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ENWT Chapter 3 Summary
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3.1 Chapter 3 begins by motivating the Euler-Lagrange formalism with an example of using Hamilton's principle to derive $y(t) = -\frac{1}{2}gt^2$. The result depends on invoking the condition that $y(t) \leq 0$ for $t > 0$, which is in essence a boundary condition; therefore, the connection between extremals and differential equations is made.

The Euler-Lagrange equation is given as follows:

**Euler-Lagrange Equations:** Let $\Gamma$ be the functional:

$$\Gamma = \int_a^b L(t, q^\mu, \dot{q^\mu})dt$$

Whose Lagrangian $L$ depends on one independent variable $t$ and N dependent variables $q^\mu(t)$ and their first derivatives $\dot{q^\mu}(t)$ for $\mu = 1, 2, ..., N$.

When all N dependent variables are independent of each other, then there are $N$ **degrees of freedom**. Regardless, the $N$ set of **generalized coordinates** form an $N$ dimensional vector in an N-dimensional vector space. Then, the set $q^\mu(t)$, or alternatively the components of $\vec{q}$, which make $\Gamma$ an extremal, are solutions to the $N$ Euler-Lagrange equations:

$$\frac{\partial L}{\partial q^\mu} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q^\mu}},\ \mu = 1, 2, ..., N$$

The proof of this theorem assumes the existence of a $\vec{q}$ which makes $\Gamma$ extremal, and then imagines varying each $q^\mu$ through some arbitrary path and parameter. This transforms the functional $\Gamma$ to a function of that parameter; then, that $\Gamma$ is extremal requires that the derivative of $\Gamma$ with respect to that parameter vanishes when the parameter is zero.

The term $\frac{\partial L}{\partial \dot{q^\mu}}$ is called the **canonical momentum**, $p^\mu = \frac{\partial L}{\partial \dot{q^\mu}}$. We can also define another function, the **Hamiltonian**, which is given by:

$$H(t, q^\mu, p^\mu) = p_\mu \dot{q^\mu} - L$$

In terms of the Hamiltonian, the Euler-Lagrange equations can be written:

$$\frac{\partial L}{\partial t} = -\dot{H}$$

This follows by evaluating $\frac{dL}{dt}$, the total derivative of the Lagrangian with respect to time.

$$
\begin{aligned}
\frac{dL}{dt} =& \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q^\mu}\frac{\partial q^\mu}{\partial t} + \frac{\partial L}{\partial \dot{q^\mu}}\frac{\partial \dot{q^\mu}}{\partial t} \\
\frac{dL}{dt} =& \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q^\mu}\dot{q^\mu} + p^\mu \ddot{q^\mu}, \text{ as } \dot{p^\mu} = \frac{\partial L}{\partial q^\mu}: \\
\frac{dL}{dt} =& \frac{\partial L}{\partial t} + \dot{p^\mu}\dot{q^\mu} + p^\mu \ddot{q^\mu}, \text{ by the product rule, } \dot{p^\mu}\dot{q^\mu} + p^\mu\ddot{q^\mu} = \frac{d}{dt}\left[p^\mu \dot{q^\mu}\right], \text{ so:} \\
\frac{dL}{dt} -& \frac{d}{dt}\left[p^\mu \dot{q^\mu}\right] = \frac{\partial L}{\partial t}, \text{ or:} \\
\frac{\partial L}{\partial t} =& -\dot{H}
\end{aligned}
$$

While the Hamiltonian often numerically equals the total mechanical energy of a system, it is important to note that it is a function not a number. Also, as the Lagrangian and the Hamiltonian have the same dimensions, if the Lagrangian has dimensions of energy, the the Hamiltonian will always be the energy of something, even if that something is not necessarily the total energy of the system.

3.2 A number of conservation laws directly follow the Euler-Lagrange equations. First, from $\frac{\partial L}{\partial q^\mu} = \dot{p^\mu}$, it follows that if $\frac{\partial L}{\partial q^\mu} = 0$, the canonical momentum $p^\mu$ is conserved.

Similarly, as $\frac{\partial L}{\partial t} = -\dot{H}$, if $\frac{\partial L}{\partial t} = 0$ (if the Lagrangian does not depend explicitly on time), then the Hamiltonian is conserved.

3.3 This section shows that Hamilton's Principle and Newton's Second Law are equivalent.

**Hamilton's Principle:** Of all possible trajectories a particle might take between two fixed times $a$ and $b$, the path actually taken is the one for which the time integral of the difference between kinetic and potential energies is minimized.

That is, in one dimension:

$$\int_a^b \frac{1}{2}m \cdot x^2 - U(x)dt = min$$

Or, applying the Euler-Lagrange equation:

$$-\frac{\partial U}{\partial x} = m\ddot{x}$$

Which is Newton's Second Law.

Conversely, starting with Newton's Second Law in rectangular coordinates:

$$-\frac{\partial U}{\partial x^\mu} = m\ddot{x}$$

We can make a coordinate change to the generalize coordinates $q^\mu$. We know that each $x^\mu$ can be written as a function of each $q^\mu$. Then, from the chain rule:

$$\dot{x^\mu} = \frac{\partial x^\mu}{\partial q^\nu}\dot{q^\mu}$$

Then, differentiating with respect to $\dot{q^\rho}$ and noting that $\frac{\partial \dot{q^\nu}}{\partial \dot{q^\rho}} = \delta^\nu_\rho$ as each $q^\mu$ is independent, it follows that:

$$\frac{\partial \dot{x^\mu}}{\dot{q^\rho}} = \frac{\partial x^\mu}{\partial q^\rho}$$

Finally:

$$\begin{aligned}
-\frac{\partial U}{\partial x_\nu}\cdot\frac{\partial x^\mu}{\partial q^\nu} =& m\ddot{x}_\mu\cdot\frac{\partial x^\mu}{\partial q^\nu}, \text{ by the chain rule:}\\
-\frac{\partial U}{\partial q^\nu} =& m\ddot{x}_\mu\cdot\frac{\partial x^\mu}{\partial q^\nu}, \text{by the product rule:}\\
-\frac{\partial U}{\partial q^\nu} =& \frac{d}{dt}\left[m\dot{x}_\mu\frac{\partial x^\mu}{\partial q^\nu}\right] - m\dot{x}_\mu\frac{d}{dt}\frac{\partial x^\mu}{\partial q^\nu}, \text{ recalling that } \frac{\partial x^\mu}{\partial q^\nu} = \frac{\partial \dot{x^\mu}}{\partial \dot{q^\nu}}:\\
-\frac{\partial U}{\partial q^\nu} =& \frac{d}{dt}m\dot{x}_\mu\frac{\partial \dot{x^\mu}}{\partial \dot{q^\nu}} - m\dot{x}_\mu\frac{d}{dt}\frac{\partial x^\mu}{\partial q^\nu}\\
-\frac{\partial U}{\partial q^\nu} =& \frac{d}{dt}(\frac{\partial K}{\partial \dot{q^\nu}}) - \frac{\partial K}{\partial q^\nu}\\
\frac{\partial (K-U)}{\partial q^\nu} =& \frac{d}{dt}\frac{\partial (K-U)}{\partial \dot{q^\nu}}
\end{aligned}$$

I skipped a few steps there, but the result is that Newton's Law implies Hamilton's Principle. However, Hamilton's Principle can be generalized to more general systems; Newton's Second Law is in essence a special case of Hamilton's Principle.

3.6 This section treats **constraints** between the $N$ independent variables. First, if there $A$ equations of constraint, then there are only $N - A$ truly independent variables.

To treat the constrained variables as part of the Lagrangian, the constraints must be added to the Lagrangian to form a **constrained Lagrangian**, $L_c$. To do this, write an equation of constraint:

$$h(t, x^\mu) = 0$$

And then the constrained Lagrangian is:

$$L_c = L + \lambda h$$

Where $\lambda$ is called a Lagrange multiplier, and will have dimensions so that the $\lambda h$ has the same dimensions as the Lagrangian. Any number of constraint functions (less than $N$ of course) can be added with corresponding Lagrange multipliers. Otherwise, the constrained Lagrangian can be treated the same as for a normal Lagrangian.