Liam Keeley
ENWT Chapter 4 Summary
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4.1 Chapter 4 begins by introducing the notion of a continuous transformation between coordinate systems. Such transformations are built using a mapping T that transforms the original independent variable and mappings Q^μ which transform the original dependent variables:

$$\begin{aligned} t \rightarrow t' =& T(t, q^\mu, \epsilon) \\ q^\mu \rightarrow q^{\mu\prime} =& Q^\mu(t, q^\nu, \epsilon) \end{aligned}$$

Where epsilon is a a parameter which can be continuously varied.

In general, a functional is covariant under a transformation. That is, if Γ is the old functional, under some transformation, the new functional Γ' is:

$$\Gamma' = \int_a^{\prime b} {}' L(t', q^{\mu\prime}(t'), \frac{dq^{\mu\prime}(t')}{dt'}) dt'$$

And the form of the Lagrangian remains the same. We can define an **infinitesimal transformation** of any transformation by expanding the original transformation in a Taylor Series about $\epsilon = 0$:

$$\begin{aligned} t' =& t + \epsilon(\frac{dT}{d\epsilon})_0 + ... \\ q^{\mu\prime} =& q^\mu + \epsilon(\frac{dQ^\mu}{d\epsilon})_0 + ... \end{aligned}$$

And we can further define $\tau \equiv (\frac{dT}{d\epsilon})_0$ and $\zeta \equiv (\frac{dQ^\mu}{d\epsilon})_0$. These τ and ζ are know as **generators** of the transformation.

A set of continuously parameterized transformations, if the set contains and identity, and each element has an inverse, forms a **Lie group**. Finally, the definition of invariance is given as:

Invariance: The functional:

$$\Gamma = \int_a^b L dt$$

is invariant under the infintesimal transformation:

$$\begin{aligned} t' =& t + t + ... \\ q^{\mu\prime} =& q^\mu + \zeta^\mu + ... \end{aligned}$$

If and only if $\Gamma' - \Gamma \sim \epsilon^s$

Or equivently, the functional is invariant if:

$$L' \frac{dt'}{dt} - L \sim \epsilon^s$$

4.2 Section 4.2 gives the Invariance Identity. That is:

Invariance Identity The functional:

$$\Gamma = \int_a^b L dt$$

is invariant under the infintesimal transformation:

$$\begin{aligned} t' =& t + t + ... \\ q^{\mu\prime} =& q^\mu + \zeta^\mu + ... \end{aligned}$$

if and only if the following identity holds:

$$\frac{\partial L}{\partial q^\mu} \zeta^\mu + p_\mu \dot{\zeta}^\mu + \frac{\partial L}{\partial t} \tau - H \dot{\tau} = 0$$

The proof of this starts by differentiating the definition of of invariance given above and proceeding from there. The following form is equivalent:

$$-(\zeta^\mu - \dot{q^\mu} \tau) \left[\frac{\partial L}{\partial q^m u} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q^\mu}} \right] = \frac{d}{dt} \left[p_\mu \zeta^\mu - H \tau \right]$$

4.3 Because the funtional:

$$\Gamma = \int_a^b L dt$$

is unchanged by adding $\frac{dS}{dt}$ as long as S(b) = S(a) (the term will be lost integration), so in the definition of invariance, this term must be accounted for. The invariance identity can the be stated in its **divergence form** as:

$$\frac{\partial L}{\partial q^{\mu}}\zeta^{\mu} + p_{\mu}\dot{\zeta}^{\mu} + \frac{\partial L}{\partial t}\tau - H\dot{\tau} - \frac{dF}{dt} = 0$$