A Brief Introduction to Lagrangians

Simply put, the Lagrangian for a system is the difference between the kinetic energy, K, and the potential energy, U,

$$\mathcal{L} = K - U\,. \tag{1}$$

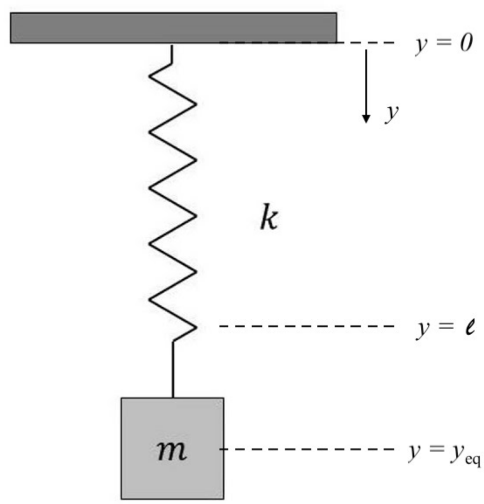
The Lagrangian gives us a very slick but powerful way find the *equation of motion* of a object by plugging it into Lagrange's equation,

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0\,. \tag{2}$$

Here, q represents a coordinate that appears in the Lagrangian. It could be a displacement such as x or y, for example, or it could be an angle or a distance along a curved path. The dot over the q in the second term designates a time derivative, $\dot{q} \equiv \frac{dq}{dt}$. Notice that, if our coordinate is x (for example), then $\dot{x} = \frac{dx}{dt}$ is the velocity in the x-direction, and $\ddot{x} = \frac{d^2x}{dt^2}$ is the acceleration.

Example: Mass on a Vertical Spring

Lagrangians are rather abstract, so let's apply it to a familiar example. Consider a mass m hanging on a vertical spring (with spring constant k and unstretched length ℓ). We know from experience that the mass should oscillate up and down around an equilibrium position where the spring force and gravity balance.



First, we need to choose a coordinate system. Let y describe the vertical position of the mass, with $y = 0$ at the top end of the spring where it attaches to the support. Define the downward direction as positive. Then the kinetic energy of the mass at any given point in its oscillation cycle will be

$$K = \frac{1}{2}m\dot{y}^2\,. \tag{3}$$

The potential energy will have two terms, one for the spring potential energy and one for the gravitational potential energy:

$$U = \frac{1}{2}k(y-\ell)^2 - mgy\,, \tag{4}$$

where $(y-\ell)$ is the amount the spring is stretched and g is the acceleration due to gravity. Notice that the gravitational potential energy term is negative so that it "goes down" as the mass goes down (remembering that down is positive, so y increases as the mass goes down).

The Lagrangian for this system is then

$$\mathcal{L} = K - U = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}k(y-\ell)^2 + mgy\,. \tag{5}$$

When we plug this into Lagrange's equation, we'll need to take some derivatives. It will be simpler to compute those separately first. To take the derivative of $\mathcal{L}$ with respect to y, everything else (including $\dot{y}$) is treated as a constant,

$$\frac{\partial \mathcal{L}}{\partial y} = -\frac{1}{2}k\left[2(y-\ell)\right] + mg = -k(y-\ell) + mg\,. \tag{6}$$

To find the derivative of $\mathcal{L}$ with respect to $\dot{y}$, we again treat everything else (including y) as a constant,

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{1}{2}m\left[2\dot{y}\right] = m\dot{y}\,. \tag{7}$$

Now we need to take another time derivative of the term we just found to get

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{d}{dt}\left(m\dot{y}\right) = m\ddot{y}\,. \tag{8}$$

Notice that this is mass times acceleration.

Now we plug everything into Lagrange's equation:

$$0 = \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = -k(y-\ell) + mg - m\ddot{y}\,. \tag{9}$$

Re-arranging a bit gives

$$-ky + k\ell + mg = m\ddot{y}\,. \tag{10}$$

This is the equation of motion for a mass on a vertical spring under gravity. We could get the same thing from Newton's second law, of course, but the Lagrangian method can be used in many situations where the forces are much more complicated than they are here.

For completeness, I'll point out that the solution to this equation is

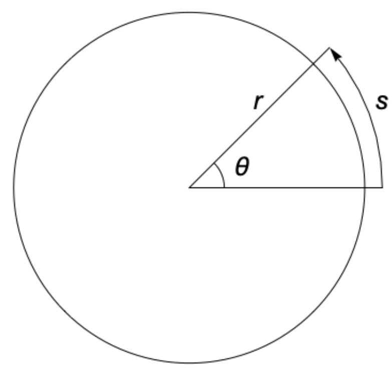
$$y(t) = \ell + \frac{mg}{k} + A\sin(\omega t + \phi)\,, \tag{11}$$

where A is the amplitude of the oscillation, $\omega = \sqrt{k/m}$ is the oscillation frequency, and ϕ is an arbitrary phase. Plug this into the equation of motion to convince yourself that this is, in fact, a solution. Remember that ℓ is the unstretched length of the spring. The quantity $\frac{mg}{k}$ is the amount the spring has stretched due to the weight of the mass, so $y_{\text{eq}} = \ell + \frac{mg}{k}$ is the equilibrium position of the spring with the mass on it. The sinusoidal oscillation occurs around that point. (In introductory texts, the coordinate system is usually chosen so that the origin is at the equilibrium position. In that case, the solution would simply be the sinusoidal term alone.)

Example: Circular Orbits

Now let's look at a case that is more directly relevant to our project. Consider the case of a particle (of mass m) moving in a region with large mass M exerting a gravitational force, $F = GMm/r^2$.

Let's start with the simplest case, the small particle traveling in a circle around the central mass. The position of the particle can be described by its distance around the circle, the arclength $s = r\theta$. Because the radius r is constant, we can equivalently describe the position of the particle by the angle θ.



The kinetic energy for this particle would be

$$K = \frac{1}{2}m\dot{s}^2 = \frac{1}{2}mr^2\dot{\theta}^2\,, \tag{12}$$

where we have applied the time derivative to the arclength $\dot{s} = r\dot{\theta}$. The gravitational potential energy in this case is

$$U(r) = -\frac{GMm}{r} = -\frac{k}{r}\,, \tag{13}$$

where $U = 0$ at $r = \infty$ and, for simplicity, $k = GMm$.

Then the Lagrangian is

$$\mathcal{L} = K - U = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{k}{r}\,. \tag{14}$$

Now we need to take the derivative of $\mathcal{L}$ with respect to θ, but θ (without the dot) doesn't appear in the equation. As a result,

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0\,. \tag{15}$$

The derivative of $\mathcal{L}$ with respect to $\dot{\theta}$, treating everything else as a constant, is

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}\,. \tag{16}$$

The Lagrange equation gives

$$0 - \frac{d}{dt}(mr^2\dot{\theta}) = 0\,. \tag{17}$$

This equation implies that the quantity inside the parentheses must be constant. Recall that angular momentum is given by $L = I\omega$, where I is the moment of inertia and ω is the angular velocity. The moment of inertia for a point particle is $I = mr^2$, and ω is the time derivative of the angular position θ (i.e., $\omega = \dot{\theta}$). So, $L = I\omega = mr^2\dot{\theta}$. In other words, Lagrange's equation has not just given us the equation of motion of this particle, but it is telling us that angular momentum is conserved! (Note: Marion & Thornton use the lower case l for angular momentum and upper-case L for the Lagrangian.)

The way that conservation of angular momentum just fell out of Lagrange's equation is actually quite profound. It comes from the fact that this system is symmetric under rotations (i.e., it doesn't matter which way you rotate this system – it always looks the same). [Side note: The connection between symmetries and conservation laws was proven by a German Jew named Emmy Noether, who was expelled from her academic position when the Nazis came to power in 1933. Her theorem is considered one of the most important mathematical developments in theoretical physics.]

The solution to this equation of motion (17) is the familiar $\theta = \theta_0 + \omega t$ (where ω is constant).

Notes on Elliptical Orbits

When applying this Lagrangian technique to elliptical orbits, we can no longer assume that the radius is constant. As a result, the kinetic energy term will actually be two terms, one for radial velocity and one for angular velocity:

$$K = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 , \tag{18}$$

The potential energy is the same as in the case of circular orbits, so the Lagrangian will be

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{k}{r} . \tag{19}$$

We are still assuming that one of the masses is far larger than the other, but that assumption isn't going to be valid in our project. In that case, both objects will orbit around the center of mass of the system, and the mass m in Eq. (19) becomes the "reduced mass,"

$$\mu = \frac{m_1 m_2}{m_1 + m_2} , \tag{20}$$

giving

$$\mathcal{L} = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + \frac{k}{r} . \tag{21}$$

This is the same as Eq. (7.7) in Marion & Thornton. They then use conservation of angular momentum ($l = \mu r^2\dot{\theta} = \text{constant}$) to eliminate $\dot{\theta}$ from the Lagrangian and proceed to find the equation of motion using Lagrange's equation for the coordinate r.