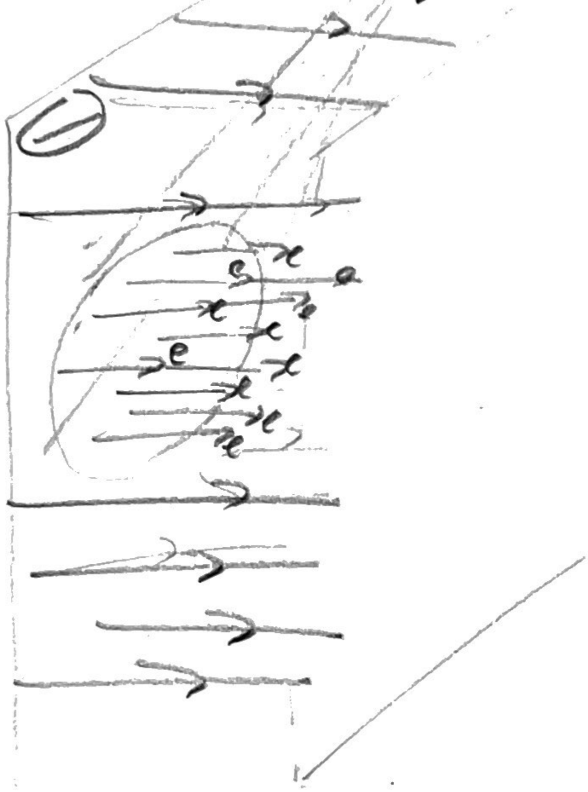
First, we describe the physical set-up of the pre-breakdown dynamics. A voltage bias is applied to one electrode—say the cathode; this bias is variable and depends on the RC timing of the driving circuit. Then, ultraviolet light is shown on the cathode emitting an electron current due to the photo-electric effect. While this may or may not be the case, we shall suppose the current is constant.



We would like to describe the ensuing electron dynamics. First, we shall try to describe the dynamics using two fluid equations; let $n_e(\vec{x})$ be the electron density as a function of position, $\vec{u}_e(\vec{x})$ be the mean electron velocity as a function of position, and $n_0(\vec{x})$ be the background neutral density, which we will assume has no mean velocity. Then, the two fluid equation of motion is:

$$n_e m_e\left[\frac{\partial\vec{u}_e}{\partial t}+\vec{u}_e\cdot\nabla\vec{u}_e\right]=n_e e(\vec{E}+\vec{u}_e\times\vec{B})-\frac{\partial}{\partial\vec{x}}\cdot\overleftrightarrow{P}_e-\vec{R}_{eo}$$

To simplify, we begin by noting that $\vec{E}>>\vec{B}$ due to the applied field, and so we assume that $\vec{B}=0$. For now, suppose that f_e—the vlasov distribution of the electrons—is isotropic. Then, $\frac{\partial}{\partial\vec{x}}\cdot\overleftrightarrow{P}=\nabla P$, where P is the scalar pressure. Our equation of motion is then:

$$n_e m_e\left[\frac{\partial\vec{u}_e}{\partial t}+\vec{u}_e\cdot\nabla\vec{u}_e\right]=n_e e\vec{E}-\nabla P-\vec{R}_{eo}$$

Now, we will further assume that ∇P is negligible—which is a reasonable assumption as $\vec{E}$ is large. Then:

$$n_e m_e\left[\frac{\partial\vec{u}_e}{\partial t}+\vec{u}_e\cdot\nabla\vec{u}_e\right]=en_e\vec{E}-R_{eo}$$

The mean velocity of the neutrals is zero in the lab frame, so say $R_{eo}=\nu_{eo}m_e n_e\vec{u}_e$; then:

$$\cancel{n_e}m_e\left[\frac{\partial\vec{u}_e}{\partial t}+\vec{u}_e\cdot\nabla\vec{u}_e\right]=e\cancel{n_e}\vec{E}-\nu_{eo}m_e\cancel{n_e}\vec{u}_e$$

$$m_e\left[\frac{\partial\vec{u}_e}{\partial t}+\vec{u}_e\cdot\nabla\vec{u}_e\right]=e\vec{E}-\nu_{eo}m_e\vec{u}_e$$

$$\frac{\partial\vec{u}_e}{\partial t}+\vec{u}_e\cdot\nabla\vec{u}_e+\nu_{eo}\vec{u}_e=\frac{e\vec{E}}{m_e},$$ or in terms of the electric potential:

$$\frac{\partial\vec{u}_e}{\partial t}+\vec{u}_e\cdot\nabla\vec{u}_e+\nu_{eo}\vec{u}_e=-\frac{e}{m_e}\nabla V$$

Now, we shall solve this for ~~under~~ certain simplified conditions; the simplist ought to reduce to the

Langmuir-Child space charge limited current. That is, suppose that we reach a steady state current, there are no neutrals so no frictional drag, and we have a one dimensional problem. Then:

$$u\frac{du_e}{dx} = -\frac{e}{m_e}\frac{dV}{dx}, \text{ by the chain rule, } \frac{d}{dx}\left[u^2\right] = 2u\frac{du}{dx}, \text{ so:}$$

$$\frac{1}{2}\frac{d}{dx}\left[u_e^2\right] = -\frac{e}{m_e}\frac{dV}{dx}$$

$$\frac{d}{dx}\left[u_e^2\right] = -\frac{2e}{m_e}\frac{dV}{dx}$$

$$\int \frac{d}{dx}\left[u_e^2\right]dx = -\frac{2e}{m_e}\int\frac{dV}{dx}dx, \text{ and applying the Leibniz Integral Rule:}$$

$$u_e^2 = -\frac{2e}{m_e}V + C, \text{ where } C \text{ is an integration constant}$$

$$u_e = \pm\sqrt{\frac{2e}{m_e}V + C}$$

We would like to relate this to the current density; this is easy enough though as the current density is the amount of charge density moving at a point in space, so:

$$J = -en_eu_e, \text{ where } n_e \text{ is the electron density distribution}$$

And it follows that $n_e = -\frac{J}{eu_e}$; Poisson's Equation gives:

$$\frac{d^2V}{dx^2} = -\frac{en_e}{\varepsilon_0} = \left(-\frac{\cancel{e}}{\varepsilon_0}\right)\cdot\left(\frac{J}{\cancel{e}u_e}\right) = \frac{J}{\varepsilon_0}\left(\frac{2e}{m_e}V + C\right)^{-1/2}$$

For now, we will assume that $u_{e0} = 0$ and $V = 0$ when the electrons leave the cathode so that $C = 0$.

Then, we seek to solve:

$$\frac{d^2V}{dx^2} - \lambda V^{-\frac{1}{2}} = 0, \text{ where } \lambda = \frac{J}{\varepsilon_1}\sqrt{\frac{m_e}{2e}}$$

$\frac{d}{dx}\left[V^{\frac{1}{2}}\right]$

$\frac{1}{2}\frac{dV}{dx}V^{-\frac{1}{2}}$

Introducing the integrating factor $\frac{dV}{dx}$, we have:

$$\frac{dV}{dx}\cdot\frac{d^2V}{dx^2} - \lambda\frac{dV}{dx}V^{-\frac{1}{2}} = 0, \text{ by the chain rule,}$$

$$\frac{dV}{dx}V^{-\frac{1}{2}} = 2\frac{d}{dx}\left[V^{\frac{1}{2}}\right], \text{ so:}$$

$$\frac{dV}{dx}\cdot\frac{d^2V}{dx^2} - 2\lambda\frac{d}{dx}\left[V^{\frac{1}{2}}\right] = 0, \text{ and } \frac{1}{2}\frac{d}{dx}\left[\left(\frac{dV}{dx}\right)^2\right] = \frac{dV}{dx}\frac{d^2V}{dx^2};$$

$$\frac{1}{2}\frac{d}{dx}\left[\left(\frac{dV}{dx}\right)^2\right] - 2\lambda\frac{d}{dx}\left[V^{\frac{1}{2}}\right] = 0$$

$$\int_0^d \frac{d}{dx}\left[\frac{1}{2}\left(\frac{dV}{dx}\right)^2 - 2\lambda V^{\frac{1}{2}}\right]dx = 0, \text{ by the fundamental theorem of calculus;}$$

$$\frac{1}{2}\left(\frac{dV}{dx}\right)^2 - 2\lambda V^{\frac{1}{2}} = 0$$

$$\left(\frac{dV}{dx}\right)^2 - 4\lambda V^{\frac{1}{2}} = 0$$

$$\frac{dV}{dx} - 2\sqrt{\lambda}\, V^{\frac{1}{4}} = 0$$

$$\int_0^V \frac{dV}{V^{\frac{1}{4}}} = \int_0^d 2\sqrt{\lambda}\, dx \quad \text{(where } d \text{ is the gap spacing)}$$

$$\frac{4}{3}V^{\frac{3}{4}}\Big|_0^V = 2\sqrt{\lambda}\, x\Big|_0^d$$

$$\frac{4}{3}V^{\frac{3}{4}} = 2\sqrt{\lambda}\, d$$

$$\frac{16}{9} V^{\frac{3}{2}} = 4 \lambda d^2 \text{, recalling } \lambda = \frac{J}{\varepsilon_0}\sqrt{\frac{m_e}{2e}}:$$

$$\frac{4}{9} V^{\frac{3}{2}} = \frac{J}{\varepsilon_0}\sqrt{\frac{m_e}{2e}}\, d^2$$

$$J = \frac{4}{9}\varepsilon_0 \sqrt{\frac{2e}{m_e}}\,\frac{V^{\frac{3}{2}}}{d^2},$$ which is the space charge limited current

Next, we will remove the requirement that there are no neutrals, but maintain that we achieve a steady state current and that the problem has one dimension. Then, our fluid equation is:

$$u_e \frac{du_e}{dx} + \nu_{e6} u_e = -\frac{e}{m_e}\frac{dV}{dx}$$, using the chain rule:

$$\frac{1}{2}\frac{d}{dx}[u_e^2] + \nu_{e6} u_e = -\frac{e}{m_e}\frac{dV}{dx}$$

Which is coupled to Poisson's equation $\frac{d^2V}{dx^2} = -\frac{n_e}{\varepsilon_0}$. We can decouple this system:

$$\frac{1}{2}\frac{d^2}{dx^2}[u_e^2] + \nu_{e6}\frac{du_e}{dx} = -\frac{e}{m_e}\frac{d^2V}{dx^2}$$

$$\frac{1}{2}\frac{d^2}{dx^2}[u_e^2] + \nu_{e6}\frac{du_e}{dx} = \frac{e}{m_e}\frac{n_e}{\varepsilon_0},$$ where, if we assume a steady current is reached, then $n_e = -\frac{J}{eu_e}$, so:

$$\frac{1}{2}\frac{d^2}{dx^2}[u_e^2] + \nu_{e6}\frac{du_e}{dx} = -\frac{J}{m_e}\cdot u_e^{-1}$$

$$\frac{1}{2}\frac{d^2}{dx^2}[u_e^2] + \nu_{e6}\frac{du_e}{dx} + \frac{J}{m_e}u_e^{-1} = 0$$

$$\frac{d}{dx}\left[u_e\frac{du}{dx}\right] + \nu_{e6}\frac{du_e}{dx} + \frac{J}{m_e}u_e^{-1} = 0$$

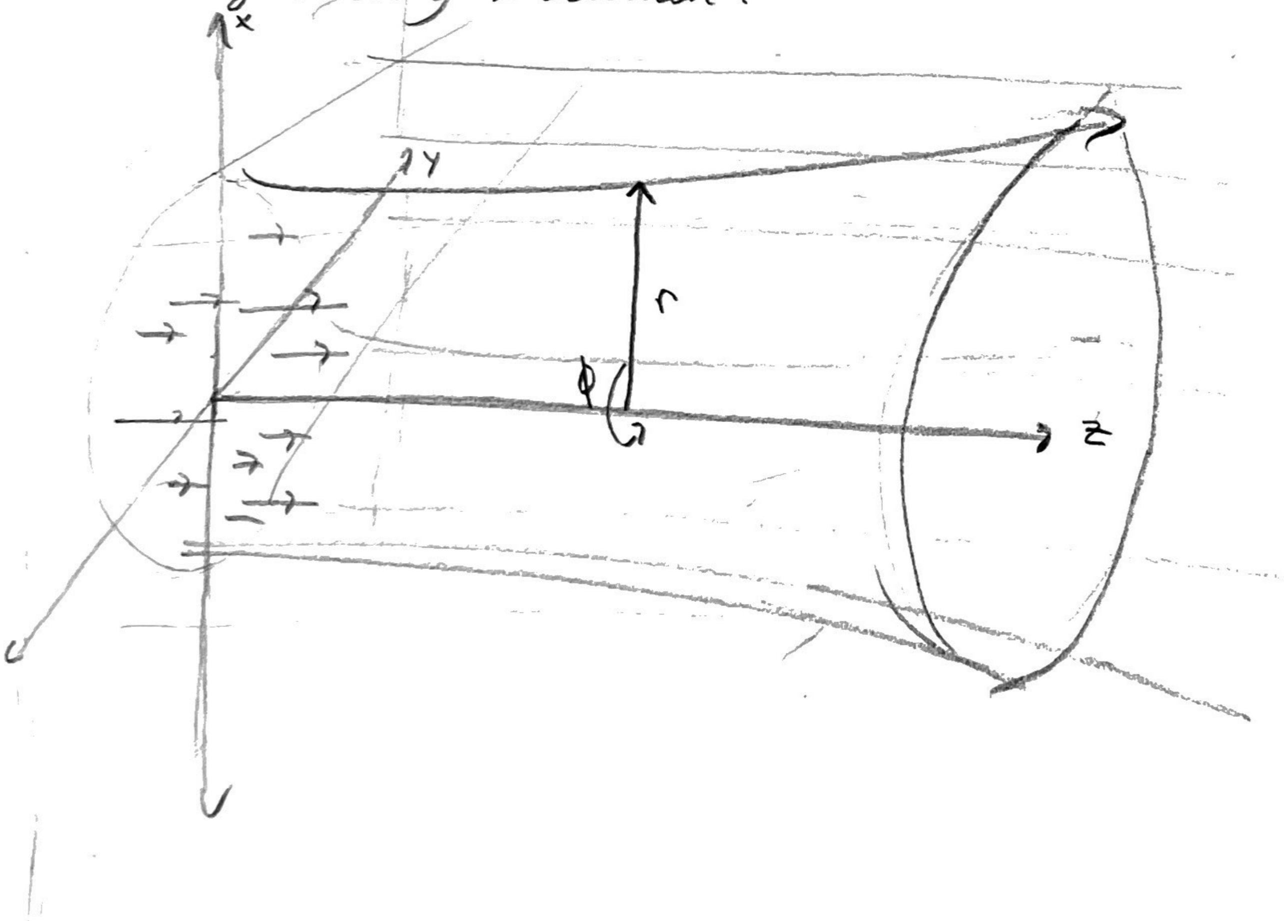
$$\left(\frac{du}{dx}\right)^2 + \frac{d^2u}{dx}u_e + \nu_{e6}\frac{du_e}{dx} + \frac{J}{m_e}u_e^{-1} = 0$$

$$\frac{16}{9} V^{\frac{3}{2}} = 4 \lambda d^2$$

$$\frac{4}{9} \frac{V^{3/2}}{d^2} = \frac{J}{\varepsilon_0} \sqrt{\frac{m_e}{2e}}$$

$$\frac{4}{9} \cdot \varepsilon_0 \sqrt{\frac{2e}{m_e}} \frac{V^{3/2}}{d^2} = J$$, which is the Child-Langmuir Law, as required

Now, lets set about solving our general equation numerically. This will be done in spherical coordinates with spherical symmetry assumed:



Our system is:

$$\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e \cdot \nabla \vec{u}_e + \nu_{es} \vec{u}_e = -\frac{e}{m_e} \nabla V$$

$$\nabla^2 V = \frac{e n_e}{\varepsilon_0} \quad \text{(poisson's Equation)}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{u}_e) = 0 \quad \text{(Species Continuity Equation)}$$

However, we wish to treat this in cylindrical coordinates; it will help to explicitly write what this means. It helps that we assume spherical symmetry so that $\frac{\partial n_e}{\partial \phi} = \frac{\partial V}{\partial \phi} = \frac{\partial \vec{u}_e}{\partial \phi} = 0$. Also, we will assume that $u_{e\phi} = 0$.

$$\frac{\partial \vec{u}_e}{\partial t} + \hat{r}\left(u_{er}\frac{\partial u_{er}}{\partial r} + \cancel{\frac{u_{e\phi}}{r}\frac{\partial u_{er}}{\partial \phi}}^{0} + u_{ez}\frac{\partial u_r}{\partial z} - \cancel{\frac{u_{e\phi}^2}{r}}^{0} \right)$$
$$+ \hat{\phi}\left(\cancel{u_{er}\frac{\partial u_{e\phi}}{\partial r}}^{0} + \cancel{\frac{u_{e\phi}}{r}\frac{\partial V_\phi}{\partial \phi}}^{0} + \cancel{u_{ez}\frac{\partial u_{e\phi}}{\partial z}}^{0} + \cancel{\frac{u_{e\phi} u_r}{r}}^{0} \right)$$
$$+ \hat{z}\left(u_{er}\frac{\partial u_{ez}}{\partial r} + \cancel{\frac{u_{e\phi}}{r}\frac{\partial u_{ez}}{\partial \phi}}^{0} + u_{ez}\frac{\partial u_{ez}}{\partial z} \right)$$
$$+ \nu_{es}\vec{u}_e = -\frac{e}{m_e}\left[\hat{r}\frac{\partial V}{\partial r} + \hat{\phi}\cancel{\frac{1}{r}\frac{\partial V}{\partial \phi}}^{0} + \hat{z}\frac{\partial V}{\partial z} \right]$$

$$\begin{pmatrix} \frac{\partial u_{er}}{\partial t} + u_{er}\frac{\partial u_{er}}{\partial r} + u_{ez}\frac{\partial u_r}{\partial z} + \nu_{es} u_{er} + \frac{e}{m_e}\frac{\partial V}{\partial r} \\ 0 \\ \frac{\partial u_{ez}}{\partial t} + u_{er}\frac{\partial u_{ez}}{\partial r} + u_{ez}\frac{\partial u_{ez}}{\partial z} + \nu_{es}\vec{u}_{ez} + \frac{e}{m_e}\frac{\partial V}{\partial z} \end{pmatrix} = \vec{0}$$

Poisson's Equation is given as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r} \right] + \cancel{\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}}^{0} + \frac{\partial^2 V}{\partial z^2} = \frac{e n_e}{\varepsilon_0}$$

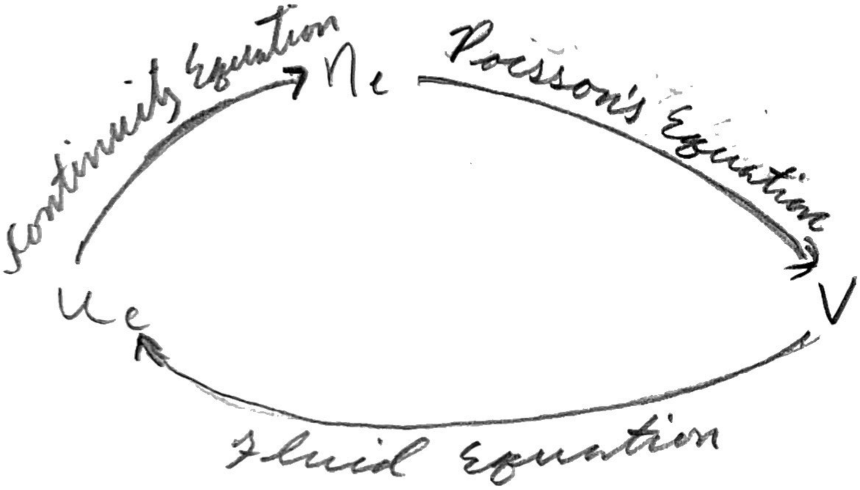
$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r} \right] + \frac{\partial^2 V}{\partial z^2} = \frac{e n_e}{\varepsilon_0}$$

And the continuity Equation is:

$$\frac{\partial n_e}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left[r n_e u_{er}\right] + \cancel{\frac{1}{r}\frac{\partial}{\partial r}\left[n_e u_{e\phi}\right]}^{0} + \frac{\partial}{\partial z}\left[n_e u_{ez}\right] = 0$$

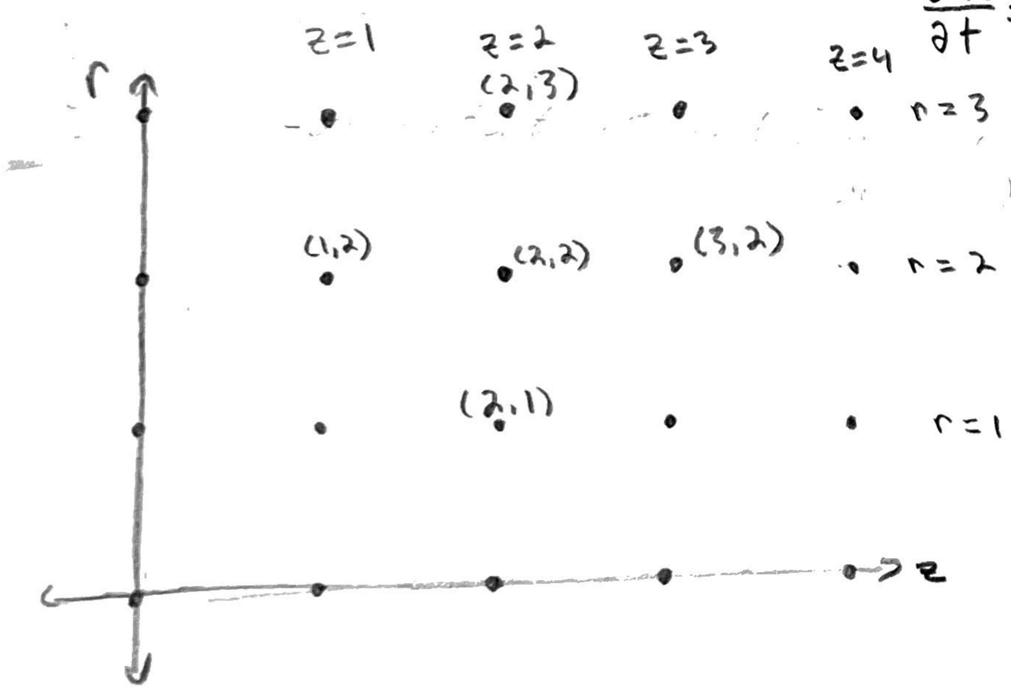
$$\frac{\partial n_e}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left[r n_e u_{er}\right] + \frac{\partial}{\partial z}\left[n_e u_{ez}\right] = 0$$

Our computational loop will generally look like:



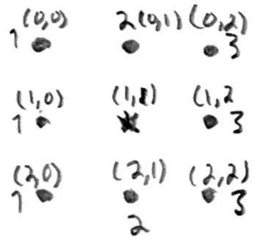
We will begin by developing a solution to the continuity Equation

$$\frac{\partial n_e}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left[r n_e u_{er}\right] - \frac{\partial}{\partial z}\left[n_e u_{ez}\right]$$



$$\vec{V}_n = \begin{bmatrix} V(z_0, r_0) \\ V(z_0, r_1) \\ V(z_0, r_2) \\ \vdots \\ V(z_1, r_0) \\ V(z_1, r_1) \\ \vdots \\ V(z_5, r_5) \end{bmatrix}$$ is a $N = i \cdot j$ dimensional vector

Lets do a very simple example of a 3×3 box;



(Boundaries are known) $j = 0,1,2 \ (J=2)$

(Solving $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \rho$) $\ell = 0,1,2 \ (L=2)$

Say Boundaries are $V=1$. We map the problem onto $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} - \rho$ and relax to $\frac{\partial V}{\partial t} \to 0$. Then, using Jacobi gives:

$$V^{n+1}_{j,\ell} = \frac{1}{4}\left[V^n_{j+1,\ell} + V^n_{j-1,\ell} + V^n_{j,\ell+1} + V^n_{j,\ell-1} \right] - \frac{\Delta x \Delta y}{4} \rho_{j,\ell}$$

If we say that $i = j(L+1) + \ell = 3j + \ell$;

$$\vec{V} = (V_{0,0}, V_{0,1}, V_{0,2}, V_{1,0}, V_{1,1}, V_{1,2}, V_{2,0}, V_{2,1}, V_{2,2})$$

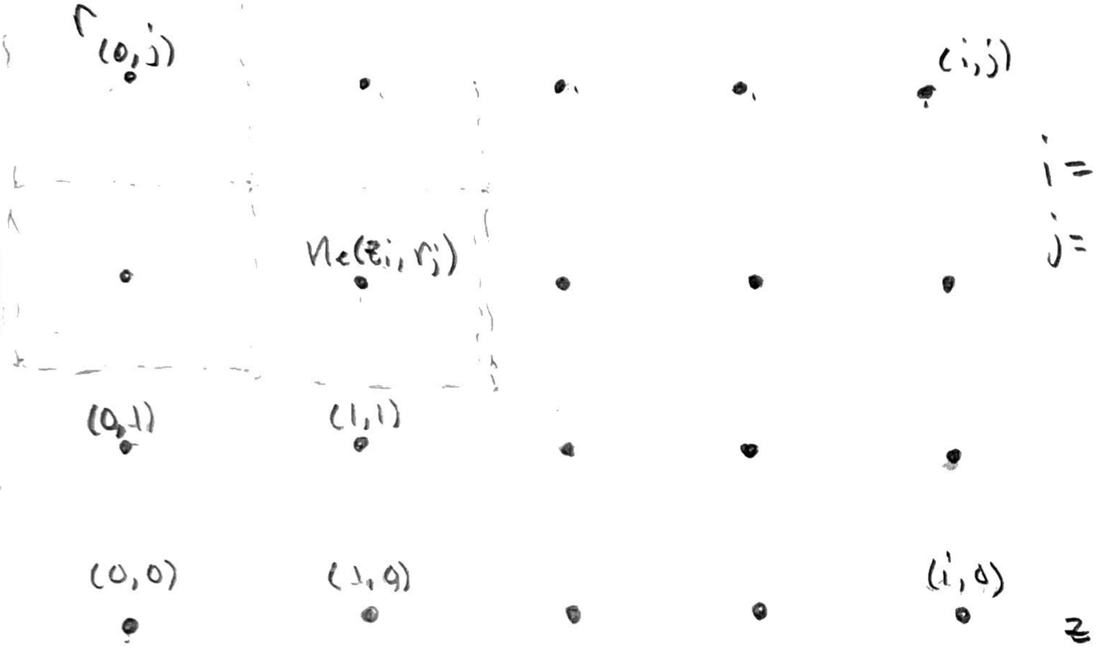
We can use this to form our matrix:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & \frac{1}{4} & 1 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

and $\vec{b} = -\frac{\Delta x \Delta y}{4}(\rho_{0,0}, \rho_{0,1}, \dots, \rho_{2,2})$

Poisson's Equation: Here, we seek a finite difference solution to Poisson's Equation in cylindrical coordinates with no ϕ-dependence:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right]+\frac{\partial^2 V}{\partial z^2}=\frac{e n_e}{\varepsilon_0}$$



$i = 0, 1, 2, 3, 4$

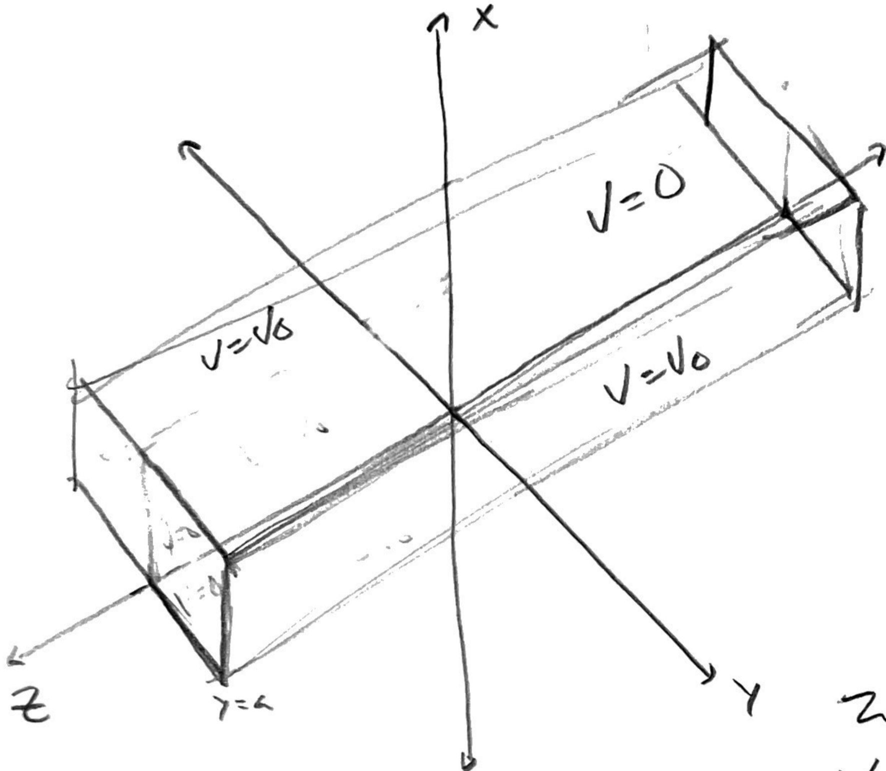
$j = 0, 1, 2, 3, 4$

Jacobi's Method:

$$V_{n+1}(z_i, r_j) = \frac{1}{4}\left[V_n(z_{i+1}, r_j) - V_n(z_{i-1}, r_j)\right]$$
$$+\frac{1}{4r}\left[rV_n(z_i, r_{j+1}) - rV(z_i, r_{j-1})\right]$$
$$-\frac{\Delta z \Delta r}{4} n_e(z_i, r_j)$$

We let $k \equiv i(J+1)+j = 5i+j$ so that $\vec{V}_n^k$ is:

We test our poisson solver in xy against a simple example problem:



The potential is $V=0$ at $x=0$, $x=a$ and $V=V_0$ at $y=0$, $y=a$ (V_0 is a constant). We seek to solve Laplace's Equation:

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$$

We guess that $V(x,y)=X(x)Y(y)$ for some X and Y. Then:

$$\frac{\partial^2}{\partial x^2}\left[X(x)Y(y)\right]+\frac{\partial^2}{\partial y^2}\left[X(x)Y(y)\right]=0$$

$$Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}=0$$, dividing by XY:

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0$$

$$\frac{1}{X}\frac{d^2X}{dx^2}=-\frac{1}{Y}\frac{d^2Y}{dy^2}=K^2$$

X and Y constant, this gives the general solutions:

$$X(x) = A\sin(kx) + B\cos(kx)$$

$$Y(y) = C\sinh(ky) + D\cosh(ky)$$

As Y is symmetric, we have $Y(y) = D\cosh(ky)$; As $V = 0$ at $x = 0$, we have $X(x) = A\sin(kx)$; as $V = 0$ at $x = a$, we also require $k = \frac{n\pi}{a}$ where n is an integer. Then:

$$V(x,y) = C\cosh\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n\pi}{a}x\right),$$ so generally:

$$V(x,y) = \sum C_n \cosh\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n\pi}{a}x\right)$$

And we require that $V(x,a) = V_0$, so:

$$\sum C_n \cosh(n\pi)\sin\left(\frac{n\pi}{a}x\right) = V_0$$

Each coefficient is given by:

$$C_n\cosh(n\pi) = \int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy,$$ as $V_0(y) = V_0$;

$$C_n\cosh(n\pi) = \frac{2V_0}{a}(1-\cos(n\pi))$$

$$C_n\cosh(n\pi) = \begin{cases} 0, & n \text{ is even} \\ \frac{4V_0}{a}, & n \text{ is odd} \end{cases}$$

So:

$$V(x,y) = \sum_{n=1,3,5\ldots}^{\infty} \frac{4V_0}{a\cosh(n\pi)} \cdot \cosh\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n\pi}{a}x\right)$$

We will test this for $a=1$, $V_0=1$. Then:

$$V(x,y) = \sum_{n=1,3,5\ldots}^{\infty} \frac{4}{\cosh(n\pi)} \cdot \cosh(n\pi y)\sin(n\pi x)$$

Our grid is defined on $y:[-1,1]$, $x:[0,1]$

We really are trying to find a solution in cylindrical coordinates with no ϕ dependence; then Laplace's equation is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] + \frac{\partial^2 V}{\partial z^2} = 0$$

Assuming this is seperable as $V(r,z) = R(r)Z(z)$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left[R(r)Z(z)\right]\right] + \frac{\partial^2}{\partial z^2}\left[R(r)Z(z)\right] = 0$$

$$\frac{Z(z)}{r}\frac{\partial}{\partial r}\left[r\frac{\partial R}{\partial r}\right] + R(r)\frac{\partial^2 Z}{\partial z^2} = 0$$, dividing through by $R(r)Z(z)$:

$$\frac{1}{rR}\frac{\partial}{\partial r}\left[r\frac{\partial R}{\partial r}\right] + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

Which gives the seperable equations:

$$\frac{d}{dr}\left[r\frac{dR}{dr}\right] = KrR$$

$$\frac{dR}{dr} + r\frac{d^2R}{dr^2} \pm KrR = 0$$

The solutions to the R equation is:

$R = J_0(Kr), N_0(Kr)$, where J_0 and N_0 are Bessel functions

and the solution to the z equation is:

$$Z = Ae^{kz} + Be^{-kz} \quad \text{or} \quad Z = Ae^{ikz} + Be^{-ikz}$$

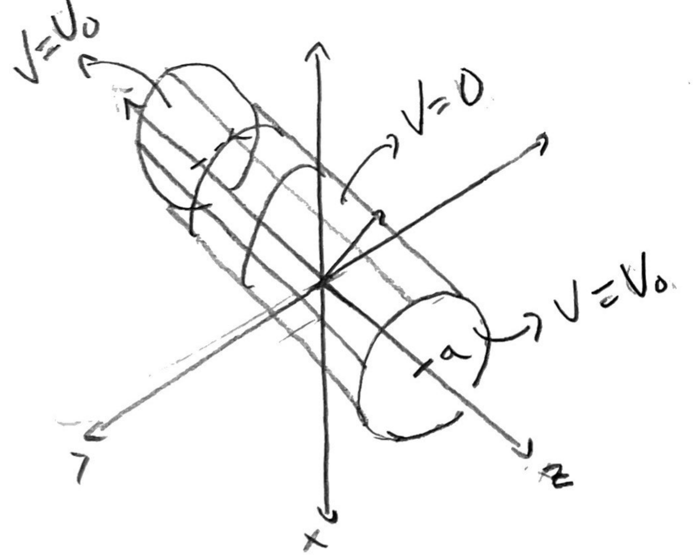
$J_0(Kr)$ is given by:

$$J_0(kr) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!}\left(\frac{kr}{2}\right)^{2n},$$ so, for instance;

$$R(r) = -\frac{1}{2}\left(\frac{Kr}{2}\right)^2 = -\frac{K^2}{8}r^2$$ should be a solution

$$\frac{d^2}{dr^2}\left[-\frac{k^2}{8}r^2\right] + \frac{1}{r}\frac{d}{dr}\left[-\frac{k^2}{8}r^2\right] \pm KK\left(-\frac{k^2}{8}r^2\right) = 0$$

As a simple example, consider the following example:



We require $V = V_0$ at $z = -a$ and $z = a$ and $V = 0$ at $r = a$.

Our solution is $V(r,z) = R(r)Z(z)$ where:

$$Z(z) = A\sinh(Kz) + B\cosh(Kz)$$

As $Z = 0$ at $-a$ and a, $A = 0$. R is given by:

$$R(r) = J_0(Kr)$$

Our general solution is then:

$$V(r,z) = \sum_{n=0}^{\infty} B_n J_0(K_n r)\cosh(K_n z), \quad K_n = \frac{a_n}{a} \text{ where}$$

a_n is nth zero of J_0

At $z=a$, we require $V(r,a) = V_0$. The orthogonality condition for Bessel functions is:

$$\int_0^a r\,dr\, J_\nu\left(\frac{a_{\nu k}r}{a}\right) J_\nu\left(\frac{a_{\nu k'}r}{a}\right) = \frac{a^2}{2}\left[J_{\nu+1}(a_{\nu k})\right]^2 \delta_{kk'}$$

where $a_{\nu k}$ are the zeros of the Bessel functions.
So, for $\nu = 0$:

$$\int_0^a r\,dr\, J_0\left(\frac{a_k r}{a}\right) J_0\left(\frac{a_{k'} r}{a}\right) = \frac{a^2}{2}\left[J_1(a_n)\right]^2 \delta_{kk'}$$

We then can use Fourier's Trick:

$$\int_0^a V_0 J_0\left(\frac{a_n r}{a}\right) r\,dr = B_n \cosh(a_n) \frac{a^2}{2}\left[J_1(a_n)\right]^2$$

$u = r, \quad dv = J_0\left(\frac{a_n r}{a}\right)$

$$B_n = \frac{2V_0}{\left[J_1(a_n)\right]^2 a^2 \cosh(a_n)} \int_0^a J_0\left(\frac{a_n r}{a}\right) r\,dr,$$ where a_n is the nth root of J_0

Finite differencing Poisson's Equation in cylindrical coordinates:

We essentially wish to drive the diffusion equation:

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] - \frac{e n_e}{\varepsilon_0}$$

One method is to define $s = \int \frac{dr}{r} = \ln(r)$ $(r = e^s)$. Then, we evaluate:

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial V(s)}{\partial r} - \frac{e n_e}{\varepsilon_0}, \text{ where } \frac{\partial V}{\partial r} = \frac{\partial V}{\partial s}\frac{\partial s}{\partial r};$$

Not sure about this step

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left[\cancel{r}\,\cancel{\frac{1}{r}}\frac{\partial V}{\partial s}\right] - \frac{e n_e}{\varepsilon_0}, \text{ again, } \frac{\partial}{\partial r}\left[\frac{\partial V}{\partial s}\right] = \frac{\partial^2 V}{\partial s^2}\frac{\partial s}{\partial r}\Big|_{r=s}$$

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial z^2} + e^{-s}\cdot\frac{1}{s}\frac{\partial^2 V}{\partial s^2} - \frac{e n_e}{\varepsilon_0}$$

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial z^2} + \frac{e^{-s}}{s}\frac{\partial^2 V}{\partial s^2} - \frac{e n_e}{\varepsilon_0},$$ which may be finite differenced for $V(z, s)$

Specifically:

$$V^{n+1}_{j,\ell} = V^{n}_{j,\ell} + \frac{\Delta t}{\Delta z^2}\left[V^{n}_{j,\ell+1} - 2V^{n}_{j,\ell} + V^{n+1}_{j,\ell-1}\right] + \frac{\Delta t}{\Delta s^2}\left[\frac{e^{-s_{j,\ell}}}{s_{j,\ell}}\left[V^{n}_{j+1,\ell} - 2V^{n}_{j,\ell} + V^{n+1}_{j,\ell-1}\right] - \frac{e\, n_{e\,j,\ell}}{\varepsilon_0}\Delta t\right.$$

If we set $\Delta t = \frac{\Delta s^2}{4} = \frac{\Delta z^2}{4}$;

$$V_{j,l}^{n+1} = V_{j,l}^{n} - \frac{1}{2}V_{j,l}^{n} + \frac{1}{4}\left[V_{j,l+1}^{n} + V_{j,l-1}^{n+1}\right] + \frac{1}{4}\frac{e^{-s_{j,l}}}{s_{j,l}}\left[V_{j+1,l}^{n} - 2V_{j,l}^{n} + V_{j,l-1}^{n+1}\right]$$

$$- \frac{e n_{e\,j,l}}{\varepsilon_0}\frac{\Delta z \Delta s}{4}$$

This could be made to work; however, upon arriving at a solution and converting back to r using $s = \ln(r)$, the grid will not transform nicely:

We see that the spacing at large r is small, which is the exact opposite of what we would like. However there exists another method we can use.

$$\frac{V_{j,l}^{n+1} - V_{j,l}^{n}}{\Delta t} = \frac{1}{\Delta z^2}\left[V_{j,l+1}^{n} - 2_{j,l}^{n} + V_{j,l+1}^{n+1}\right]$$

$$+ \frac{1}{r_{j,l}}\frac{1}{\Delta r}\left[\frac{r_{j+\frac{1}{2},l}\left[V_{j+1,l}^{n} - V_{j,l}^{n}\right] - r_{j-\frac{1}{2},l}\left[V_{j,l}^{n} - V_{j-1,l}^{n+1}\right]}{\Delta r}\right]$$

$$- \frac{e n_{e\,j,l}}{\varepsilon_0}$$

As before, taking $\Delta t = \frac{\Delta r^2}{4} = \frac{\Delta z^2}{4}$:

$$V_{j,l}^{n+1} = V_{j,l}^{n} + \frac{1}{4}\left[V_{j,l+1}^{n} - 2V_{j,l}^{n} + V_{j,l+1}^{n+1}\right]$$

$$+ \frac{1}{4r_{j,l}}\left[r_{j+\frac{1}{2},l}\left[V_{j+1,l}^{n} - V_{j,l}^{n}\right] - r_{j-\frac{1}{2},l}\left[V_{j,l}^{n} - V_{j-1,l}^{n+1}\right]\right]$$

$$- \frac{e n_{e j,l}}{4\varepsilon_0}\Delta z \Delta s$$

$$= \cancel{V_{j,l}^{n}} - \frac{1}{2}V_{j,l}^{n} + \cancel{\frac{1}{4}V_{j,l+1}^{n}} + \cancel{\frac{1}{4}V_{j,l+1}^{n+1}} - \frac{r_{j+\frac{1}{2},l}}{4r_{j,l}}V_{j,l}^{n} - \frac{r_{j-\frac{1}{2},l}}{4r_{j,l}}V_{j,l}^{n}$$

$$+ \frac{r_{j+\frac{1}{2},l}}{4r_{j,l}}V_{j+1,l}^{n} + \frac{r_{j-\frac{1}{2},l}}{4r_{j,l}}V_{j-1,l}^{n+1} - \frac{e n_{e j,l}}{4\varepsilon_0}\Delta z \Delta r$$

Grouping terms:

$$= \left(\frac{1}{2} - \frac{r_{j+\frac{1}{2},l} + r_{j-\frac{1}{2},l}}{4r_{j,l}}\right)V_{j,l}^{n} + \frac{1}{4}V_{j,l+1}^{n} + \frac{1}{4}V_{j,l-1}^{n+1}$$

$$+ \frac{r_{j+\frac{1}{2},l}}{4r_{j,l}}V_{j+1,l}^{n} + \frac{r_{j-\frac{1}{2},l}}{4r_{j,l}}V_{j-1,l}^{n+1} - \frac{e n_{e j,l}}{4\varepsilon_0}\Delta z \Delta r$$

On a grid with equal spacing, $r_{j+\frac{1}{2},l} = r_{j,l} + \frac{\Delta r}{2}$ and $r_{j-\frac{1}{2},l} = r_{j,l} - \frac{\Delta r}{2}$, so $r_{j+\frac{1}{2},l} + r_{j-\frac{1}{2},l} = 2r_{j,l}$ and:

$$= \frac{1}{4}V_{j,l+1}^{n} + \frac{1}{4}V_{j,l-1}^{n+1} + \frac{r_{j,l} + \frac{\Delta r}{2}}{4r_{j,l}}V_{j+1,l}^{n} + \frac{r_{j,l} - \frac{\Delta r}{2}}{4r_{j,l}}V_{j-1,l}^{n+1}$$

$$- \frac{e n_{e j,l}}{4\varepsilon_0}\Delta z \Delta r$$

And multiplying by 4:

$$4V_{j,\ell}^{n+1} = V_{j,\ell+1}^{n} + V_{j,\ell-1}^{n+1} + \frac{r_{j,\ell} + \frac{\Delta r}{2}}{r_{j,\ell}} V_{j+1,\ell}^{n} + \frac{r_{j,\ell} - \frac{\Delta r}{2}}{r_{j,\ell}} V_{j-1,\ell}^{n+1}$$
$$- \frac{e n_{e_{j,\ell}}}{\varepsilon_0} \Delta z \Delta r$$

Then, we can calculate our weight matrices:

$$a = b = 1$$

$$c = \begin{bmatrix} \frac{r_0 + \frac{\Delta r}{2}}{r_0} & \frac{r_1 + \frac{\Delta r}{2}}{r_1} & \cdots & \frac{r_n + \frac{\Delta r}{2}}{r_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{r_0 + \frac{\Delta r}{2}}{r_0} & \frac{r_1 + \frac{\Delta r}{2}}{r_1} & \cdots & \frac{r_n + \frac{\Delta r}{2}}{r_n} \end{bmatrix}$$

$$d = \begin{bmatrix} \frac{r_0 - \frac{\Delta r}{2}}{r_0} & \frac{r_1 + \frac{\Delta r}{2}}{r_1} & \cdots & \frac{r_n + \frac{\Delta r}{2}}{r_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{r_0 - \frac{\Delta r}{2}}{r_0} & \frac{r_1 + \frac{\Delta r}{2}}{r_1} & \cdots & \frac{r_n + \frac{\Delta r}{2}}{r_n} \end{bmatrix}$$

$$e = -4$$

$$f = \frac{e}{\varepsilon_0} \Delta z \Delta r \, n_{e_{j,\ell}}$$

Finite differencing continuity equation (in cylindrical

$$\frac{\partial n_e}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left[r n_e \vec{u}_{er}\right] + \frac{\partial}{\partial z}\left[n_e \vec{u}_{ez}\right] = 0$$

$$\frac{n^{n+1}_{ej,\ell} - n^{n-1}_{ej,\ell}}{\Delta t} = -\frac{1}{r^n_{j,\ell}}\frac{1}{\Delta r}\left[r^n_{j+1,\ell}\, n^n_{ej+1,\ell}\, u^n_{erj+1,\ell} - r^n_{j-1,\ell}\, n^n_{ej-1,\ell}\, u^n_{erj-1,\ell}\right]$$

$$- \frac{1}{\Delta z}\left[n^n_{ej,\ell+1} u^n_{ezj,\ell+1} - n^n_{ej,\ell-1} u^n_{ezj,\ell-1}\right]$$

$$n^{n+1}_{ej,\ell} = n^{n-1}_{ej,\ell} - \frac{1}{r^n_{j,\ell}}\frac{\Delta t}{\Delta r}\left[r^n_{j+1,\ell}\, n^n_{ej+1,\ell}\, u^n_{erj+1,\ell} - r^n_{j-1,\ell}\, n^n_{ej-1,\ell}\, u^n_{erj-1,\ell}\right]$$

$$- \frac{\Delta t}{\Delta z}\left[n^n_{ej,\ell+1} u^n_{ezj,\ell+1} - n^n_{ej,\ell-1} u^n_{ezj,\ell-1}\right]$$

Finally, we must finite diffeunce our fluid equation:

$$\frac{\partial u_{er}}{\partial t} = -u_{er}\frac{\partial u_{er}}{\partial r} - u_{ez}\frac{\partial u_{er}}{\partial z} - \nu_{eo} u_{er} - \frac{e}{m_e}\frac{\partial V}{\partial r}$$

$$\frac{\partial u_{ez}}{\partial t} = -u_{er}\frac{\partial u_{ez}}{\partial r} - u_{ez}\frac{\partial u_{ez}}{\partial z} - \nu_{eo} u_{ez} - \frac{e}{m_e}\frac{\partial V}{\partial z}$$

$$\frac{u^{n+1}_{erj,\ell} - u^{n-1}_{erj,\ell}}{\Delta t} = -u^n_{erj,\ell}\frac{1}{\Delta r}\left[u^n_{erj+1,\ell} - u^n_{erj-1,\ell}\right]$$

$$- u^n_{ezj,\ell}\frac{1}{\Delta z}\left[u^n_{erj,\ell+1} - u^n_{erj,\ell-1}\right] - \nu_{eo} u_{erj,\ell}$$

$$- \frac{e}{m_e}\frac{1}{\Delta r}\left[V^n_{j+1,\ell} - V^n_{j-1,\ell}\right]$$

$$u_{erj,l}^{n+1} = u_{erj,l}^{n-1} - u_{erj,l}^{n} \frac{\Delta t}{\Delta r}\left[u_{erj+1,l}^{n} - u_{erj-1,l}^{n}\right]$$

$$- u_{ezj,l}^{n} \frac{\Delta t}{\Delta z}\left[u_{erj,l+1}^{n} - u_{erl-1}^{n}\right] - \nu_{e\delta} u_{erj,l} \Delta t$$

$$- \frac{e}{m_e} \frac{\Delta t}{\Delta r}\left[V_{j+1,l}^{n} - V_{j-1,l}^{n}\right]$$

We treat the u_{ez} equation similarly:

$$\frac{u_{ezj,l}^{n+1} - u_{ezj,l}^{n-1}}{\Delta t} = - u_{erj,l}^{n} \frac{1}{\Delta r}\left[u_{ezj+1,l}^{n} - u_{ezj-1,l}^{n}\right]$$

$$- u_{ezj,l} \frac{1}{\Delta z}\left[u_{ezj,l+1}^{n} - u_{ezj,l-1}^{n}\right] - \nu_{e\delta} u_{ezj,l}^{n}$$

$$- \frac{e}{m_e} \frac{1}{\Delta z}\left[V_{j,l+1}^{n} - V_{j,l-1}^{n}\right]$$

$$u_{ezj,l}^{n+1} = u_{ezj,l}^{n-1} - u_{erj,l}^{n} \frac{\Delta t}{\Delta r}\left[u_{ezj+1,l}^{n} - u_{ezj-1,l}^{n}\right]$$

$$- u_{ezj,l} \frac{\Delta t}{\Delta z}\left[u_{ezj,l+1}^{n} - u_{ezj,l-1}^{n}\right] - \nu_{e\delta} u_{ezj,l}^{n}$$

$$- \frac{e}{m_e} \frac{\Delta t}{\Delta z}\left[V_{j,l+1}^{n} - V_{j,l-1}^{n}\right]$$