Gravitational Waves Block 4, 2022 Reflection

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December 22, 2022

1 Introduction

This block I worked on simulating the gravitational wave signal from a binary star system. First, I used work done by Hugo Wahlquist to animate the gravitational wave signal as a function of the orbital angle. This was done for an arbitrarily oriented binary star system, and could accept any numerically feasible star masses, orbital eccentricity, and orbital period. Next, I numerically solved Keppler's ψ-function to relate the orbital angle to time, which I then used to determine the time dependent gravitational wave signal. In this way, I was able to generate a realistic signal that we would observe on Earth due to a distant binary system; however, this solution assumed a static system. In reality, gravitational waves carry energy, and so the binary system loses energy over time. Using work done by P.C. Peters, I was able to account for the energy lost in the binary system over a relatively short time period, but I could not achieve a full time evolution solution because of numerical and computational constraints.

In my final paper and presentation, I will give a short conceptual introduction to gravitational waves, but mostly quote important results which use general relativity. This is simply because I do not have the requisite knowledge to give a deeper description of these phenomena; that said, I will review derivations which use only classical mechanics where it seems appropriate.

2 Gravitational Wave Background

In our everyday lives where objects move much slower than the speed of light, we are used to speaking of the distance between objects in three dimensions, and this distance is useful because it is the same for any frame of reference. However, it is a consequence of the Special Theory of Relativity that this three dimensional distance is not the same for all observers; in fact, the distance is smaller for observers moving at high velocities. To account this change in distance, the moving observer must experience less time than a stationairy observer so that physics is valid in both frames: in fact, the quantity:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

Is the same for both observers. For this reason, we use it as a replacement for our conventional sense of distance in the spacetime of relativity.

Similar to how distances on Earth are warped because we must travel on the surface of earth, General Relativity predicts that the distance between objects in spacetime is warped due gravity. The cause of gravity is mass, and we conclude that massive objects cause spacetime how to curve.

In electrodynamics, we know that the cause of electric and magnetic fields are charged particle; however, a moving charge distribution can produce a propagating electromagnetic wave (light) whose fields can have a large effect on objects far from the original charged particle. In an analogous fashion, the warping of spacetime due to a moving, asymmetric mass distribution can propagate as gravitational waves, significantly effecting the spacetime distance between objects far from the original mass distribution which produced these waves.

Gravitational waves come in two linearly indpendent polarizations: if the first effects the distance of objects upwards and sideways, the second effects objects at 45°. These polarizations are termed h_+ and h_x.

It is interesting to notice that, while these polarizations are independent, there names are not really important: we can always rotate our coordinate system 45°, and our names have switched!

3 The Wahlquist Equations

One moving, asymmetrical mass distribution which should produce gravitational waves is a binary star system. To determine the gravitational wave signal we must consider how to describe such a system. First, there is the problem of orientation: from earth, the orbital plane of a binary star system might be tilted and rotated in any way. We define the inclination i to be the angle from the normal plane of the position vector of the system relative to earth to the orbital plane of the system and ϕ to be the orientation of the line defined as the intersection of the orbital plane with the position normal. Additionally, we must consider how the orbit is oriented within the orbital plane: if we track the orbit angle θ, then we define a constant θ_p to be the angle at pariapsis and θ_n to be the angle of the nodes. In general, we can always choose one angle to be zero, but the other will be nonzero. Other parameters we must consider to describe the graviational wave are the masses of the stars, m_1 and m_2, the eccentricity ε of the orbit, and the distance from Earth to the binary stars, R.

Using linearized Einstein theory of gravitational waves, Wahlquist was able to determine the h_p and h_x gravitational wave signals as functions of the orbital position θ and parameters introduced above:

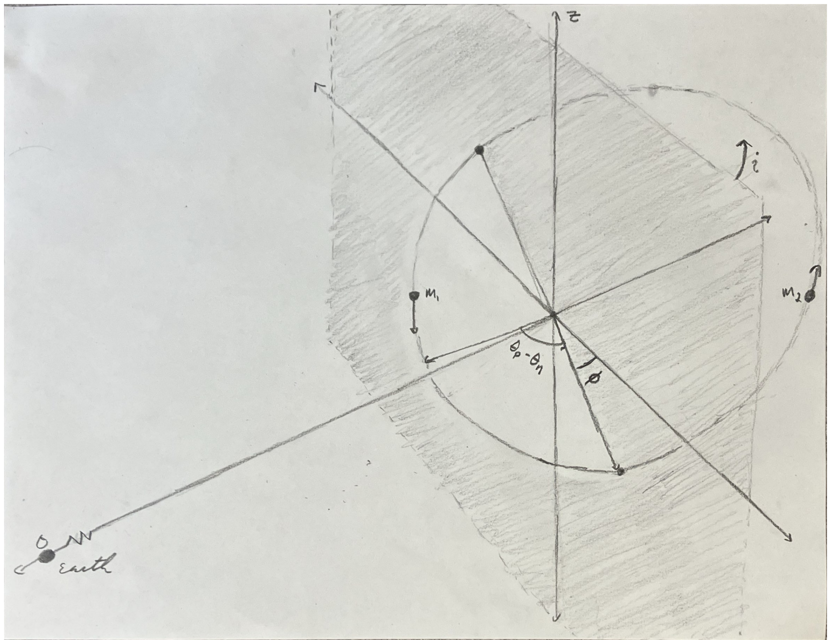

Figure 1: Sketching showing the orientation angles included in the Wahlquist equations. The shaded region is orthogonal to the position vector of the orbit relative to earth; the inclination is the angle between this plane and the orbital plane; ϕ describes the orientation of the node line, the line where the orbital plane and the shaded region intersect; and $\theta_p - \theta_n$ is the angular difference between pariapsis and the node line.

$$h_+(\theta) = H\left[\cos(2\phi)\left[A_0 + \varepsilon A_1 + \varepsilon^2 A_2\right] - \sin(2\phi)\left[B_0 + \varepsilon B_1 + \varepsilon^2 B_2\right]\right]$$
$$h_x(\theta) = H\left[\sin(2\phi)\left[A_0 + \varepsilon A_1 + \varepsilon^2 A_2\right] + \cos(2\phi)\left[A_0 + \varepsilon A_1 + \varepsilon^2 A_2\right]\right]$$

Where:

$$H = \frac{4G^2 m_1 m_2}{c^4 a(1-\varepsilon^2)R}$$
$$A_0 = -\frac{1}{2}\left[1+\cos^2(i)\right]\cos\left[2(\theta - \theta_n)\right]$$
$$B_0 = -\cos(i)\sin\left[2(\theta - \theta_n)\right]$$
$$A_1 = \frac{1}{4}\sin^2(i)\cos(\theta - \theta_p) - \frac{1}{8}\left[1+\cos^2(i)\right]\left[5\cos(\theta - 2\theta_n - \theta_p)\right]$$
$$B_1 = -\frac{1}{4}\cos(i)\left[5\sin(\theta - 2\theta_n + \theta_p) + sin(3\theta - 2\theta_n - \theta_p)\right]$$
$$A_2 = \frac{1}{4}\sin^2(i) - \frac{1}{4}\left[1+\cos^2(i)\right]\cos\left[2(\theta_n - \theta_p)\right]$$
$$B_2 = \frac{1}{2}\cos(i)\sin\left[2(\theta_n - \theta_p)\right]$$

Unfortunately, I was not able to understand his derivation, and I needed to take his equations more or less on faith. Regardless, the signal given by the Wahlquist equations would not in general be the signal observed on Earth: we would see a time dependent signal, not a signal which depends upon the position of the stars in the binary system. For this reason, we must determine the time dependence of θ to understand the signal which we would get on Earth.

4 Keppler's ψ-function

To determine the time dependence of the gravitational signal we must determine the orbital angle θ as a function of time. Using Kepler's second and relating the area swept out in a period–which must be the areal velocity always–leads to the relationship:

$$\frac{\pi ab}{T}t = \frac{1}{2}\int_{\theta_0}^{\theta_f} r^2(\theta)d\theta$$

Where $r(\theta)$ is the shape equation for for our orbit, which can be derived from principles of central force motion. However, the resulting expression cannot be easily inverted. Instead, we relate θ to a parameter ψ, and when we transform coordinates, we can more easily solve our integral in terms of ψ. The resulting equations are:

$$\frac{2\pi}{T}t = \psi - \varepsilon \sin\psi$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\psi}{2}$$

While this equation needs to be inverted approximately as well, I was able to do it relatively easily with Newton's Method.

5 Initial Results

After combining the Wahlquist equations with Kepler's equation, we should take a moment to appreciate the results before moving on. While there are countless possible orbital configurations and corresponding signals, we can begin to get an intuition for how different signals are produced.

Figure 2 shows a number of gravitational wave signals over one orbital period when we vary the eccentricity of the orbit while keeping other parameters constant. We find that for a circular orbit we receive sinusoidal signals with half the period of our orbit. For an equal mass system, this is what we expect from symmetry considerations–the system is identical after each mass has undergone half a orbit. This turns out to be true for any circular orbit binary. As we increase the eccentricity of the orbit, we find that our gravitational waves have higher peak amplitudes, and the signal is compressed: that is, it changes rapidly, and then relaxes to about zero, before changing rapidly again. The rapidly changing peaks correspond to when the the stars are nearest each other and moving the most rapidly, a result of Kepler's Second Law which states that the reduced mass orbit sweeps out equal areas in equal times.

Figure 3 shows the result of varying the node orientation of the orbit. There are two ways to interpret this effect: first, we could imagine that we are observing a physically different system. On the other hand, we could imagine that we are rotating our axes about Earth's position vector. Recalling that the distinction between the h_+ and h_x polarizations is nothing more than a choice of coordinates, it is unsurprising that when we vary ϕ, we see the strength of the h_+ and h_x polarizations change, but the total magnitude of the wave remain constant. If we change *phi* at steps of $\frac{\pi}{4}$, than we find that the shapes of our waves do not change at all, except that the polarizations switch or the graphs are reflected.

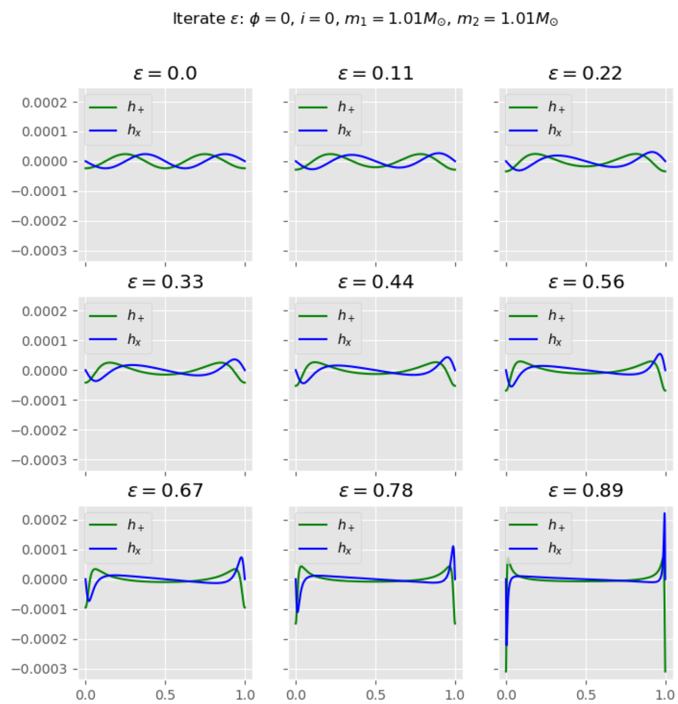

Figure 2: Gravitatioanl wave signal from orbits with various eccentricities.

Finally, Figure 4 demonstrates the effect of varying the inclination of the orbit. Again, we could think of this as changing the physical binary star system we are hypothetically observing, but it is more interesting to consider that we are changing coordinate systems. In this case, we are essentially changing the position of Earth relative to our hypothetical Earth; that is, we are changing where we are observing the gravitational way. In this way, we see that we are essentially tracing out the relative strength of gravitational waves in various directions.

With this in mind, we find that gravitational waves are emitted with the highest intensity directly out of the orbital plane. Then, as we move downwards from $i = 0$ to $i = \frac{\pi}{2}$–which is in the orbital plane–the gravitational wave amplitude decreases to a minimum. Additionally, in the plane, the h_x signal vanishes: the observed wave only stretches distances perpendicular to the plane while squeezing distances in the plane, or vice versa.

6 Peter's Equations

While the Wahlquist Equations and Keppler's ψ-function are sufficient to give the gravitational signals in most situations, it turns out that gravitational waves carry energy. It follows that the binary orbit must lose energy over time: the result is orbit circularization, that is, the orbit eccentricity tends towards zero as time passes, and a decrease in the semimajor axis, that is, the stars eventually collide.

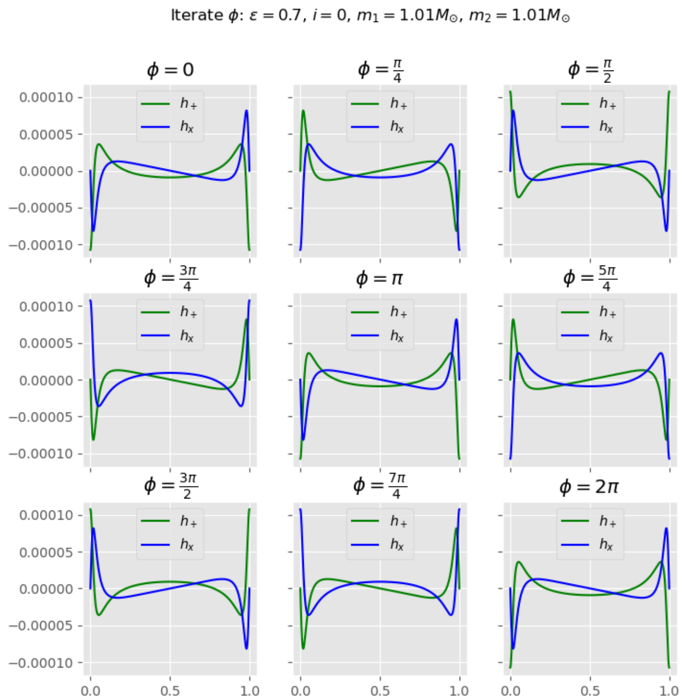

Figure 3: Effect of changing node orientation on gravitational wave signal.

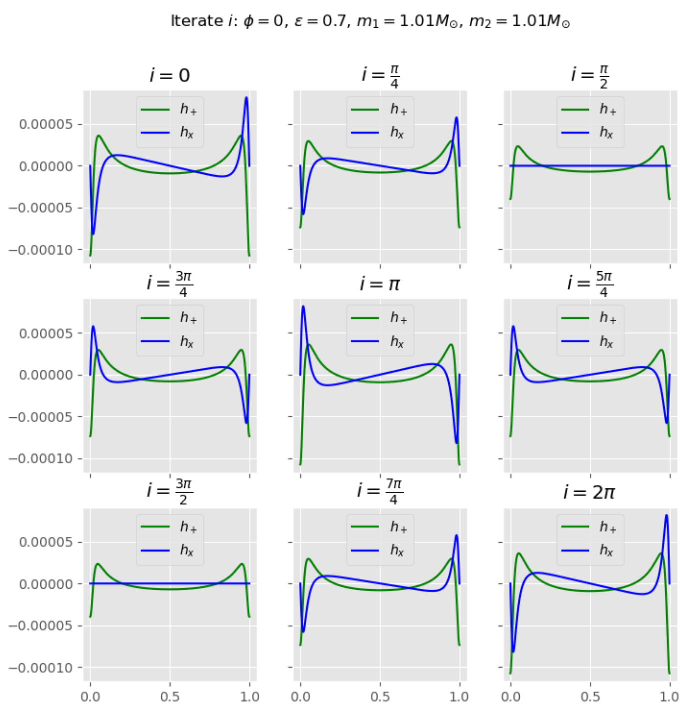

Figure 4: Change of inclination on gravitational wave signal

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From considerations of conservation of energy and angular momentum, Peters worked out the time evolution of binary system orbits as a set of coupled differential equations relating the eccentricity ε and semimajor axis a. Specifically, the equations are given as:

$$\left\langle \frac{da}{dt} \right\rangle = -\frac{64}{5}\frac{G^3 m_1^2 m_2^2 (m_1+m_2)}{c^5 a^3 (1-\varepsilon^2)^{\frac{7}{2}}}(1+\frac{73}{24}\varepsilon^2+\frac{37}{96}\varepsilon^4)$$

$$\left\langle \frac{d\varepsilon}{dt} \right\rangle = -\frac{304}{15}\varepsilon\frac{G^3 m_1 m_2 (m_1+m_2)}{c^5 a^4 (1-\varepsilon^2)^{\frac{5}{2}}}(1+\frac{121}{304}\varepsilon^2)$$

Dividing these equations and integrating gives the semimajor axis as a function of the eccentricity:

$$a(\varepsilon) = \frac{c_0 \varepsilon^{\frac{12}{19}}}{(1-\varepsilon^2)}\left[1+\frac{121}{304}\varepsilon^2\right]^{\frac{870}{2299}} \quad (c_0 \text{ chosen so that } e = e_0 \text{ when } a = a_0)$$

While this does allow our equations to be decoupled, neither of the resulting equations can be solved analytically.

7 Numerical Considerations

Using the analytic solution $a(\varepsilon)$, we only need to solve the differential equation for the eccentricity. This proves to be a more difficult problem then one might expect: because the eccentricity changes very slowly in time, we would need to integrate a very large region to obtain a complete solution, that is, to determine the gravitational wave signal at every point from our initial orbit to coalescence. Even a partial solution, that is, solving the equations from a given initial eccentricity, is not trivial.

I tried a number of options to integrate; for all of them, I ended up using units where $c = G = 1$, although I experimented with a system of dimensionless units as well. First, I tried to integrate this expression using Euler's method. Not surprisingly, this did not work: the results were divergent, and taking smaller and smaller time steps never approached a single solution.

Next, I tried a fourth order Runga-Kutta. Fourth order Runga-Kutta works by first taking a single Euler step, and then using this result to work backwards to an intermediate value; this intermediate value is used to work backwards again, and this answer to work backwards once more, generating a total of four answers. An optimized linear combination of the four results is used as the final answer at each step.

The results from fourth order Runga-Kutta were convergent, and ended up being the most successful method. However, I did try a few other algorithms. I coded a higher order Runga-Kutta method with adaptive step size. This method works similar to fourth order Runga-Kutta, except it uses more intermediate steps (the one I chose used 6); then, their are multiple ways to combine these six solutions to give a final solution, except one combination gives a solution with error proportional to Δt^6 and the other proportional to Δt^5. The difference of these solutions estimates the error of the Δt^5 solution; then, if the error is greater than what we want to limit the error to, this solution is used to solve for a smaller step size which algebraically should maintain this level of error. On the other hand, if the error is much smaller than what we want, a new, larger step size is solved for. In this way, we hope to use large step sizes where the function is changing slowly, and small step sizes where the function is changing rapidly, and efficiently solve our differential equation.

I also used a variety of algorithms provided with the scipy.integrate library. Unfortunately, neither the Runga-Kutta algorithm I coded nor those provided with scipy.integrate worked. My own algorithm was taking to many steps to solve for a new step size, making it inefficient; those provided with scipy.integrate broke after reporting too small of a step size to efficiently solve the problem. I expect both are due to the massive integration region, and thus large step sizes, involved with the problem.

So, in the end, the relatively simple fourth order Runga-Kutta method was the most successful. While this was accurate, it is hopeless to use to get a full solution to the problem; adaptive step size would be needed to do this. Still, the results from the fourth order Runga-Kutta method, while not complete, do offer some interesting insights into the time evolution of the gravitational wave signal.

8 Results

First, it seemed important to check the results of the fourth order Runge-Kutta method if possible. To highest order, Perters' differential equation for the time evolution of our binary system is proportional to ε^{-2}, and so we would expect the most rapid change in ε to be directly before coalescence. Therefore, by focusing in on the limit as $\varepsilon \to 0$, we should be able to find where our numerical routine begins to fail. Stated another way, if the numerical routine is stable a little before coalescence, then it should be stable everywhere else. The limit $\varepsilon \to 0$ is also a convenient point to check our results because our orbit becomes circular, for which there is an analytical solution for the time evolution of the orbital frequency. Specifically:

$$f_{gw}(\tau) = \frac{1}{\pi}(\frac{5}{256}\frac{1}{\tau})^{\frac{3}{8}}(\frac{GM_c}{c^3})^{-\frac{5}{8}}$$

Where $M_c = \frac{(m_1 m_2)^{\frac{3}{5}}}{(m_1+m_2)^{\frac{1}{5}}}$ is the chirp mass of the system and $\tau = t_{coal} - t$ is the time to coallescence (see e.g. Maggoire). My code tracks the orbital period, and $f_{orbit} = \frac{1}{T}$ is the orbital frequency; in the circular orbit limit, the gravitational wave frequency is $f_{gw} = 2f_{orbit}$. So, we can compare the time evolution of the orbital frequency with the expected time dependent gravitational wave frequency; the results of this comparison are show in 5.

Now that we are confident that our solution should be reasonably accurate, at least until a short time before before coalescence, we can use our routine to make some observations about the inspiral of a binary system; however, we do suffer some limitations. First, we are unable to integrate for appreciable periods of time, so we cannot determine a fully time dependent gravitational wave signal. Second, our equations are derived with the assumption of flat spacetime, which is not a

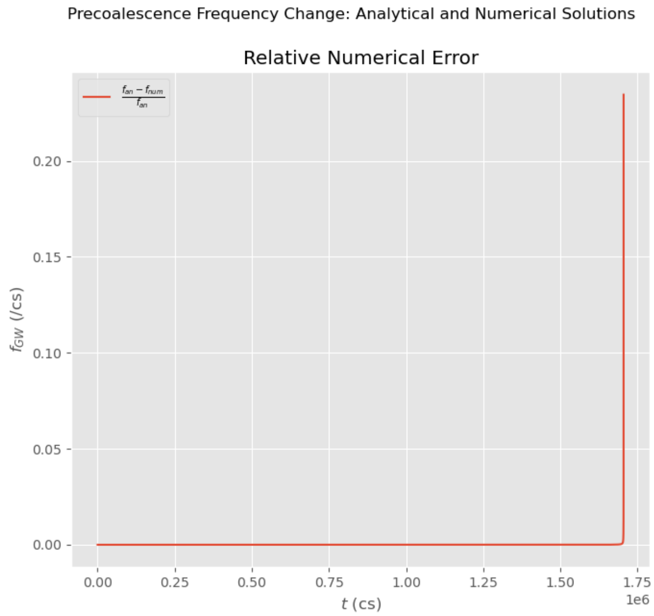

Figure 5: Relative error between analytical gravitational wave frequency and numerical gravitational wave frequency, calculated from the time evolution of the orbital period. We see that the error is negligible until directly before coalescence, where the numerical routine ends.

valid assumption near coalescence. Still, we can observe general trends in the way in which the gravitational wave magnitude will change in time, and also get an approximate solution at coalescence, which is interesting, if not completely accurate.

Figure 6 shows how the gravitational wave magnitude changes with increasing eccentricity; as we know that the eccentricity is decreasing in time, we see that the gravitational wave magnitude will increase in time. While this does not exactly show how the wave magnitude will increase in time, we know that the eccentricity increases more and more at later times, so the increase in the wave magnitude with time will be even greater than what is shown here. I'll be honest, this is not a very satisfying result, but it does show the general trend that we would expect.
A more interesting result is the gravitational wave signal directly before coalescence.

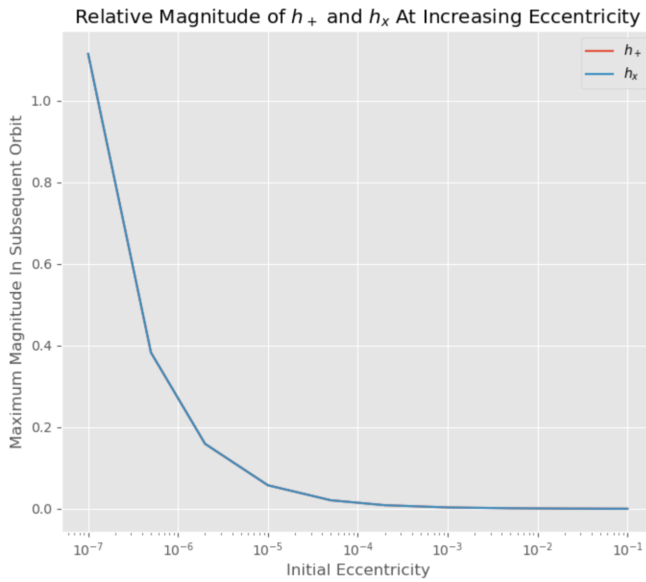

Figure 6: Relative magnitude of gravitational waves at increasing eccentricity. For a given set of eccentricities, time evolution code was used to get the maximum magnitude of h_+ and h_x near that eccentricity.

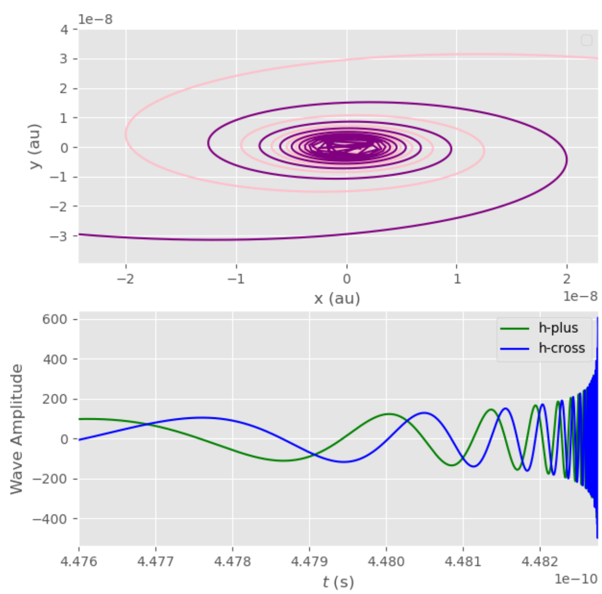

Figure 7: Caption