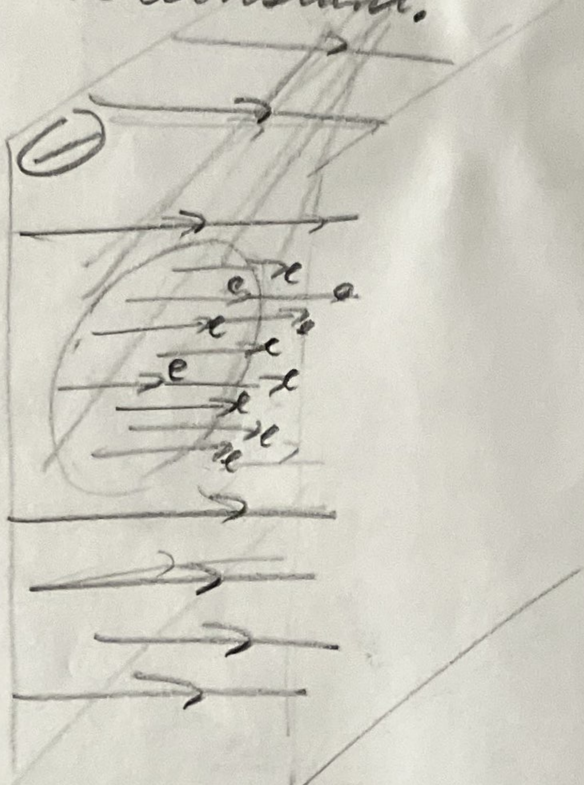
First, we describe the physical set-up of the pre-breakdown dynamics. A voltage bias is applied to one electrode—say the cathode; this bias is variable and depends on the RC timing of the driving circuit. Then, ultraviolet light is shown on the cathode emitting an electron current due to the photo-electric effect. While this may or m not be the case, we shall suppose the current is constant.



We would like to describe the ensuing electron dynamics. First, we shall try to describe the dynamics using two fluid equations; let $n_e(\vec{x})$ be the electron density as a function of position, $\vec{u}_e(\vec{x})$ be the mean electron velocity as a function of position, and $n_b(\vec{x})$ be the background neutral density, which we will assume has no mean velocity. Then, the two fluid equation of motion is:

$$n_e m_e\left[\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e\cdot\nabla\vec{u}_e\right] = n_e e(\vec{E} + \vec{u}_e\times\vec{B}) - \frac{\partial}{\partial \vec{x}}\cdot\overleftrightarrow{P}_e - \vec{R}_{e0}$$

To simplify, we begin by noting that $\vec{E} \gg \vec{B}$ due to the applied field, and so we assume that $\vec{B}=0$. For now, suppose that f_e—the vlasov distribution of the electrons—is isotropic. Then, $\frac{\partial}{\partial \vec{x}}\cdot\overleftrightarrow{P} = \nabla P$, where P is the scalar pressure. Our equation of motion is then:

$$n_e m_e\left[\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e\cdot\nabla\vec{u}_e\right] = n_e e\vec{E} - \nabla P - \vec{R}_{e0}$$

Now, we will further assume that ∇P is negligible—which is a reasonable assumption as $\vec{E}$ is large. Then:

$$n_e m_e\left[\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e\cdot\nabla\vec{u}_e\right] = e n_e\vec{E} - R_{e0}$$

The mean velocity of the neutrals is zero in the lab frame, so say $R_{e0} = \nu_{e0} m_e n_e \vec{u}_e$; then:

$$\cancel{n_e} m_e\left[\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e\cdot\nabla\vec{u}_e\right] = e\cancel{n_e}\vec{E} - \nu_{e0} m_e \cancel{n_e}\vec{u}_e$$

$$m_e\left[\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e\cdot\nabla\vec{u}_e\right] = e\vec{E} - \nu_{e0} m_e\vec{u}_e$$

$$\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e\cdot\nabla\vec{u}_e + \nu_{e0}\vec{u}_e = \frac{e\vec{E}}{m_e},$$ or in terms of the electric potential:

$$\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e\cdot\nabla\vec{u}_e + \nu_{e0}\vec{u}_e = -\frac{e}{m_e}\nabla V$$

Now, we shall solve this for under certain simplified conditions; the simplist ought to reduce to the

Langmuir-Child space charge limited current. That is, suppose that we reach a steady state current, and there are no neutrals so no frictional drag, and we have a one dimensional problem. Then:

$u\frac{du_e}{dx} = -\frac{e}{m_e}\frac{dV}{dx}$, by the chain rule, $\frac{d}{dx}[u^2] = 2u\frac{du}{dx}$, so:

$$\frac{1}{2}\frac{d}{dx}[u_e^2] = -\frac{e}{m_e}\frac{dV}{dx}$$

$$\frac{d}{dx}[u_e^2] = -\frac{2e}{m_e}\frac{dV}{dx}$$

$\int \frac{d}{dx}[u_e^2]\,dx = -\frac{2e}{m_e}\int \frac{dV}{dx}\,dx$, and applying the Leibniz Integral Rule:

$u_e^2 = -\frac{2e}{m_e}V + C$, where C is an integration constant

$$u_e = \pm\sqrt{\frac{2e}{m_e}V + C}$$

We would like to relate this to the current density; this is easy enough though as the current density is the amount of charge density moving at a point in space, so:

$J = -en_eu_e$, where n_e is the electron density distribution

And it follows that $n_e = -\frac{J}{eu_e}$; Poisson's Equation gives:

$$\frac{d^2V}{dx^2} = -\frac{en_e}{\varepsilon_0} = \left(-\frac{e}{\varepsilon_0}\right)\cdot\left(\frac{J}{eu_e}\right) = \frac{J}{\varepsilon_0}\left(\frac{2e}{m_e}V + C\right)^{-1/2}$$

For now, we will assume that $u_{e0} = 0$ and $V = 0$ where the electrons leave the cathode so that $C = 0$.

Then, we seek to solve:

$$\frac{d^2V}{dx^2} - \lambda V^{-\frac{1}{2}} = 0, \text{ where } \lambda = \frac{J}{\varepsilon_1}\sqrt{\frac{m_e}{2e}}$$

$\frac{d}{dx}[V^{\frac{1}{2}}]$
$= \frac{1}{2}\frac{dV}{dx}V^{-\frac{1}{2}}$

Introducing the integrating factor $\frac{dV}{dx}$, we have:

$$\frac{dV}{dx}\cdot\frac{d^2V}{dx^2} - \lambda\frac{dV}{dx}V^{-\frac{1}{2}} = 0,$$ by the chain rule,

$$\frac{dV}{dx}V^{-\frac{1}{2}} = 2\frac{d}{dx}\left[V^{\frac{1}{2}}\right],$$ so:

$$\frac{dV}{dx}\cdot\frac{d^2V}{dx^2} - 2\lambda\frac{d}{dx}\left[V^{\frac{1}{2}}\right] = 0,$$ and $\frac{1}{2}\frac{d}{dx}\left[\left(\frac{dV}{dx}\right)^2\right] = \frac{dV}{dx}\frac{d^2V}{dx^2}$;

$$\frac{1}{2}\frac{d}{dx}\left[\left(\frac{dV}{dx}\right)^2\right] - 2\lambda\frac{d}{dx}\left[V^{\frac{1}{2}}\right] = 0$$

$$\int_0^d \frac{d}{dx}\left[\frac{1}{2}\left(\frac{dV}{dx}\right)^2 - 2\lambda V^{\frac{1}{2}}\right]dx = 0,$$ by the fundamental theorem of calculus;

$$\frac{1}{2}\left(\frac{dV}{dx}\right)^2 - 2\lambda V^{\frac{1}{2}} = 0$$

$$\left(\frac{dV}{dx}\right)^2 - 4\lambda V^{\frac{1}{2}} = 0$$

$$\frac{dV}{dx} - 2\sqrt{\lambda}\,V^{\frac{1}{4}} = 0$$

$$\int_0^V \frac{dV}{V^{\frac{1}{4}}} = \int_0^d 2\sqrt{\lambda}\,dx$$ (where d is the gap spacing)

$$\frac{4}{3}V^{\frac{3}{4}}\Big|_0^V = 2\sqrt{\lambda}\,x\Big|_0^d$$

$$\frac{4}{3}V^{\frac{3}{4}} = 2\sqrt{\lambda}\,d$$

$$\frac{16}{9} V^{\frac{3}{2}} = 4 \lambda d^2, \text{ recalling } \lambda = \frac{J}{\varepsilon_0}\sqrt{\frac{m_e}{2e}}:$$

$$\frac{4}{9} V^{\frac{3}{2}} = \frac{J}{\varepsilon_0}\sqrt{\frac{m_e}{2e}}\, d^2$$

$$J = \frac{4}{9}\varepsilon_0 \sqrt{\frac{2e}{m_e}} \frac{V^{\frac{3}{2}}}{d^2},$$ which is the space charge limited current

Next, we will remove the requirement that there are no neutrals, but maintain that we achieve a steady state current and that the problem has one dimension. Then, our fluid equation is:

$$u_e \frac{du_e}{dx} + \nu_{e0} u_e = -\frac{e}{m_e}\frac{dV}{dx},$$ using the chain rule;

$$\frac{1}{2}\frac{d}{dx}[u_e^2] + \nu_{e0} u_e = -\frac{e}{m_e}\frac{dV}{dx}$$

Which is coupled to Poisson's equation $\frac{d^2V}{dx^2} = -\frac{n_e}{\varepsilon_0}$.
We can decouple this system:

$$\frac{1}{2}\frac{d^2}{dx^2}[u_e^2] + \nu_{e0}\frac{du_e}{dx} = -\frac{e}{m_e}\frac{d^2V}{dx^2}$$

$$\frac{1}{2}\frac{d^2}{dx^2}[u_e^2] + \nu_{e0}\frac{du_e}{dx} = \frac{e\, n_e}{m_e \varepsilon_0},$$ where, if we assume a steady current is reached, then $n_e = -\frac{J}{e u_e}$, so:

$$\frac{1}{2}\frac{d^2}{dx^2}[u_e^2] + \nu_{e0}\frac{du_e}{dx} = -\frac{J}{m_e}\cdot u_e^{-1}$$

$$\frac{1}{2}\frac{d^2}{dx^2}[u_e^2] + \nu_{e0}\frac{du_e}{dx} + \frac{J}{m_e}u_e^{-1} = 0$$

$$\frac{d}{dx}\left[u_e\frac{du}{dx}\right] + \nu_{e0}\frac{du_e}{dx} + \frac{J}{m_e}u_e^{-1} = 0$$

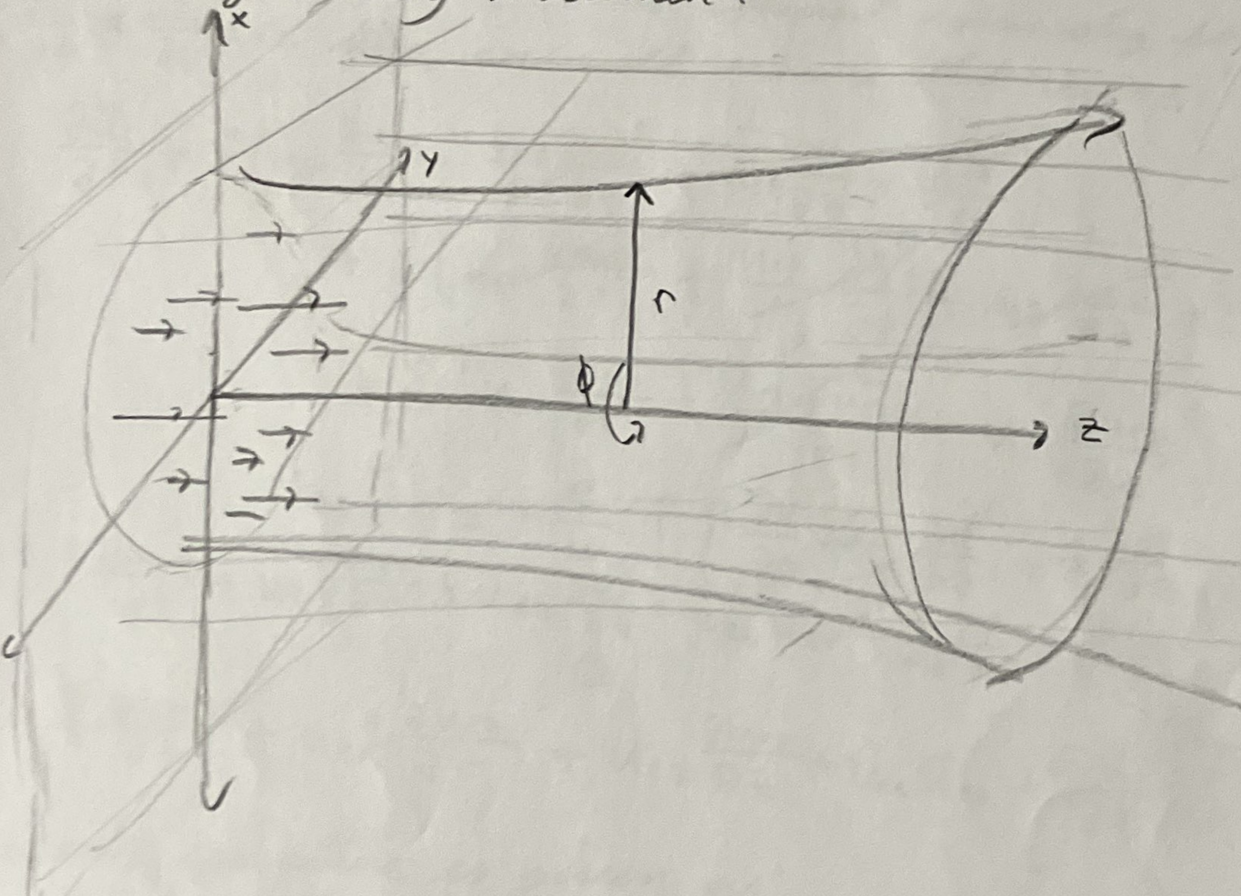
$$\left(\frac{du}{dx}\right)^2 + \frac{d^2u}{dx}u_e + \nu_{e0}\frac{du_e}{dx} + \frac{J}{m_e}u_e^{-1} = 0$$

$$\frac{16}{9} V^{\frac{3}{2}} = 4\lambda d^2$$

$$\frac{4}{9}\frac{V^{3/2}}{d^2} = \frac{J}{\varepsilon_0}\sqrt{\frac{m_e}{2e}}$$

$$\frac{4}{9}\cdot\varepsilon_0\sqrt{\frac{2e}{m_e}}\,\frac{V^{3/2}}{d^2} = J$$, which is the Child-Langmuir Law, as required

Now, lets set about solving our general equation numerically. This will be done in spherical coordinates with spherical symmetry assumed:



Our system is:

$$\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e \cdot \nabla \vec{u}_e + \nu_{e6} \vec{u}_e = -\frac{e}{m_e} \nabla V$$

$$\nabla^2 V = \frac{e n_e}{\varepsilon_0} \quad \text{(poisson's Equation)}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{u}_e) = 0 \quad \text{(Species Continuity Equation)}$$

However, we wish to treat this in cylindrical coordinates; it will help to explicitly write what this means. It helps that we assume spherical symmetry so that $\frac{\partial n_e}{\partial \phi} = \frac{\partial V}{\partial \phi} = \frac{\partial \vec{u}_e}{\partial \phi} = 0$. Also, we will assume that $u_{e\phi} = 0$.

$$\frac{\partial \vec{u}_e}{\partial t} + \hat{r}\left(u_{er}\frac{\partial u_{er}}{\partial r} + \cancel{\frac{u_{e\phi}}{r}\frac{\partial u_{er}}{\partial \phi}}^{0} + u_{ez}\frac{\partial u_r}{\partial z} - \cancel{\frac{u_{e\phi}^2}{r}}^{0}\right)$$
$$+ \hat{\phi}\left(\cancel{u_{er}\frac{\partial u_{e\phi}}{\partial r}}^{0} + \cancel{\frac{u_{e\phi}}{r}\frac{\partial V_\phi}{\partial \phi}}^{0} + \cancel{u_{ez}\frac{\partial u_{e\phi}}{\partial z}}^{0} + \cancel{\frac{u_{e\phi} u_r}{r}}^{0}\right)$$
$$+ \hat{z}\left(u_{er}\frac{\partial u_{ez}}{\partial r} + \cancel{\frac{u_{e\phi}}{r}\frac{\partial u_{ez}}{\partial \phi}}^{0} + u_{ez}\frac{\partial u_{ez}}{\partial z}\right)$$
$$+ \nu_{e6}\vec{u}_e = -\frac{e}{m_e}\left[\hat{r}\frac{\partial V}{\partial r} + \cancel{\hat{\phi}\frac{1}{r}\frac{\partial V}{\partial \phi}}^{0} + \hat{z}\frac{\partial V}{\partial z}\right]$$

$$\begin{pmatrix} \frac{\partial u_{er}}{\partial t} + u_{er}\frac{\partial u_{er}}{\partial r} + u_{ez}\frac{\partial u_r}{\partial z} + \nu_{e6} u_{er} + \frac{e}{m_e}\frac{\partial V}{\partial r} \\ 0 \\ \frac{\partial u_{ez}}{\partial t} + u_{er}\frac{\partial u_{ez}}{\partial r} + u_{ez}\frac{\partial u_{ez}}{\partial z} + \nu_{e6}\vec{u}_{ez} + \frac{e}{m_e}\frac{\partial V}{\partial z} \end{pmatrix} = \vec{0}$$

Poisson's Equation is given as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] + \cancel{\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}}^{0} + \frac{\partial^2 V}{\partial z^2} = \frac{e n_e}{\varepsilon_0}$$

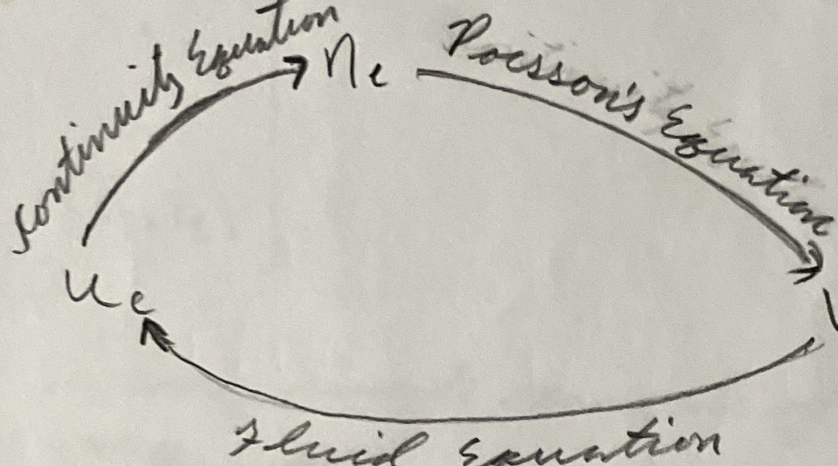
$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] + \frac{\partial^2 V}{\partial z^2} = \frac{e n_e}{\varepsilon_0}$$

And the continuity Equation is:

$$\frac{\partial n_e}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left[r n_e u_{er}\right] + \overset{\to 0}{\cancel{\frac{1}{r}\frac{\partial}{\partial r}\left[n_e u_{e\phi}\right]}} + \frac{\partial}{\partial z}\left[n_e u_{ez}\right] = 0$$

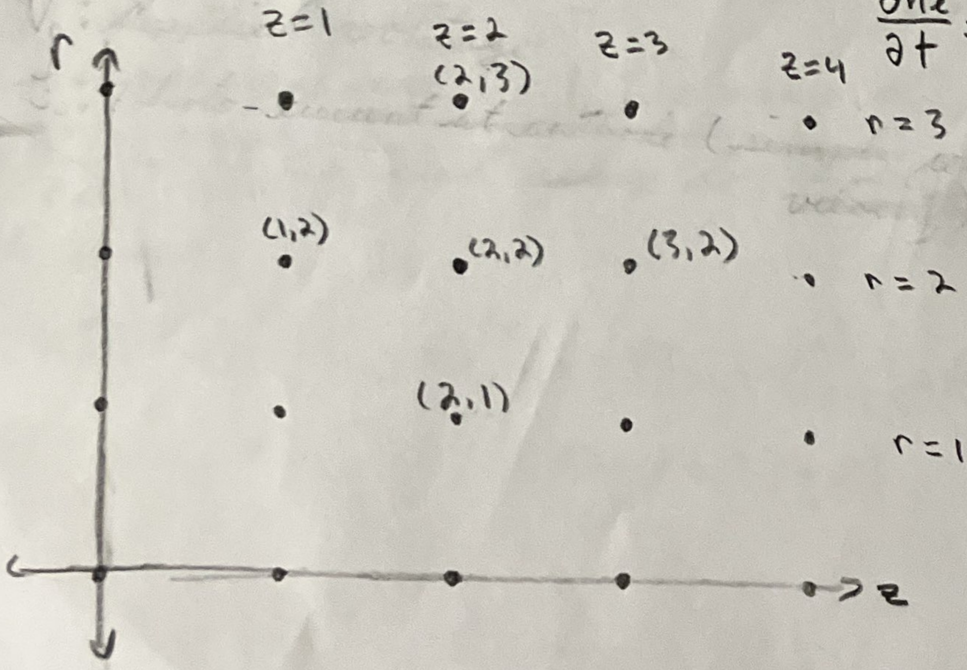
$$\frac{\partial n_e}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left[r n_e u_{er}\right] + \frac{\partial}{\partial z}\left[n_e u_{ez}\right] = 0$$

Our computational Loop will generally look like:



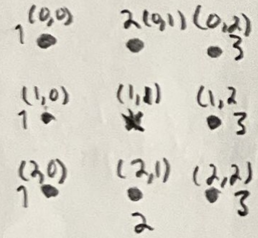
We will begin by developing a solution to the continuity Equation

$$\frac{\partial n_e}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left[r n_e u_{er}\right] - \frac{\partial}{\partial z}\left[n_e u_{ez}\right]$$



$$\vec{V}_n = \begin{bmatrix} V(z_0, r_0) \\ V(z_0, r_1) \\ V(z_0, r_2) \\ \vdots \\ V(z_1, r_0) \\ V(z_1, r_1) \\ \vdots \\ V(z_5, r_5) \end{bmatrix}$$ is a $N = i \cdot j$ dimensional vector

Lets do a very simple example of a 3×3 box;



(Boundaries are known) $j = 0,1,2$ $(J=2)$

(Solving $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \rho$) $l = 0,1,2$ $(L=2)$

Say Boundaries are $V=1$, We map the problem onto $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} - \rho$ and relax to $\frac{\partial V}{\partial t} \to 0$. Then, using Jacobi gives:

$$V_{j,l}^{n+1} = \frac{1}{4}\left[V_{j+1,l}^{n} + V_{j-1,l}^{n} + V_{j,l+1}^{n} + V_{j,l-1}^{n} \right] - \frac{\Delta x \Delta y}{4} \rho_{j,l}$$

If we say that $i = j(L+1) + l = 3j + l$;

$$\vec{V} = (V_{0,0}, V_{0,1}, V_{0,2}, V_{1,0}, V_{1,1}, V_{1,2}, V_{2,0}, V_{2,1}, V_{2,2})$$

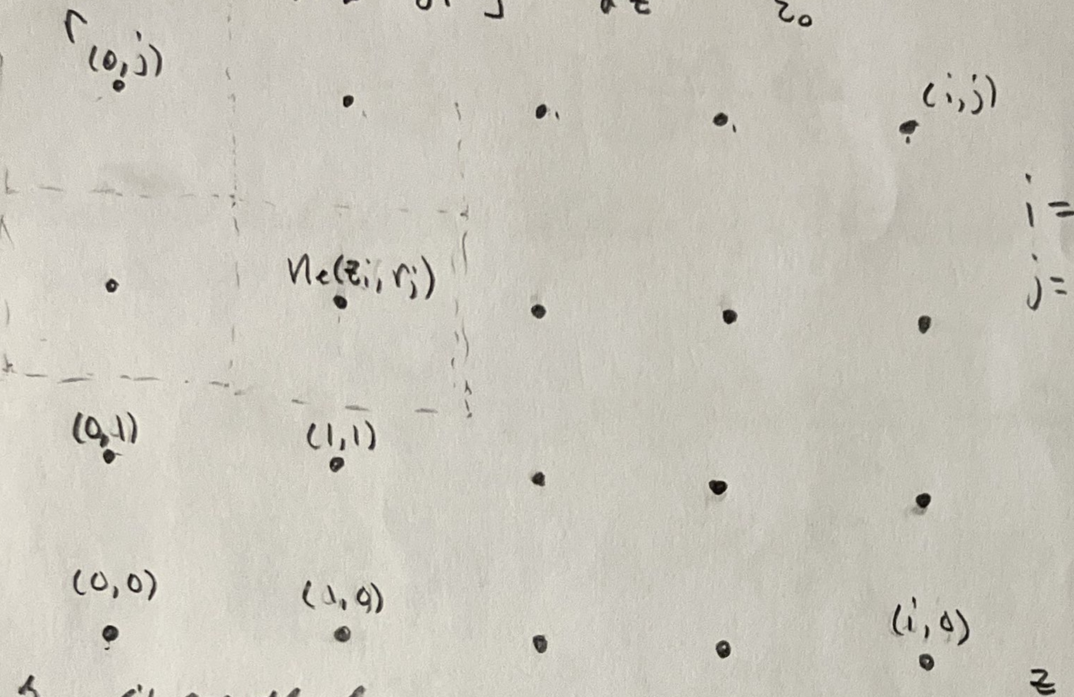
We can use this to form our matrix:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & \frac{1}{4} & 1 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

and $\vec{b} = -\frac{\Delta x \Delta y}{4}(\rho_{0,0}, \rho_{0,1}, \dots, \rho_{2,2})$

Poisson's Equation: Here, we seek a finite difference solution to Poisson's Equation in cylindrical coordinates with no ϕ-dependence:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right]+\frac{\partial^2 V}{\partial z^2}=\frac{e n_e}{\varepsilon_0}$$



$i = 0, 1, 2, 3, 4$

$j = 0, 1, 2, 3, 4$

Jacobi's Method:

$$V_{n+1}(z_i, r_j)=\frac{1}{4}\left[V_n(z_{i+1}, r_j)-V_n(z_{i-1}, r_j)\right]$$
$$+\frac{1}{4r}\left[rV_n(z_i, r_{j+1})-rV(z_i, r_{j-1})\right]$$
$$-\frac{\Delta z\,\Delta r}{4}n_e(z_i, r_j)$$

We let $k \equiv i(J+1)+j=5i+j$ so that $\vec{V}_n^k$ is:

Finite differencing Poisson's Equation in cylindrical coordinates:

We essentially wish to drive the diffusion equation:

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] - \frac{e n_e}{\varepsilon_0}$$

One method is to define $s = \int \frac{dr}{r} = \ln(r)$ $(r = e^s)$. Then, we evaluate:

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial V(s)}{\partial r} - \frac{e n_e}{\varepsilon_0}, \text{ where } \frac{\partial V}{\partial r} = \frac{\partial V}{\partial s}\frac{\partial s}{\partial r};$$

Not sure about this step

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left[\cancel{r}\,\frac{1}{r}\frac{\partial V}{\partial s}\right] - \frac{e n_e}{\varepsilon_0}, \text{ again, } \frac{\partial}{\partial r}\left[\frac{\partial V}{\partial s}\right] = \frac{\partial^2 V}{\partial s^2}\frac{\partial s}{\partial r}\bigg|_{r=s}$$

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial z^2} + e^{-s}\cdot\frac{1}{s}\frac{\partial^2 V}{\partial s^2} - \frac{e n_e}{\varepsilon_0}$$

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial z^2} + \frac{e^{-s}}{s}\frac{\partial^2 V}{\partial s^2} - \frac{e n_e}{\varepsilon_0},$$ which may be finite differenced for $V(z, s)$

Specifically:

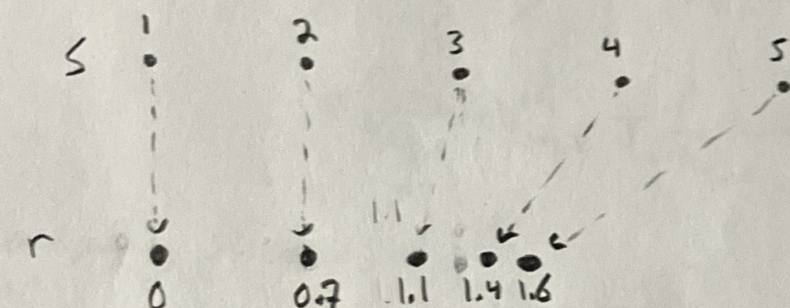
$$V_{j,l}^{n+1} = V_{j,l}^{n} + \frac{\Delta t}{\Delta z^2}\left[V_{j,l+1}^{n} - 2V_{j,l}^{n} + V_{j,l-1}^{n+1}\right] + \frac{\Delta t}{\Delta s^2}\left[\frac{e^{-s_{j,l}}}{s_{j,l}}\left[V_{j+1,l}^{n} - 2V_{j,l}^{n} + V_{j,l-1}^{n+1}\right]\right] - \frac{e\, n_{e\,j,l}}{\varepsilon_0}\Delta t$$

If we set $\Delta t = \frac{\Delta s^2}{4} = \frac{\Delta z^2}{4}$;

$$V^{n+1}_{j,\ell} = V^n_{j,\ell} - \frac{1}{2}V^n_{j,\ell} + \frac{1}{4}\left[V^n_{j,\ell+1} + V^{n+1}_{j,\ell-1}\right] + \frac{1}{4}\frac{e^{-s_{j,\ell}}}{s_{j,\ell}}\left[V^n_{j+1,\ell} - 2V^n_{j,\ell} + V^{n+1}_{j,\ell-1}\right]$$

$$- \frac{e n_{e j,\ell}}{\epsilon_0}\frac{\Delta z \Delta s}{4}$$

This could be made to work; however, upon arriving at a solution and converting back to r using $s = \ln(r)$, the grid will not transform nicely:



We see that the spacing at large r is small, which is the exact opposite of what we would like. However there exists another method we can use.

$$\frac{V^{n+1}_{j,\ell} - V^n_{j,\ell}}{\Delta t} = \frac{1}{\Delta z^2}\left[V^n_{j,\ell+1} - 2^n_{j,\ell} + V^{n+1}_{j,\ell+1}\right]$$

$$+ \frac{1}{r_{j,\ell}}\frac{1}{\Delta r}\left[\frac{r_{j+\frac{1}{2},\ell}\left[V^n_{j+1,\ell} - V^n_{j,\ell}\right] - r_{j-\frac{1}{2},\ell}\left[V^n_{j,\ell} - V^{n+1}_{j-\frac{1}{2},\ell}\right]}{\Delta r}\right]$$

$$- \frac{e n_{e j,\ell}}{\epsilon_0}$$

As before, taking $\Delta t = \frac{\Delta r^2}{4} = \frac{\Delta z^2}{4}$:

$$V^{n+1}_{j,l} = V^{n}_{j,l} + \frac{1}{4}\left[V^{n}_{j,l+1} - 2V^{n}_{j,l} + V^{n+1}_{j,l+1}\right]$$

$$+ \frac{1}{4r_{j,l}}\left[r_{j+\frac{1}{2},l}\left[V^{n}_{j+1,l} - V^{n}_{j,l}\right] - r_{j-\frac{1}{2},l}\left[V^{n}_{j,l} - V^{n+1}_{j-1,l}\right]\right]$$

$$- \frac{e n_{e_{j,l}}}{4\varepsilon_0}\Delta z \Delta s$$

$$= \cancel{V^{n}_{j,l}} - \frac{1}{2}V^{n}_{j,l} + \cancel{\frac{1}{4}V^{n}_{j,l+1}} + \cancel{\frac{1}{4}V^{n+1}_{j,l+1}} - \cancel{\frac{r_{j+\frac{1}{2},l}}{4r_{j,l}}}V^{n}_{j,l} - \cancel{\frac{r_{j-\frac{1}{2},l}}{4r_{j,l}}}V^{n}_{j,l}$$

$$+ \frac{r_{j+\frac{1}{2},l}}{4r_{j,l}}V^{n}_{j+1,l} + \frac{r_{j-\frac{1}{2},l}}{4r_{j,l}}V^{n+1}_{j-1,l} - \frac{e n_{e_{j,l}}}{4\varepsilon_0}\Delta z \Delta r$$

Grouping terms:

$$= \left(\frac{1}{2} - \frac{r_{j+\frac{1}{2},l} + r_{j-\frac{1}{2},l}}{4r_{j,l}}\right)V^{n}_{j,l} + \frac{1}{4}V^{n}_{j,l+1} + \frac{1}{4}V^{n+1}_{j,l-1}$$

$$+ \frac{r_{j+\frac{1}{2},l}}{4r_{j,l}}V^{n}_{j+1,l} + \frac{r_{j-\frac{1}{2},l}}{4r_{j,l}}V^{n+1}_{j-1,l} - \frac{e n_{e_{j,l}}}{4\varepsilon_0}\Delta z \Delta r$$

On a grid with equal spacing, $r_{j+\frac{1}{2},l} = r_{j,l} + \frac{\Delta r}{2}$ and $r_{j-\frac{1}{2},l} = r_{j,l} - \frac{\Delta r}{2}$, so $r_{j+\frac{1}{2},l} + r_{j-\frac{1}{2},l} = 2r_{j,l}$ and:

$$= \frac{1}{4}V^{n}_{j,l+1} + \frac{1}{4}V^{n+1}_{j,l-1} + \frac{r_{j,l} + \frac{\Delta r}{2}}{4r_{j,l}}V^{n}_{j+1,l} + \frac{r_{j,l} - \frac{\Delta r}{2}}{4r_{j,l}}V^{n+1}_{j-1,l}$$

$$- \frac{e n_{e_{j,l}}}{4\varepsilon_0}\Delta z \Delta r$$

And multiplying by 4:

$$4V_{j,\ell}^{n+1} = V_{j,\ell+1}^{n} + V_{j,\ell-1}^{n+1} + \frac{r_{j,\ell} + \frac{\Delta r}{2}}{r_{j,\ell}} V_{j+1,\ell}^{n} + \frac{r_{j,\ell} - \frac{\Delta r}{2}}{r_{j,\ell}} V_{j-1,\ell}^{n+1}$$

$$- \frac{e n_{e_{j,\ell}}}{\varepsilon_0} \Delta z \Delta r$$

Then, we can calculate our weight matrices:

$$a = b = 1$$

$$c = \begin{bmatrix} \frac{r_0 + \frac{\Delta r}{2}}{r_0} & \frac{r_1 + \frac{\Delta r}{2}}{r_1} & \cdots & \frac{r_n + \frac{\Delta r}{2}}{r_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{r_0 + \frac{\Delta r}{2}}{r_0} & \frac{r_1 + \frac{\Delta r}{2}}{r_1} & \cdots & \frac{r_n + \frac{\Delta r}{2}}{r_n} \end{bmatrix}$$

$$d = \begin{bmatrix} \frac{r_0 - \frac{\Delta r}{2}}{r_0} & \frac{r_1 - \frac{\Delta r}{2}}{r_1} & \cdots & \frac{r_n - \frac{\Delta r}{2}}{r_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{r_0 - \frac{\Delta r}{2}}{r_0} & \frac{r_1 - \frac{\Delta r}{2}}{r_1} & \cdots & \frac{r_n - \frac{\Delta r}{2}}{r_n} \end{bmatrix}$$

$$e = -4$$

$$f = \frac{e}{\varepsilon_0} \Delta z \Delta r \, n_{e_{j,\ell}}$$

We must treat the $r=0$ case specially; here, symmetry dictates that $\frac{\partial V}{\partial r}=0$; then:

$$\left.\frac{\partial V}{\partial t}\right|_{r=0} = \frac{\partial^2 V}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right]\Bigg|_{r=0} - \frac{en_e}{\varepsilon_0}$$, by the product rule:

$$= \frac{\partial^2 V}{\partial z^2} + \frac{1}{r}r\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{en_e}{\varepsilon_0}$$

$$= \frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial r^2} + \lim_{r\to 0}\left[\frac{1}{r}\frac{\partial V}{\partial r}\right] - \frac{en_e}{\varepsilon_0}$$, by L'Hôspital's Rule:

$$= \frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial r^2} - \frac{en_e}{\varepsilon_0}$$

$$= \frac{\partial^2 V}{\partial z^2} + 2\frac{\partial^2 V}{\partial r^2} - \frac{en_e}{\varepsilon_0}.$$

Now, finite differencing this we have:

$$\frac{V^{n+1}_{j,0} - V^{n}_{j,0}}{\Delta t} = \frac{1}{\Delta z^2}\left[V^{n}_{j+1,0} - 2V^{n}_{j,0} + V^{n+1}_{e-1,0}\right]$$

$$+ \frac{2}{\Delta r^2}\left[V^{n+1}_{j,1} - 2V^{n}_{j,0} + V^{n+1}_{e,-1}\right]$$ (note that we have switched to looping backwards over ℓ)

$$- \frac{en^{n}_{e j,\ell}}{\varepsilon_0}$$

Then, we can require the fictitious $V^{n}_{-1,e}$ to be:

$$V^{n}_{j,-1} = V^{n+1}_{j,1} \quad //$$

Due to symmetry.

$$\frac{V_{j,0}^{n+1} - V_{j,0}^{n}}{\Delta t} = \frac{1}{\Delta z^2}\left[V_{j+1,0}^{n} - 2V_{j,0}^{n} + V_{j-1,0}^{n+1}\right]$$

$$+ \frac{2}{\Delta r^2}\left[2V_{j,1}^{n+1} - 2V_{j,0}^{n}\right] - \frac{e n_{i0,e}^{n}}{\varepsilon_0}$$

$$V_{j,0}^{n+1} = \frac{\Delta t}{\Delta z^2}\left[V_{j+1,0}^{n} - 2V_{j,0}^{n} + V_{j-1,0}^{n+1}\right]$$

$$+ \frac{4\Delta t}{\Delta r^2}\left[V_{j,1}^{n+1} - V_{j,0}^{n}\right] - \frac{e n_{e0,e}^{n}}{\varepsilon_0}\Delta t + V_{0,e}^{n}$$

Here, we can take $\Delta t = \frac{\Delta r^2}{4} = \frac{\Delta z^2}{4}$;

$$V_{j,0}^{n+1} = \frac{1}{4}\left[V_{j+1,0}^{n} - 2V_{j,0}^{n} + V_{j-1,0}^{n+1}\right]$$

$$+ V_{j,1}^{n+1} - \cancel{V_{j,0}^{n}} - \frac{e n_{e\,j,0}}{4\varepsilon_0}\Delta r \Delta z + \cancel{V_{0,e}^{n}}$$

Finally, combining terms:

$$V_{j,0}^{n+1} = \frac{1}{4}V_{j-1,0}^{n+1} + V_{j,1}^{n+1} + \frac{1}{4}V_{j+1,0}^{n} - \frac{1}{2}V_{j,0}^{n} - \frac{e n_{e\,j,0}}{4\varepsilon_0}\Delta r \Delta z$$

Multiplying by 4:

$$4V_{j,0}^{n+1} = V_{j-1,0}^{n+1} + 4V_{j,1}^{n+1} + V_{j+1,0}^{n} - 2V_{j,0}^{n} - \frac{e n_{e\,j,0}}{\varepsilon_0}\Delta r \Delta z$$

So each row of our matrix is:

$$V_{j+1,0}^{n} + V_{j-1,0}^{n+1} + 4V_{j,1}^{n+1} - 2V_{j,0}^{n} = \frac{e n_{e\,j,e}}{\varepsilon}\Delta r \Delta z$$

So:

$a = 1$ $\quad g = -2$

$b = 1$

$c = 4$

$d = 0$

$e = -4$

Lets reconsider the condition $V^n_{-1,\ell} = V^{n+1}_{1,\ell}$ and instead take $V^n_{-1,\ell} = V^n_{1,\ell}$. Then, we have:

$$\frac{V^{n+1}_{0,\ell} - V^n_{0,\ell}}{\Delta t} = \frac{1}{\Delta z^2}\left[V^n_{0,\ell+1} - 2V^n_{0,\ell} + V^{n+1}_{0,\ell-1}\right]$$
$$+ \frac{2}{\Delta r^2}\left[V^{n+1}_{1,\ell} - 2V^n_{0,\ell} + V^n_{1,\ell}\right] - \frac{e n^n_{e0,\ell}}{\varepsilon_0}$$

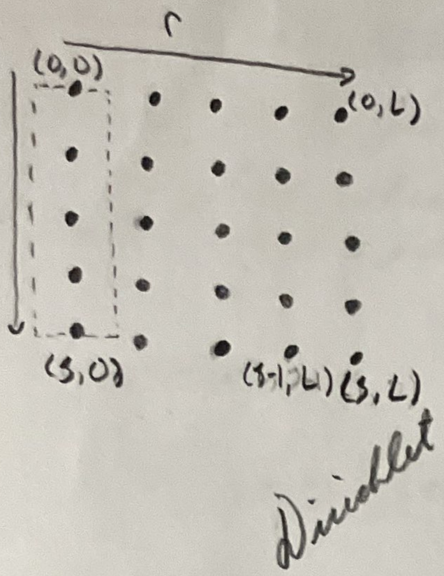
$$V^{n+1}_{0,\ell} - V^n_{0,\ell} = \frac{\Delta t}{\Delta z^2}\left[V^n_{0,\ell+1} - 2V^n_{0,\ell} + V^{n+1}_{0,\ell-1}\right]$$
$$+ \frac{2\Delta t}{\Delta r^2}\left[V^{n+1}_{1,\ell} - 2V^n_{0,\ell} + V^n_{1,\ell}\right] - \frac{e n^n_{e0,\ell}}{\varepsilon_0}\Delta t$$

As before, taking $\Delta t = \frac{\Delta r^2}{4} = \frac{\Delta z^2}{4}$:

$$V^{n+1}_{0,\ell} = \frac{1}{4}V^n_{0,\ell+1} - \frac{1}{2}V^n_{0,\ell} + \frac{1}{4}V^{n+1}_{0,\ell-1} + \frac{1}{2}V^{n+1}_{1,\ell} - \cancel{V^n_{0,\ell}} + \frac{1}{2}V^n_{1,\ell}$$
$$- \frac{e n^n_{0,\ell}}{\varepsilon_0}\frac{\Delta r \Delta z}{4} + \cancel{V^n_{0,\ell}}$$

$$V^{n+1}_{0,\ell} = \frac{1}{4}V^n_{0,\ell+1} - \frac{1}{2}V^n_{0,\ell} + \frac{1}{4}V^{n+1}_{0,\ell-1} + \frac{1}{2}V^{n+1}_{1,\ell} + \frac{1}{2}V^n_{1,\ell} - \frac{e n^n_{0,\ell}}{\varepsilon_0}\frac{\Delta r \Delta z}{4}$$

$$4V^{n+1}_{0,\ell} = V^n_{0,\ell+1} - 2V^n_{0,\ell} + V^{n+1}_{0,\ell-1} + 2V^{n+1}_{1,\ell} + 2V^n_{1,\ell} - \frac{e n^n_{0,\ell}}{\varepsilon_0}\Delta r \Delta z$$



It is the $L=0$ case which we must be careful with

J-loop: We may loop J normally

L-loop: It is the L loop we must be careful with. We must loop from the Dirichlet boundary at L (so the loop starts at $L-1$) to the von Neuman condition at $L=0$, which we must treat specially

So, our outer loop is over j:

FOR $1 \le j \le J-1$:

and the inner loop is backwards over l:

FOR $L-1 \ge l \ge 1$:

$\mathrm{RES} = \mathrm{RES}(j,l)$

$V[j,l] \mathrel{-}= \omega \cdot \dfrac{\mathrm{res}}{e(j,l)}$

Now, we treat the $l=0$ case:

$\mathrm{RES} = \mathrm{RES\_B}(j,0)$

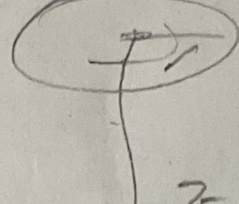
$V[j,0] \mathrel{-}= \omega \cdot \dfrac{\mathrm{res}}{e(j,l)}$

Analytic Solution to Laplace's Equation: Cylindrical Coordinates (RZ-coordinates)

We would like to test our Poisson Solver against some analytic solutions, but first, we must find those solutions. In full cylindrical coordinates, Laplace Equation is:

$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$



Assuming Azimuthal Symmetry means $\frac{\partial^2 V}{\partial \phi^2} = 0$, so:

$$\frac{\partial^2 V}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] = 0$$

Next, we seek to solve this by seperation of variables. We guess that $V(r,z) = R(r)Z(z)$. Then:

$$\frac{\partial^2}{\partial z^2}\left[R(r)Z(z)\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left[R(r)Z(z)\right]\right] = 0$$

$$R\frac{\partial^2 Z}{\partial z^2} + Z\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial R}{\partial r}\right] = 0,$$ dividing through by RZ gives:

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + \frac{1}{rR}\frac{\partial}{\partial r}\left[r\frac{\partial R}{\partial r}\right] = 0.$$

$$\frac{1}{rR}\frac{\partial}{\partial r}\left[r\frac{\partial R}{\partial r}\right] = -\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}$$

As the left has no z-dependence and the right no r-dependence - and this must hold for all r, z, this equality must be some constant K, which gives:

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -k, \text{ or } \frac{\partial^2 Z}{\partial z^2} = -kZ$$

$$\frac{1}{rR}\frac{\partial}{\partial r}\left[r\frac{\partial R}{\partial r}\right] = -k, \text{ or } \frac{\partial}{\partial r}\left[r\frac{\partial R}{\partial r}\right] = -krR$$

$$\frac{1}{rR}\left[r\frac{\partial^2 R}{\partial r^2} + \frac{\partial R}{\partial r}\right] = -k$$

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{rR}\frac{\partial R}{\partial r} = -k$$

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\frac{\partial R}{\partial r} + kr^2 = 0$$

Here, lets make the assumption that $k = \alpha^2 > 0$ (so the solution will be exponential in z). Then:

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\frac{\partial R}{\partial r} + \alpha^2 r^2 = 0$$

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + \alpha^2 R = 0$$

$$\frac{1}{\alpha^2}\frac{\partial^2 R}{\partial r^2} + \frac{1}{r\alpha^2}\frac{\partial R}{\partial r} + R = 0$$

$$\frac{\partial^2 R}{\partial (r\alpha)^2} + \frac{1}{r\alpha}\frac{\partial R}{\partial (r\alpha)} + R = 0$$

Now, we define $\rho = r\alpha$, and take $R(r) = \tilde{R}(\rho)$. Then:

$$\frac{\partial^2 \tilde{R}}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \tilde{R}}{\partial \rho} + \tilde{R} = 0$$

This may be solved with the method of Frobenius. Assume that $\tilde{R} = \sum_{n=0}^{\infty} c_n \rho^{n+s}$ for some coefficients c_n. Then:

$$\frac{\partial^2}{\partial \rho^2}\left[\sum_{n=0}^{\infty} c_n \rho^{n+s}\right] + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\sum_{n=0}^{\infty} c_n \rho^{n+s}\right] + \sum_{n=0}^{\infty} c_n \rho^{n+s} = 0$$

As derivatives are distributive:

$$\sum_{n=0}^{\infty} c_n \frac{\partial^2}{\partial \rho^2}\left[\rho^{n+s}\right] + \sum_{n=0}^{\infty} c_n \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\sum_{n=0}^{\infty} \rho^{n+s}\right] + \sum_{n=0}^{\infty} c_n \rho^{n+s} = 0$$

$$\sum_{n=0}^{\infty} c_n (n+s)(n+s-1)\rho^{n+s-2} + \sum_{n=0}^{\infty} c_n \frac{1}{\rho}(n+s)\rho^{n+s-1} + \sum_{n=0}^{\infty} c_n \rho^{n+s} = 0$$

$$\sum_{n=0}^{\infty} c_n (n+s)(n+s-1)\rho^{n+s-2} + \sum_{n=0}^{\infty} c_n (n+s)\rho^{n+s-2} + \sum_{n=0}^{\infty} c_n \rho^{n+s} = 0$$

$$\sum_{n=0}^{\infty} c_n \left[\left[(n+s)(n+s-1)+(n+s)\right]\rho^{n+s-2} + \rho^{n+s}\right] = 0$$

Multiplying through by ρ^2:

$$\sum_{n=0}^{\infty} c_n \left[\left[(n+s)(n+s-1)+(n+s)\right]\rho^{n+s} + \rho^{n+s+2}\right] = 0$$

$$\sum_{n=0}^{\infty} c_n \left[(n+s)(n+s-1)+(n+s)\right]\rho^{n+s} + \sum_{n=2}^{\infty} c_{n-2}\rho^{n+s} = 0$$

$$c_0\left[s(s-1)+s\right]\rho^s + c_1\left[(1+s)s+(1+s)\right]\rho^{s+1}$$
$$+ \sum_{n=2}^{\infty} c_n\left[(n+s)(n+s-1)+(n+s)\right]\rho^{n+s} + c_{n-2}\rho^{n+s} = 0$$

$$c_0 s^2 \rho^s + c_1\left[s^2+2s+1\right]\rho^{s+1} + \sum_{n=2}^{\infty} c_n\left[(n+s)(n+s-1)+(n+s)\right]\rho^{n+s}$$
$$+ c_{n-2}\rho^{n+s} = 0$$

For this to be true for all ρ, each coefficient must vanish. So:

$$c_0 s^2 = 0$$

$$c_1[s^2+2s+1] = 0$$

If we assume $c_0 \neq 0$, then $s=0$ (and $c_1=0$).
For all other c_n, we have:

$$c_n[(n+s)(n+s-1)+(n+s)]\rho^{n+s} + c_{n-2}\rho^{n+s} = 0, \text{ as } s=0:$$

$$c_n[n(n-1)+n]\cancel{\rho^n} + c_{n-2}\cancel{\rho^n} = 0$$

$$n^2 c_n + c_{n-2} = 0, \text{ which gives the recurrence relation:}$$

$$c_n = -\frac{c_{n-2}}{n^2}$$

And in general:

$$\tilde{R} = \sum_{n=0}^{\infty} c_n \rho^n$$

If we take $c_0=0$, then say $c_1 \neq 0$ so:

$$s^2+2s+1=0$$

$$(s+1)^2=0, \text{ so } s=-1$$

Then, for all other n:

$$c_n[(n-1)(n-2)+(n-1)]\cancel{\rho^{n-1}} + c_{n-2}\cancel{\rho^{n-1}} = 0$$

$$c_n[n^2-3n+2+n-1] = -c_{n-2}$$

$$c_n = -\frac{c_{n-2}}{n^2-2n+1}$$

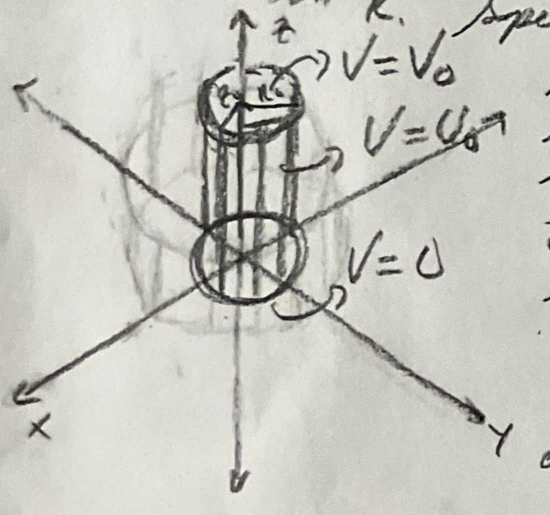
and so:

$$\tilde{R} = \sum_{n=0}^{\infty} c_n \rho^{n-1}$$

Lets focus on the first solution as it relates to R; specifically, $R = \sum_{n=0}^{\infty} c_n (\alpha r)^n$; this is a 0th order Bessel Function of the first kind. Therefore, in R our solution is:

$$R(r) = C_R J_0(\alpha r)$$

Lets construct a problem which uses only this solution in R. Specifically, we take a soda can



that is grounded on the bottom ~~and held at a potential V_0 at $z=z_0$ and $V=0$ at~~ The $V=0$ at $z=0$ solutions leaves us only with a sinh solution in z:

$$Z(z) = C_z \sinh(\alpha z)$$

In r we have:

$$R(r) = C_R J_0(\alpha r)$$

As $V=0$ at $r=r_0$, $J_0(\alpha r_0)=0$; this requires that $\alpha r_0 = \alpha_{01}$ (there is no reason the solution we decrease then increase again in r_0).

Therefore, $\alpha = \frac{\alpha_{01}}{r_0}$ and so:

$$Z(z) = C_z \sinh\left(\frac{\alpha_{01}}{r_0} z\right)$$

and:

$$C_z = \frac{V_0}{\sinh\left(\frac{\alpha_{01}}{r_0} z_0\right)}$$

and we will take a solution:

$$V(r,z) = RZ = \frac{V_0}{\sinh\left(\frac{\alpha_{01}}{r_0} z_0\right)} \sinh\left(\frac{\alpha_{01}}{r_0} z\right) J_0(\alpha r)$$

(The z-boundary is actually $V_0 J_0(\alpha r)$, but that will work). Also, lets take $r_0 = z_0 = 1$ and $V_0 = 1$. Then:

$$V(r,z) = \sinh^{-1}(\alpha_{01}) \cdot \sinh(\alpha_{01} z) J_0(\alpha_{01} r)$$ //

Finite differencing continuity equation (in cylindrical coordinates)

$$\frac{\partial n_e}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left[r n_e \vec{u}_{er}\right] + \frac{\partial}{\partial z}\left[n_e \vec{u}_{ez}\right] = 0$$

$$\frac{n^{n+1}_{ej,l} - n^{n-1}_{ej,l}}{\Delta t} = -\frac{1}{r^n_{j,l}}\frac{1}{\Delta r}\left[r^n_{j+1,l}\, n^n_{ej+1,l}\, u^n_{ej+1,l} - r^n_{j-1,l}\, n^n_{ej-1,l}\, u^n_{ej-1,l}\right]$$

$$-\frac{1}{\Delta z}\left[n^n_{ej,l+1}\, u^n_{ezj,l+1} - n^n_{ej,l-1}\, u^n_{ezj,l-1}\right]$$

$$n^{n+1}_{ej,l} = n^{n-1}_{ej,l} - \frac{1}{r^n_{j,l}}\frac{\Delta t}{\Delta r}\left[r^n_{j+1,l}\, n^n_{ej+1,l}\, u^n_{erj+1,l} - r^n_{j-1,l}\, n^n_{ej-1,l}\, u^n_{erj-1,l}\right]$$

$$-\frac{\Delta t}{\Delta z}\left[n^n_{ej,l+1}\, u^n_{ezj,l+1} - n^n_{ej,l-1}\, u^n_{ezj,l-1}\right]$$

Finally, we must finite difference our fluid equation:

$$\frac{\partial u_{er}}{\partial t} = -u_{er}\frac{\partial u_{er}}{\partial r} - u_{ez}\frac{\partial u_{er}}{\partial z} - \nu_{e0} u_{er} - \frac{e}{m_e}\frac{\partial V}{\partial r}$$

$$\frac{\partial u_{ez}}{\partial t} = -u_{er}\frac{\partial u_{ez}}{\partial r} - u_{ez}\frac{\partial u_{ez}}{\partial z} - \nu_{e0} u_{ez} - \frac{e}{m_e}\frac{\partial V}{\partial z}$$

$$\frac{u^{n+1}_{erj,l} - u^{n-1}_{erj,l}}{\Delta t} = -u^n_{erj,l}\frac{1}{\Delta r}\left[u^n_{erj+1,l} - u^n_{erj-1,l}\right]$$

$$-u^n_{ezj,l}\frac{1}{\Delta z}\left[u^n_{erj,l+1} - u^n_{erj,l-1}\right] - \nu_{e0} u_{erj,l}$$

$$-\frac{e}{m_e}\frac{1}{\Delta r}\left[V^n_{j+1,l} - V^n_{j-1,l}\right]$$

$$u^{n+1}_{erj,l} = u^{n-1}_{erj,l} - u^{n}_{erj,l}\frac{\Delta t}{\Delta r}\left[u^{n}_{erj+1,l} - u^{n}_{erj-1,l}\right]$$

$$- u^{n}_{ezj,l}\frac{\Delta t}{\Delta z}\left[u^{n}_{erj,l+1} - u^{n}_{erl-1}\right] - \nu_{eo} u_{erj,l}\Delta t$$

$$- \frac{e}{m_e}\frac{\Delta t}{\Delta r}\left[V^{n}_{j+1,l} - V^{n}_{j-1,l}\right]$$

We treat the u_{ez} equation similarly:

$$\frac{u^{n+1}_{ezj,l} - u^{n-1}_{ezj,l}}{\Delta t} = - u^{n}_{erj,l}\frac{1}{\Delta r}\left[u^{n}_{ezj+1,l} - u^{n}_{ezj-1,l}\right]$$

$$- u_{ezj,l}\frac{1}{\Delta z}\left[u^{n}_{ezj,l+1} - u^{n}_{ezj,l-1}\right] - \nu_{eo} u^{n}_{ezj,l}$$

$$- \frac{e}{m_e}\frac{1}{\Delta z}\left[V^{n}_{j,l+1} - V^{n}_{j,l-1}\right]$$

$$u^{n+1}_{ezj,l} = u^{n-1}_{ezj,l} - u^{n}_{erj,l}\frac{\Delta t}{\Delta r}\left[u^{n}_{ezj+1,l} - u^{n}_{ezj-1,l}\right]$$

$$- u_{ezj,l}\frac{\Delta t}{\Delta z}\left[u^{n}_{ezj,l+1} - u^{n}_{ezj,l-1}\right] - \nu_{eo} u^{n}_{ezj,l}$$

$$- \frac{e}{m_e}\frac{\Delta t}{\Delta z}\left[V^{n}_{j,l+1} - V^{n}_{j,l-1}\right]$$