{I} - \hat{R}\hat{R}) = (m_1 m_2/M)[\mathbf{r}^{\mathrm{T}}\mathbf{r}^{\mathrm{T}} - \tfrac{1}{2}(\mathbf{r}^{\mathrm{T}} \cdot \mathbf{r}^{\mathrm{T}})(\underset{\sim}{I} - \hat{R}\hat{R})] \tag{14}$$

4. EXPLICIT CALCULATION OF THE WAVE METRIC

In order to calculate the metric tensor of the wave explicitly in an observational frame, we define the following sets of basis triads. First, let

$\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ be the fundamental celestial frame with $\hat{\imath}$ directed to the first point of Aries and $\hat{k}$ to the north celestial pole. Then

$$\hat{R} = \cos(\delta)[\cos(\alpha)\hat{\imath} + \sin(\alpha)\hat{\jmath}] + \sin(\delta)\hat{k} \tag{15}$$

where $\{\alpha, \delta\}$ are the right ascension and declination of the binary source. Define orthonormal vectors on the celestial sphere by

$$\begin{aligned} \hat{\alpha} &\equiv -\sin(\alpha)\hat{\imath} + \cos(\alpha)\hat{\jmath} \\ \hat{\delta} &\equiv -\sin(\delta)[\cos(\alpha)\hat{\imath} + \sin(\alpha)\hat{\jmath}] + \cos(\delta)\hat{k} \end{aligned} \tag{16}$$

so that $\hat{\alpha}$ points to increasing right ascension (tangent to the circle of constant declination), and $\hat{\delta}$ points to increasing declination (tangent to the meridian of constant right ascension), at the position of the source. We adopt independent polarization tensors based on these celestial coordinates, i.e.,

$$\underset{\sim}{e}_{+} = (\hat{\alpha}\hat{\alpha} - \hat{\delta}\hat{\delta}) \qquad \underset{\sim}{e}_{x} = (\hat{\alpha}\hat{\delta} + \hat{\delta}\hat{\alpha}) \tag{17}$$

Since $\hat{p} = -\hat{R}$, these satisfy the TT conditions

$$\hat{p} \cdot \underset{\sim}{e}_{+} = \hat{p} \cdot \underset{\sim}{e}_{x} = 0 \tag{18}$$

and

$$\mathrm{Tr}(\underset{\sim}{e}_{+}) = \mathrm{Tr}(\underset{\sim}{e}_{x}) = 0 \tag{19}$$

Now, let $\hat{u}$ be a unit vector which lies in the orbital plane of the binary along the line of nodes, which we define to be the intersection of the orbital plane with the tangent plane of the sky. Then $\hat{u} \cdot \hat{R} = 0$, and we may write

$$\hat{u} = \cos(\phi)\hat{\alpha} + \sin(\phi)\hat{\delta} \tag{20}$$

where ϕ defines the orientation of the line of nodes in the sky. The orthogonal to the line of nodes in the orbital plane is

$$\hat{v} = \cos(i)[-\sin(\phi)\hat{\alpha} + \cos(\phi)\hat{\delta}] + \sin(i)\hat{R} \tag{21}$$

where i is the angle of inclination of the orbital plane to the tangent plane of the sky.

The relative separation vector $\mathbf{r}$ can now be written

$$\mathbf{r} = r\hat{r} = r[\cos(\theta - \theta_n)\hat{u} + \sin(\theta - \theta_n)\hat{v}] \tag{22}$$

where θ_n is the value of θ at the line of nodes. The transverse projection of $\mathbf{r}$ into the tangent plane of the sky is

$$\mathbf{r}^{\mathrm{T}} = (\underset{\sim}{I} - \hat{R}\hat{R}) \cdot \mathbf{r} = r\{\cos(\theta - \theta_n)\hat{u} + \sin(\theta - \theta_n)[\hat{v} - \sin(i)\hat{R}]\} \tag{23}$$

or in terms of $\hat{\alpha}$ and $\hat{\delta}$ from (20) and (21)

$$\mathbf{r}^{\mathrm{T}} = r\{[\cos(\phi)\cos(\theta - \theta_n) - \sin(\phi)\cos(i)\sin(\theta - \theta_n)]\hat{\alpha} + [\sin(\phi)\cos(\theta - \theta_n) + \cos(\phi)\cos(i)\sin(\theta - \theta_n)]\hat{\delta}\} \tag{24}$$

Inserting this expression into (14) for $\underset{\sim}{Q}^{\mathrm{TT}}$ we obtain

$$\underset{\sim}{Q}^{\mathrm{TT}} = Q_+ \underset{\sim}{e}_+ + Q_x \underset{\sim}{e}_x \tag{25}$$

where

$$Q_+ = (m_1 m_2/2M)\, r^2\{\cos(2\phi)[\cos^2(\theta - \theta_n) - \cos^2(i)\sin^2(\theta - \theta_n)] - \sin(2\phi)\cos(i)\sin 2(\theta - \theta_n)\} \tag{26}$$

$$Q_x = (m_1 m_2/2M)\, r^2\{\sin(2\phi)[\cos^2(\theta - \theta_n) - \cos^2(i)\sin^2(\theta - \theta_n)] + \cos(2\phi)\cos(i)\sin 2(\theta - \theta_n)\} \tag{27}$$

To calculate the metric, we now use (9) in the form

$$\underset{\sim}{h}^{\mathrm{TT}}(t) = \frac{2G}{c^4 R}\ddot{\underset{\sim}{Q}}^{\mathrm{TT}} = \frac{2G}{c^4 R}\left[\dot{\theta}\frac{d}{d\theta}\left(\dot{\theta}\frac{d\underset{\sim}{Q}^{\mathrm{TT}}}{d\theta}\right)\right]\Bigg|_{t(\theta) = t - R} \tag{28}$$

treating θ as the independent variable with $\dot{\theta}$ given by (7) and $r(\theta)$ by (5). The polarization amplitudes

$$\underset{\sim}{h}^{\mathrm{TT}} = h_+ \underset{\sim}{e}_+ + h_x \underset{\sim}{e}_x \tag{29}$$

then are

$$h_+(\theta) = H\{\cos(2\phi)[A_0 + eA_1 + e^2 A_2] - \sin(2\phi)[B_0 + eB_1 + e^2 B_2]\} \tag{30}$$

and

$$h_x(\theta) = H\{\sin(2\phi)[A_0 + eA_1 + e^2 A_2] + \cos(2\phi)[B_0 + eB_1 + e^2 B_2]\} \tag{31}$$

where

$$
\begin{aligned}
H &\equiv \frac{4G^2 m_1 m_2}{c^4 a(1-e^2)R} \\
A_0 &= -\tfrac{1}{2}[1+\cos^2(i)]\cos 2(\theta-\theta_n) \\
B_0 &= -\cos(i)\sin 2(\theta-\theta_n) \\
A_1 &= \tfrac{1}{4}\sin^2(i)\cos(\theta-\theta_{\mathrm{p}}) - \tfrac{1}{8}[1+\cos^2(i)][5\cos(\theta-2\theta_n+\theta_{\mathrm{p}}) \\
&\quad + \cos(3\theta-2\theta_n-\theta_{\mathrm{p}})] \qquad (32)\\
B_1 &= -\tfrac{1}{4}\cos(i)[5\sin(\theta-2\theta_n+\theta_{\mathrm{p}}) + \sin(3\theta-2\theta_n-\theta_{\mathrm{p}})] \\
A_2 &= \tfrac{1}{4}\sin^2(i) - \tfrac{1}{4}[1+\cos^2(i)]\cos 2(\theta_n-\theta_{\mathrm{p}}) \\
B_2 &= \tfrac{1}{2}\cos(i)\sin 2(\theta_n-\theta_{\mathrm{p}})
\end{aligned}
$$

Again it is understood that $h(t) = h(\theta)|_{t(\theta)=t-R}$ with $t(\theta)$ given by (8).

5. THE DOPPLER SIGNAL FROM A BINARY

The spacecraft Doppler signal produced by a plane-fronted linearized gravitational wave is given by [4]

$$y(t) \equiv \Delta v/v = -\tfrac{1}{2}(1-\mu)\,\psi[t] - \mu\psi[t-(1+\mu)T] + \tfrac{1}{2}(1+\mu)\,\psi[t-2T] \tag{33}$$

where the function ψ is the amplitude of the gravitational wave metric projected onto the Doppler link

$$\psi(t) \equiv (1-\mu^2)^{-1}[\hat{n}\cdot \underset{\sim}{h}^{\mathrm{TT}}(t)\cdot \hat{n}] \tag{34}$$

Here, $\mu = \hat{p}\cdot\hat{n}$ with $\hat{p}$ the unit propagation vector of the plane wave and $\hat{n}$ the unit direction vector of the Doppler link to a spacecraft situated at light-time distance T from Earth.

The three terms in (33) correspond, respectively, to the projected amplitude of the wave at the event of reception of the Doppler tracking signal at Earth, $\psi[t-\hat{p}\cdot\mathbf{r}_{\mathrm{E}}(t)/c]$; the transponding event at the spacecraft, $\psi[t-T-\hat{p}\cdot\mathbf{r}_{\mathrm{s/c}}(t-T)/c] = \psi[t-T-(\hat{p}\cdot\hat{n})T-\hat{p}\cdot\mathbf{r}_{\mathrm{E}}(t-T)/c]$; and the emission event of the tracking signal from Earth, $\psi[t-2T-\hat{p}\cdot\mathbf{r}_{\mathrm{E}}(t-2T)/c]$. To obtain the simplified expressions in (33), the origin 0 is taken coincident with the Earth's position and their relative motion is neglected; i.e., $\mathbf{r}_{\mathrm{E}}(t) \cong 0$ for all t. Over a long observation period, the relative motions could become significant and so require using the true phases at the three events as given above.

Using the right ascension and declination of the spacecraft $\{A, D\}$, we can write

$$\hat{n} = \cos(D)[\cos(A)\hat{\imath} + \sin(A)\hat{\jmath}] + \sin(D)\hat{k} \tag{35}$$

but for calculations it is convenient to express $\hat{n}$ in the $\{\hat{\alpha}, \hat{\delta}, \hat{R}\}$ frame associated with the source position

$$\hat{n} = \sin(\eta)[\cos(\lambda)\hat{\alpha} + \sin(\lambda)\hat{\delta}] + \cos(\eta)\hat{R} \tag{36}$$

where we have introduced the auxiliary angle variables $\{\eta, \lambda\}$. With $\hat{p} = -\hat{R}$, we have

$$\mu = \hat{p} \cdot \hat{n} = -\hat{R} \cdot \hat{n} = -\cos(\eta) \tag{37}$$

and from (15) and (35)

$$\cos(\eta) = \sin(D)\sin(\delta) + \cos(D)\cos(\delta)\cos(\alpha - A) \tag{38}$$

Further, using (16) and (35)

$$\tan(\lambda) = \frac{\hat{n} \cdot \hat{\delta}}{\hat{n} \cdot \hat{\alpha}} = \frac{\cos(D)\sin(\delta)\cos(\alpha - A) - \sin(D)\cos(\delta)}{\cos(D)\sin(\alpha - A)} \tag{39}$$

In these auxiliary variables, using (36)

$$\hat{n} \cdot \underset{\sim}{e}_+ \cdot \hat{n} = (\hat{n} \cdot \hat{\alpha})^2 - (\hat{n} \cdot \hat{\delta})^2 = \sin^2(\eta)\cos(2\lambda) \tag{40}$$

and

$$\hat{n} \cdot \underset{\sim}{e}_x \cdot \hat{n} = 2(\hat{n} \cdot \hat{\alpha})(\hat{n} \cdot \hat{\delta}) = \sin^2(\eta)\sin(2\lambda) \tag{41}$$

Since $1 - \mu^2 = \sin^2(\eta)$, (34) becomes simply

$$\psi(\theta) = \cos(2\lambda)\, h_+(\theta) + \sin(2\lambda)\, h_x(\theta) \tag{42}$$

To summarize, then, the set of equations (30)–(32) for h_+ and h_x; (37) and (38) for μ; (39) for λ; (42) for ψ; (33) for y; and (8) for $t(\theta)$—provide an algorithm for computing the Doppler signal observed with a spacecraft located at A, D, T, responding to the gravitational wave from a binary having intrinsic parameters m_1, m_2, a, e; orbital orientation parameters i, ϕ, θ_n, θ_p, t_p; and celestial coordinates α, δ, R.

Figures 2 through 5 illustrate some of the variety of signals that can result. In these examples, only the eccentricity and Doppler link geometry (λ, $\mu = -\cos(\eta)$, T) are varied, since the other parameters affect primarily amplitude and phase rather than the intrinsic waveforms. The periapse is always positioned at $\theta_p = 180°$, or $t_p = 0.5$ P (1.5 P).

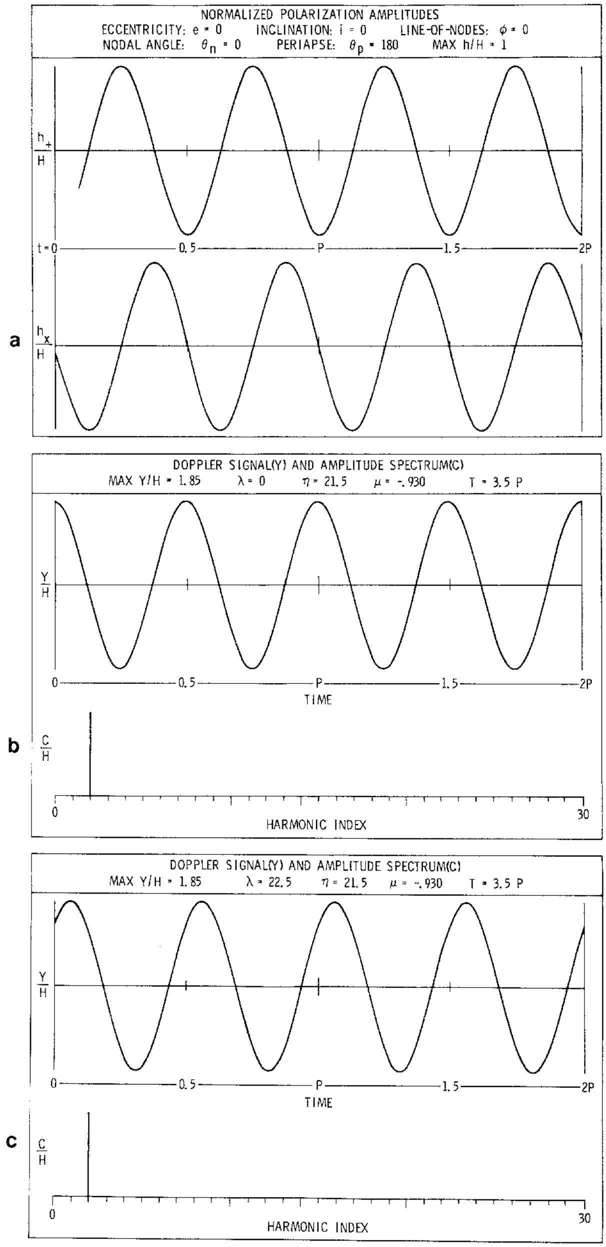

Fig. 2. (a) A circular binary orbit produces elliptically polarized monochromatic gravitational radiation at twice the orbital frequency (pure $n = 2$ harmonic). (b) For $\lambda = 0$, the Doppler link responds only to the h_+ polarization component. With the particular geometry used here the Doppler transfer function, Eq. 33, amplifies the signal close to the maximum possible for a monochromatic gravitational wave; i.e., $Y/H = 2$. (c) With $\lambda = 22.5°$, the Doppler link senses both polarization components equally. The only difference in this case is a phase shift, of course.

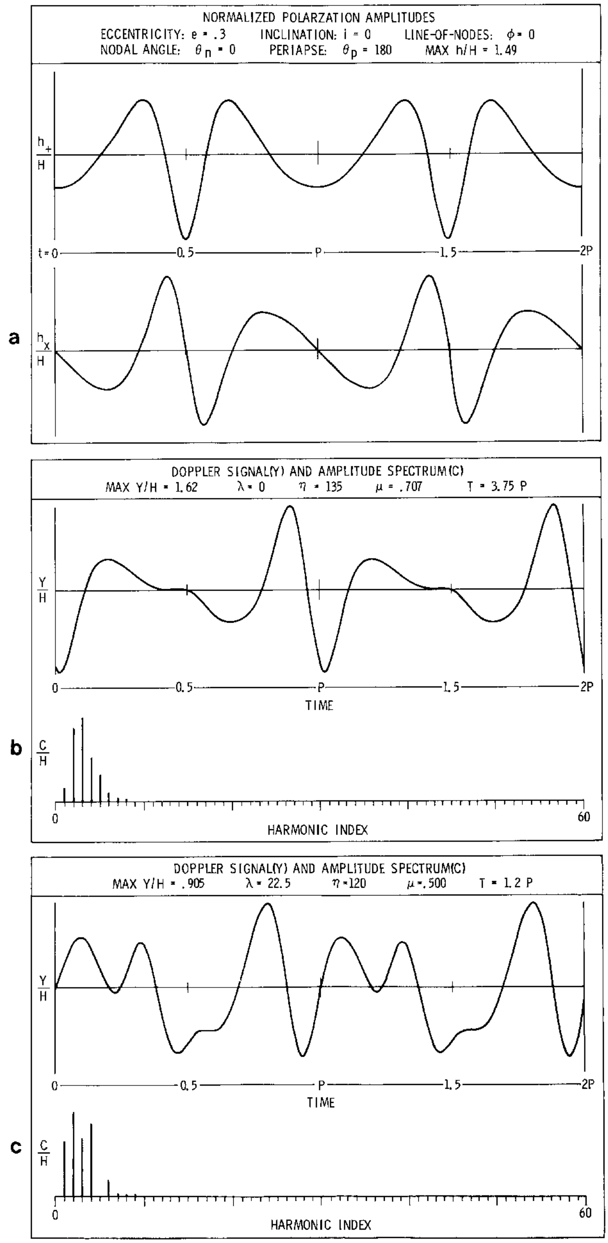

Fig. 3. (a) A binary orbit with low eccentricity radiates power at the fundamental and at a few harmonics of the orbital frequency. (b) Response is to h_+ polarization only. Note that the transfer function completely changes the h_+ waveform and so modifies the harmonic amplitudes. (c) In this link geometry, the "3-pulse character" of the transfer function is quite noticeable. It also clearly gives a null in the response at the $n=5$ harmonic.

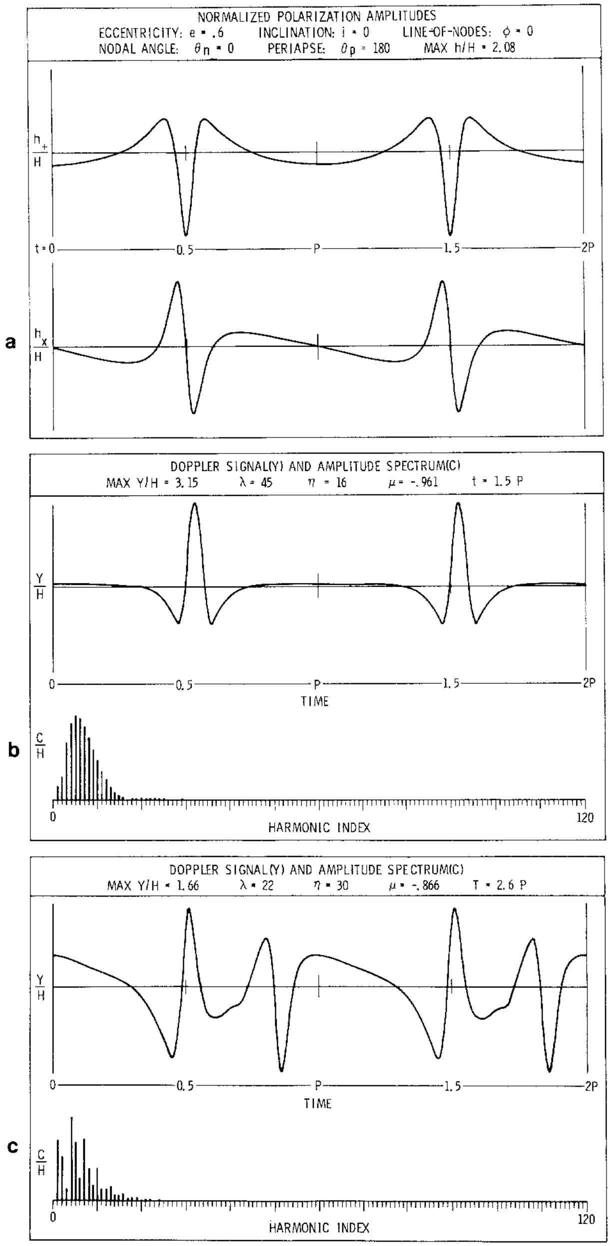

Fig. 4. (a) A binary with eccentricity comparable to that of the binary pulsar, PSR 1913 + 16. (b) The link geometry senses only the h_x polarization component ($\lambda = 45°$). The other parameters have been deliberately chosen to "mask" the three-pulse character, so the Doppler signal appears simple. It is entirely different from the h_x gravitational waveform, and in fact mimics the h_+ component inverted. (c) Link geometry is sensitive to both polarizations and gives a more typical multiply humped signal.

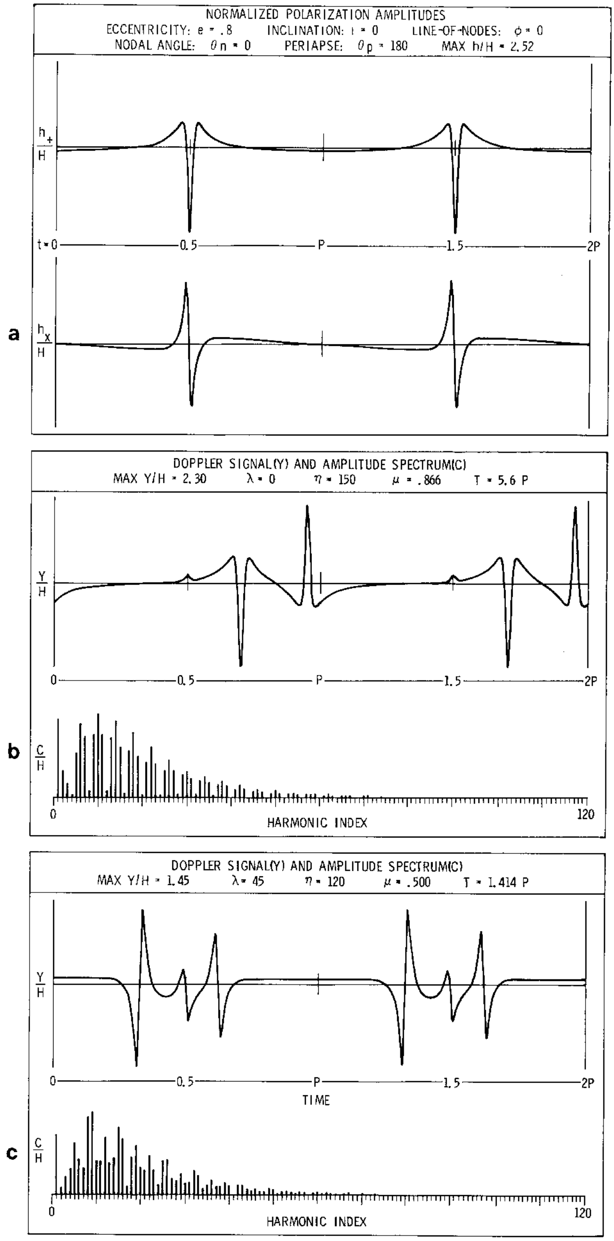

Fig. 5. (a) The gravitational waveform of a high eccentricity binary orbit is virtually a single narrow pulse at periapse giving power in high harmonics. (b) Typical response to the h_+ component. Three repetitions of the pulse are evident with relative amplitudes, senses, and spacing determined by the geometry. Note the regularly spaced near-nulls in the spectrum. (c) Response to the h_x component for a different geometry, again with three modified repetitions of the gravitational waveform.

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REFERENCES

1. Blandford, R. D., and Thorne, K. S. (1979). In *General Relativity: An Einstein Centenary Survey*, Hawking, S. W., and Israel, W., eds. (Cambridge University Press, New York).
2. Bond, J. R., and Carr, B. J. (1984). *Mon. Not. R. Astron. Soc.*, **207**, 585–609.
3. Peters, P. C., and Mathews, J. (1963). *Phys. Rev.*, **131**, 435–440; Peters, P. C. (1964). *Phys. Rev. B*, **136**, 1224–1232.
4. Estabrook, F. B., and Wahlquist, H. D. (1975). *Gen. Rel. Gravitation*, **6**, 439–447.
5. Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation* (W. H. Freeman, San Francisco), p. 999.