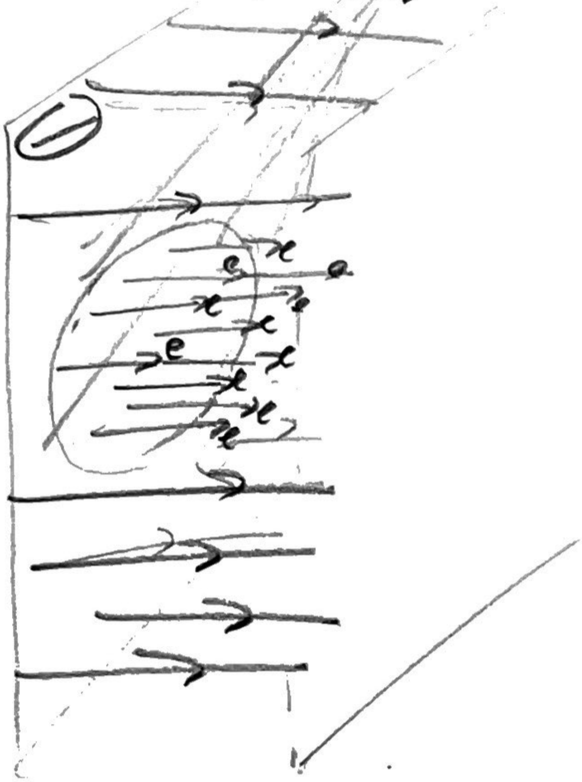
First, we describe the physical set up of the pre-breakdown dynamics. A voltage bias is applied to one electrode — say the cathode; this bias is variable and depends on the RC timing of the driving circuit. Then, ultraviolet light is shown on the cathode emitting an electron current due to the photo-electric effect. While this may or may not be the case, we shall suppose the current is constant.



We would like to describe the ensuing electron dynamics. First, we shall try to describe the dynamics using two fluid equations; let $n_e(\vec{x})$ be the electron density as a function of position, $\vec{u}_e(\vec{x})$ be the mean electron velocity as a function of position, and $n_0(\vec{x})$ be the background neutral density, which we will assume has no mean velocity. Then, the two fluid equation of motion is:

$$n_e m_e\left[\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e\cdot\nabla\vec{u}_e\right] = n_e e(\vec{E} + \vec{u}_e\times\vec{B}) - \frac{\partial}{\partial\vec{x}}\cdot\overleftrightarrow{P}_e - \vec{R}_{e0}$$

To simplify, we begin by noting that $\vec{E} >> \vec{B}$ due to the applied field, and so we assume that $\vec{B}=0$. For now, suppose that f_e — the vlasov distribution of the electrons — is isotropic. Then, $\frac{\partial}{\partial\vec{x}}\cdot\overleftrightarrow{P} = \nabla P$, where P is the scalar pressure. Our equation of motion is then:

$$n_e m_e\left[\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e\cdot\nabla\vec{u}_e\right] = n_e e\vec{E} - \nabla P - \vec{R}_{e0}$$

Now, we will further assume that ∇P is negligible — which is a reasonable assumption as $\vec{E}$ is large. Then:

$$n_e m_e\left[\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e\cdot\nabla\vec{u}_e\right] = e n_e\vec{E} - R_{e0}$$

The mean velocity of the neutrals is zero in the lab frame, so say $R_{e0} = \nu_{e0} m_e n_e \vec{u}_e$; then:

$$\cancel{n_e} m_e\left[\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e\cdot\nabla\vec{u}_e\right] = e\cancel{n_e}\vec{E} - \nu_{e0} m_e \cancel{n_e}\vec{u}_e$$

$$m_e\left[\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e\cdot\nabla\vec{u}_e\right] = e\vec{E} - \nu_{e0} m_e\vec{u}_e$$

$$\frac{\partial \vec{u}_e}{\partial t} + (\vec{u}_e\cdot\nabla)\vec{u}_e + \nu_{e0}\vec{u}_e = \frac{e\vec{E}}{m_e}$$, or in terms of the electric potential:

$$\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e\cdot\nabla\vec{u}_e + \nu_{e0}\vec{u}_e = -\frac{e}{m_e}\nabla V$$

Now, we shall solve this ~~for under~~ certain simplified conditions; the simplist ought to reduce to the

Langmuir-Child space charge limited current. That is, suppose that we reach a steady state current, there are no neutrals so no frictional drag, and we have a one dimensional problem. Then:

$$u\frac{du_e}{dx} = -\frac{e}{m_e}\frac{dV}{dx},$$ by the chain rule, $\frac{d}{dx}[u^2] = 2u\frac{du}{dx}$, so:

$$\frac{1}{2}\frac{d}{dx}[u_e^2] = -\frac{e}{m_e}\frac{dV}{dx}$$

$$\frac{d}{dx}[u_e^2] = -\frac{2e}{m_e}\frac{dV}{dx}$$

$$\int\frac{d}{dx}[u_e^2]\,dx = -\frac{2e}{m_e}\int\frac{dV}{dx}\,dx,$$ and applying the Leibniz Integral Rule:

$$u_e^2 = -\frac{2e}{m_e}V + C,$$ where C is an integration constant

$$u_e = \pm\sqrt{\frac{2e}{m_e}V + C}$$

We would like to relate this to the current density; this is easy enough though as the current density is the amount of charge density moving at a point in space, so:

$$J = -en_eu_e,$$ where n_e is the electron density distribution

And it follows that $n_e = -\frac{J}{eu_e}$; Poisson's Equation gives:

$$\frac{d^2V}{dx^2} = -\frac{en_e}{\varepsilon_0} = \left(-\frac{e}{\varepsilon_0}\right)\cdot\left(\frac{J}{eu_e}\right) = \frac{J}{\varepsilon_0}\left(\frac{2e}{m_e}V + C\right)^{-1/2}$$

For now, we will assume that $u_{e0} = 0$ and $V = 0$ where the electrons leave the cathode so that $C = 0$.

Then, we seek to solve:

$$\frac{d^2V}{dx^2} - \lambda V^{-\frac{1}{2}} = 0, \text{ where } \lambda = \frac{J}{\varepsilon_1}\sqrt{\frac{m_e}{2e}}$$

$\frac{d}{dx}[V^{\frac{1}{2}}] = \frac{1}{2}\frac{dV}{dx}V^{-\frac{1}{2}}$

Introducing the integrating factor $\frac{dV}{dx}$, we have:

$$\frac{dV}{dx}\cdot\frac{d^2V}{dx^2} - \lambda\frac{dV}{dx}V^{-\frac{1}{2}} = 0, \text{ by the chain rule,}$$

$$\frac{dV}{dx}V^{-\frac{1}{2}} = 2\frac{d}{dx}\left[V^{\frac{1}{2}}\right], \text{ so:}$$

$$\frac{dV}{dx}\cdot\frac{d^2V}{dx^2} - 2\lambda\frac{d}{dx}\left[V^{\frac{1}{2}}\right] = 0, \text{ and } \frac{1}{2}\frac{d}{dx}\left[\left(\frac{dV}{dx}\right)^2\right] = \frac{dV}{dx}\frac{d^2V}{dx^2};$$

$$\frac{1}{2}\frac{d}{dx}\left[\left(\frac{dV}{dx}\right)^2\right] - 2\lambda\frac{d}{dx}\left[V^{\frac{1}{2}}\right] = 0$$

$$\int_0^d \frac{d}{dx}\left[\frac{1}{2}\left(\frac{dV}{dx}\right)^2 - 2\lambda V^{\frac{1}{2}}\right]dx = 0, \text{ by the fundamental theorem of calculus:}$$

$$\frac{1}{2}\left(\frac{dV}{dx}\right)^2 - 2\lambda V^{\frac{1}{2}} = 0$$

$$\left(\frac{dV}{dx}\right)^2 - 4\lambda V^{\frac{1}{2}} = 0$$

$$\frac{dV}{dx} - 2\sqrt{\lambda}\,V^{\frac{1}{4}} = 0$$

$$\int_0^V \frac{dV}{V^{\frac{1}{4}}} = \int_0^d 2\sqrt{\lambda}\,dx \quad \text{(where } d \text{ is the gap spacing)}$$

$$\frac{4}{3}V^{\frac{3}{4}}\Big|_0^V = 2\sqrt{\lambda}\,x\Big|_0^d$$

$$\frac{4}{3}V^{\frac{3}{4}} = 2\sqrt{\lambda}\,d$$

$\frac{16}{9} V^{\frac{3}{2}} = 4 \lambda d^2$, recalling $\lambda = \frac{J}{\varepsilon_0}\sqrt{\frac{m_e}{2e}}$:

$$\frac{4}{9} V^{\frac{3}{2}} = \frac{J}{\varepsilon_0}\sqrt{\frac{m_e}{2e}}\, d^2$$

$J = \frac{4}{9}\varepsilon_0 \sqrt{\frac{2e}{m_e}}\frac{V^{\frac{3}{2}}}{d^2}$, which is the space charge limited current

Next, we will remove the requirement that there are no neutrals, but maintain that we achieve a steady state current and that the problem has one dimension. Then, our fluid equation is:

$u_e \frac{du_e}{dx} + \nu_{e0} u_e = -\frac{e}{m_e}\frac{dV}{dx}$, using the chain rule:

$$\frac{1}{2}\frac{d}{dx}[u_e^2] + \nu_{e0} u_e = -\frac{e}{m_e}\frac{dV}{dx}$$

Which is coupled to Poisson's equation $\frac{d^2V}{dx^2} = -\frac{n_e}{\varepsilon_0}$.

We can decouple this system:

$$\frac{1}{2}\frac{d^2}{dx^2}[u_e^2] + \nu_{e0}\frac{du_e}{dx} = -\frac{e}{m_e}\frac{d^2V}{dx^2}$$

$\frac{1}{2}\frac{d^2}{dx^2}[u_e^2] + \nu_{e0}\frac{du_e}{dx} = \frac{e\, n_e}{m_e \varepsilon_0}$, where, if we assume a steady current is reached, then $n_e = -\frac{J}{e u_e}$, so:

$$\frac{1}{2}\frac{d^2}{dx^2}[u_e^2] + \nu_{e0}\frac{du_e}{dx} = -\frac{J}{m_e}\cdot u_e^{-1}$$

$$\frac{1}{2}\frac{d^2}{dx^2}[u_e^2] + \nu_{e0}\frac{du_e}{dx} + \frac{J}{m_e}u_e^{-1} = 0$$

$$\frac{d}{dx}\left[u_e\frac{du}{dx}\right] + \nu_{e0}\frac{du_e}{dx} + \frac{J}{m_e}u_e^{-1} = 0$$

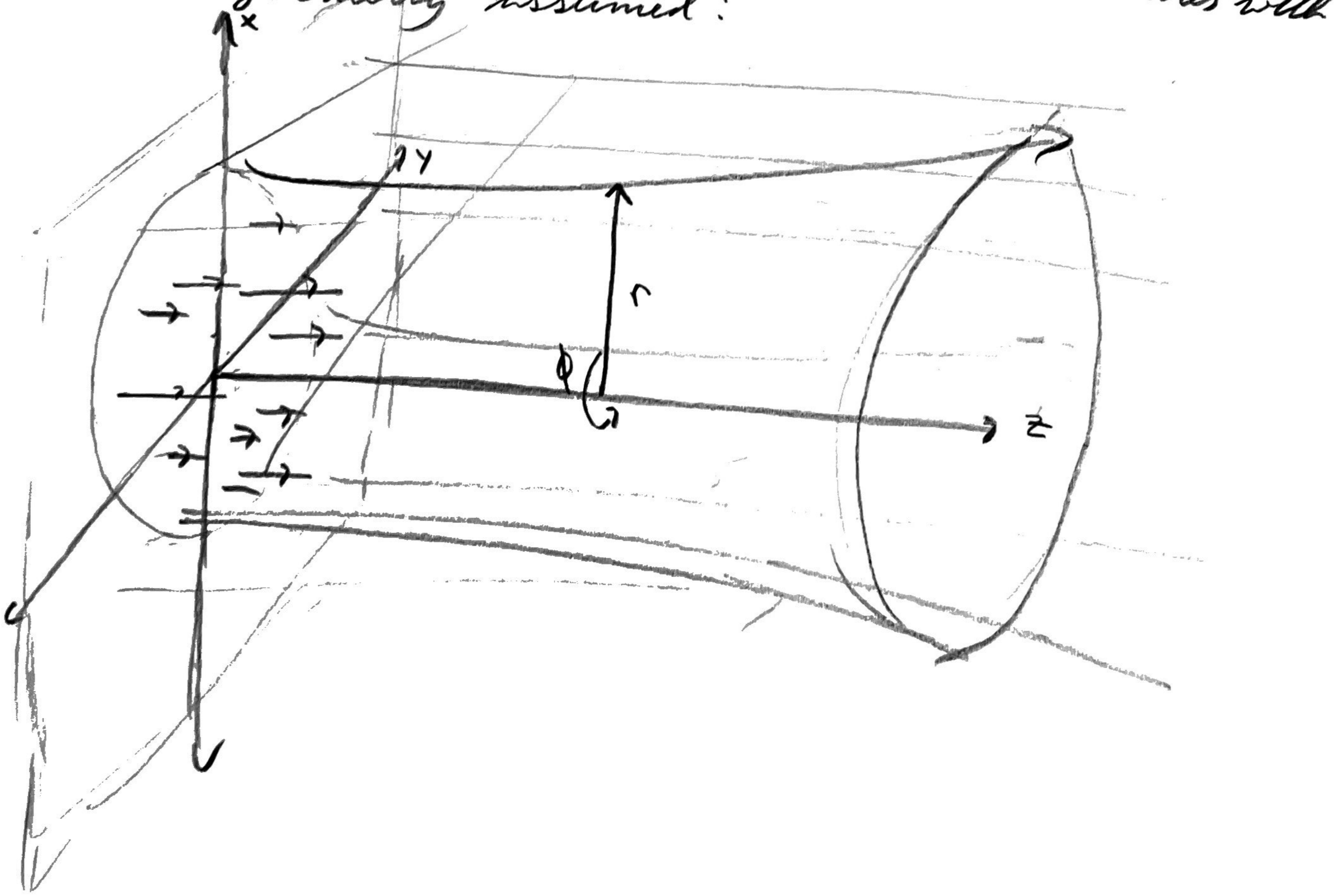
$$\left(\frac{du}{dx}\right)^2 + \frac{d^2u}{dx}u_e + \nu_{e0}\frac{du_e}{dx} + \frac{J}{m_e}u_e^{-1} = 0$$

$$\frac{16}{9} V^{\frac{3}{2}} = 4 \lambda d^2$$

$$\frac{4}{9} \frac{V^{3/2}}{d^2} = \frac{J}{\varepsilon_0} \sqrt{\frac{m_e}{2e}}$$

$$\frac{4}{9} \cdot \varepsilon_0 \sqrt{\frac{2e}{m_e}} \frac{V^{3/2}}{d^2} = J$$, which is the Child-Langmuir Law, as required

Now, lets set about solving our general equation numerically. This will be done in spherical coordinates with spherical symmetry assumed:



Our system is [illegible]

$$\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e \cdot \nabla \vec{u}_e + \nu_{e6} \vec{u}_e = -\frac{e}{m_e} \nabla V$$

$$\nabla^2 V = \frac{e n_e}{\varepsilon_0} \quad \text{(poisson's Equation)}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{u}_e) = 0 \quad \text{(Species Continuity Equation)}$$

However, we wish to treat this in cylindrical coordinates; it will help to explicitly write what this means. It helps that we assume spherical symmetry so that $\frac{\partial n_e}{\partial \phi} = \frac{\partial V}{\partial \phi} = \frac{\partial \vec{u}_e}{\partial \phi} = 0$. Also, we will assume that $u_{e\phi} = 0$.

$$\frac{\partial \vec{u}_e}{\partial t} + \hat{r}\left(u_{er}\frac{\partial u_{er}}{\partial r} + \cancelto{0}{\frac{u_{e\phi}}{r}\frac{\partial u_{er}}{\partial \phi}} + u_{ez}\frac{\partial u_{er}}{\partial z} - \cancelto{0}{\frac{u_{e\phi}^2}{r}} \right)$$

$$+ \hat{\phi}\left(\cancelto{0}{u_{er}\frac{\partial u_{e\phi}}{\partial r}} + \cancelto{0}{\frac{u_{e\phi}}{r}\frac{\partial V_\phi}{\partial \phi}} + \cancelto{0}{u_{ez}\frac{\partial u_{e\phi}}{\partial z}} + \cancelto{0}{\frac{u_{e\phi} u_{er}}{r}} \right)$$

$$+ \hat{z}\left(u_{er}\frac{\partial u_{ez}}{\partial r} + \cancelto{0}{\frac{u_{e\phi}}{r}\frac{\partial u_{ez}}{\partial \phi}} + u_{ez}\frac{\partial u_{ez}}{\partial z} \right)$$

$$+ \nu_{e6}\vec{u}_e = -\frac{e}{m_e}\left[\hat{r}\frac{\partial V}{\partial r} + \cancelto{0}{\hat{\phi}\frac{1}{r}\frac{\partial V}{\partial \phi}} + \hat{z}\frac{\partial V}{\partial z} \right]$$

$$\begin{pmatrix} \frac{\partial u_{er}}{\partial t} + u_{er}\frac{\partial u_{er}}{\partial r} + u_{ez}\frac{\partial u_{er}}{\partial z} + \nu_{e6} u_{er} + \frac{e}{m_e}\frac{\partial V}{\partial r} \\ 0 \\ \frac{\partial u_{ez}}{\partial t} + u_{er}\frac{\partial u_{ez}}{\partial r} + u_{ez}\frac{\partial u_{ez}}{\partial z} + \nu_{ec}\vec{u}_{ez} + \frac{e}{m_e}\frac{\partial V}{\partial z} \end{pmatrix} = \vec{0}$$

Poisson's Equation is given as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] + \cancelto{0}{\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}} + \frac{\partial^2 V}{\partial z^2} = \frac{e n_e}{\varepsilon_0}$$

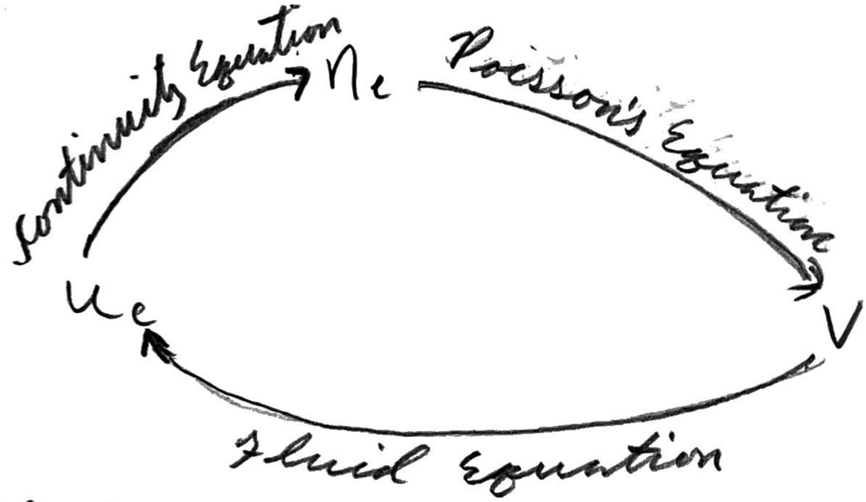
$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] + \frac{\partial^2 V}{\partial z^2} = \frac{e n_e}{\varepsilon_0}$$

And the continuity Equation is:

$$\frac{\partial n_e}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left[r n_e u_{er}\right] + \cancelto{0}{\frac{1}{r}\frac{\partial}{\partial r}\left[n_e u_{e\phi}\right]} + \frac{\partial}{\partial z}\left[n_e u_{ez}\right] = 0$$

$$\frac{\partial n_e}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left[r n_e u_{er}\right] + \frac{\partial}{\partial z}\left[n_e u_{ez}\right] = 0$$

Our computational loop will generally look like:



We will begin by developing a solution to the continuity Equation

$$\frac{\partial n_e}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left[r n_e u_{er}\right] - \frac{\partial}{\partial z}\left[n_e u_{ez}\right]$$