CHAPTER 7

Central-Force Motion

7.1 Introduction

The motion of a system consisting of two bodies affected by a force directed along the line connecting the centers of the two bodies (i.e., a *central force*) is an extremely important physical problem—one we can solve completely. The importance of such a problem lies in large measure in two quite different realms of physics: the motion of celestial bodies—planets, moons, comets, double stars, and the like—and certain two-body nuclear interactions, such as the scattering of α particles by nuclei. In the prequantum-mechanics days, physicists also described the hydrogen atom in terms of a classical two-body central force. Although such a description is still useful in a qualitative sense, the quantum-theoretical approach must be used for a detailed description. In addition to some general considerations regarding motion in central-force fields, we discuss in this and the following chapter several of the problems of two bodies encountered in celestial mechanics and nuclear physics.

7.2 The Reduced Mass

Describing a system consisting of two particles requires the specification of six quantities; for example, the three components of each of the two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ for the particles.* Alternatively, we may choose the three components of the center-of-mass vector $\mathbf{R}$ and the three components of $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$ (see Figure 7-1a). Here we restrict our attention to systems without frictional losses and for which the potential energy is a function only of $r = |\mathbf{r}_1 - \mathbf{r}_2|$. The Lagrangian

*The orientation of the particles is assumed to be unimportant; i.e., they are spherically symmetrical (or are point particles).

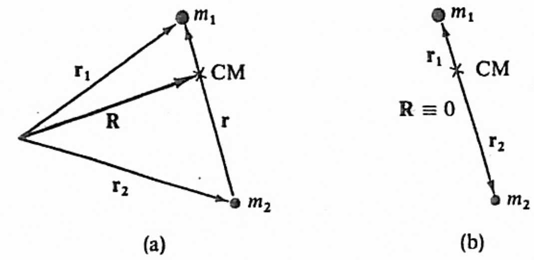


Figure 7-1

for such a system may be written as

$$L = \tfrac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \tfrac{1}{2}m_2|\dot{\mathbf{r}}_2|^2 - U(r) \tag{7.1}$$

Since translational motion of the system as a whole is uninteresting from the standpoint of the particle orbits with respect to one another, we may choose the origin for the coordinate system to be the particles' center of mass—that is, $\mathbf{R} \equiv 0$ (see Figure 7-1b). Then (see Section 8.2)

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0 \tag{7.2}$$

This equation, combined with $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, yields

$$\left.\begin{aligned} \mathbf{r}_1 &= \frac{m_2}{m_1 + m_2}\mathbf{r} \\ \mathbf{r}_2 &= -\frac{m_1}{m_1 + m_2}\mathbf{r} \end{aligned}\right\} \tag{7.3}$$

Substituting Equations 7.3 into the expression for the Lagrangian gives

$$\boxed{L = \tfrac{1}{2}\mu|\dot{\mathbf{r}}|^2 - U(r)} \tag{7.4}$$

where μ is the **reduced mass,**

$$\boxed{\mu \equiv \frac{m_1 m_2}{m_1 + m_2}} \tag{7.5}$$

We have therefore formally reduced the problem of the motion of two bodies to an *equivalent one-body problem* in which we must determine only the motion of a "particle" of mass μ in the central field described by the potential function $U(r)$. Once we obtain the solution for $\mathbf{r}(t)$ by applying the Lagrange equations to Equation 7.4, we can find the individual motions of the particles, $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, by using Equations 7.3. This latter step is not necessary if only the orbits relative to one another are required.

7.3 Conservation Theorems—First Integrals of the Motion

The system we wish to discuss consists of a particle of mass μ moving in a central-force field described by the potential function $U(r)$. Since the potential energy depends only on the distance of the particle from the force center and not on the orientation, the system possesses **spherical symmetry**; that is, the system's rotation about any fixed axis through the center of force cannot affect the equations of motion. We have already shown (see Section 6.9) that under such conditions the angular momentum of the system is conserved:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \text{const.} \tag{7.6}$$

From this relation, it should be clear that both the radius vector and the linear momentum vector of the particle lie always in a plane normal to the angular momentum vector $\mathbf{L}$, which is fixed in space (see Figure 7-2). Therefore we have only a two-dimensional problem, and the Lagrangian may then be conveniently expressed in plane polar coordinates:

$$\boxed{L = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)} \tag{7.7}$$

Since the Lagrangian is cyclic in θ, the angular momentum conjugate to the coordinate θ is conserved:

$$\dot{p}_\theta = \frac{\partial L}{\partial \theta} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} \tag{7.8}$$

or

$$p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta} = \text{const.} \tag{7.9}$$

The system's symmetry has therefore permitted us to integrate immediately one of the equations of motion. The quantity p_θ is a *first integral* of the motion, and we denote its constant value by the symbol l:

$$\boxed{l \equiv \mu r^2\dot{\theta} = \text{const.}} \tag{7.10}$$

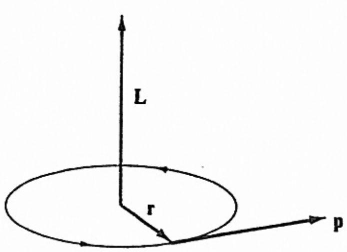


Figure 7-2

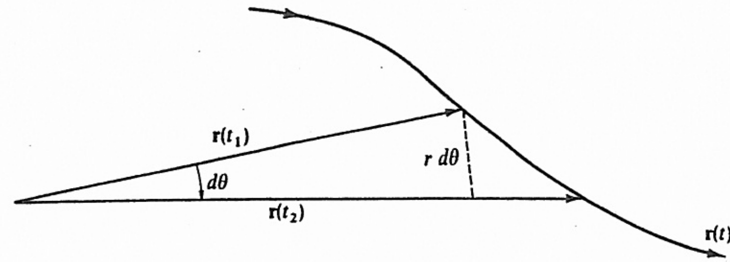


Figure 7-3

That l is constant has a simple geometrical interpretation. Referring to Figure 7-3, we see that in describing the path $\mathbf{r}(t)$, the radius vector sweeps out an area $\frac{1}{2}r^2\,d\theta$ in a time interval dt:

$$dA = \tfrac{1}{2}r^2\,d\theta \tag{7.11}$$

Upon dividing by the time interval, the **areal velocity** is shown to be

$$\begin{aligned} \frac{dA}{dt} &= \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta} \\ &= \frac{l}{2\mu} = \text{const.} \end{aligned} \tag{7.12}$$

Thus the areal velocity is constant in time. This result was obtained empirically by Kepler for planetary motion, and it is known as **Kepler's Second Law.*** It is important to note that the conservation of the areal velocity is not limited to an inverse-square-law force (the case for planetary motion) but is a general result for central-force motion.

Since we have eliminated from consideration the uninteresting uniform motion of the system's center of mass, the conservation of linear momentum adds nothing new to the description of the motion. The conservation of energy is thus the only remaining first integral of the problem. The conservation of the total energy E is automatically assured since we have limited the discussion to nondissipative systems. Thus

$$T + U = E = \text{const.} \tag{7.13}$$

and

$$E = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r)$$

*Published by Johannes Kepler (1571–1630) in 1609 after an exhaustive study of the compilations made by Tycho Brahe (1546–1601) of the positions of the planet Mars. Kepler's First Law deals with the shape of planetary orbits (see Section 7.7).

or

$$\boxed{E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)} \tag{7.14}$$

7.4 Equations of Motion

When $U(r)$ is specified, Equation 7.14 completely describes the system, and the integration of this equation gives the general solution of the problem in terms of the parameters E and l. Solving Equation 7.14 for $\dot{r}$, we have

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U) - \frac{l^2}{\mu^2 r^2}} \tag{7.15}$$

This equation can be solved for dt and integrated to yield the solution $t = t(r)$. An inversion of this result then gives the equation of motion in the standard form $r = r(t)$. At present, however, we are interested in the equation of the path in terms of r and θ. We can write

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr \tag{7.16}$$

Into this relation we can substitute $\dot{\theta} = l/\mu r^2$ (Equation 7.10) and the expression for $\dot{r}$ from Equation 7.15. Integrating, we have

$$\theta(r) = \int \frac{(l/r^2)\,dr}{\sqrt{2\mu\left(E - U - \dfrac{l^2}{2\mu r^2}\right)}} \tag{7.17}$$

Furthermore, since l is constant in time, $\dot{\theta}$ cannot change sign and therefore $\theta(t)$ must increase monotonically with time.

Although we have reduced the problem to the formal evaluation of an integral, the actual solution can be obtained only for certain specific forms of the force law. If the force is proportional to some power of the radial distance, $F(r) \propto r^n$, then the solution can be expressed in terms of elliptic integrals for certain integer and fractional values of n. Only for $n = 1, -2$, and -3 are the solutions expressible in terms of circular functions.* The case $n = 1$ is just that of the harmonic oscillator (see Chapter 3), and the case $n = -2$ is the important inverse-square-law force treated in Sections 7.6 and 7.7. These two cases, $n = 1, -2$, are of prime importance in physical situations. Details of some other cases of interest will be found in the Problems at the end of this chapter.

We have therefore solved the problem in a formal way by combining the equations that express the conservation of energy and angular momentum into a single result, which gives the equation of the orbit $\theta = \theta(r)$. We can also attack the problem using Lagrange's equation for the coordinate r:

*See, for example, Goldstein (Go80, pp. 88–90).

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

Using Equation 7.7 for L, we find

$$\mu(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial U}{\partial r} = F(r) \tag{7.18}$$

Equation 7.18 can be cast in a form more suitable for certain types of calculations by making a simple change of variable:

$$u \equiv \frac{1}{r}$$

First we compute

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}}$$

But from Equation 7.10, $\dot{\theta} = l/\mu r^2$, so

$$\frac{du}{d\theta} = -\frac{\mu}{l}\dot{r}$$

Next we write

$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{\mu}{l}\dot{r}\right) = \frac{dt}{d\theta}\frac{d}{dt}\left(-\frac{\mu}{l}\dot{r}\right) = -\frac{\mu}{l\dot{\theta}}\ddot{r}$$

and with the same substitution for $\dot{\theta}$ we have

$$\frac{d^2u}{d\theta^2} = -\frac{\mu^2}{l^2}r^2\ddot{r}$$

Therefore, solving for $\ddot{r}$ and $r\dot{\theta}^2$ in terms of u, we find

$$\left.\begin{aligned} \ddot{r} &= -\frac{l^2}{\mu^2}u^2\frac{d^2u}{d\theta^2} \\ r\dot{\theta}^2 &= \frac{l^2}{\mu^2}u^3 \end{aligned}\right\} \tag{7.19}$$

Substituting Equations 7.19 into Equation 7.18, we obtain the transformed equation of motion:

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2}\frac{1}{u^2}F(u) \tag{7.20}$$

which we may also write as

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2}F(r)} \tag{7.21}$$

This form of the equation of motion is particularly useful if we wish to find the force law that gives a particular known orbit $r = r(\theta)$.

EXAMPLE 7.1

Find the force law for a central-force field that allows a particle to move in a logarithmic spiral orbit given by $r = ke^{\alpha\theta}$, where k and α are constants.

Solution: We use Equation 7.21 to determine the force law $F(r)$. First we determine

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{e^{-\alpha\theta}}{k}\right) = \frac{-\alpha e^{-\alpha\theta}}{k}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{\alpha^2 e^{-\alpha\theta}}{k} = \frac{\alpha^2}{r}$$

From Equation 7.21, we now determine $F(r)$:

$$F(r) = \frac{-l^2}{\mu r^2}\left(\frac{\alpha^2}{r} + \frac{1}{r}\right)$$

$$F(r) = \frac{-l^2}{\mu r^3}(\alpha^2 + 1) \tag{7.22}$$

Thus the force law is an attractive inverse cube.

EXAMPLE 7.2

Determine $r(t)$ and $\theta(t)$ for the problem in Example 7.1.

Solution: From Equation 7.10, we find

$$\dot{\theta} = \frac{l}{\mu r^2} = \frac{l}{\mu k^2 e^{2\alpha\theta}} \tag{7.23}$$

Rearranging Equation 7.23 gives

$$e^{2\alpha\theta}\, d\theta = \frac{l}{\mu k^2}\, dt$$

and integrating gives

$$\frac{e^{2\alpha\theta}}{2\alpha} = \frac{lt}{\mu k^2} + C'$$

where C' is an integration constant. Multiplying by 2α and letting $C = 2\alpha C'$ gives

$$e^{2\alpha\theta} = \frac{2\alpha l t}{\mu k^2} + C \tag{7.24}$$

We solve for $\theta(t)$ by taking the natural logarithm of Equation 7.24:

$$\theta(t) = \frac{1}{2\alpha}\ln\left(\frac{2\alpha l t}{\mu k^2} + C\right) \tag{7.25}$$

We can similarly solve for $r(t)$ by examining Equations 7.23 and 7.24:

$$\frac{r^2}{k^2} = e^{2\alpha\theta} = \frac{2\alpha l t}{\mu k^2} + C$$

$$r(t) = \left[\frac{2\alpha l}{\mu}t + k^2 C\right]^{\frac{1}{2}} \tag{7.26}$$

The integration constant C and angular momentum l needed for Equations 7.25 and 7.26 are determined from the initial conditions.

EXAMPLE 7.3

What is the total energy of the orbit of the previous two examples?

Solution: The energy is found from Equation 7.14. In particular, we need $\dot{r}$ and $U(r)$.

$$U(r) = -\int F\, dr = \frac{+l^2}{\mu}(\alpha^2 + 1)\int r^{-3}\, dr$$

$$U(r) = -\frac{l^2(\alpha^2 + 1)}{2\mu}\frac{1}{r^2} \tag{7.27}$$

where we have let $U(\infty) = 0$.

We rewrite Equation 7.10 to determine $\dot{r}$:

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{d\theta}{dr}\frac{dr}{dt} = \frac{l}{\mu r^2}$$

$$\dot{r} = \frac{dr}{d\theta}\frac{l}{\mu r^2} = \alpha k e^{\alpha\theta}\frac{l}{\mu r^2} = \frac{\alpha l}{\mu r} \tag{7.28}$$

Substituting Equations 7.27 and 7.28 into Equation 7.14 gives

$$E = \frac{1}{2}\mu\left(\frac{\alpha l}{\mu r}\right)^2 + \frac{l^2}{2\mu r^2} - \frac{l^2(\alpha^2 + 1)}{2\mu r^2}$$

$$E = 0 \tag{7.29}$$

The total energy of the orbit is zero if $U(r = \infty) = 0$.

7.5 Orbits in a Central Field

The radial velocity of a particle moving in a central field is given by Equation 7.15. This equation indicates that $\dot{r}$ vanishes at the roots of the radical, that is, at points for which

$$E - U(r) - \frac{l^2}{2\mu r^2} = 0 \tag{7.30}$$

The vanishing of $\dot{r}$ implies that a *turning point* in the motion has been reached (see Section 2.6). In general, Equation 7.30 possesses two roots: r_{max} and r_{min}. The motion of the particle is therefore confined to the annular region specified by $r_{max} \geq r \geq r_{min}$. Certain combinations of the potential function $U(r)$ and the parameters E and l produce only a single root for Equation 7.30. In such a case, $\dot{r} = 0$ for all values of the time; hence, $r =$ const., and the orbit is circular.

If the motion of a particle in the potential $U(r)$ is periodic, then the orbit is *closed*; that is, after a finite number of excursions between the radial limits r_{min} and r_{max}, the motion exactly repeats itself. But if the orbit does not close upon itself after a finite number of oscillations, the orbit is said to be *open* (Figure 7-4). From Equation 7.17 we can compute the change in the angle θ that results from one complete transit of r from r_{min} to r_{max} and back to r_{min}. Since the motion is symmetrical in time, this angular change is twice that which would result from the passage from r_{min} to r_{max}; thus

$$\Delta\theta = 2\int_{r_{min}}^{r_{max}} \frac{(l/r^2)\,dr}{\sqrt{2\mu\left(E - U - \dfrac{l^2}{2\mu r^2}\right)}} \tag{7.31}$$

The path is closed only if $\Delta\theta$ is a rational fraction of 2π—that is, if $\Delta\theta = 2\pi \cdot (a/b)$, where a and b are integers. Under these conditions, after b periods the radius vector of the particle will have made a complete revolutions and will have returned to its original position. We can show (see Problem 7-37) that if the

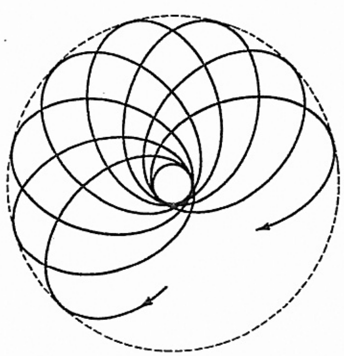


Figure 7-4

potential varies with some integer power of the radial distance, $U(r) \propto r^{n+1}$, then a closed noncircular path can result *only** if $n = -2$ or $+1$. The case $n = -2$ corresponds to an inverse-square-law force—for example, the gravitational or electrostatic force. The $n = +1$ case corresponds to the harmonic oscillator potential. For the two-dimensional case discussed in Section 3.4, we found that a closed path for the motion resulted if the ratio of the angular frequencies for the x and y motions were rational.

7.6 Centrifugal Energy and the Effective Potential

In the preceding expressions for $\dot{r}$, $\Delta\theta$, and so forth, a common term is the radical

$$\sqrt{E - U - \frac{l^2}{2\mu r^2}}$$

The last term in the radical has the dimensions of energy and, according to Equation 7.10, can also be written as

$$\frac{l^2}{2\mu r^2} = \frac{1}{2}\mu r^2\dot{\theta}^2$$

If we interpret this quantity as a "potential energy,"

$$U_c \equiv \frac{l^2}{2\mu r^2} \tag{7.32}$$

then the "force" that must be associated with U_c is

$$F_c = -\frac{\partial U_c}{\partial r} = \frac{l^2}{\mu r^3} = \mu r\dot{\theta}^2 \tag{7.33}$$

This quantity is traditionally called the **centrifugal force**,† although it is not a force in the ordinary sense of the word.‡ We shall, however, continue to use this unfortunate terminology, since it is customary and convenient.

We see that the term $l^2/2\mu r^2$ can be interpreted as the *centrifugal potential energy* of the particle and, as such, can be included with $U(r)$ in an *effective potential energy* defined by

$$\boxed{V(r) \equiv U(r) + \frac{l^2}{2\mu r^2}} \tag{7.34}$$

*Certain fractional values of n also lead to closed orbits, but in general these cases are uninteresting from a physical standpoint.

†The expression is more readily recognized in the form $F_c = mr\omega^2$. The first real appreciation of centrifugal force was by Huygens, who made a detailed examination in his study of the conical pendulum in 1659.

‡See Section 9.3 for a more critical discussion of centrifugal force.

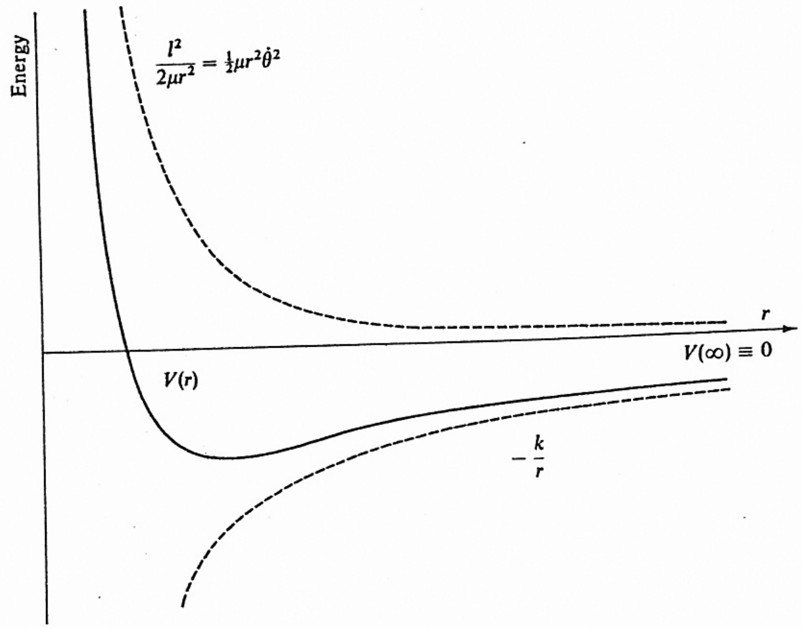


Figure 7-5

$V(r)$ is therefore a *fictitious* potential that combines the real potential function $U(r)$ with the energy term associated with the angular motion about the center of force. For the case of inverse-square-law central-force motion, the force is given by

$$F(r) = -\frac{k}{r^2} \tag{7.35}$$

from which

$$U(r) = -\int F(r)\,dr = -\frac{k}{r} \tag{7.36}$$

The effective potential function for gravitational attraction is therefore

$$V(r) = -\frac{k}{r} + \frac{l^2}{2\mu r^2} \tag{7.37}$$

This effective potential and its components are shown in Figure 7-5. The value of the potential is arbitrarily taken to be zero at $r = \infty$. (This is implicit in Equation 7.36, where we omitted the constant of integration.)

We may now draw conclusions similar to those in Section 2.6 on the motion of a particle in an arbitrary potential well. If we plot the total energy E of the particle on a diagram similar to Figure 7-5, we may identify three regions of

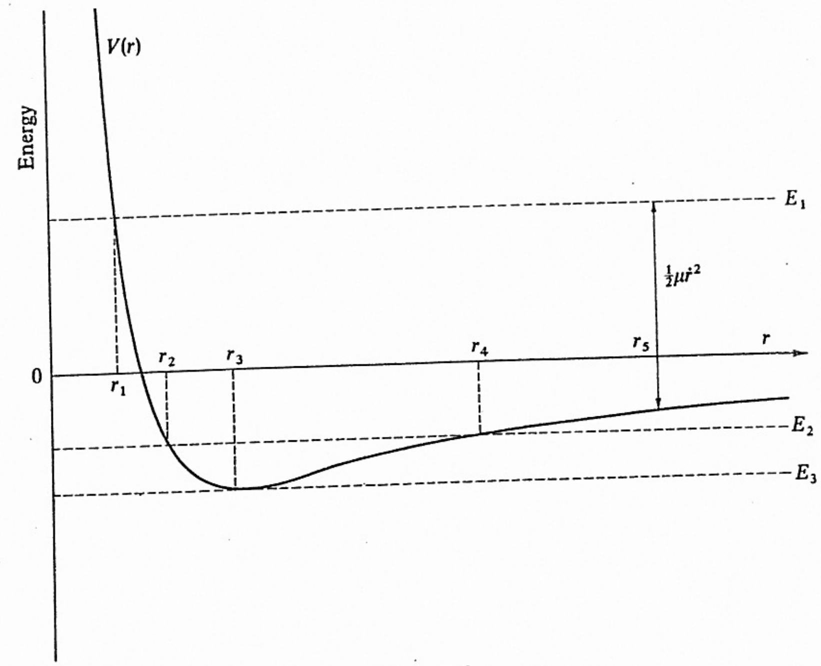


Figure 7-6

interest (see Figure 7-6). If the total energy is positive or zero (e.g., $E_1 \geq 0$), then the motion is unbounded; the particle moves toward the force center (located at $r = 0$) from infinitely far away until it "strikes" the potential barrier at the *turning point* $r = r_1$ and is reflected back toward infinitely large r. Note that the height of the constant total energy line above $V(r)$ at any r, such as r_5 in Figure 7-6, is equal to $\frac{1}{2}\mu\dot{r}^2$. Thus the radial velocity $\dot{r}$ vanishes and changes sign at the turning point (or points).

If the total energy is negative* and lies between zero and the minimum value of $V(r)$, as does E_2, then the motion is bounded, with $r_2 \leq r \leq r_4$. The values r_2 and r_4 are the turning points, or the **apsidal distances**, of the orbit. If E equals the minimum value of the effective potential energy (see E_3 in Figure 7-6), then the radius of the particle's path is limited to the single value r_3, and then $\dot{r} = 0$ for all values of the time; hence the motion is circular.

Values of E less than $V_{min} = -(\mu k^2/2l^2)$ do not result in physically real motion; for such cases, $\dot{r}^2 < 0$ and the velocity is imaginary.

The methods discussed in this section are often used in present-day research in general fields, especially atomic, molecular, and nuclear physics. For example,

*Note that negative values of the total energy arise only because of the arbitrary choice of $V(r) = 0$ at $r = \infty$.

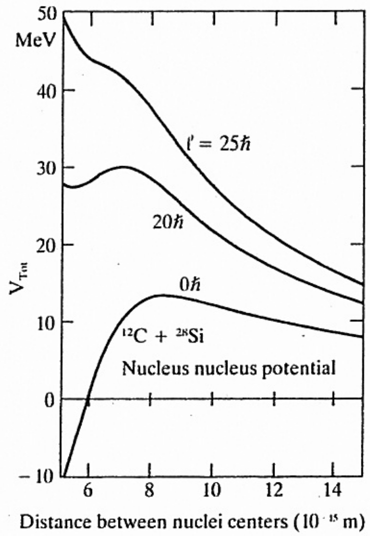


Figure 7-7

Figure 7-7 shows effective total nucleus–nucleus potentials for the scattering of ^{28}Si and ^{12}C. The total potential includes the coulomb, nuclear, and the centrifugal contributions. The potential for $l = 0\hbar$ indicates the potential with no centrifugal term. For a relative angular momentum value of $l = 20\hbar$, a "pocket" exists where the two scattering nuclei may be bound together (even if only for a short time). For $l = 25\hbar$, the centrifugal "barrier" dominates, and the nuclei are not attracted to each other.

7.7 Planetary Motion—Kepler's Problem

The equation for the path of a particle moving under the influence of a central force whose magnitude is inversely proportional to the square of the distance between the particle and the force center can be obtained (see Equation 7.17) from

$$\theta(r) = \int \frac{(l/r^2)\,dr}{\sqrt{2\mu\left(E + \dfrac{k}{r} - \dfrac{l^2}{2\mu r^2}\right)}} + \text{const.} \tag{7.38}$$

The integral can be evaluated if the variable is changed to $u \equiv l/r$ (see Problem 7-2). If we define the origin of θ so that the integration constant is zero, we find

$$\cos\theta = \frac{\dfrac{l^2}{\mu k}\cdot\dfrac{1}{r} - 1}{\sqrt{1 + \dfrac{2El^2}{\mu k^2}}} \tag{7.39}$$

Let us now define the following constants:

$$\left.\begin{aligned} \alpha &\equiv \frac{l^2}{\mu k} \\ \varepsilon &\equiv \sqrt{1 + \frac{2El^2}{\mu k^2}} \end{aligned}\right\} \tag{7.40}$$

Equation 7.39 can thus be written as

$$\boxed{\frac{\alpha}{r} = 1 + \varepsilon\cos\theta} \tag{7.41}$$

This is the equation of a conic section with one focus at the origin*. The quantity ε is called the **eccentricity**, and 2α is termed the **latus rectum** of the orbit. Conic sections are formed by the intersection of a plane and a cone. A conic section is formed by the loci of points (formed in a plane) where the ratio of the distance from a fixed point (the focus) to a fixed line (called the directrix) is a constant. The directrix for the parabola is shown in Figure 7-8 by the vertical dashed line, drawn so that $r/r' = 1$.

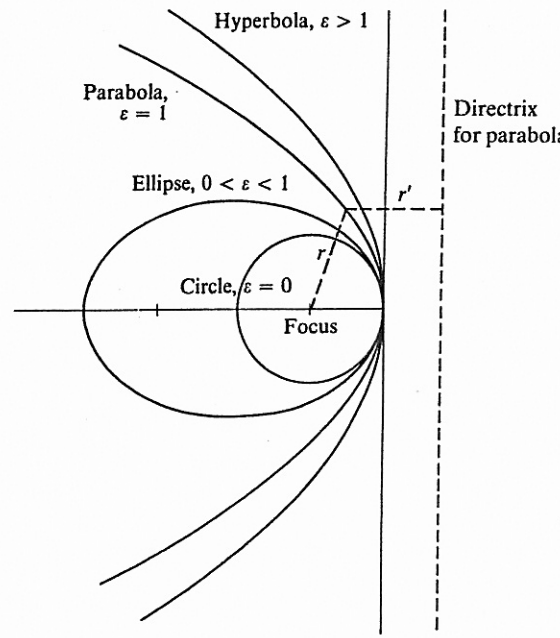


Figure 7-8

*Johann Bernoulli (1667–1748) appears to have been the first to prove that *all* possible orbits of a body moving in a potential proportional to $1/r$ are conic sections (1710).

The minimum value for r in Equation 7.41 occurs when $\cos\theta$ is a maximum—that is, for $\theta = 0$. Thus the choice of *zero* for the constant in Equation 7.38 corresponds to measuring θ from $r_{\min}$, which position is called the **pericenter**; $r_{\max}$ corresponds to the **apocenter.*** The general term for turning points is **apsides.**

Various values of the eccentricity (and hence of the energy E) classify the orbits according to different conic sections (see Figure 7-8):

$\varepsilon > 1,$	$E > 0$	Hyperbola
$\varepsilon = 1,$	$E = 0$	Parabola
$0 < \varepsilon < 1,$	$V_{\min} < E < 0$	Ellipse
$\varepsilon = 0,$	$E = V_{\min}$	Circle
$\varepsilon < 0,$	$E < V_{\min}$	Not allowed

For planetary motion, the orbits are ellipses with major and minor axes (equal to $2a$ and $2b$, respectively) given by

$$a = \frac{\alpha}{1-\varepsilon^2} = \frac{k}{2|E|} \tag{7.42}$$

$$b = \frac{\alpha}{\sqrt{1-\varepsilon^2}} = \frac{l}{\sqrt{2\mu|E|}} \tag{7.43}$$

Thus the major axis depends only on the energy of the particle, whereas the minor axis is a function of both first integrals of the motion, E and l. The geometry of elliptic orbits in terms of the parameters α, ε, a, and b is shown in Figure 7-9; P and P' are the foci. From this diagram we see that the apsidal distances ($r_{\min}$ and $r_{\max}$ as measured from the foci to the orbit) are given by

$$\left.\begin{aligned} r_{\min} &= a(1-\varepsilon) = \frac{\alpha}{1+\varepsilon} \\ r_{\max} &= a(1+\varepsilon) = \frac{\alpha}{1-\varepsilon} \end{aligned}\right\} \tag{7.44}$$

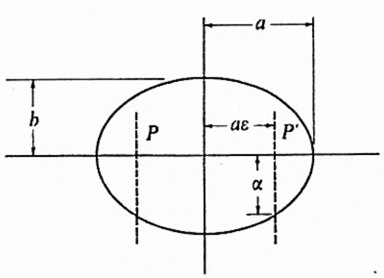


Figure 7-9

*The corresponding terms for motion about the sun are *perihelion* and *aphelion*, and for motion about the earth, *perigee* and *apogee*.

To find the period for elliptic motion, we rewrite Equation 7.12 for the areal velocity as

$$dt = \frac{2\mu}{l}\,dA$$

Since the entire area A of the ellipse is swept out in one complete period τ,

$$\int_0^\tau dt = \frac{2\mu}{l}\int_0^A dA$$

$$\tau = \frac{2\mu}{l}A \tag{7.45}$$

The area of an ellipse is given by $A = \pi ab$, and using a and b from Equations 7.42 and 7.43, we find

$$\tau = \frac{2\mu}{l}\cdot\pi ab = \frac{2\mu}{l}\cdot\pi\cdot\frac{k}{2|E|}\cdot\frac{l}{\sqrt{2\mu|E|}}$$

$$= \pi k\sqrt{\frac{\mu}{2}}\cdot|E|^{-\frac{3}{2}} \tag{7.46}$$

We also note from Equations 7.42 and 7.43 that the semiminor axis* can be written as

$$b = \sqrt{\alpha a} \tag{7.47}$$

Therefore, since $\alpha = l^2/\mu k$, the period τ can also be expressed as

$$\boxed{\tau^2 = \frac{4\pi^2\mu}{k}a^3} \tag{7.48}$$

This result, that the square of the period is proportional to the cube of the semimajor axis of the elliptic orbit, is known as **Kepler's Third Law.**† Note that this result is concerned with the equivalent one-body problem, so account must be taken of the fact that it is the *reduced* mass μ that occurs in Equation 7.48. Kepler actually concluded that the squares of the periods of the planets were proportional to the cubes of the major axes of their orbits—with the same proportionality constant for all planets. In this sense, the statement is only approximately correct, since the reduced mass is different for each planet. In particular, since the gravitational force is given by

$$F(r) = -\frac{Gm_1m_2}{r^2} = -\frac{k}{r^2}$$

*The quantities a and b are called *semimajor* and *semiminor axes*, respectively.

†Published by Kepler in 1619. Kepler's Second Law is stated in Section 7.3. The First Law (1609) dictates that the planets move in elliptical orbits with the sun at one focus. It should be noted that Kepler's work preceded by almost 80 years Newton's enunciation of his general laws of motion. Indeed, Newton's conclusions were based to a great extent on Kepler's pioneering studies (and on those of Galileo and Huygens).

we identify $k = Gm_1m_2$. The expression for the square of the period therefore becomes

$$\tau^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)} \cong \frac{4\pi^2 a^3}{Gm_2}, \qquad m_1 \ll m_2 \tag{7.49}$$

and Kepler's statement is correct only if the mass m_1 of a planet can be neglected with respect to the mass m_2 of the sun. (But note, for example, that the mass of Jupiter is about 1/1000 of the mass of the sun, so the departure from the approximate law is not difficult to observe in this case.)

Kepler's laws can now be summarized:

I. *Planets move in elliptical orbits about the sun with the sun at one focus.*
II. *The area per unit time swept out by a radius vector from the sun to a planet is constant.*
III. *The square of a planet's period is proportional to the cube of the major axis of the planet's orbit.*

EXAMPLE 7.4

Halley's comet, which passed around the sun early in 1986, moves in a highly elliptical orbit with an eccentricity of 0.967 and a period of 76 yr. Calculate its minimum and maximum distances from the sun.

Solution: Equation 7.49 relates the period of motion with the semimajor axes. Since m (Halley's comet) $\ll m_{\text{sun}}$,

$$a = \left(\frac{Gm_{\text{sun}}\tau^2}{4\pi^2}\right)^{\frac{1}{3}}$$

$$= \left[\frac{\left(6.67 \times 10^{-11} \dfrac{\text{Nm}^2}{\text{kg}^2}\right)(1.99 \times 10^{30}\ \text{kg})\left(76\ \text{yr}\dfrac{365\ \text{day}}{\text{yr}}\dfrac{24\ \text{hr}}{\text{day}}\dfrac{3600\ \text{s}}{\text{hr}}\right)^2}{4\pi^2}\right]^{\frac{1}{3}}$$

$$a = 2.68 \times 10^{12}\ \text{m}$$

Using Equations 7.44 we can determine r_{min} and r_{max}.

$$r_{\text{min}} = 2.68 \times 10^{12}\ \text{m}(1 - 0.967) = 8.8 \times 10^{10}\ \text{m}$$

$$r_{\text{max}} = 2.68 \times 10^{12}\ \text{m}(1 + 0.967) = 5.27 \times 10^{12}\ \text{m}$$

This orbit takes the comet inside the path of Venus, almost to Mercury, and out past even the orbit of Neptune and sometimes even to the moderately eccentric orbit of Pluto. Edmond Halley is generally given the credit for bringing Newton's work on gravitational and central forces to the attention of the world. After observing the comet personally in 1682, Halley became interested. Partly as a result of a bet between Christopher Wren and Robert Hooke, Halley asked Newton in 1684 what paths the planets must follow if the sun pulled them with a force inversely proportional to the square of their distances. To the astonishment

of Halley, Newton replied, "Why, in ellipses, of course." Newton had worked it out twenty years previously but had not published the result. With painstaking effort, Halley was able in 1705 to predict the next occurrence of the comet, now bearing his name, to be in 1758.

7.8 Kepler's Equation (optional)

We have found in Equation 7.41 the relationship between the coordinates r and θ describing the motion of a particle attracted toward a center by a force varying inversely with r^2. But for astronomical calculations, it is not $r(\theta)$ that is desired but the function $\theta(t)$, so that the direction of the body (a planet, comet, etc.) may be found at any time. Furthermore, we wish to have an expression giving θ (called the *true anomaly**) as a function of time that involves as parameters only the two fundamental observable constants of the orbit—that is, the period τ and the eccentricity ε. We can make such a calculation in the following way. Since it requires a time τ for the radius vector of the body to sweep out the entire area πab of an elliptical orbit, and since the areal velocity is a constant of the motion, in a time t an area $(\pi ab/\tau)t$ is swept out. We can equate this expression to the integral of the area by writing

$$\frac{\pi ab}{\tau}t = \int dA \tag{7.50}$$

According to Equation 7.11, if we take $\theta = 0$ at $t = 0$, we have

$$\frac{\pi ab}{\tau}t = \frac{1}{2}\int_0^\theta r^2\, d\theta \tag{7.51}$$

Equation 7.41 can be written as

$$r = \frac{\alpha}{1 + \varepsilon\cos\theta} \tag{7.52}$$

Therefore (see Equation E.16, Appendix E)

$$\frac{\pi ab}{\tau}t = \frac{\alpha^2}{2}\int_0^\theta \frac{d\theta}{(1 + \varepsilon\cos\theta)^2}$$

$$= \frac{\alpha^2}{2(1 - \varepsilon^2)}\left[\frac{2}{\sqrt{1 - \varepsilon^2}}\tan^{-1}\left(\frac{(1 - \varepsilon)\tan(\theta/2)}{\sqrt{1 - \varepsilon^2}}\right) - \frac{\varepsilon\sin\theta}{1 + \varepsilon\cos\theta}\right]$$

Noting that $ab = \alpha^2(1 - \varepsilon^2)^{-\frac{3}{2}}$, we can simplify this expression to

$$\frac{2\pi t}{\tau} = 2\tan^{-1}\left(\sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}}\tan\frac{\theta}{2}\right) - \frac{\varepsilon\sqrt{1 - \varepsilon^2}\sin\theta}{1 + \varepsilon\cos\theta} \tag{7.53}$$

---

*The historical term "anomaly" is not used in the sense of "strange" but rather as "deviation," since θ measures the angular deviation from some fixed point, which in astronomical calculations is usually the perihelion.

Table 7-1

SOME PROPERTIES OF THE PRINCIPAL OBJECTS IN THE SOLAR SYSTEM

Name	Semimajor axis of orbit (in astronomical units[a])	Period (yr)	Eccentricity	Mass (in units of the mass of the earth[b])
Sun	—	—	—	333,480
Mercury	0.3871	0.2408	0.2056	0.0543
Venus	0.7233	0.6152	0.0068	0.8137
Earth	1.0000	1.0000	0.0167	1.000
Eros (asteroid)	1.4583	1.7610	0.2230	2×10^{-9} (?)
Mars	1.5237	1.8809	0.0934	0.1071
Ceres (asteroid)	c	4.6035	0.0765	1/8000 (?)
Jupiter	5.2028	c	0.0484	318.35
Saturn	9.5388	29.458	0.0557	c
Uranus	19.182	84.013	0.0472	14.58
Neptune	30.058	164.794	0.0086	17.26
Pluto	39.518	248.430	0.2486	<0.1
Halley (comet)	18	76	0.967	

[a]One astronomical unit (A.U.) is the length of the semimajor axis of the earth's orbit. One A.U. $\cong 1.495 \times 10^{11}$ m $\cong 93 \times 10^6$ miles.

[b]The mass of the earth is approximately 5.976×10^{24} kg.

[c]See Problem 7-19.

This is a formidable equation and one that is certainly not easy to use. To make matters worse, it is not $t(\theta)$ that is necessary but rather $\theta(t)$; that is, the equation must be inverted. Clearly, this cannot be done in any simple way; only a series expansion is possible:

$$\theta(t) = \frac{2\pi t}{\tau} + 2\varepsilon \sin\frac{2\pi t}{\tau} + \frac{5}{4}\varepsilon^2 \sin\frac{4\pi t}{\tau} + \frac{1}{12}\varepsilon^3\left(13\sin\frac{6\pi t}{\tau} - 3\sin\frac{2\pi t}{\tau}\right) + \cdots \tag{7.54}$$

If ε is sufficiently small, then the terms in ε^2 and higher powers can be neglected and an easily handled expression results. But for planetary studies, such an approximation is not permissible, since the eccentricities of most of the planets are greater than 0.04. (Table 7-1 gives some of the pertinent data regarding the major objects of the solar system.) Comets, of course, have eccentricities that are close to unity. Therefore, if Equation 7.54 is used, it will in general be necessary to take many terms in the series to achieve an accuracy comparable with that of astronomical observations (which is exceedingly high!). This is at best a tedious procedure, so we wish to find a less laborious method for calculating $\theta(t)$. The solution to this problem was sought by Kepler (although, of course, he did not know the mathematical relations stated by the preceding equations), who devised an ingenious geometrical method to calculate the anomaly as a function of time.

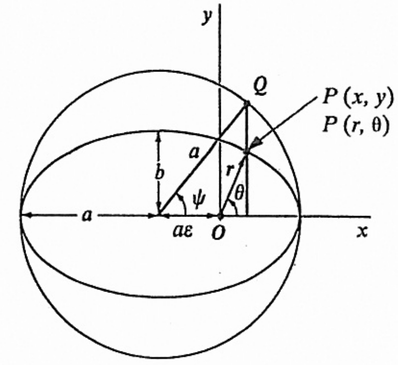


Figure 7-10

We shall not give his geometrical solution here but instead obtain the same result by algebraic means.

Figure 7-10 shows Kepler's construction. The motion takes place in the elliptical orbit with the force center located at the focus O, which is also the origin for a rectangular coordinate system. In this system the equation of the orbit, point $P(x, y)$, is

$$\frac{(x + a\varepsilon)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{7.55}$$

Next we circumscribe the ellipse with a circle of radius a and project the point P (defined by r and θ) onto the circle at point Q. The angle between the x-axis and the line connecting the center of the circle with the point Q is called the **eccentric anomaly** ψ and is defined by

$$\left.\begin{aligned} \cos\psi &= \frac{x + a\varepsilon}{a} \\ \sin\psi &= \frac{y}{b} \end{aligned}\right\} \tag{7.56}$$

where the value for $\sin\psi$ follows from Equation 7.55, which now appears as $\cos^2\psi + \sin^2\psi = 1$.

From these relations we can write

$$\left.\begin{aligned} x &= a(\cos\psi - \varepsilon) \\ y &= b\sin\psi \\ &= a\sqrt{1 - \varepsilon^2}\sin\psi \end{aligned}\right\} \tag{7.57}$$

Squaring these equations and adding, we find

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= a^2(1 - \varepsilon\cos\psi)^2 \end{aligned}$$

and, in terms of the eccentric anomaly, the quantity r is

$$r = a(1 - \varepsilon\cos\psi) \tag{7.58}$$

We now wish to obtain an explicit relationship between ψ and θ. First we rewrite Equation 7.52 as

$$\varepsilon r\cos\theta = a(1 - \varepsilon^2) - r \tag{7.59}$$

If we add εr to both sides of this equation, we have

$$\varepsilon r(1 + \cos\theta) = (1 - \varepsilon)[a(1 + \varepsilon) - r]$$

Substituting for r from Equation 7.58 in the right-hand side of this expression, we find

$$\varepsilon r(1 + \cos\theta) = (1 - \varepsilon)[a(1 + \varepsilon) - a(1 - \varepsilon\cos\psi)]$$

or

$$r(1 + \cos\theta) = a(1 - \varepsilon)(1 + \cos\psi) \tag{7.60a}$$

If we subtract εr from both sides of Equation 7.59 and simplify, we obtain

$$r(1 - \cos\theta) = a(1 + \varepsilon)(1 - \cos\psi) \tag{7.60b}$$

Dividing Equation 7.60b by Equation 7.60a, we find

$$\frac{1 - \cos\theta}{1 + \cos\theta} = \frac{1 + \varepsilon}{1 - \varepsilon}\cdot\frac{1 - \cos\psi}{1 + \cos\psi}$$

We can use the half-angle formula for the tangent to write this equation as

$$\tan\frac{\theta}{2} = \sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}}\tan\frac{\psi}{2} \tag{7.61}$$

which gives ψ uniquely in terms of θ (if we confine ourselves to principal values of the tangent functions). Therefore, $\theta(t)$ can be obtained once $\psi(t)$ is found. To calculate $\psi(t)$, we can transform Equation 7.51 into an equation for ψ by computing the integrand in terms of ψ. Differentiating Equation 7.61 yields

$$d\theta = \sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}}\cdot\frac{\cos^2(\theta/2)}{\cos^2(\psi/2)}\,d\psi \tag{7.62}$$

From Equation 7.60a we can write

$$\begin{aligned} r &= a(1 - \varepsilon)\frac{1 + \cos\psi}{1 + \cos\theta} \\ &= a(1 - \varepsilon)\frac{\cos^2(\psi/2)}{\cos^2(\theta/2)} \end{aligned} \tag{7.63}$$

where we have used the half-angle formula for the cosine functions.

To express $r^2\,d\theta$ in terms of ψ, we take one factor of r from Equation 7.58, the other factor of r from Equation 7.63, and $d\theta$ from Equation 7.62. Thus

$$\begin{aligned} r^2\,d\theta &= [a(1 - \varepsilon\cos\psi)]\left[a(1 - \varepsilon)\frac{\cos^2(\psi/2)}{\cos^2(\theta/2)}\right] \\ &\quad\cdot\left[\sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}}\cdot\frac{\cos^2(\theta/2)}{\cos^2(\psi/2)}\,d\psi\right] \\ &= a^2\sqrt{1 - \varepsilon^2}(1 - \varepsilon\cos\psi)\,d\psi \end{aligned} \tag{7.64}$$

In view of this result, Equation 7.51 can now be written as

$$\frac{\pi ab}{\tau}t = \frac{a^2\sqrt{1 - \varepsilon^2}}{2}\int_0^\psi (1 - \varepsilon\cos\psi)\,d\psi$$

Integrating, and again using $b = a\sqrt{1 - \varepsilon^2}$, we have the result

$$\frac{2\pi t}{\tau} = \psi - \varepsilon\sin\psi \tag{7.65}$$

The quantity $2\pi t/\tau$ is called the **mean anomaly**, because it measures the angular deviation of a body moving in a circular orbit with a period τ. Following astronomical practice,* we denote the mean anomaly by M:

$$\boxed{M = \psi - \varepsilon\sin\psi} \tag{7.66}$$

This is **Kepler's equation.** To find $\psi(t)$, this result must be inverted by some approximation procedure. Then, since Equation 7.61 relates ψ and θ, the time dependence of the true anomaly can be found.

We can use Kepler's equation to obtain a simple expression for the velocity of a body in its orbit in terms of the magnitude of the radius vector. Referring to Figure 7-10, we can write

$$v^2 = \dot{x}^2 + \dot{y}^2 \tag{7.67}$$

Using Equations 7.57 for x and y, the square of the velocity becomes

$$\begin{aligned} v^2 &= a^2\dot{\psi}^2\sin^2\psi + a^2(1 - \varepsilon^2)\dot{\psi}^2\cos^2\psi \\ &= a^2\dot{\psi}^2(1 - \varepsilon^2\cos^2\psi) \end{aligned} \tag{7.68}$$

If we differentiate Kepler's equation (Equation 7.65) with respect to the time, we have

$$\frac{2\pi}{\tau} = \dot{\psi}(1 - \varepsilon\cos\psi) \tag{7.69}$$

*It is also customary to denote the eccentric anomaly by E and the true anomaly by v or f.

Solving this equation for $\dot{\psi}$ and substituting into Equation 7.68, we obtain

$$\begin{aligned} v^2 &= \left(\frac{2\pi}{\tau}\right)^2 a^2 \frac{1 - \varepsilon^2 \cos^2 \psi}{(1 - \varepsilon \cos \psi)^2} \\ &= \left(\frac{2\pi}{\tau}\right)^2 a^2 \frac{1 + \varepsilon \cos \psi}{1 - \varepsilon \cos \psi} \\ &= \left(\frac{2\pi}{\tau}\right)^2 a^2 \frac{2 - (1 - \varepsilon \cos \psi)}{1 - \varepsilon \cos \psi} \end{aligned} \tag{7.70}$$

Substituting $r/a = 1 - \varepsilon \cos \psi$ from Equation 7.58, there results

$$v^2 = \left(\frac{2\pi}{\tau}\right)^2 a^3 \left(\frac{2}{r} - \frac{1}{a}\right) \tag{7.71}$$

Finally, Kepler's Third Law (Equation 7.48) can be used to reduce this expression to

$$\boxed{v^2 = \frac{k}{\mu}\left(\frac{2}{r} - \frac{1}{a}\right)} \tag{7.72}$$

If we wish to calculate $\theta(t)$ for the motion of a body whose orbit has an eccentricity not too large (say, $\varepsilon \approx 0.1$), and if we wish to achieve an accuracy of, say, 1 part in 10^6, then many terms in Equation 7.54 are necessary. The use of Kepler's equation in such a situation is somewhat easier. Astronomical calculations of orbits are almost always based on Kepler's equation. Details of approximation procedures for Kepler's equation can be found in various texts on celestial mechanics.*

7.9 Apsidal Angles and Precession (optional)

If a particle executes bounded, noncircular motion in a central-force field, then the radial distance from the force center to the particle must always be in the range $r_{max} \geq r \geq r_{min}$, that is, r must be bounded by the apsidal distances. Figure 7-6 indicates that only *two* apsidal distances exist for bounded, noncircular motion. But in executing one complete revolution in θ, the particle may not return to its original position (see Figure 7-4). The angular separation between two successive values of $r = r_{max}$ depends on the exact nature of the force. The angle between any two consecutive apsides is called the **apsidal angle**, and, since a closed orbit must be symmetrical about any apsis, it follows that all apsidal angles for such motion must be equal. The apsidal angle for elliptical motion,

*More than 120 methods of obtaining approximate solutions to Kepler's equation are discussed in the literature. See, for example, Moulton (Mo58, p. 164ff) for details.

for example, is just π. If the orbit is not closed, the particle reaches the apsidal distances at different points in each revolution; the apsidal angle is not then a rational fraction of 2π, as is required for a closed orbit. If the orbit is *almost* closed, the apsides *precess*, or rotate slowly in the plane of the motion. This effect is exactly analogous to the slow rotation of the elliptical motion of a two-dimensional harmonic oscillator whose natural frequencies for the x- and y-motions are almost equal (see Section 3.4).

Since an inverse-square-law force requires that all elliptical orbits be exactly closed, the apsides must stay fixed in space for all time. If the apsides move with time, however slowly, this indicates that the force law under which the body moves does not vary exactly as the inverse square of the distance. This important fact was realized by Newton, who pointed out that any advance or regression of a planet's perihelion would require the radial dependence of the force law to be slightly different from $1/r^2$. Thus, Newton argued, the observation of the time dependence of the perihelia of the planets would be a sensitive test of the validity of the form of the universal gravitation law.

In point of fact, for planetary motion within the solar system, one expects that, because of the perturbations introduced by the existence of all of the other planets, the force experienced by any planet does not vary exactly as $1/r^2$, if r is measured from the sun. This effect is small, however, and only slight variations of planetary perihelia have been observed. The perihelion of Mercury, for example, which shows the largest effect, advances only about 574" of arc per century.* Detailed calculations of the influence of the other planets on the motion of Mercury predict that the rate of advance of the perihelion should be approximately 531" per century. The uncertainties in this calculation are considerably less than the difference of 43" between observation and calculation,† and for a considerable time this discrepancy was the outstanding unresolved difficulty in the Newtonian theory. We now know that the modification introduced into the equation of motion of a planet by the general theory of relativity almost exactly accounts for the difference of 43". This result is one of the major triumphs of relativity theory.

We next indicate the way the advance of the perihelion can be calculated from the modified equation of motion. To perform this calculation, it is convenient to use the equation of motion in the form of Equation 7.20. If we use

*This precession is in addition to the general precession of the equinox with respect to the "fixed" stars, which amounts to 5025.645" ± 0.050" per century.

†In 1845, the French astronomer Urbain Jean Joseph Le Verrier (1811–1877) first called attention to the irregularity in the motion of Mercury. Similar studies by LeVerrier and by the English astronomer John Couch Adams of irregularities in the motion of Uranus led to the discovery of the planet Neptune in 1846. An interesting account of this episode is given by Turner (Tu04, Chapter 2). We must note, in this regard, that perturbations may be either *periodic* or *secular* (i.e., ever increasing with time). Laplace showed in 1773 (published, 1776) that any perturbation of a planet's mean motion that is caused by the attraction of another planet must be periodic, although the period may be extremely long. This is the case for Mercury; the precession of 531" per century is periodic, but the period is so long that the change from century to century is small compared to the residual effect of 43".

the universal gravitational law for $F(r)$, we can write

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{1}{u^2}F(u)$$
$$= \frac{Gm^2M}{l^2} \tag{7.73}$$

where we consider the motion of a body of mass m in the gravitational field of a body of mass M. The quantity u is therefore the reciprocal of the distance between m and M.

The modification of the gravitational force law required by the general theory of relativity introduces into the force a small component that varies as $1/r^4(=u^4)$. Thus we have

$$\frac{d^2u}{d\theta^2} + u = \frac{Gm^2M}{l^2} + \frac{3GM}{c^2}u^2 \tag{7.74}$$

where c is the velocity of propagation of the gravitational interaction and is identified with the velocity of light.* To simplify the notation, we define

$$\frac{1}{\alpha} \equiv \frac{Gm^2M}{l^2}$$
$$\delta \equiv \frac{3GM}{c^2} \tag{7.75}$$

and we can write Equation 7.74 as

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{\alpha} + \delta u^2 \tag{7.76}$$

This is a nonlinear equation, and we use a successive approximation procedure to obtain a solution. We choose the first trial solution to be the solution of Equation 7.76 in the case that the term δu^2 is neglected†:

$$u_1 = \frac{1}{\alpha}(1 + \varepsilon\cos\theta) \tag{7.77}$$

This is the familiar result for the pure inverse-square-law force (see Equation 7.41). Note that α is here the same as that defined in Equation 7.40 except

*One-half of the relativistic term results from effects understandable in terms of special relativity, viz., time dilation (1/3) and the relativistic momentum effect (1/6); the velocity is greatest at perihelion and least at aphelion (see Chapter 14). The other half of the term arises from general relativistic effects and is associated with the finite propagation time of gravitational interactions. Thus the agreement between theory and experiment confirms the prediction that the gravitational propagation velocity is the same as that for light.

†We eliminate the necessity of introducing an arbitrary phase into the argument of the cosine term by choosing to measure θ from the position of perihelion; i.e., u_1 is a maximum (and hence r_1 is a minimum) at $\theta = 0$.

that μ has been replaced by m. If we substitute this expression into the right-hand side of Equation 7.76 we find

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{\alpha} + \frac{\delta}{\alpha^2}[1 + 2\varepsilon\cos\theta + \varepsilon^2\cos^2\theta]$$
$$= \frac{1}{\alpha} + \frac{\delta}{\alpha^2}\left[1 + 2\varepsilon\cos\theta + \frac{\varepsilon^2}{2}(1 + \cos 2\theta)\right] \tag{7.78}$$

where $\cos^2\theta$ has been expanded in terms of $\cos 2\theta$. The first trial function u_1, when substituted into the left-hand side of Equation 7.76, reproduces only the first term on the right-hand side: $1/\alpha$. We can therefore construct a second trial function by adding to u_1 a term that reproduces the remainder of the right-hand side (in Equation 7.78). We can verify that such a particular integral is

$$u_p = \frac{\delta}{\alpha^2}\left[\left(1 + \frac{\varepsilon^2}{2}\right) + \varepsilon\theta\sin\theta - \frac{\varepsilon^2}{6}\cos 2\theta\right] \tag{7.79}$$

The second trial function is therefore

$$u_2 = u_1 + u_p$$

If we stop the approximation procedure at this point, we have

$$u \cong u_2 = u_1 + u_p$$
$$= \left[\frac{1}{\alpha}(1 + \varepsilon\cos\theta) + \frac{\delta\varepsilon}{\alpha^2}\theta\sin\theta\right]$$
$$+ \left[\frac{\delta}{\alpha^2}\left(1 + \frac{\varepsilon^2}{2}\right) - \frac{\delta\varepsilon^2}{6\alpha^2}\cos 2\theta\right] \tag{7.80}$$

where we have regrouped the terms in u_1 and u_p.

Consider the terms in the second set of brackets in Equation 7.80: the first of these is just a constant, and the second is only a small and periodic disturbance of the normal Keplerian motion. Therefore, on a long time scale neither of these terms contributes, on the average, to any change in the positions of the apsides. But in the first set of brackets, the term proportional to θ produces secular and therefore observable effects. Let us consider the first set of brackets:

$$u_{\text{secular}} = \frac{1}{\alpha}\left[1 + \varepsilon\cos\theta + \frac{\delta\varepsilon}{\alpha}\theta\sin\theta\right] \tag{7.81}$$

Next we can expand the quantity

$$1 + \varepsilon\cos\left(\theta - \frac{\delta}{\alpha}\theta\right) = 1 + \varepsilon\left(\cos\theta\cos\frac{\delta}{\alpha}\theta + \sin\theta\sin\frac{\delta}{\alpha}\theta\right)$$
$$\cong 1 + \varepsilon\cos\theta + \frac{\delta\varepsilon}{\alpha}\theta\sin\theta \tag{7.82}$$

where we have used the fact that δ is small to approximate

$$\cos\frac{\delta}{\alpha}\theta \cong 1, \qquad \sin\frac{\delta}{\alpha}\theta \cong \frac{\delta}{\alpha}\theta$$

Hence we can write u_{secular} as

$$u_{\text{secular}} \cong \frac{1}{\alpha}\left[1 + \varepsilon\cos\left(\theta - \frac{\delta}{\alpha}\theta\right)\right] \tag{7.83}$$

We have chosen to measure θ from the position of perihelion at $t = 0$. Successive appearances at perihelion result when the argument of the cosine term in u_{secular} increases to $2\pi, 4\pi, \ldots$, and so forth. But an increase of the argument by 2π requires that

$$\theta - \frac{\delta}{\alpha}\theta = 2\pi$$

or

$$\theta = \frac{2\pi}{1 - (\delta/\alpha)} \cong 2\pi\left(1 + \frac{\delta}{\alpha}\right)$$

Therefore, the effect of the relativistic term in the force law is to displace the perihelion in each revolution by an amount

$$\Delta \cong \frac{2\pi\delta}{\alpha} \tag{7.84a}$$

that is, the apsides rotate slowly in space. If we refer to the definitions of α and δ (Equations 7.75), we find

$$\Delta \cong 6\pi\left(\frac{GmM}{cl}\right)^2 \tag{7.84b}$$

From Equations 7.40 and 7.42 we can write $l^2 = \mu ka(1 - \varepsilon^2)$; then, since $k = GmM$ and $\mu \cong m$, we have

$$\boxed{\Delta \cong \frac{6\pi GM}{ac^2(1 - \varepsilon^2)}} \tag{7.84c}$$

We see therefore that the effect is enhanced if the semimajor axis a is small and if the eccentricity is large. Mercury, which is the planet nearest the sun and which has the most eccentric orbit of any planet (except Pluto), provides the most sensitive test of the theory.* The calculated value of the precessional rate for

*Alternatively, we can say that the relativistic advance of the perihelion is a maximum for Mercury because the orbital velocity is greatest for Mercury and the relativistic parameter v/c largest (see Chapter 14 and Problem 14-35).

Table 7-2

PRECESSIONAL RATES FOR THE PERIHELIA OF SOME PLANETS

	Precessional rate (seconds of arc/century)	
Planet	Calculated	Observed
Mercury	43.03 ± 0.03	43.11 ± 0.45
Venus	8.63	8.4 ± 4.8
Earth	3.84	5.0 ± 1.2
Mars	1.35	—
Jupiter	0.06	—

Mercury is 43.03″ ± 0.03″ of arc per century. The observed value (corrected for the influence of the other planets) is 43.11″ ± 0.45″,* so the prediction of relativity theory is confirmed in striking fashion. The precessional rates for some of the planets are given in Table7-2.

7.10 Stability of Circular Orbits

In Section 7.6 we pointed out that the orbit is circular if the total energy equals the minimum value of the effective potential energy, $E = V_{\text{min}}$. More generally, however, a circular orbit is allowed for *any* attractive potential, since the attractive force can *always* be made to just balance the centrifugal force by the proper choice of radial velocity. Although circular orbits are therefore always possible in a central, attractive force field, such orbits are not necessarily stable. A circular orbit at $r = \rho$ exists if $\dot{r}|_{r=\rho} = 0$ for all t; this is possible if $(\partial V/\partial r)|_{r=\rho} = 0$. But only if the effective potential has a *true minimum* does stability result. All other equilibrium circular orbits are unstable.

Let us consider an attractive central force with the form

$$F(r) = -\frac{k}{r^n} \tag{7.85}$$

The potential function for such a force is

$$U(r) = -\frac{k}{n-1}\cdot\frac{1}{r^{(n-1)}} \tag{7.86}$$

and the effective potential function is

$$V(r) = -\frac{k}{n-1}\cdot\frac{1}{r^{(n-1)}} + \frac{l^2}{2\mu r^2} \tag{7.87}$$

*R. L. Duncombe, *Astron. J. 61*, 174 (1956); see also G. M. Clemence, *Rev. Mod. Phys. 19*, 361 (1947).

The conditions for a minimum of $V(r)$ and hence for a stable circular orbit with a radius ρ are

$$\left.\frac{\partial V}{\partial r}\right|_{r=\rho} = 0 \quad \text{and} \quad \left.\frac{\partial^2 V}{\partial r^2}\right|_{r=\rho} > 0 \tag{7.88}$$

Applying these criteria to the effective potential of Equation 7.87, we have

$$\left.\frac{\partial V}{\partial r}\right|_{r=\rho} = \frac{k}{\rho^n} - \frac{l^2}{\mu\rho^3} = 0$$

or

$$\rho^{(n-3)} = \frac{\mu k}{l^2} \tag{7.89}$$

and

$$\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=\rho} = -\frac{nk}{\rho^{(n+1)}} + \frac{3l^2}{\mu\rho^4} > 0$$

so

$$-\frac{nk}{\rho^{(n-3)}} + \frac{3l^2}{\mu} > 0 \tag{7.90}$$

Substituting $\rho^{(n-3)}$ from Equation 7.89 into Equation 7.90, we have

$$(3-n)\frac{l^2}{\mu} > 0 \tag{7.91}$$

The condition that a stable circular orbit exist is thus $n < 3$.

Next we apply a more general procedure and inquire about the frequency of oscillation about a circular orbit in a general force field. We write the force as

$$F(r) = -\mu g(r) = -\frac{\partial U}{\partial r} \tag{7.92}$$

Equation 7.18 can now be written as

$$\ddot{r} - r\dot{\theta}^2 = -g(r) \tag{7.93}$$

Substituting for $\dot{\theta}$ from Equation 7.10,

$$\ddot{r} - \frac{l^2}{\mu^2 r^3} = -g(r) \tag{7.94}$$

We now consider the particle to be initially in a circular orbit with radius ρ and apply a perturbation of the form $r \to \rho + x$, where x is small. Since $\rho = \text{const.}$, we also have $\ddot{r} \to \ddot{x}$. Thus

$$\ddot{x} - \frac{l^2}{\mu^2\rho^3[1 + (x/\rho)]^3} = -g(\rho + x) \tag{7.95}$$

But by hypothesis $(x/\rho) \ll 1$, so we can expand the quantity:

$$[1 + (x/\rho)]^{-3} = 1 - 3(x/\rho) + \cdots \tag{7.96}$$

We also assume that $g(r) = g(\rho + x)$ can be expanded in a Taylor series about the point $r = \rho$:

$$g(\rho + x) = g(\rho) + xg'(\rho) + \cdots \tag{7.97}$$

where

$$g'(\rho) \equiv \left.\frac{dg}{dr}\right|_{r=\rho}$$

If we neglect all terms in x^2 and higher powers, then the substitution of Equations 7.96 and 7.97 into Equation 7.95 yields

$$\ddot{x} - \frac{l^2}{\mu^2\rho^3}[1 - 3(x/\rho)] \cong -[g(\rho) + xg'(\rho)] \tag{7.98}$$

Recall that we assumed the particle to be initially in a circular orbit with $r = \rho$. Under such a condition, no radial motion occurs—that is, $\dot{r}|_{r=\rho} = 0$. Then, also, $\ddot{r}|_{r=\rho} = 0$. Therefore, evaluating Equation 7.94 at $r = \rho$, we have

$$g(\rho) = \frac{l^2}{\mu^2\rho^3} \tag{7.99}$$

Substituting this relation into Equation 7.98, we have, approximately,

$$\ddot{x} - g(\rho)[1 - 3(x/\rho)] \cong -[g(\rho) + xg'(\rho)]$$

or

$$\ddot{x} + \left[\frac{3g(\rho)}{\rho} + g'(\rho)\right]x \cong 0 \tag{7.100}$$

If we define

$$\omega_0^2 \equiv \frac{3g(\rho)}{\rho} + g'(\rho) \tag{7.101}$$

then Equation 7.100 becomes the familiar equation for the undamped harmonic oscillator:

$$\ddot{x} + \omega_0^2 x = 0 \tag{7.102}$$

The solution to this equation is

$$x(t) = Ae^{+i\omega_0 t} + Be^{-i\omega_0 t} \tag{7.103}$$

If $\omega_0^2 < 0$ so that ω_0 is imaginary, then the second term becomes $B\exp(|\omega_0|t)$, which clearly increases without limit as time increases. The condition for oscillation is therefore $\omega_0^2 > 0$, or

$$\frac{3g(\rho)}{\rho} + g'(\rho) > 0 \tag{7.104a}$$

Since $g(\rho) > 0$ (see Equation 7.99), we can divide through by $g(\rho)$ and write this inequality as

$$\frac{g'(\rho)}{g(\rho)} + \frac{3}{\rho} > 0 \tag{7.104b}$$

or, since $g(r)$ and $F(r)$ are related by a constant multiplicative factor, stability results if

$$\boxed{\frac{F'(\rho)}{F(\rho)} + \frac{3}{\rho} > 0} \tag{7.105}$$

We now compare the condition on the force law imposed by Equation 7.105 with that previously obtained for a power-law force:

$$F(r) = -\frac{k}{r^n} \tag{7.106}$$

Equation 7.105 becomes

$$\frac{nk\rho^{-(n+1)}}{-k\rho^{-n}} + \frac{3}{\rho} > 0$$

or

$$(3 - n) \cdot \frac{1}{\rho} > 0 \tag{7.107}$$

and we are led to the same condition as before—that is, $n < 3$. (We must note, however, that the case $n = 3$ needs further examination; see Problem 7-24.)

EXAMPLE 7.5

Investigate the stability of circular orbits in a force field described by the potential function

$$U(r) = \frac{-k}{r} e^{-(r/a)} \tag{7.108}$$

where $k > 0$ and $a > 0$.

Solution: This potential is called the **screened Coulomb potential** (when $k = Ze^2$, where Z is the atomic number and e is the electron charge) since it falls off with distance more rapidly than $1/r$ and hence approximates the electrostatic potential of the atomic nucleus in the vicinity of the nucleus by taking into account the partial "cancellation" or "screening" of the nuclear charge by the atomic electrons. The force is found from

$$F(r) = -\frac{\partial U}{\partial r} = -k\left[\frac{1}{ar} + \frac{1}{r^2}\right]e^{-(r/a)}$$

and

$$\frac{\partial F}{\partial r} = k\left[\frac{1}{a^2 r} + \frac{2}{ar^2} + \frac{2}{r^3}\right]e^{-(r/a)}$$

The condition for stability (see Equation 7.105) is

$$3 + \rho\frac{F'(\rho)}{F(\rho)} > 0$$

Therefore

$$3 + \frac{\rho k\left[\frac{1}{a^2\rho} + \frac{2}{a\rho^2} + \frac{2}{\rho^3}\right]}{-k\left[\frac{1}{a\rho} + \frac{1}{\rho^2}\right]} > 0$$

which simplifies to

$$a^2 + a\rho - \rho^2 > 0$$

We may write this as

$$\frac{a^2}{\rho^2} + \frac{a}{\rho} - 1 > 0$$

Stability thus results for all $q \equiv a/\rho$ that exceed the value satisfying the equation

$$q^2 + q - 1 = 0$$

The positive (and therefore the only physically meaningful) solution is

$$q = \tfrac{1}{2}(\sqrt{5} - 1) \cong 0.62$$

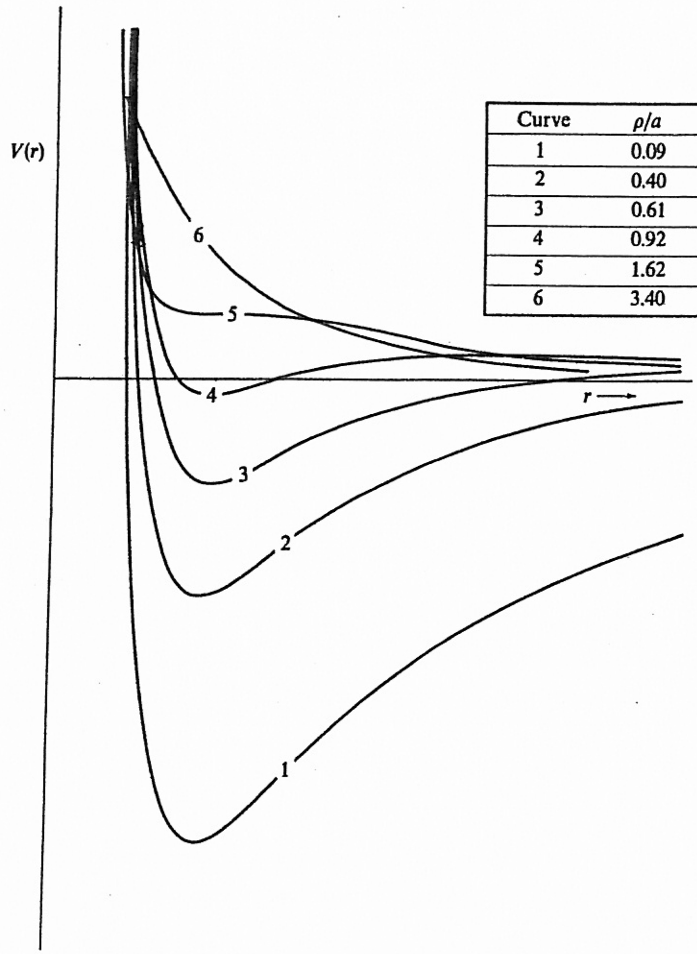
If, then, the angular momentum and energy allow a circular orbit at $r = \rho$, the motion is stable if

$$\frac{a}{\rho} \gtrsim 0.62$$

or

$$\rho \lesssim 1.62a \tag{7.109}$$

The stability condition for orbits in a screened potential is illustrated graphically in Figure 7-11, which shows the potential $V(r)$ for various values of ρ/a. The force constant k is the same for all of the curves, but $l^2/2\mu$ has been adjusted to maintain the minimum of the potential at the same value of the radius as a is changed. For $\rho/a < 1.62$ a true minimum exists for the potential, indicating that the circular orbit is stable with respect to small oscillations. For $\rho/a > 1.62$ there is no minimum, so circular orbits cannot exist. For $\rho/a = 1.62$ the potential has zero slope at the position that a circular orbit would occupy. The orbit



Curve	ρ/a	$l^2/2\mu$
1	0.09	$1.93V_0$
2	0.40	$1.82V_0$
3	0.61	$1.75V_0$
4	0.92	$1.50V_0$
5	1.62	$\equiv V_0$
6	3.40	$0.59V_0$

Figure 7-11

is unstable at this position, because ω_0^2 is zero in Equation 7.102 and the displacement x increases linearly with time.

An interesting feature of this potential function is that under certain conditions there can exist bound orbits for which the total energy is positive (see, for example, curve 4 in Figure 7-11).

EXAMPLE 7.6

Determine whether a particle moving on the surface of a cone (see Example 6.3) can have a stable circular orbit.

Solution: In Example 6.3 we found that the angular momentum about the z-axis was a constant of the motion:

$$l = mr^2\dot{\theta} = \text{const.}$$

We also found the equation of motion for the coordinate r:

$$\ddot{r} - r\dot{\theta}^2 \sin^2\alpha + g\sin\alpha\cos\alpha = 0 \tag{7.110}$$

If the initial conditions are appropriately selected, the particle can move in a circular orbit about the vertical axis with the plane of the orbit at a constant height z_0 above the horizontal plane passing through the apex of the cone. Although this problem does not involve a central force, certain aspects of the motion are the same as for the central-force case. Thus we may discuss, for example, the stability of circular orbits for the particle. To do this, we perform a perturbation calculation.

First we assume that a circular orbit exists for $r = \rho$. Then we apply the perturbation $r \to \rho + x$. The quantity $r\dot{\theta}^2$ in Equation 7.110 can be expressed as

$$\begin{aligned} r\dot{\theta}^2 &= r \cdot \frac{l^2}{m^2r^4} = \frac{l^2}{m^2r^3} \\ &= \frac{l^2}{m^2}(\rho + x)^{-3} = \frac{l^2}{m^2\rho^3}\left(1 + \frac{x}{\rho}\right)^{-3} \\ &\cong \frac{l^2}{m^2\rho^3}\left(1 - 3\frac{x}{\rho}\right) \end{aligned}$$

where we have retained only the first term in the expansion, since x/ρ is by hypothesis a small quantity.

Then, since $\ddot{\rho} = 0$, Equation 7.110 becomes, approximately,

$$\ddot{x} - \frac{l^2\sin^2\alpha}{m^2\rho^3}\left(1 - 3\frac{x}{\rho}\right) + g\sin\alpha\cos\alpha = 0$$

or

$$\ddot{x} + \left(\frac{3l^2\sin^2\alpha}{m^2\rho^4}\right)x - \frac{l^2\sin\alpha}{m^2\rho^3} + g\sin\alpha\cos\alpha = 0 \tag{7.111}$$

If we evaluate Equation 7.110 at $r = \rho$, then $\ddot{r} = 0$, and we have

$$\begin{aligned} g\sin\alpha\cos\alpha &= \rho\dot{\theta}^2\sin^2\alpha \\ &= \frac{l^2}{m^2\rho^3}\sin^2\alpha \end{aligned}$$

In view of this result, the last two terms in Equation 7.111 cancel, and there remains

$$\ddot{x} + \left(\frac{3l^2\sin^2\alpha}{m^2\rho^4}\right)x = 0 \tag{7.112}$$

The solution to this equation is just a harmonic oscillation with a frequency ω, where

$$\omega = \frac{\sqrt{3}\, l}{m\rho^2} \sin \alpha \tag{7.113}$$

Thus the circular orbit is stable.

7.11 Orbital Dynamics

The use of central-force motion is nowhere more useful, important, and interesting than in space dynamics. Although space dynamics is actually quite complex because of the gravitational attraction of a spacecraft to various bodies and the orbital motion involved, we examine two rather simple aspects: a proposed trip to Mars and flybys past comets and planets.

Orbits are changed by single or multiple thrusts of the rocket engines. The simplest maneuver is a single thrust applied in the orbital plane that does not change the direction of the angular momentum but does change the eccentricity and energy simultaneously. The most economical method of interplanetary transfer consists of moving from one circular heliocentric (sun-oriented motion) orbit to another in the same plane. Earth and Mars represent such a system reasonably well, and a Hohmann transfer (Figure 7-12) represents the path of minimum total energy expenditure.* Two engine burns are required: (1) the first burn injects the spacecraft from the circular earth orbit to an elliptical transfer orbit that intersects Mars' orbit; (2) the second burn transfers the spacecraft from the elliptical orbit into Mars' orbit.

We can calculate the velocity changes needed for a Hohmann transfer by calculating the velocity of a spacecraft moving in the orbit of the earth around the sun (r_1 in Figure 7-12) and the velocity needed to "kick" it into an elliptical transfer orbit that can reach Mars' orbit. We are considering only the gravitational attraction of the sun and not that of the earth and Mars.

For circles and ellipses we have, from Equation 7.42,

$$E = \frac{-k}{2a} \tag{7.114}$$

For a circular path around the sun, this becomes

$$E = \frac{-k}{2r_1} = \frac{1}{2} m v_1^2 - \frac{k}{r_1} \tag{7.115}$$

*See Kaplan (Ka76, Chapter 3) for the proof. Walter Hohmann, a German pioneer in space travel research, proposed in 1925 the most energy-efficient method of transferring between elliptical (planetary) orbits in the same plane using only two velocity changes.

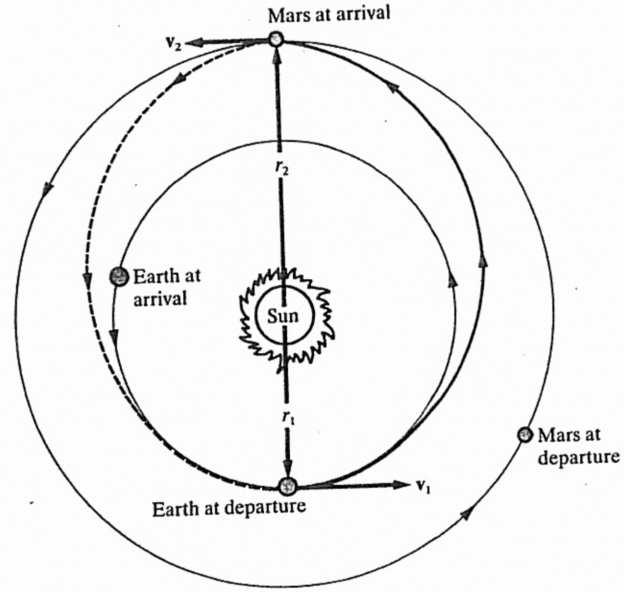


Figure 7-12

where we have $E = T + U$. We solve Equation 7.115 for v_1:

$$v_1 = \sqrt{\frac{k}{m r_1}} \tag{7.116}$$

We denote the semimajor axis of the transfer ellipse by a_t:

$$2a_t = r_1 + r_2 \tag{7.117}$$

If we calculate the energy at the perihelion for the transfer ellipse, we have

$$E_t = \frac{-k}{r_1 + r_2} = \frac{1}{2} m v_{t1}^2 - \frac{k}{r_1} \tag{7.118}$$

where v_{t1} is the perihelion transfer velocity. The direction of v_{t1} is along v_1 in Figure 7-12. Solving Equation 7.118 for v_{t1} gives

$$v_{t1} = \sqrt{\frac{2k}{m r_1}\left(\frac{r_2}{r_1 + r_2}\right)} \tag{7.119}$$

The velocity transfer Δv_1 needed is just

$$\Delta v_1 = v_{t1} - v_1 \tag{7.120}$$

Similarly, for the transfer from the ellipse to the circular orbit of radius r_2, we have

$$\Delta v_2 = v_2 - v_{t2} \tag{7.121}$$

where

$$v_2 = \sqrt{\frac{k}{mr_2}} \tag{7.122}$$

and

$$v_{t2} = \sqrt{\frac{2}{m}\left(E_t + \frac{k}{r_2}\right)}$$

$$v_{t2} = \sqrt{\frac{2k}{mr_2}\left(\frac{r_1}{r_1 + r_2}\right)} \tag{7.123}$$

The direction of v_{t2} is along v_2 in Figure 7-12. The total speed increment can be determined by adding the speed changes, $\Delta v = \Delta v_1 + \Delta v_2$.

The total time required to make the transfer T_t is a half-period of the transfer orbit. From Equation 7.48, we have

$$T_t = \frac{\tau_t}{2}$$

$$T_t = \pi\sqrt{\frac{m}{k}}\,a_t^{\frac{3}{2}} \tag{7.124}$$

EXAMPLE 7.7

Calculate the time needed for a spacecraft to make a Hohmann transfer from the earth to Mars and the heliocentric transfer speed required assuming both planets are in coplanar orbits.

Solution: We need to insert the appropriate constants into Equation 7.124.

$$\frac{m}{k} = \frac{m}{GmM_{\text{sun}}} = \frac{1}{GM_{\text{sun}}}$$

$$= \frac{1}{(6.67 \times 10^{-11}\ \text{m}^3/\text{s}^2 \cdot \text{kg})(1.99 \times 10^{30}\ \text{kg})}$$

$$= 7.53 \times 10^{-21}\ \text{s}^2/\text{m}^3 \tag{7.125}$$

Since k/m occurs so often in solar system calculations, we write it as well.

$$\frac{k}{m} = 1.33 \times 10^{20}\ \text{m}^3/\text{s}^2 \tag{7.126}$$

$$a_t = \tfrac{1}{2}(r_{\text{earth-sun}} + r_{\text{Mars-sun}})$$

$$= \tfrac{1}{2}(1.50 \times 10^{11}\ \text{m} + 2.28 \times 10^{11}\ \text{m})$$

$$= 1.89 \times 10^{11}\ \text{m}$$

$$T_t = \pi(7.53 \times 10^{-21}\ \text{s}^2/\text{m}^3)^{\frac{1}{2}}(1.89 \times 10^{11}\ \text{m})^{\frac{3}{2}}$$

$$= 2.24 \times 10^7\ \text{s}$$

$$= 259\ \text{days}$$

The heliocentric speed needed for the transfer is given in Equation 7.119.

$$v_{t1} = \left[\frac{2(1.33 \times 10^{20}\ \text{m}^3/\text{s}^2)(2.28 \times 10^{11}\ \text{m})}{(1.50 \times 10^{11}\ \text{m})(3.78 \times 10^{11}\ \text{m})}\right]^{\frac{1}{2}}$$

$$= 3.27 \times 10^4\ \text{m/s} = 32.7\ \text{km/s}$$

We can compare v_{t1} to the orbital speed of the earth (Equation 7.116).

$$v_1 = \left[\frac{1.33 \times 10^{20}\ \text{m}^3/\text{s}^2}{1.50 \times 10^{11}\ \text{m}}\right]^{\frac{1}{2}} = 29.8\ \text{km/s}$$

It is important to note that for transfers to the outer planets, the spacecraft should be launched in the direction of the earth's orbit in order to gain the earth's orbital velocity. To transfer to the inner planets (e.g., to Venus), the spacecraft should be launced opposite the earth's motion. In each case it is the relative velocity Δv_1 that is important to the spacecraft (i.e., relative to the earth).

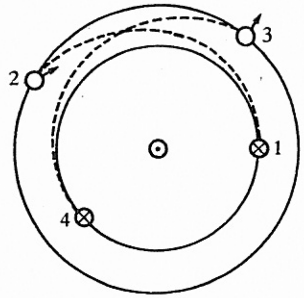
Although the Hohmann transfer path represents the least energy expenditure, it does not represent the shortest time. For a round trip from the earth to Mars, the spacecraft would have to remain on Mars for 460 days until the earth and Mars were positioned correctly for the return trip (see Figure 7-13a). The total trip (259 + 460 + 259 = 978 days = 2.7 yr.) would probably be too long. Other schemes use more fuel to gain velocity by the slingshot effect of flybys. Such a flyby mission past Venus (see Figure 7-13c) could be done in less than two years with only a few weeks near (or on)Mars.

Several spacecraft in recent years have escaped the earth's gravitational attraction to explore our solar system. Such **interplanetary transfer** can be divided into three segments: (1) the escape from the earth, (2) a heliocentric transfer to the area of interest, and (3) an encounter with another body—so far, either a planet or a comet. The spacecraft fuel required for such missions can be enormous, but a clever trick has been designed to "steal" energy from other solar system bodies. Since the mass of a spacecraft is so much smaller than the planets (or their moons), the energy loss of the heavenly body is negligible.

We examine a simple version of this flyby or slingshot effect. A spacecraft coming from infinity approaches a body (labeled B), interacts with B, and recedes. The path is a hyperbola (Figure 7-14). The initial and final velocities, *with respect to B*, are denoted by v_i' and v_f', respectively. The net effect on the spacecraft is a deflection angle of δ with respect to B.

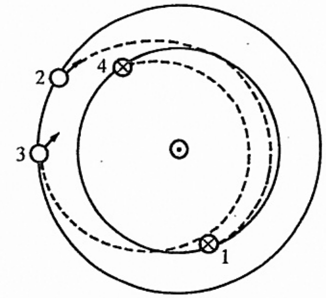
If we examine the system in some inertial frame in which the motion of B occurs, the velocities of the spacecraft can be quite different *because of the motion*

1. Earth departure
2. Mars arrival
3. Mars departure
4. Earth arrival



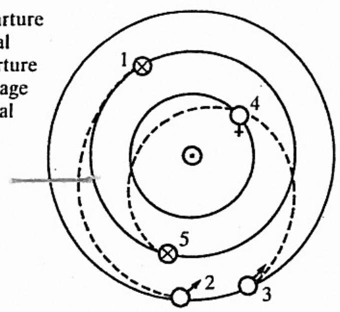
(a) Minimum energy mission requires long stayover on Mars before returning to Earth.

1. Earth departure
2. Mars arrival
3. Mars departure
4. Earth arrival



(b) Shorter mission requires more fuel and a closer orbit to the sun.

1. Earth departure
2. Mars arrival
3. Mars departure
4. Venus passage
5. Earth arrival



(c) The shorter mission of (b) can be further improved if Venus is positioned for a gravity assist during flyby.

Figure 7-13

of B. The initial velocity v_i is shown in Figure 7-15a, and both v_i and v_f are shown in Figure 7-15b. Notice that the spacecraft has increased its speed as well as changed its direction. An increase in velocity occurs when the spacecraft passes *behind* B's direction of motion. Similarly, a decrease in velocity occurs when the spacecraft passes in *front* of B's motion.

During the 1970s scientists at the Jet Propulsion Laboratory of the National Aeronautics and Space Administration (NASA) realized that the four largest planets of our solar system would be in a fortuitous position to allow a spacecraft to fly past them and many of their 32 known moons in a single, relatively short "Grand Tour" mission using the gravity-assist method just discussed. This

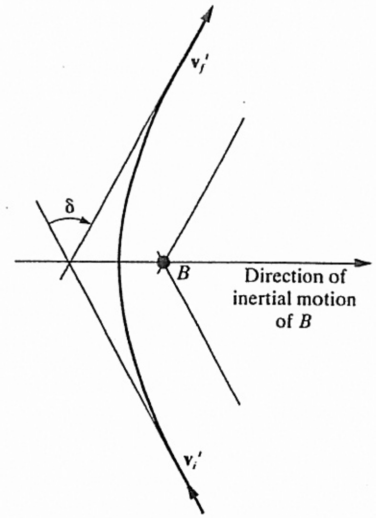


Figure 7-14

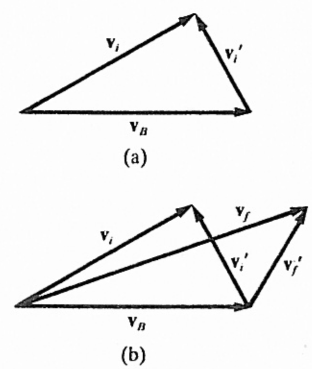


Figure 7-15

opportunity of the planets' alignment would not occur again for 175 years. Because of budget constraints, there was not time to develop the new technology needed, and a mission to last only four years to visit just Jupiter and Saturn was approved and planned. No special equipment was put on board the twin Voyager spacecrafts for an encounter with Uranus and Neptune. Voyagers 1 and 2 were launched in 1977 for visits to Jupiter in 1979 and Saturn in 1980 (Voyager 1) and 1981 (Voyager 2). Because of the success of these visits to Jupiter and Saturn, funding was later approved to extend Voyager 2's mission to include Uranus and

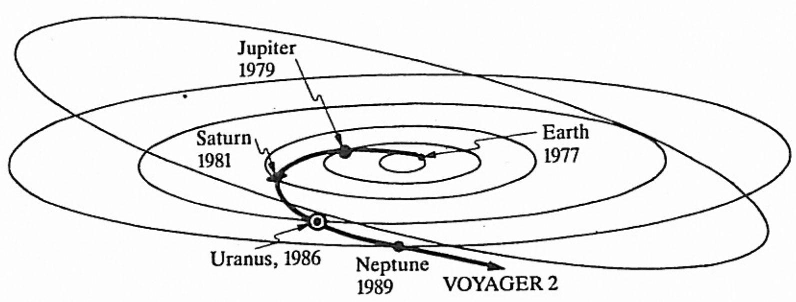


Figure 7-16

Neptune. Voyager 1 is now on its way out of our solar system.

The path of Voyager 2 is shown in Figure 7-16. The slingshot effect of gravity allowed the path of Voyager 2 to be redirected, for example, toward Uranus as it passed Saturn by the method shown in Figure 7-14. The gravitational attraction from Saturn was used to pull the spacecraft off its straight path and redirect it at a different angle. The effect of the orbital motion of Saturn allows an increase in the spacecraft's speed. It was only by using this gravity-assist technique that the spectacular mission of Voyager 2 was made possible in only a brief 12-year period. Voyager 2 passed Uranus early in 1986 and encounters Neptune in 1989 before proceeding into interstellar space in one of the most successful space missions ever undertaken.

A spectacular display of flybys occurred in the years 1982–1985 by a spacecraft initially called the International Sun–Earth Explorer 3 (ISEE-3). Launched in 1978, its mission was to monitor the solar wind between the sun and

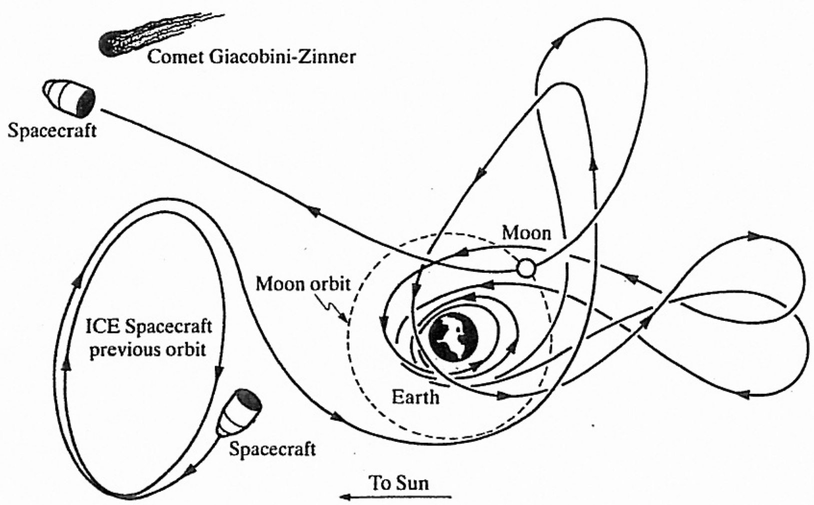


Figure 7-17

the earth. For four years the spacecraft circled in the ecliptical plane about 2 million miles from the earth. In 1982—because the United States had decided not to participate in a joint European, Japanese, and Soviet spacecraft investigation of Halley's comet in 1986—NASA decided to reprogram the ISEE-3, rename it the International Cometary Explorer (ICE), and send it through the Giacobini–Zinner comet in September 1985, some six months before the flybys of other spacecraft with Halley's comet. The subsequent three-year journey of ICE was spectacular (Figure 7-17). The path of ICE included two close trips to the earth and five flybys of the moon along its billion-mile trip to the comet. During one flyby the satellite came within 75 miles of the lunar surface. The entire path could be planned precisely because the force law is very well known. The eventual interaction with the comet, some 44 million miles from the earth, included a 20-minute trip through the comet—about 5,000 miles behind the comet's nucleus.

Problems

7-1. In Section 7.2 we showed that the motion of two bodies interacting only with each other by central forces could be reduced to an equivalent one-body problem. Show by explicit calculation that such a reduction is also possible for bodies moving in an external uniform gravitational field.

7-2. Perform the integration of Equation 7.38 to obtain Equation 7.39.

7-3. A particle moves in a circular orbit in a force field given by

$$F(r) = -k/r^2$$

Show that, if k suddenly decreases to half its original value, the particle's orbit becomes parabolic.

7-4. Perform an explicit calculation of the time average (i.e., the average over one complete period) of the potential energy for a particle moving in an elliptical orbit in a central inverse-square-law force field. Express the result in terms of the force constant of the field and the semimajor axis of the ellipse. Perform a similar calculation for the kinetic energy. Compare the results and thereby verify the virial theorem for this case.

7-5. Two particles moving under the influence of their mutual gravitational force describe circular orbits about one another with a period τ. If they are suddenly stopped in their orbits and allowed to gravitate toward each other, show that they will collide after a time $\tau/4\sqrt{2}$.

7-6. Two gravitating masses m_1 and m_2 ($m_1 + m_2 = M$) are separated by a distance r_0 and released from rest. Show that when the separation is $r(<r_0)$, the velocities are

$$v_1 = m_2\sqrt{\frac{2G}{M}\left(\frac{1}{r}-\frac{1}{r_0}\right)}, \qquad v_2 = m_1\sqrt{\frac{2G}{M}\left(\frac{1}{r}-\frac{1}{r_0}\right)}$$

7-7. Show that the areal velocity is constant for a particle moving under the influence of an attractive force given by $F(r) = -kr$. Calculate the time averages of the kinetic and potential energies and compare with the results of the virial theorem.

7-8. Investigate the motion of a particle *repelled* by a force center according to the law $F(r) = kr$. Show that the orbit can only be hyperbolic.

7-9. A communications satellite is in a circular orbit around the earth at radius R and velocity v. A rocket accidentally fires quite suddenly, giving the rocket an outward radial velocity v in addition to its original velocity.
(a) Calculate the ratio of the new energy and angular momentum to the old.
(b) Describe the subsequent motion of the satellite and plot $T(r)$, $V(r)$, $U(r)$, and $E(r)$ after the rocket fires.

7-10. Assume that the earth's orbit to be circular and that the sun's mass suddenly decreases by half. What orbit does the earth then have? Will the earth escape the solar system?

7-11. A particle moves under the influence of a central force given by $F(r) = -k/r^n$. If the particle's orbit is circular and passes through the force center, show that $n = 5$.

7-12. Consider a comet moving in a parabolic orbit in the plane of the earth's orbit. If the distance of closest approach of the comet to the sun is βr_e, where r_e is the radius of the earth's (assumed) circular orbit and where $\beta < 1$, show that the time the comet spends within the orbit of the earth is given by

$$\sqrt{2(1-\beta)} \cdot (1 + 2\beta)/3\pi \times 1 \text{ year}$$

If the comet approaches the sun to the distance of the perihelion of Mercury, how many days is it within the earth's orbit?

7-13. Discuss the motion of a particle in a central inverse-square-law force field for a superimposed force whose magnitude is inversely proportional to the cube of the distance from the particle to the force center; that is,

$$F(r) = -\frac{k}{r^2} - \frac{\lambda}{r^3}, \qquad k, \lambda > 0$$

Show that the motion is described by a precessing ellipse. Consider the cases $\lambda < l^2/\mu$, $\lambda = l^2/\mu$, and $\lambda > l^2/\mu$.

7-14. Find the force law for a central-force field that allows a particle to move in a spiral orbit given by $r = k\theta^2$, where k is a constant.

7-15. A particle of unit mass moves from infinity along a straight line that, if continued, would allow it to pass a distance $b\sqrt{2}$ from a point P. If the particle is attracted toward P with a force varying as k/r^5, and if the angular momentum about the point P is $\sqrt{k}/b$, show that the trajectory is given by

$$r = b \coth(\theta/\sqrt{2})$$

7-16. A particle executes elliptical (but almost circular) motion about a force center. At some point in the orbit a *tangential* impulse is applied to the particle, changing the velocity from v to $v + \delta v$. Show that the resulting relative change in the major and minor axes of the orbit is twice the relative change in the velocity and that the axes are *increased* if $\delta v < 0$.

7-17. A particle moves in an elliptical orbit in an inverse-square-law central-force field. If the ratio of the maximum angular velocity to the minimum angular velocity of the particle in its orbit is n, then show that the eccentricity of the orbit is

$$\varepsilon = \frac{\sqrt{n} - 1}{\sqrt{n} + 1}$$

7-18. Use Kepler's results (i.e., his first and second laws) to show that the gravitational force must be central and that the radial dependence must be $1/r^2$. Thus, perform an inductive derivation of the gravitational force law.

7-19. Calculate the missing entries denoted by c in Table 7-1.

7-20. Show that the product of the maximum and minimum (linear) velocities of a body moving in an elliptical orbit is $(2\pi a/\tau)^2$.

7-21. If η is defined as the angle between a planet's direction of motion (in an elliptical orbit) and the direction perpendicular to the planet's radius vector, show that

$$\tan \eta = \frac{\varepsilon \sin \psi}{\sqrt{1 - \varepsilon^2}}$$

where ψ is the eccentric anomaly.

7-22. For a particle moving in an elliptical orbit with semimajor axis a and eccentricity ε, show that

$$\langle (a/r)^4 \cos \theta \rangle = \varepsilon/(1 - \varepsilon^2)^{\frac{5}{2}}$$

where the slanted brackets denote a time average over one complete period.

7-23. Consider the family of orbits in a central potential for which the total energy is a constant. Show that if a stable circular orbit exists, the angular momentum associated with this orbit is larger than that for any other orbit of the family.

7-24. Discuss the motion of a particle moving in an attractive central force field described by $F(r) = -k/r^3$.* Sketch some of the orbits for different values of the total energy. Can a circular orbit be stable in such a force field?

7-25. An earth satellite moves in an elliptical orbit with a period τ, eccentricity ε, and semimajor axis a. Show that the maximum radial velocity of the satellite is $2\pi a\varepsilon/(\tau\sqrt{1-\varepsilon^2})$

7-26. An earth satellite has a perigee of 300 km and an apogee of 3500 km above the earth's surface. How far is the satellite above the earth when (a) it has rotated 90° around the earth from perigee and (b) it has moved halfway from perigee to apogee?

7-27. An earth satellite has a speed of 25,000 km/hr when it is at its perigee of 220 km above the earth's surface. Find the apogee distance, its speed at apogee, and its period of revolution.

7-28. Show that the most efficient way to change the energy of an elliptical orbit for a single short engine thrust is by firing the rocket along the direction of travel at perigee.

7-29. A spacecraft in an orbit about earth has the speed of 10,160 m/s at a perigee of 6680 km above the earth's surface. What speed does the spacecraft have at apogee of 42,200 km?

7-30. What is the minimum escape velocity of a spacecraft from the moon?

7-31. The minimum and maximum velocities of a moon rotating around Uranus are $v_{min} = v - v_0$ and $v_{max} = v + v_0$.
(a) Find the eccentricity in terms of v and v_0.
(b) Show that $v_{min}v_{max} = (2\pi a/\tau)^2$.

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*This particular force law was extensively investigated by Roger Cotes (1682–1716), and the orbits are known as **Cotes' spirals**.

7-32. A spacecraft is placed in orbit 200 km above the earth in a circular orbit. Calculate the minimum escape speed from the earth. Sketch the escape trajectory, showing the earth and the circular orbit. What is the spacecraft's trajectory with respect to the earth?

7-33. Consider a force law of the form

$$F(r) = -\frac{k}{r^2} - \frac{k'}{r^4}$$

Show that if $\rho^2 k > k'$, then a particle can move in a stable circular orbit at $r = \rho$.

7-34. Consider a force law of the form $F(r) = -(k/r^2)\exp(-r/a)$. Investigate the stability of circular orbits in this force field.

7-35. Consider a particle of mass m constrained to move on the surface of a paraboloid whose equation (in cylindrical coordinates) is $r^2 = 4az$. If the particle is subject to a gravitational force, show that the frequency of small oscillations about a circular orbit with radius $\rho = \sqrt{4az_0}$ is

$$\omega = \sqrt{\frac{2g}{a + z_0}}$$

7-36. Consider the problem of the particle moving on the surface of a cone, as discussed in Examples 6.3 and 7.6. Show that the effective potential is

$$V(r) = \frac{l^2}{2mr^2} + mgr \cot \alpha$$

(Note that here r is the radial distance in cylindrical coordinates, not spherical coordinates; see Figure 6-1.) Show that the turning points of the motion can be found from the solution of a cubic equation in r. Show further that only two of the roots are physically meaningful, so that the motion is confined to lie within two horizontal planes that cut the cone.

7-37. An almost circular orbit (i.e., $\varepsilon \ll 1$) can be considered to be a circular orbit to which a small perturbation has been applied. Then the frequency of the radial motion is given by Equation 7.101. Consider a case where the force law is $F(r) = -k/r^n$ (where n is an integer), and show that the apsidal angle is $\pi/\sqrt{3-n}$. Thus, show that a closed orbit generally results only for the harmonic oscillator force and the inverse-square-law force (if values of n equal to or smaller than -6 are excluded).

7-38. A particle moves in an almost circular orbit in a force field described by $F(r) = -(k/r^2)\exp(-r/a)$. Show that the apsides advance by an amount approximately equal to $\pi\rho/a$ in each revolution, where ρ is the radius of the circular orbit and where $\rho \ll a$.

7-39. A communications satellite is in a circular orbit around the earth at a distance above the earth equal to the earth's radius. Find the minimum velocity Δv required to double the height of the satellite and put it in another circular orbit.

7-40. Calculate the minimum Δv required to place a satellite already in the earth's heliocentric orbit (assumed circular) into the orbit of Venus (also assumed circular and coplanar with the earth). Consider only the gravitational attraction of the sun. What time of flight would such a trip take?

7-41. Assuming a rocket engine can be fired only once from a low earth orbit, does a Mars flyby or a Venus flyby require a larger Δv? Explain.

7-42. A spacecraft is being designed to dispose of nuclear waste either by carrying it out of

the solar system or crashing into the sun. Assume that no planetary flybys are permitted and that thrusts occur only in the orbital plane. Which mission requires the least energy? Explain.

7-43. A spacecraft is parked in a circular orbit 200 km above the earth's surface. We want to use a Hohmann transfer to send the spacecraft to the moon's orbit. What are the total Δv and the transfer time required?

7-44. A spacecraft of mass 10,000 kg is parked in a circular orbit 200 km above the earth's surface. What is the minimum energy required (neglect the fuel mass burned) to place the satellite in a synchronous orbit (i.e., $\tau = 24$ hr.)?