使用深度强化学习实现无人艇集群合围算法的证明

2023 年 3 月 10 日

1 背景介绍与引理

1.1

考虑一个由 n 艘无人艇组成的多无人艇系统，每一个艇的运动学模型可以表示为:

$$\dot{x}_i = u_i \cos\varphi_i - v_i \sin\varphi_i$$

$$\dot{y}_i = u_i \sin\varphi_i + v_i \cos\varphi_i$$

$$\dot{\varphi}_i = r_i$$

其动力学模型可以表示为:

$$\dot{u}_i = k_1 u_i + k_2 v_i r_i + k_3 \tau_{1,i}$$

$$\dot{r}_i = k_4 r_i + k_5 \tau_{2,i}$$

$$\dot{v}_i = k_6 v_i + k_7 u_i r_i$$

另外:

$$\rho_i = \sqrt{(x_i - x_o)^2 + (y_i - y_o)^2}$$

$$\theta_i = \arctan 2(y_i - y_o, x_i - x_o) + 2k\pi \in [0, 2\pi)$$

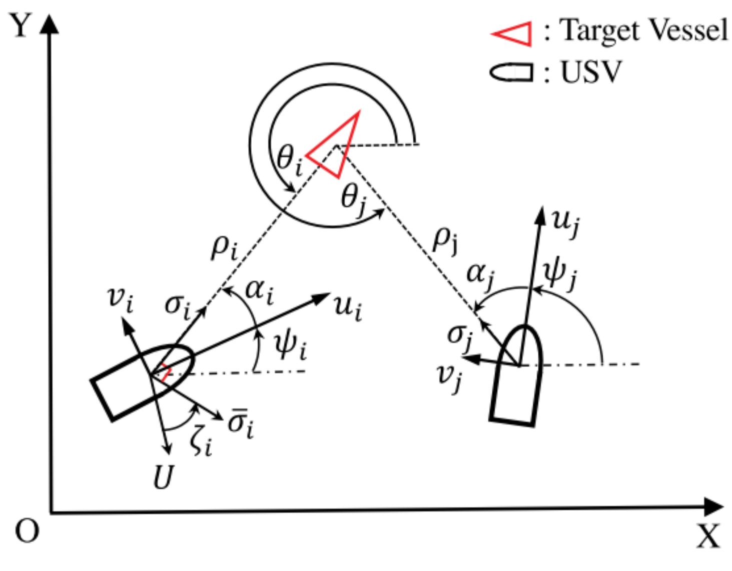

Figure 1: 对无人艇系统各轴承方向角的说明

2 控制过程设计

2.1 Step1

定义单个无人艇的合围误差 z_1 为:

$$z_1 = (\rho_i - \rho_o)^2 + (2\theta_i - \theta_{i+} - \theta_{i-})^2 + \left(\varphi_i + \frac{\pi}{2} - \theta_i\right)^2 \tag{1}$$

令其对时间 t 求导，得:

$$\dot{z}_1 = (\alpha_1 + \beta_1 - \gamma_2)\, Q(\varphi_i) \begin{pmatrix} u_i \\ v_i \end{pmatrix} + \gamma_1 r_i - (\alpha_1 + \beta_1 - \beta_2 - \beta_3) \begin{pmatrix} v_{ox} \\ v_{oy} \end{pmatrix} - \delta \tag{2}$$

其中:

$$\alpha_1 = 2\,(\rho_i - \rho_o)\,(\cos\theta_i \quad \sin\theta_i)$$

$$\beta_1 = \frac{4}{\rho_i}\,(2\theta_i - \theta_{i+} - \theta_{i-})\,(-\sin\theta_i \quad \cos\theta_i)$$

$$\beta_2 = \frac{4}{\rho_{i+}}\,(2\theta_i - \theta_{i+} - \theta_{i-})\,(-\sin\theta_{i+} \quad \cos\theta_{i+})$$

$$\beta_3 = \frac{4}{\rho_{i-}}\,(2\theta_i - \theta_{i+} - \theta_{i-})\,(-\sin\theta_{i-} \quad \cos\theta_{i-})$$

$$\gamma_1 = 2\left(\varphi_i + \frac{\pi}{2} - \theta_i\right)$$

$$\gamma_2 = \frac{2}{\rho_i}\left(\varphi_i + \frac{\pi}{2} - \theta_i\right)(-\sin\theta_i \quad \cos\theta_i)$$

$$\delta = \beta_2 \cdot Q\left(\varphi_{i+}\right) \begin{pmatrix} u_{i+} \\ v_{i+} \end{pmatrix} + \beta_3 \cdot Q\left(\varphi_{i-}\right) \begin{pmatrix} u_{i-} \\ v_{i-} \end{pmatrix}$$

综上所述，我们可以得到:

$$\dot{z}_1 = \Lambda_1 \boldsymbol{v} - \Lambda_2 \tag{3}$$

其中:

$$\Lambda_1 = \left((\alpha_1 + \beta_1 - \gamma_1)\, Q(\varphi_i) \quad \gamma_1\right)$$

$$\Lambda_2 = (\alpha_1 + \beta_1) \begin{pmatrix} v_{ox} \\ v_{oy} \end{pmatrix} + \delta$$

将 $\boldsymbol{v}$ 设为最优虚拟控制目标，有 $\boldsymbol{v} = a^*(z_1) \in R^3$，因此最优值函数可以定义为:

$$V_1^*(z_1) = \min_{\alpha \in \psi(\Omega_1)} \left(\int_t^\infty R_1(z_1, \alpha)\, ds\right) = \int_t^\infty R_1(z_1, \alpha^*)\, ds \tag{4}$$

其中 $R_1(z_1, \alpha) = z_1^T z_1 + \alpha^T R_{ii} \alpha$ 表示代价函数，$\psi(\Omega_1)$ 是可接受控制策略的集合，Ω_1 是一个紧集，为便于实现最优跟踪控制，可将最优值函数分解为:

$$V_1^*(z_1) = \zeta_1 \|z_1\|^2 + V_1^0(z_1) \tag{5}$$

其中 ζ_1 表示正常数。由 (3) 式可得哈密顿-雅可比-贝尔曼方程如下:

$$H_1\left(z_1, \alpha^*, \frac{\partial V_1^*}{\partial z_1}\right) = R_1(z_1, \alpha^*) + \left(\frac{\partial V_1^*}{\partial z_1}\right)^T \dot{z}_1 = z_1^T z_1 + \alpha^{*T}\alpha^* + \left(\frac{\partial V_1^*}{\partial z_1}\right)^T [\Lambda_1 \boldsymbol{v} - \Lambda_2] = 0 \tag{6}$$

通过求解 $\partial H_1\left(z_1, \alpha^*, \frac{\partial V_1^*}{\partial z_1}\right) / \partial \alpha^* = 0$，得到最优虚拟控制为:

$$\alpha^* = R_{ii}^{-1} \Lambda_1^T \left[-\zeta_1 z_1 - \frac{1}{2}\frac{\partial V_1^0}{\partial z_1}\right] \tag{7}$$

另一方面，连续函数 $\partial V_1^0/\partial z_1$ 可以通过神经网络近似为：

$$\frac{\partial V_1^0}{\partial z_1}=W_1^{*T}S_1\left(z_1\right)+\varepsilon_1\left(z_1\right) \tag{8}$$

因此，$\partial V_1^*\left(z_1\right)/\partial z_1$ 和 α^* 可以被分别表示为：

$$\frac{\partial V_1^*\left(z_1\right)}{\partial z_1}=2\zeta_1 z_1+W_1^{*T}S_1\left(z_1\right)+\varepsilon_1\left(z_1\right) \tag{9}$$

$$\alpha^*=R_{ii}^{-1}\Lambda_1^T\left[-\zeta_1 z_1-\frac{1}{2}W_1^{*T}S_1\left(z_1\right)-\frac{1}{2}\varepsilon_1\left(z_1\right)\right] \tag{10}$$

因此，我们有：

$$H_1\left(z_1,\alpha^*,\frac{\partial V_1^*}{\partial z_1}\right)=-\left(a\zeta_1^2-1\right)\left\|z_1\right\|^2-\frac{1}{4}a\left\|W_1^{*T}S_1\right\|^2-W_1^{*T}S_1\left(\zeta_1 a z_1+\Lambda_2\right)+\epsilon_1-2\zeta_1 z_1\Lambda_2 \tag{11}$$

其中：

$$\epsilon_1=\varepsilon_1\Lambda_1\alpha^*-\varepsilon_1\Lambda_2+\frac{1}{4}a\varepsilon_1^2$$

$$a=\left(R_{ii}^{-1}\Lambda_1^T\right)^T\cdot R_{ii}\cdot\left(R_{ii}^{-1}\Lambda_1^T\right)=\Lambda_1 R_{ii}^{-1}\Lambda_1^T$$

由于权值 W_1^* 实际上是未知的，$\partial V_1^*\left(z_1\right)/\partial z_1$ 和 α^* 是不可用的。为实现最优跟踪控制，定义批评家 $Critic\ NN$ 和行动家 $Actor\ NN$ 如下：

$$\frac{\partial \hat{V}_1^*}{\partial z_1}=2\zeta_1 z_1+\hat{W}_{1c}^T S_1\left(z_1\right) \tag{12}$$

$$\hat{\alpha}=R_{ii}^{-1}\Lambda_1^T\left[-\zeta_1 z_1-\frac{1}{2}\hat{W}_{1a}^T S_1\left(z_1\right)\right] \tag{13}$$

所以，HJB 方程的近似值可表示为：

$$H_1\left(z_1,\hat{\alpha},\frac{\partial \hat{V}_1^*}{\partial z_1}\right)=\left\|z_1\right\|^2+a\left\|\zeta_1 z_1+\frac{1}{2}\hat{W}_{1a}^T S_1\left(z_1\right)\right\|^2+\left(2\zeta_1 z_1+\hat{W}_{1c}^T S_1\left(z_1\right)\right)\times\left[\Lambda_1\hat{\alpha}-\Lambda_2\right] \tag{14}$$

其中：

$$\left[\Lambda_1\hat{\alpha}-\Lambda_2\right]=-a\zeta_1 z_1-\frac{1}{2}a\hat{W}_{1a}^T S_1\left(z_1\right)-\Lambda_2$$

定义 Bellman 误差与误差函数：

$$e_1(t)=H_1\left(z_1,\hat{\alpha},\frac{\partial \hat{V}_1^*}{\partial z_1}\right)-H_1\left(z_1,\alpha^*,\frac{\partial V_1^*}{\partial z_1}\right)\ ;\ E_1(t)=\frac{1}{2}e_1^2(t)$$

为使 Bellman 误差最小化，采用临界神经网络自适应律：

$$\begin{aligned}\dot{\hat{W}}_{1c}(t)=&-\frac{\eta_{1c}}{\phi_1}\frac{\partial E_1(t)}{\partial \hat{W}_{1c}(t)}\\=&-\frac{\eta_{1c}}{\phi_1}\varpi_1\left[-2\zeta_1 z_1^T\Lambda_2-\left(a\zeta_1^2-1\right)\left\|z_1\right\|^2+\frac{1}{4}a\left\|\hat{W}_{1a}^T(t)S_1\left(z_1\right)\right\|^2+\varpi_1^T\hat{W}_{1c}(t)\right]\end{aligned} \tag{15}$$

Actor 神经网络自适应律被设计为：

$$\dot{\hat{W}}_{1a}(t)=\frac{1}{2}S_1^T\left(z_1\right)z_1+\frac{\eta_{1c}}{4\phi_1}S_1\left(z_1\right)S_1^T\left(z_1\right)\hat{W}_{1a}\varpi_1^T\hat{W}_{1c}(t)-\eta_{1a}S_1\left(z_1\right)S_1^T\left(z_1\right)\hat{W}_{1a}(t) \tag{16}$$

引入误差变量 $z_2=\boldsymbol{v}-\hat{\alpha}$，则误差动态 (3) 可以改写为：

$$\dot{z}_1=\Lambda_1\left(z_2(t)+\hat{\alpha}\right)-\Lambda_2 \tag{17}$$

设计李雅普诺夫函数为:

$$L_1(t)=\frac{1}{2}\left\|z_1\right\|^2+\frac{1}{2}\tilde{W}_{1c}^T(t)\tilde{W}_{1c}(t)+\frac{1}{2}\tilde{W}_{1a}^T(t)\tilde{W}_{1a}(t) \tag{18}$$

其中:

$$\tilde{W}_{1c}=\hat{W}_{1c}-W_1^*$$
$$\tilde{W}_{1a}=\hat{W}_{1a}-W_1^*$$

通过一系列的化简，可得:

$$\begin{aligned}\dot{L}_1(t)=&-a\zeta_1\left\|z_1\right\|^2+z_1^T\Lambda_1z_2-z_1^T\Lambda_2z_2-\frac{1}{2}az_1W_{1a}^*(t)S_1\left(z_1\right)\\&-\frac{\eta_{1a}}{2}\tilde{W}_{1a}^TS_1\left(z_1\right)S_1^T\left(z_1\right)\tilde{W}_{1a}-\frac{\eta_{1a}}{2}\hat{W}_{1a}^TS_1\left(z_1\right)S_1^T\left(z_1\right)\hat{W}_{1a}+\frac{\eta_{1a}}{2}W_{1a}^*S_1\left(z_1\right)S_1^T\left(z_1\right)W_{1a}^*\\&+\frac{\eta_{1c}}{4\phi}a\tilde{W}_{1a}^T(t)S_1\left(z_1\right)S_1^T\left(z_1\right)\varpi_1^T\hat{W}_{1c}\\&-\frac{\eta_{1c}}{\phi_1}\tilde{W}_{1c}^T\varpi_1\times\left[\varpi_1\hat{W}_{1c}-\left(a\zeta_1^2-1\right)\left\|z_1\right\|^2-2\zeta_1z_1\Lambda_2+\frac{1}{4}a\|\hat{W}_{1a}S_1\left(z_1\right)\|^2\right]\end{aligned} \tag{19}$$

利用柯西不等式和杨氏不等式，有:

$$\begin{aligned}z_1^T(t)\Lambda_1z_2(t)&\leq\frac{1}{2}\left\|z_1(t)\right\|^2+\frac{1}{2}\left\|\Lambda_1z_2(t)\right\|^2\\z_1^T(t)\Lambda_2&\leq\frac{1}{2}\left\|z_1(t)\right\|^2+\frac{1}{2}\left\|\Lambda_2\right\|^2\\-\frac{1}{2}az_\eta^T(t)W_{1a}^*S_1(z_1)&\leq a\left\|z_1(t)\right\|^2+\frac{\eta_{1a}}{2}W_{1a}^*S_1\left(z_1\right)S_1^T\left(z_1\right)W_{1c}^*\end{aligned} \tag{20}$$

代入并化简可得:

$$\dot{L}_1(t)\leq-B^TAB+C+\frac{1}{2}\left\|\Lambda_1z_2\right\|^2-\left(\frac{\eta_{1a}}{2}-\frac{\eta_{1c}}{2}\right)\hat{W}_{1a}^TS_1^T\left(z_1\right)S_1\left(z_1\right)\hat{W}_{1a} \tag{21}$$

其中:

$$A=\begin{bmatrix}k_1&0&0\\0&k_2&0\\0&0&k_3\end{bmatrix}$$

$$B=\begin{bmatrix}z_1\\\tilde{W}_{1a}\\\tilde{W}_{1c}\end{bmatrix}$$

$$C=\left(a+\frac{\eta_{1a}}{2}\right)W_1^{*T}S_1^T\left(z_1\right)S_1\left(z_1\right)W_1^*+\frac{\eta_{1c}}{2\phi}\epsilon_1^2+\frac{1}{2}\left\|\Lambda_2\right\|^2$$

$$k_1=a\zeta-2$$

$$k_2=\frac{\eta_{1a}}{2}-\frac{a\eta_{1c}^2}{2}-\frac{a}{32}W_1^{*T}\varpi_1\varpi_1^TW_1^*$$

$$k_3=\frac{1}{\phi}\left(\frac{\eta_{1c}}{2}-\frac{a}{32}W_1^{*T}S_1^T\left(z_1\right)S_1\left(z_1\right)W_1^*\right)$$

通过选择设计参数 $a\zeta$，η_{1a}，η_{1c}，满足以下三项条件，可以使矩阵 A 成为正定矩阵:

$$\zeta>2/a$$

$$\eta_{1a}>a\eta_{1c}^2+\frac{a}{16}W_1^{*T}\varpi_1\varpi_1^TW_1^*$$

$$\eta_{1c}>\frac{a\phi}{16}\sup_{t\geq0}\left(W_1^{*T}S_1^T\left(z_1\right)S_1\left(z_1\right)W_1^*\right)$$

所以，我们有:

$$\dot{L}_1(t)\leq\frac{1}{2}\left\|z_1(t)\right\|^2-a_1\left\|B(t)\right\|^2+c_1 \tag{22}$$

其中:

$$a_1 = \inf_{t\geq 0}\left\{\lambda_{\min}\left\{A(t)\right\}\right\}$$

$$c_1 = \sup_{t\geq 0}\left\{C(t)\right\}$$

根据引理与不等式 (23) 式可知，跟踪误差可以达到预期的精度，从而使水面舰艇能够跟踪到预期的精度。

2.2 Step2

由于误差变量 $z_2 = \boldsymbol{v} - \hat{\alpha}$，令其对 t 求导，我们有:

$$\dot{z_2} = f\left(\boldsymbol{v}\right) + u - \dot{\hat{\alpha}} \tag{23}$$

同理，定义最优代价函数为:

$$V_2^*\left(z_2\right) = \min_{u\in\psi(\Omega_2)}\left(\int_t^\infty R_2\left(z_2, u\right)ds\right) = \int_t^\infty R_2\left(z_2, u^*\right)ds \tag{24}$$

同理可得，其中 $R_2\left(z_2, u\right) = z_2^T z_2 + u^T u$ 表示代价函数，$\psi\left(\Omega_2\right)$ 是可接受控制策略的集合，Ω_2 是一个紧集，为便于实现最优跟踪控制，可将最优值函数分解为:

$$V_2^*\left(z_2\right) = \zeta_2\left\|z_2\right\|^2 + V_2^0\left(z_2\right) \tag{25}$$

其中 ζ_2 表示正常数。由 (23) 式可得哈密顿-雅可比-贝尔曼方程如下:

$$H_2\left(z_2, u^*, \frac{\partial V_2^*}{\partial z_2}\right) = R_2\left(z_2, u^*\right) + \left(\frac{\partial V_2^*}{\partial z_2}\right)^T\dot{z}_2 = z_2^T z_2 + u^{*T}u^* + \left(\frac{\partial V_2^*}{\partial z_2}\right)^T\left[f\left(\boldsymbol{v}\right) + u - \dot{\hat{\alpha}}\right] = 0 \tag{26}$$

同理，通过求解 $\partial H_2\left(z_2, u^*, \frac{\partial V_2^*}{\partial z_2}\right)/\partial u^* = 0$，得到最优虚拟控制为:

$$u^* = -\zeta_2 z_2 - \frac{1}{2}\frac{\partial V_2^0}{\partial z_2} \tag{27}$$

另一方面，连续函数 $\partial V_2^0/\partial z_2$ 可以通过神经网络近似为:

$$\frac{\partial V_2^0}{\partial z_2} = W_2^{*T}S_2\left(z_2\right) + \varepsilon_2\left(z_2\right) \tag{28}$$

因此，$\partial V_2^*\left(z_2\right)/\partial z_2$ 和 u^* 可以同样被分别表示为:

$$\frac{\partial V_2^*\left(z_2\right)}{\partial z_2} = 2\zeta_2 z_2 + W_2^{*T}S_2\left(z_2\right) + \varepsilon_2\left(z_2\right) \tag{29}$$

$$u^* = -\zeta_2 z_2 - \frac{1}{2}W_2^{*T}S_2\left(z_2\right) - \frac{1}{2}\varepsilon_2\left(z_2\right) \tag{30}$$

因此，我们有:

$$\begin{aligned} H_2\left(z_2, u^*, \frac{\partial V_2^*}{\partial z_2}\right) = & -\left(a\zeta_2^2 - 1\right)\left\|z_2\right\|^2 - \frac{1}{4}\left\|W_2^{*T}S_2\right\|^2 \\ & - W_2^{*T}S_2\left(f\left(\boldsymbol{v}\right) - \dot{\hat{\alpha}} - \zeta_2 z_2\right) + 2\zeta_2 z_2\left(f\left(\boldsymbol{v}\right) - \dot{\hat{\alpha}}\right) + \sigma \end{aligned} \tag{31}$$

其中:

$$\sigma = \varepsilon_2 u^* + \varepsilon_2\left(f\left(\boldsymbol{v}\right) - \dot{\hat{\alpha}}\right) + \frac{1}{4}\left\|\varepsilon_2\right\|^2$$

由于权值 W_2^* 实际上是未知的，$\partial V_2^*\left(z_2\right)/\partial z_2$ 和 u^* 是不可用的。为实现最优跟踪控制，定义批评家 $Critic\ NN$ 和行动家 $Actor\ NN$ 如下:

$$\frac{\partial\hat{V}_2^*}{\partial z_2} = 2\zeta_2 z_2 + \hat{W}_{2c}^T S_2\left(z_2\right) \tag{32}$$

$$\hat{u} = -\zeta_2 z_2 - \frac{1}{2}\hat{W}_{2a}^T S_2(z_2) \tag{33}$$

所以，HJB 方程的近似值同样可表示为：

$$\begin{aligned} H_2\left(z_2, \hat{u}, \frac{\partial \hat{V}_2^*}{\partial z_2}\right) = &\ \|z_2\|^2 + \left\|\zeta_2 z_2 + \frac{1}{2}\hat{W}_{2a}^T S_2(z_2)\right\|^2 \\ &+ \left(2\zeta_2 z_2 + \hat{W}_{2c}^T S_2(z_2)\right) \times \left[\left(f(\boldsymbol{v}) - \dot{\hat{\alpha}}\right) - \zeta_2 z_2 - \frac{1}{2} S_2^T \hat{W}_{2a}\right] \end{aligned} \tag{34}$$

同样的，定义 Bellman 误差与误差函数：

$$e_2(t) = H_2\left(z_2, \hat{u}, \frac{\partial \hat{V}_2^*}{\partial z_2}\right) - H_2\left(z_2, u^*, \frac{\partial V_2^*}{\partial z_2}\right) \ ; \ E_2(t) = \frac{1}{2} e_2^2(t)$$

同理，为使 Bellman 误差最小化，采用临界神经网络自适应律：

$$\begin{aligned} \dot{\hat{W}}_{2c}(t) &= -\frac{\eta_{2c}}{\phi_2}\frac{\partial E_2(t)}{\partial \hat{W}_{2c}(t)} \\ &= -\frac{\eta_{2c}}{\phi_2}\varpi_2\left[2\zeta_2 z_2^T\left(f(\boldsymbol{v}) - \dot{\hat{\alpha}}\right) - \left(\zeta_2^2 - 1\right)\|z_2\|^2 + \frac{1}{4}\left\|\hat{W}_{2a}^T(t) S_2(z_2)\right\|^2 + \varpi_2^T \hat{W}_{2c}(t)\right] \end{aligned} \tag{35}$$

Actor 神经网络自适应律被设计为：

$$\dot{\hat{W}}_{2a}(t) = \frac{1}{2} S_2^T(z_2) z_2 + \frac{\eta_{2c}}{4\phi_2} S_2(z_2) S_2^T(z_2) \hat{W}_{2a} \varpi_2^T \hat{W}_{2c}(t) - \eta_{2a} S_2(z_2) S_2^T(z_2) \hat{W}_{2a}(t) \tag{36}$$

同理，设计李雅普诺夫函数为：

$$L_2(t) = L_1(t) + \frac{1}{2}\|z_2\|^2 + \frac{1}{2}\tilde{W}_{2c}^T(t)\tilde{W}_{2c}(t) + \frac{1}{2}\tilde{W}_{2a}^T(t)\tilde{W}_{2a}(t) \tag{37}$$

其中：

$$\tilde{W}_{2c} = \hat{W}_{2c} - W_2^*$$
$$\tilde{W}_{2a} = \hat{W}_{2a} - W_2^*$$

通过上述式子，同理可得：

$$\begin{aligned} \dot{L}_2(t) = &\ \dot{L}_1(t) + z_2^T\left(f(\boldsymbol{v}) + u - \dot{\hat{\alpha}}\right) \\ &+ \tilde{W}_{2a}^T(t)\left(\frac{1}{2} S_2^T(z_2) z_2 + \frac{\eta_{2c}}{4\phi_2} S_2(z_2) S_2^T(z_2)\hat{W}_{2a}\varpi_2^T \hat{W}_{2c}(t) - \eta_{2a} S_2(z_2) S_2^T(z_2)\hat{W}_{2a}(t)\right) \\ &- \frac{\eta_{2c}}{\phi_2}\tilde{W}_{2c}(t)\varpi_2\left[2\zeta_2 z_2^T\left(f(\boldsymbol{v}) - \dot{\hat{\alpha}}\right) - \left(\zeta_2^2 - 1\right)\|z_2\|^2 + \frac{1}{4}\left\|\hat{W}_{2a}^T(t) S_2(z_2)\right\|^2 + \varpi_2^T \hat{W}_{2c}(t)\right] \end{aligned} \tag{38}$$

和步骤一类似，代入并化简可得：

$$\begin{aligned} \dot{L}_2(t) \le &\ \dot{L}_1(t) - (\zeta_2 - 3)\frac{1}{2}\|z_2\|^2 - \left(\frac{\eta_{2a}}{2} - \frac{\eta_{2c}^2}{2} - \frac{1}{32} W_2^{*T}\varpi_2\varpi_2^T W_2^*\right)\tilde{W}_{2a}^T S_2^T(z_2) S_2(z_2)\tilde{W}_{2a} \\ &- \frac{\eta_{2c}}{\phi_2}\left(\frac{\eta_{2c}}{2} - \frac{1}{32} S_2^T(z_1) S_2(z_2)\right)\tilde{W}_{2c}^T\varpi_2\varpi_2^T\tilde{W}_{2c} - \left(\frac{\eta_{2a}}{2} - \frac{\eta_{2c}^2}{2}\right)\hat{W}_{2a}^T S_2^T(z_2) S_2(z_2)\hat{W}_{2a} \\ &+ \left(1 + \frac{\eta_{2a}}{2}\right) W_2^{*T} S_2^T(z_2) S_2(z_2) W_2^* + \frac{1}{2} f^2(\boldsymbol{v}) + \frac{1}{2}\|\dot{a}\|^2 + \frac{\eta_{2c}}{2}\sigma^2 \end{aligned} \tag{39}$$

进一步地，代入第一步的结果，可以得到：

$$\dot{L}_2(t) \le -a_1\|B\|^2 + c_1 - E^T D E + F - \left(\frac{\eta_{2a}}{2} - \frac{\eta_{2c}^2}{2}\right)\hat{W}_{2a}^T S_2^T(z_2) S_2(z_2)\hat{W}_{2a} \tag{40}$$

其中:

$$D=\left[\begin{array}{ccc} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{array}\right]$$

$$E=\left[\begin{array}{c} z_2 \\ \tilde{W}_{2a} \\ \tilde{W}_{2c} \end{array}\right]$$

$$F=\left(1+\frac{\eta_{2a}}{2}\right) W_2^{*T} S_2^T\left(z_2\right) S_2\left(z_2\right) W_2^*+\frac{1}{2} f^2(\boldsymbol{v})+\frac{1}{2}\|\dot{a}\|^2+\frac{\eta_{2c}}{2} \sigma^2$$

$$q_1=\zeta_2-3-\frac{l}{2}$$

$$q_2=\frac{\eta_{2a}}{2}-\frac{\eta_{2c}^2}{2}-\frac{1}{32} W_2^{*T} \varpi_2 \varpi_2^T W_2^*$$

$$q_3=\frac{\eta_{2c}}{\phi_2}\left(\frac{\eta_{2c}}{2}-\frac{1}{32} S_2^T\left(z_1\right) S_2\left(z_2\right)\right) \varpi_2 \varpi_2^T$$

通过选择设计参数 ζ_2，η_{2a}，η_{2c}，满足以下三项条件，可以使矩阵 A 成为正定矩阵:

$$\zeta_2>3+\frac{l}{2}$$

$$\eta_{2a}>\eta_{2c}^2+\frac{1}{16} W_2^{*T} \varpi_2 \varpi_2^T W_2^*$$

$$\eta_{2c}>\frac{1}{16} \sup_{t \geq 0}\left(W_2^{*T} S_2^T\left(z_2\right) S_2\left(z_2\right) W_2^*\right)$$

所以，我们有:

$$\dot{L}_2(t) \leq -a_1\|B(t)\|^2-a_2\|E(t)\|^2+c_1+c_2 \tag{41}$$

其中:

$$a_2=\inf_{t \geq 0}\left\{\lambda_{\min}\{D(t)\}\right\}$$

$$c_2=\sup_{t \geq 0}\{F(t)\}$$