2.5 Coordinates and Line Element

The Euclidean Geometry of a Plane

A systematic way of labeling points is a prerequisite to a specification of the distance between nearby ones. A system of coordinates assigns unique labels to each point, and there are many systems that do so. In two dimensions, for instance, there are Cartesian coordinates (x, y), polar coordinates (r, ϕ) about some origin, etc. (Figure 2.5).

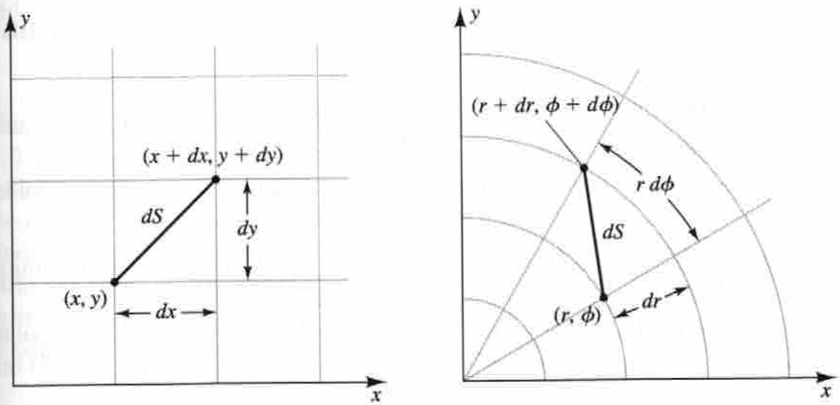

FIGURE 2.5 Cartesian and polar coordinates. Cartesian and polar coordinates are both systematic ways of labeling points in the plane, and the distance between nearby points can be expressed in terms of either.

Nearby points have nearby values of their coordinates. For example, the points (x, y) and $(x+dx, y+dy)$ are nearby when dx and dy are infinitesimal. Similarly, (r, ϕ) and $(r + dr, \phi + d\phi)$ are nearby.

In Cartesian coordinates (x, y), the distance dS between the points (x, y) and $(x + dx, y + dy)$ is (see Figure 2.5)

$$dS = \left[(dx)^2 + (dy)^2\right]^{1/2}. \tag{2.7}$$

The same rule can be expressed in polar coordinates where the distance between the nearby points (r, ϕ) and $(r + dr, \phi + d\phi)$ is (see Figure 2.5)

$$dS = \left[(dr)^2 + (r\,d\phi)^2\right]^{1/2}. \tag{2.8}$$

Expression (2.8) and others like it are valid only if dr and $d\phi$ are small. However, large distances can be built up from these infinitesimal relations by integration. Let's, for example, calculate the ratio of the circumference to the diameter of a circle of radius R. Choosing the origin at the center, the equation for such a circle in Cartesian coordinates is

$$x^2 + y^2 = R^2. \tag{2.9}$$

The circumference C is the integral of dS around the circle. Using (2.7) this is

$$C = \oint dS = \oint \left[(dx)^2 + (dy)^2\right]^{1/2} \tag{2.10a}$$

$$= 2\int_{-R}^{+R} dx \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}_{x^2+y^2=R^2} \tag{2.10b}$$

$$= 2\int_{-R}^{+R} dx \sqrt{\frac{R^2}{R^2 - x^2}}. \tag{2.10c}$$

Changing variables by writing $x = R\xi$, we have

$$C = 2R\int_{-1}^{1} \frac{d\xi}{\sqrt{1-\xi^2}} = 2\pi R. \tag{2.11}$$

This is the correct answer. The integral could even be taken to define π; by doing it numerically, one could discover that $\pi = 3.1415926535\ldots$.

Deriving the relation between radius and circumference is even easier in polar coordinates, where the equation of the circle is just $r = R$. Evaluating (2.8) on the circle and integrating the resulting dS over it gives

$$C = \oint dS = \int_0^{2\pi} R\,d\phi = 2\pi R. \tag{2.12}$$

The ease of using polar coordinates to arrive at (2.12) shows that, for a given problem, some coordinates are better than others.

By proceeding in this way we could derive all the theorems of Euclidean plane geometry. The angle between two intersecting lines, for example, can be defined as the ratio of the length ΔC of the part of a circle centered on their intersection that lies between the lines to the circle's radius R.

$$\theta \equiv \frac{\Delta C}{R} \qquad \text{(radians)}. \tag{2.13}$$

With this definition we could prove that the sum of the interior angles of a triangle is π. Indeed, we could verify the *axioms* of Euclidean plane geometry from (2.7) or (2.8). All geometry can be reduced to relations between distances; all distances can be reduced to integrals of distances between nearby points; all Euclidean plane geometry is contained in (2.7) or (2.8).

Line Element

To summarize, a geometry is specified by the *line element*, such as (2.7) or (2.8), which gives the distance between nearby points in terms of the coordinate intervals between them in some coordinate system. Conventionally, a line element is written as a quadratic relation for dS^2, e.g.,

$$dS^2 = dx^2 + dy^2 \tag{2.14}$$

with no brackets around the differentials. The form of the line element for a geometry varies from coordinate system to coordinate system [e.g., (2.7) and (2.8)], but the geometry remains the same.

The Non-Euclidean Geometry of a Sphere

An example of a non-Euclidean geometry is provided by the surface of a two-dimensional sphere of radius a. We can use the angles (θ, ϕ) of three-dimensional polar coordinates to label points on the sphere. The distance between points (θ, ϕ) and $(\theta + d\theta, \phi + d\phi)$ can be seen after a little work (Figure 2.6) to be

Line Element for a Sphere

$$dS^2 = a^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{2.15}$$

This is the line element of the surface of a sphere.

Let's use the line element (2.15) to calculate the ratio of the circumference to the radius of a circle on the sphere. By circle we mean the locus of points *on the surface* that are a constant distance (the radius) *along the surface* from a fixed point (the center) *in the surface*. Since no one point is distinguished geometrically from any other on the sphere, we may conveniently orient our polar coordinate system so that the polar axis is at the center of the circle. A circle is then a curve of constant θ. Consider the circle defined by the equation

$$\theta = \Theta \tag{2.16}$$

for constant Θ. The circumference is the distance around this curve. Nearby points along the curve are separated by $d\phi$ but have $d\theta = 0$. Thus, (2.15) gives $dS =$

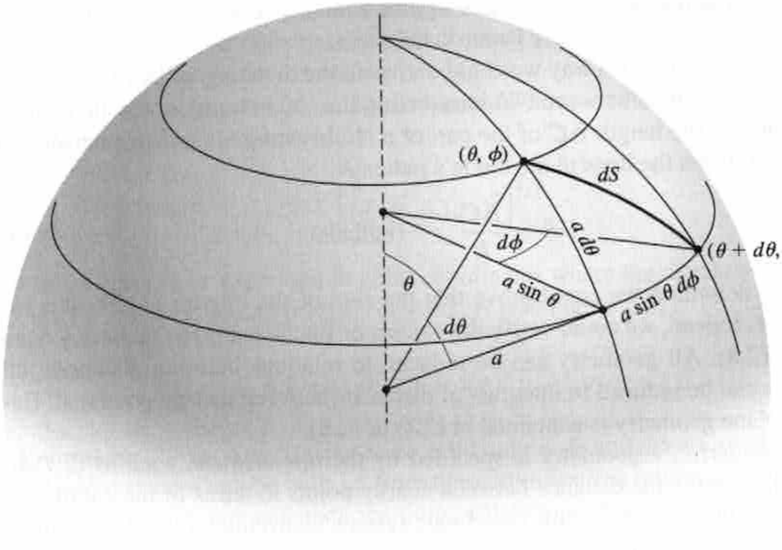

FIGURE 2.6 Deriving the line element on the sphere. The derivation makes use of the fact that the two-dimensional sphere is a surface in three-dimensional Euclidean space. Two infinitesimally separated points at locations (θ, ϕ) and $(\theta + d\theta, \phi + d\phi)$ are indicated. The construction shows that the distance between ϕ and $\phi + d\phi$ along a line of constant latitude θ is $a \sin\theta\, d\phi$. The distance between θ and $\theta + d\theta$ along a line of constant longitude is $a\, d\theta$. Because the θ and ϕ coordinate lines are orthogonal, the sum of the squares of these two differentials gives the square of the distance dS between the two points when $d\theta$ and $d\phi$ are infinitesimally small. This gives (2.15).

$a \sin\Theta\, d\phi$ along the circle, and the circumference is

$$C = \oint dS = \int_0^{2\pi} a \sin\Theta\, d\phi = 2\pi a \sin\Theta. \tag{2.17}$$

The radius is the distance from the center to the circle along a curve for which θ varies but $d\phi = 0$. Along this curve, (2.15) gives $dS = a\, d\theta$, and the radius is

$$r = \int_{\text{center}}^{\text{circle}} dS = \int_0^{\Theta} a\, d\theta = a\Theta. \tag{2.18}$$

Using (2.18) to eliminate Θ in (2.17), the relation between the circumference and radius of a circle in the non-Euclidean geometry of a sphere becomes

$$C = 2\pi a \sin\left(\frac{r}{a}\right). \tag{2.19}$$

In this expression a is a fixed number characterizing the geometry. It measures the scale on which the geometry is curved. When the radius of the circle is much smaller than the radius of the sphere, $r \ll a$, then we have approximately

$$C \approx 2\pi r, \tag{2.20}$$

which is the familiar result in Euclidean geometry. The geometry of the surface of the Earth is the same as a sphere to a good approximation.

The many different projections used to make maps of its surface are just different coordinate systems for expressing the geometry of a sphere as described in Box 2.3.