1 Implementing Delay Times

1.1 Dynamical Friction

The dynamical friction timescale is given by

$$T_{\mathrm{dyn\_fr}} = \max\left(T_{\mathrm{df},1}, T_{\mathrm{df},2}\right) \tag{1}$$

where $T_{\mathrm{df},1}$ and $T_{\mathrm{df},2}$ are the timescales for the orbital decay of the SMBH of the primary (larger bulge mass) and secondary galaxy, respectively. They are given in the following equations:

$$T_{\mathrm{df},1} = 0.06\frac{2}{\ln\Lambda'}\left(\frac{R_{\mathrm{eff}}}{10\mathrm{kpc}}\right)^2\left(\frac{\sigma}{300\mathrm{kms}^{-1}}\right)\left(\frac{10^8 M_\odot}{M_{\mathrm{bulge}}}\right)\mathrm{Gyr} \tag{2}$$

$$T_{\mathrm{df},2} = 0.15\frac{2}{\ln\Lambda'}\left(\frac{R_{\mathrm{eff}}}{10\mathrm{kpc}}\right)\left(\frac{\sigma}{300\mathrm{kms}^{-1}}\right)^2\left(\frac{100\mathrm{kms}^{-1}}{\sigma_{\mathrm{s}}}\right)^3\mathrm{Gyr} \tag{3}$$

where

- σ and σ_{s} are the velocity dispersion of stars of the primary and secondary galaxy, respectively and are given by the following $M-\sigma$ relation (see [Arxiv1211.2816])

$$\log_{10}\left(M_{\mathrm{BH}}\right) = 8.32 + 5.64\log_{10}\left(\frac{\sigma}{200\mathrm{kms}^{-1}}\right) \tag{4}$$

  From (4) we see that

$$\sigma = \left(200\mathrm{kms}^{-1}\right)\left(\frac{M_{\mathrm{BH}}}{10^{8.32}}\right)^{1/5.64}$$

- The Coulomb logarithm Λ' is given by

$$\Lambda' = 2^{2/3}\frac{\sigma}{\sigma_{\mathrm{s}}} \tag{5}$$

- R_{eff} is the effective radius of the main galaxy (see [Arxiv1505.02062])

$$R_{\mathrm{eff}} = \max\left(2.95\left(\frac{M_{\mathrm{stars}}}{10^6 M_\odot}\right)^{0.596}; 34.8\left(\frac{M_{\mathrm{stars}}}{10^6 M_\odot}\right)^{0.399}\right)\mathrm{pc} \tag{6}$$

  for elliptical galaxies, and

$$R_{\mathrm{eff}} = 2.95\left(\frac{M_{\mathrm{stars}}}{10^6 M_\odot}\right)^{0.596}\mathrm{pc} \tag{7}$$

  for bulges of spirals.

1.1.1 Distribution of galaxies across morphological type as a function of stellar mass

The type of a galaxy exhibits a dependence on the ratio between bulge and total (stellar) mass. In particular, we have (see [Arxiv0508046])

- If $\frac{M_{\rm bulge}}{M_*} \geq 0.7$ then we have *elliptical* galaxies
- If $0.03 \geq \frac{M_{\rm bulge}}{M_*} < 0.7$ then we have normal *spirals*
- If $\frac{M_{\rm bulge}}{M_*} < 0.03$ then we have pure *disks*

1.2 Hardening

This phase can be dominated either by stellar or gas hardening. The shortest timescale contributes to the total merger time.

1.2.1 Dehnen Profile

We model the galaxy with a Dehnen density profile (see [Dehnen, 1993])

$$\rho(r) = \frac{(3-\gamma)M_{\rm stars}}{4\pi} \frac{r_0}{r^\gamma \left(r + r_0\right)^{4-\gamma}} \tag{8}$$

where $\gamma \in [0, 3)$ and r_0 is a scale radius. For now we set $\gamma = 1$ (Hernquist profile)
The scale radius is related to the effective radius, $R_{\rm eff}$, defined in (6), by the following expression

$$r_0 = \frac{4}{3}\left(2^{\frac{1}{3-\gamma}} - 1\right) R_{\rm eff} \tag{9}$$

Since in the following computations we'll be interested in quantities defined at the influence radius, $r_{\rm inf}$, here we give the expressions.
The influence radius, $r_{\rm inf}$, is defined as the distance at which the mass in stars equals twice the mass of the binary

$$M_{\rm stars}(r < r_{\rm inf}) = 2M$$

where $M = M_1 + M_2$ is the total mass of the binary.
Given a Dehnen profile

$$r_{\rm inf} = \frac{r_0}{\left(\frac{M_{\rm stars}}{2M}\right)^{\frac{1}{3-\gamma}} - 1} \tag{10}$$

Given (9) and (10) it is straightforward to compute $\rho(r_{\rm inf})$.
For a Dehnen profile the velocity dispersion is given by

$$\sigma_{\rm r}^2 = G M_{\rm stars} r^\gamma \left(r + r_0\right)^{4-\gamma} \int_r^\infty dr' \frac{r'^{1-2\gamma}}{\left(r' + r_0\right)^{7-2\gamma}} \tag{11}$$

For $\gamma = 1$, things simplify and we have

$$\sigma_{\mathrm{r}}^2 = \frac{GM_{\mathrm{stars}}}{12r_0}\left\{\frac{12r\,(r+r_0)^3}{r_0^4}\ln\left(\frac{r+r_0}{r}\right) - \frac{r}{r+r_0}\left[25+52\frac{r}{r_0}+42\left(\frac{r}{r_0}\right)^2+12\left(\frac{r}{r_0}\right)^3\right]\right\} \tag{12}$$

which is the formula we implement in the calculation for the time being.

1.2.2 Stellar Hardening and GW emission

The binary separation varies with time according to

$$\frac{da}{dt} = \left.\frac{da}{dt}\right|_{\mathrm{3b}} + \left.\frac{da}{dt}\right|_{\mathrm{GW}} = -Aa^2 - \frac{B}{a^3} \tag{13}$$

with

$$A = \frac{GH\rho_{\mathrm{inf}}}{\sigma_{\mathrm{inf}}}; \qquad B = \frac{64G^3M_1M_2MF(e)}{5c^5} \tag{14}$$

We take the hardening rate $H = 15$, and

$$F(e) = \frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \tag{15}$$

Setting $a_{\mathrm{hard-GW}}$ to be the separation at which the two contributions are equal, we'll have that for $a > a_{\mathrm{hard-GW}}$ stellar hardening will prevail, and for $a < a_{\mathrm{hard-GW}}$ till coalescence the GW contribution will take over.

$$a_{\mathrm{hard-GW}} = \left(\frac{64G^2\sigma_{\mathrm{inf}}M_1M_2MF(e)}{5c^5H\rho_{\mathrm{inf}}}\right)^{1/5} \tag{16}$$

Given that, the 3-body encounters will shrink the separation according to

$$\left.\frac{da}{dt}\right|_{\mathrm{3b}} = -\frac{GH\rho_{\mathrm{inf}}}{\sigma_{\mathrm{inf}}}a^2 \tag{17}$$

Integrating between $a_{\mathrm{initial}} = r_{\mathrm{inf}}$ and $a_{\mathrm{hard-GW}}$ we get

$$t_{\mathrm{stellar-hard}} = \frac{\sigma_{\mathrm{inf}}}{GH\rho_{\mathrm{inf}}}\left(\frac{1}{a_{\mathrm{hard,GW}}} - \frac{1}{r_{\mathrm{inf}}}\right) \tag{18}$$

From now on GW emission prevails and a shrinks down to zero according to

$$\left.\frac{da}{dt}\right|_{\mathrm{GW}} = -\frac{64}{5}\frac{G^3}{c^5}M^3\frac{q}{(1+q)^2}F(e)\frac{1}{a^3} \tag{19}$$

where q is the mass ratio.
Integrating from $a_{\mathrm{initial}} = a_{\mathrm{hard-GW}}$ to $a_{\mathrm{final}} = 0$, we get

$$t_{\rm GW} = \frac{5}{256}\frac{c^5}{G^3}\frac{a^4_{\rm hard-GW}}{M^3}\frac{(1+q)^2}{q} \tag{20}$$

1.3 Gas Interaction and GW emission

Considering a steady circumbinary disk ($\dot{m} = const$), we can derive the shrinking of the separation a over time by equating

$$\frac{dL_{\rm disk}}{dt} = \frac{dL_{\rm BHB}}{dt} \tag{21}$$

where $L_{\rm disk}$ is the gaseous disk angular momentum

$$L_{\rm disk} = m\sqrt{GMr_{\rm gap}} \tag{22}$$

with $r_{\rm gap} = 2a$ and $L_{\rm BHB}$ is the black hole binary angular momentum

$$L_{\rm BHB} = \mu\sqrt{GMa} \tag{23}$$

with μ the reduced mass ($\mu = \frac{M_1 M_2}{M_1+M_2}$).
Developing (21) and assuming that there is no gas leaking ($\dot{\mu} = 0 = \dot{M}$):

$$\begin{aligned}
\frac{dL_{\rm disk}}{dt} &= \frac{dL_{\rm BHB}}{dt} \\
-\dot{m}\sqrt{2}\sqrt{GMa} &= \dot{\mu}\sqrt{GMa} + \mu\frac{1}{2\sqrt{GMa}}Ga\dot{M} + \mu\frac{1}{2\sqrt{GMa}}GM\dot{a} \\
-\dot{m}\sqrt{2}\sqrt{GMa} &= \mu\frac{1}{2\sqrt{GMa}}GM\dot{a} \\
-2\sqrt{2}\dot{m} &= \mu\frac{\dot{a}}{a}
\end{aligned} \tag{24}$$

Then

$$\implies \left.\frac{da}{dt}\right|_{\rm gas} = -2\sqrt{2}\frac{\dot{m}}{M}a\frac{(1+q)^2}{q} \tag{25}$$

where we used

$$\mu = M\frac{q}{(1+q)^2}$$

GW emission overtakes the process when

$$\left.\frac{da}{dt}\right|_{\rm gas} = \left.\frac{da}{dt}\right|_{\rm GW} \tag{26}$$

where $\left.\frac{da}{dt}\right|_{\rm GW}$ is as in (19).
Solving for $a_{\rm gas-GW}$ we obtain

$$a_{\rm gas-GW} = \left(\frac{16\sqrt{2}}{5}\frac{G^3}{c^5}M^4\frac{q^2}{(1+q)^4}F(e)\frac{1}{\dot{m}}\right)^{1/4} \tag{27}$$

Using

$$\dot{m} = \frac{L}{\epsilon c^2} = \frac{L/L_{\rm Edd}L_{\rm Edd}}{\epsilon c^2} = \frac{f_{\rm Edd}L_{\rm Edd}}{\epsilon c^2} \tag{28}$$

where L is the luminosity, $L_{\rm Edd}$ is the Eddington luminosity given by

$$L_{\rm Edd} = \frac{4\pi G c m_{\rm p} M}{\sigma_{\rm T}}$$

with $m_{\rm p}$ proton mass and $\sigma_{\rm T}$ Thompson cross section, and

$$f_{\rm Edd} = \frac{L}{L_{\rm Edd}}$$

and ϵ the radiation efficiency (we can assume $\epsilon \sim 0.1$).
Substituting in (27) we obtain

$$a_{\rm gas-GW} = \left(\frac{4\sqrt{2}}{5} \frac{G^2}{c^4} \frac{\sigma_{\rm T}}{m_{\rm p}} \epsilon \right)^{1/4} M^{3/4} \frac{q^{1/2}}{1+q} \left(F(e)\right)^{1/4} \left(f_{\rm Edd}\right)^{-1/4} \tag{29}$$

Substituting the values of the constants, we get

$$a_{\rm gas-GW} \sim 0.01 {\rm pc} \left(\frac{M}{10^8 M_\odot} \right)^{3/4} \frac{q^{1/2}}{1+q} \left(F(e)\right)^{1/4} \left(f_{\rm Edd}\right)^{-1/4} \tag{30}$$

Integrating the equation (25) for $\left.\frac{da}{dt}\right|_{\rm gas}$ from $a_{\rm initial} = r_{\rm inf}$ to $a_{\rm final} = a_{\rm gas-GW}$ one gets

$$t_{\rm gas} = \frac{\sqrt{2}}{16} \frac{\epsilon c \sigma_{\rm T}}{\pi G m_{\rm p}} \frac{q}{(1+q)^2} \ln \left(\frac{r_{\rm inf}}{a_{\rm gas-GW}} \right) \tag{31}$$

GW emission prevails from $a_{\rm initial} = a_{\rm gas-GW}$ down to zero and its governed by (19). Integrating

$$t_{\rm GW} = \frac{5}{256} \frac{c^5}{G^3} \frac{a^4_{\rm gas-GW}}{M^3} \frac{(1+q)^2}{q} \tag{32}$$