4.3

<u>Claim:</u> The two forms of the Invariance Identity:

$$\frac{\partial L}{\partial q^\mu}\zeta^\mu + p_\mu \zeta^\mu + \frac{\partial L}{\partial t}\tau - H\dot{\tau} = 0 \qquad (4.37)$$

and:

$$-(\zeta^\mu - \dot{q}^\mu \tau)\left[\frac{\partial L}{\partial q^\mu} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\mu}\right] = \frac{d}{dt}\left[p_\mu \zeta^\mu - H\tau\right] \qquad (4.38)$$

are equivalent.

<u>Proof:</u> We begin with the differentiated form of invariance 4.36:

$$L\dot{\tau} + \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q^\mu}\zeta^\mu + \frac{\partial L}{\partial \dot{q}^\mu}(\dot{\zeta}^\mu - \dot{q}^\mu \dot{\tau}) = 0$$

First, to arrive at 4.37 we note that $p^\mu = \frac{\partial L}{\partial \dot{q}^\mu}$ and $H = p_\mu \dot{q}^\mu - L$. Then:

$$\frac{\partial L}{\partial q^\mu}\zeta^\mu + p_\mu \dot{\zeta}^\mu + \frac{\partial L}{\partial t}\tau + \dot{\tau}(L - p_\mu \dot{q}^\mu) = 0$$

$$\frac{\partial L}{\partial q^\mu}\zeta^\mu + p_\mu \dot{\zeta}^\mu + \frac{\partial L}{\partial t}\tau - H\dot{\tau} = 0$$

Next, to get to 4.38, we notice that, by the product rule, $\frac{d}{dt}[p_\mu \zeta^\mu] = \dot{p}_\mu \zeta^\mu + p_\mu \dot{\zeta}^\mu$ and $\frac{d}{dt}[H\tau] = \dot{H}\tau + H\dot{\tau}$. Furthermore, $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^\mu}\right] = \dot{p}^\mu$. Then, 4.38 becomes:

$$-\zeta^\mu \frac{\partial L}{\partial q^\mu} + \cancel{\dot{p}_\mu \zeta^\mu} - \dot{q}_\mu \dot{p}^\mu \tau + \dot{q}_\mu \frac{\partial L}{\partial q^\mu}\tau = \cancel{\dot{p}_\mu \zeta^\mu} + p_\mu \dot{\zeta}^\mu - \dot{H}\tau - H\dot{\tau}$$

$$\left[\dot{H} - \dot{q}_\mu \dot{p}^\mu + \dot{q}_\mu \frac{\partial L}{\partial q^\mu}\right]\tau = \zeta^\mu \frac{\partial L}{\partial q^\mu} + p_\mu \dot{\zeta}^\mu - H\dot{\tau}$$

Here, we note that $\dot{H} = \frac{d}{dt}\left[\dot{q}_\mu p^\mu - L\right] = \ddot{q} p^\mu + \dot{q}_\mu \dot{p}^\mu - \frac{dL}{dt}$, where, by the chain rule, $\frac{dL}{dt} = \frac{\partial L}{\partial t} + \dot{q}_\mu \frac{\partial L}{\partial q^\mu} + \ddot{q}\frac{\partial L}{\partial \dot{q}^\mu}$

$$\left[\cancel{\ddot{q}_\mu p^\mu} + \cancel{\dot{q}_\mu \dot{p}^\mu} - \frac{\partial L}{\partial t} - \cancel{\dot{q}_\mu \frac{\partial L}{\partial q^\mu}} - \cancel{\dot{q}_\mu \dot{p}^\mu} - \cancel{\ddot{q}_\mu p^\mu} + \cancel{\frac{\partial L}{\partial q_\mu}}\right]\tau = \zeta^\mu \frac{\partial L}{\partial q^\mu} + p_\mu \dot{\zeta}^\mu - H\dot{\tau}$$

$$0 = \frac{\partial L}{\partial t}\tau + p_\mu \dot{\zeta}^\mu - H\dot{\tau}$$

Hence, the two identities are equivalent.

Q.E.D.

4.4

<u>Claim</u>: The functional:

$$\Gamma = \int_a^b L(t, q^\mu, \dot{q}^\mu)\, dt$$

is divergent invariant under the infintesimal transformation

$$t' = t + \varepsilon\tau + \dots$$
$$q'^\mu = q^\mu + \varepsilon\zeta^\mu + \dots$$

if and only if:

$$\frac{\partial L}{\partial q^\mu} + p_\mu \dot{\zeta}^\mu + \frac{\partial L}{\partial t}\tau - H\dot{\tau} - \frac{dF}{dt} = 0$$

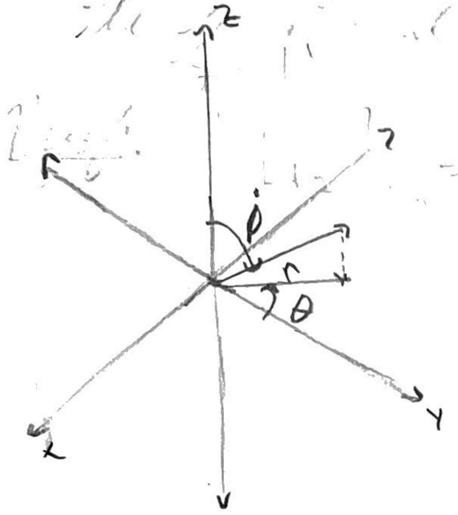
Where $F(t)$ is some function of the independent variable t.

<u>Proof</u>: We begin by considering Γ and some $\Gamma' = \int_a^b \left(L + \frac{dF}{dt}\right) dt$; then, by the Fundamental Theorem of Calculus:

$$\Gamma' = \Gamma + [F(b) - F(a)]$$

Therefore, if $F(b) = F(a)$, $\Gamma' = \Gamma$. As a and b are arbitrary, we must allow for this possibility. As F is a function of t, the Lagrangian $L^* = L + \frac{dF}{dt}$ is also a valid Lagrangian. Then, applying the definition of invariance:

4.8



A central force is a force that acts radially, and therefore can be written $F = -\nabla U(r)$, where $U(r)$ is a potential energy function of the radius r. In spherical coordinates, we can write the functional of Hamilton's Principle:

$$S = \int_a^b \left(\frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2 + (r\dot{\phi})^2\right] - U(r)\right)dt$$

We next consider a change of longitude and latitude, which essentially a rotation; in spherical coordinates, this can be represented by an infintesimal transform:

$$t' = t$$
$$\theta' = \theta + \varepsilon$$
$$\phi' = \phi + \varepsilon$$
$$r' = r$$

So that $\tau = 0$, $\zeta^\theta = 1$, $\zeta^\phi = 1$, and $\zeta^r = 0$. As $\tau = \dot{\tau} = 0$, and each $\dot{\zeta}^\mu = 0$, the invariance identity reduces to:

$$\frac{\partial L}{\partial \theta}\zeta^\theta + \frac{\partial L}{\partial \phi}\zeta^\phi + \frac{\partial L}{\partial r}\zeta^r = 0,$$

which as $U(r)$ only depends on r, is clearly satisfied