HWTC #14

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Q1: $0<x<1 \quad 0<y<1$

$\text{Pdf}_{x,y}(x,y) = cxy$

(a) $\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \text{Pdf}_{x,y}(x,y)\,dx\,dy = 1 = \int_0^1\int_0^1 cxy\,dx\,dy = \int_0^1 \Big|_0^1 cy\frac{x^2}{2} = \int_0^1 \frac{cy}{2} = \frac{c}{4} = 1$

$\hookrightarrow$ support(x)
support(y)

$C=4$

(b) $P\{\frac{X}{Y}>2 \mid X<\frac{1}{2}\} = P\{X>2Y \mid X<\frac{1}{2}\} = P\{2Y<X \mid X<\frac{1}{2}\}$

$= \displaystyle\int_0^{1/2} P\{2Y<a \mid X=a\}.\text{pdf}_X(a)\,da$

$= \displaystyle\int_0^{1/2} P\{Y<\frac{a}{2} \mid X=a\}.\text{pdf}_X(a)\,da \overset{\text{indep.}}{=} \int_0^{1/2} \frac{P\{Y<\frac{a}{2}\}.\cancel{P\{X=a\}}}{\cancel{P\{X=a\}}}.\text{Pdf}_X(a)\,da$

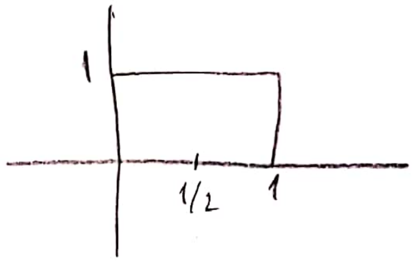
$= \displaystyle\int_0^{1/2} P\{Y<\frac{a}{2}\}.\text{Pdf}_X(a)\,da$

$\text{Pdf}_X(x) = \displaystyle\int_0^1 4xy\,dy = \underline{\underline{2X}}$

$= \displaystyle\int_0^{1/2} \text{cdf}_Y(\frac{a}{2}) \cdot \underbrace{\text{Pdf}_X(a)}_{2a}\,da$

$\text{Pdf}_Y(y) = \displaystyle\int_0^1 4xy\,dx = 2y$

Buradan gelmedi.



$X>2y$

$0<x<\frac{1}{2}$

$x=2y$

$y=\frac{x}{2}$

$Y<\frac{X}{2}$

$\displaystyle\int_0^{1/2}\int_0^{x/2} 4xy\,dy\,dx$

$\displaystyle\int_0^{1/2} \Big|_0^{x/2} 2xy^2\,dx$

$= \displaystyle\int_0^{1/2} 2x\cdot\frac{x^2}{4}\,dx = \int_0^{1/2} \frac{x^3}{2}\,dx = \Big|_0^{1/2} \frac{x^4}{16} = \frac{1}{(16)^2} = \frac{1}{256}$

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Q_2: $Z(t) = \int_{-\infty}^{+\infty} h(t-s) N(s)\, ds \Rightarrow E[Z(t)] =$

$= E\left[\int_{-\infty}^{+\infty} h(t-s)\, Z(s)\, ds\right] = E[Z(t)] = E\left[\int_{-\infty}^{+\infty} h(t-s) \cdot (X(s) + Y(s))\, ds\right]$

$\underset{\text{of expectation}}{\overset{\text{linearity}}{=}} \int_{-\infty}^{+\infty} E[h(t-s) E[Z(s)]] \cdot ds \qquad Z(s) = X(s) + Y(s)$

$\int_{-\infty}^{+\infty} E[h(t-s) \cdot X(s)]\, ds + \int_{-\infty}^{+\infty} E[h(t-s)\, Y(s)]\, ds$

$\int_{-\infty}^{+\infty} h(t-s) \underbrace{E[X(s)]}_{\mu_x} ds + \int_{-\infty}^{+\infty} h(t-s) \overbrace{E[Y(s)]}^{\mu_y} ds$

X(s) ve Y(s) WSS olduklarından $E[X(s)]$ ve $E[Y(s)]$ constant in time

$\int_{-\infty}^{+\infty} h(t-s)\, \mu_x\, ds + \int_{-\infty}^{+\infty} h(t-s)\, \mu_y \qquad t - s = v$

$-ds = dv$

$\mu_x \int_{+\infty}^{-\infty} h(v)(-dv) + \mu_y \int_{+\infty}^{-\infty} h(v)(-dv)$

$\mu_x \int_{-\infty}^{+\infty} h(v)\, dv + \mu_y \underbrace{\int_{-\infty}^{+\infty} h(v)\, dv}_{A}$

$\mathcal{F}\{h(t)\} = H(w) = \int_{-\infty}^{+\infty} h(t) \exp(-jwt)\, dt \qquad A = H(0) = \int_{-\infty}^{+\infty} h(t) \exp(0) = \underbrace{\int_{-\infty}^{+\infty} h(t)}_{A}$

$E[Z(t)] = \mu_x \cdot H(0) + \mu_y \cdot H(0) =$ constant in time ✓

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Q2.2

$$Z'(t) = \int_{-\infty}^{+\infty} h(s).Z(t-s)\,ds \qquad Z(t+\tau) = \int_{-\infty}^{+\infty} h(r).Z(t+\tau-r)\,dr$$

$$E[Z'(t)\cdot Z(t+\tau)] = R_{zz}(t,t+\tau) = E\left[\int_{-\infty}^{+\infty} dr\,h(r) . \int_{-\infty}^{+\infty} ds\,h(s)\,Z(t-s)\,ds\right]$$

linearity of integral and expectation $\Big\}$ $$\int_{-\infty}^{+\infty} dr\,h(r) \underbrace{\int_{-\infty}^{+\infty} ds\,h(s)\; \underbrace{E[Z(t-s)\,Z(t+\tau-r)]}_{R_{zz}(\tau-r+s)}}_{A}$$

$$A = \int_{+\infty}^{-\infty} -dv\,h(-v)\,R_{zz}(\tau-r-v)$$

$$= \int_{-\infty}^{+\infty} \tilde{h}(v)\,dv\,R_{zz}(\tau-r-v) = \underbrace{R_{zz}(\tau-r) * \tilde{h}(\tau-r)}_{G(\tau-r)}$$

$$R_{zz}(t,t+\tau) = \int_{-\infty}^{+\infty} dr\,h(r)\,G(\tau-r) = h(\tau) * \underbrace{G(\tau)}_{R_{zz}(\tau) * \underbrace{h(-\tau)}_{\tilde{h}(\tau)}}$$

$$\tilde{R}_{zz}(t,t+\tau) = h(\tau) * \underbrace{R_{zz}(\tau) * h(-\tau)}_{R_{zz}(\tau)}$$

$$R_{zz}(\tau) = E[Z(t) : Z(t+\tau)] = R_{zz}(t,t+\tau) \checkmark$$

then $Z(t)$ is WSS

CamScanner ile tarandı

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(3) $Pdf_X(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$ λ: Average number of suc.
x: Actual number of suc.

$P\{A\} = \frac{exp(-10) \cdot 10^x}{10!}$, $P\{B\} = \frac{exp(-20) \, 20^x}{20!}$

$X = \{$20 kasa domates ulaştırılması$\}$

$Y = \{$12 kasanın A dağıtıcısı tarafından ulaştırılması$\}$

$P\{Y|X\} = ?$ $\frac{P\{Y, X\}}{P\{X\}}$

$Z = \{$8 kasanın B dağıtıcısı tarafından ulaştırılması$\}$ Artık Z ve Y birbirinden bağımsız.

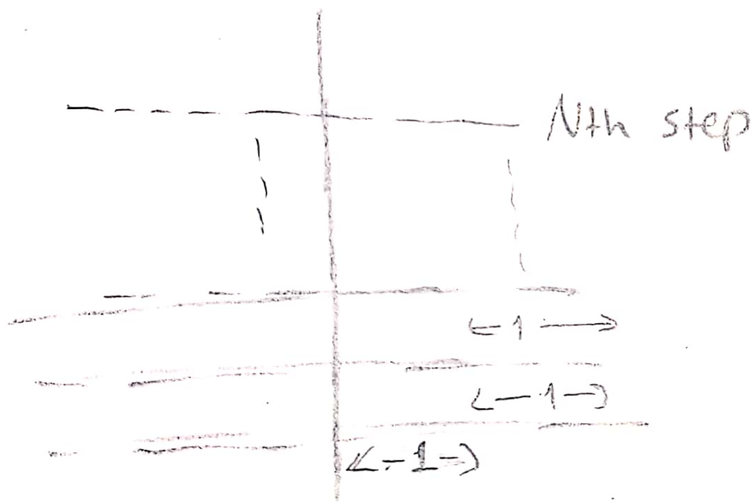
$P\{Y|X\} = P\{Y, Z\} = P\{Y\} \cdot P\{Z\}$

$P\{Y\} \Rightarrow x = 12, \lambda = 10, \; P\{Y\} = \frac{exp(-10) \cdot 10^{12}}{10!}$

$P\{Z\} \Rightarrow x = 8, \lambda = 20, \; P\{Z\} = \frac{exp(-20) \cdot 20^8}{8!}$

$$\boxed{P\{Y\} \cdot P\{Z\} = \frac{exp(-10) \cdot 10^{12}}{10!} \cdot \frac{exp(-20) \cdot 20^8}{8!}}$$

Q4:



R: Deterministic value

N_s: Deterministic

$N \sim \text{Poisson}(\lambda)$

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Binomial RV ⇒ Let us have a set of n iid Bernoulli RV as $\{X_1, X_2 \dots X_n\}$ each with success probability "p"

Recall support of a single Bernoulli RV → support$(X_i) = \{0,1\}$ → for $i \leq j \leq n$

Then the binomial RV. ⇒ $\boxed{Y = X_1 + X_2 + X_3 \dots X_n}$ $\boxed{Y = \sum_{i=1}^{n} X_i}$ support$(Y) = \{0,1,\dots,n\}$

Pmf of binomial RV

$$P\{Y=k\} = \binom{n}{k} \cdot p^k \cdot q^{n-k}, \Rightarrow P\left\{\begin{matrix}1 \text{ success} \\ \text{out of } 3\end{matrix}\right\} = \binom{3}{1} \cdot p^1 \cdot q^2 = 3pq^2$$

$\boxed{Var(Y) = npq}$

Cdf of binomial RV → $P\{Y \leq m\} = cdf_Y(m) = \sum_{k=0}^{m} pmf_Y(k) = \sum_{k=0}^{m} P\{Y=k\} = \sum_{k=0}^{m} \binom{n}{k} p^k q^{n-k}$

if the cdf is proper → $\lim_{m \to +\infty} P\{Y \leq m\} = P\{Y \leq n\} = cdf_Y(n) = \sum_{k=0}^{n} pmf_Y(k) = \sum_{k=0}^{n} P\{Y=k\} = \underbrace{\sum_{k=0}^{n} \binom{n}{k} \cdot p^k q^{n-k}}_{(p+q)^n = 1^n = 1}$ ✓

$E[Y^k] = npE[(Z+1)^{k-1}]$

$\boxed{E[Y] = np \cdot 1 = np}$ → $E[Y^2] = n \cdot p \cdot E[Y+1] = n \cdot p[\underbrace{E[Y]}_{(n-1)p} + \underbrace{E[1]}_{1}]$

$\boxed{E[Y^2] = np \cdot ((n-1)p + 1)}$

Baye's Rule

$$P\{B_j|A\} = \frac{P\{AB_j\}}{P\{A\}} = \frac{P\{A|B_j\} \cdot P\{B_j\}}{\sum_{k=1}^{N} P\{A|B_k\} P\{B_k\}}$$

Soruda verilen given that tek olmalı sorulan olasılık birden fazla durum içermeli $B_1, B_2 \dots$

Example on Baye's Rule:

iki para var biri hileli biri hilesiz. hileli (two headed). Kontrol etmeden birini seçip ~~pros~~ hileli ve hilesizi seçme olasılığı eşittir para atıldığında "head" geldiyse hilesiz seçme olasılığı nedir, $B_1 = \{$hilesiz para seçildi$\}$, $B_2 = \{$Hileli seçildi$\}$, $A\{$Head geldi$\}$

$P\{B_1|A\} = ?$

$$P\{B_1|A\} = \frac{P\{A|B_1\} \cdot P\{B_1\}}{P\{A|B_1\} \cdot P\{B_1\} + P\{A|B_2\} \cdot P\{B_2\}} = \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}} = \frac{1/4}{3/4} = \frac{1}{3}$$

Example on Baye's Rule → if a woman worker is known to have resigned recently, then what is the probability that she used to work for firm C.

	# of women	# of men
Firm A	20	80
Firm B	45	30
Firm C	50	50

$F_A: \{$resigned person from firm A$\}$
$F_B\{$ " " " " B$\}$ $G = \{$the person who has resigned is women$\}$
$F_C\{$ " " " " C$\}$

$$P\{F_C|G\} = \frac{P\{G|F_C\} \cdot P\{F_C\}}{P\{G|F_A\} \cdot P\{F_A\} + P\{G|F_B\} \cdot P\{F_B\} + P\{G|F_C\} P\{F_C\}}$$

$P\{G|F_A\} = \frac{20}{100}$ $P\{F_A\} = 1/3$
$P\{G|F_B\} = \frac{45}{75}$ $P\{F_B\} = 1/3$
$P\{G|F_C\} = \frac{50}{100}$ $P\{F_C\} = 1/3$

Ex. on Bayes Rule: 3 people toss coin first heads up wins what is the probability first, second third person wins?

solve S_k: outcome of the kth coin tossing event we have two possible events regarding the random variable.

$S_k = H$, $S_k = T$, complementary events (Mut. exc. and all inc.)

first person wins → $A_0 = \{S_1 = H\}$, $A_1 = \{S_1 = T \dots S_4 = H\}$, $A_2 = \{S_1 = T \dots S_7 = H\}$

$\{A_0 \cup A_1 \cup \dots\} = \{$first person wins$\}$

$$\{A_0 \cup \dots\} \overset{Ax.3}{=} \sum_{k=0}^{+\infty} P\{A_k\} = \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^7} + \dots = \frac{1}{2}\left(\underbrace{1 + \frac{1}{2^3} + \frac{1}{2^6} \dots}_{\underbrace{1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 \dots}_{\frac{1}{1 - \left(\frac{1}{2}\right)^3}}}\right) = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{8}} = \frac{4}{7}$$

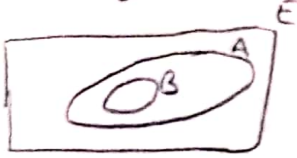
$\left(\frac{1}{2}\right)^2$ ile başlayarak 2nd person wins
$\left(\frac{1}{2}\right)^3$ " " 3rd ~~H~~ wins bulunur.

Functions of RV → If X is a RV of a known distribution, then what is the distribution of a function of the variable X, with $f(\cdot)$. For ex. $Y=f(x)$, therefore creating a new RV?

Example: Let X be a continuous RV, and let Y be another RV derived from X, $Y=|X|$.
then what's pdf and cdf of Y. $cdf_Y(y) = P\{Y \le y\} \to P\{|X| \le y\}$

$cdf_Y(y) = P\{|X| \le y\}$, $\{|X| \le y\} = \{-y \le X \le y\}$ the interval $\{-\infty < X < +\infty\}$

aralık kırpılabilir $\underbrace{\{-\infty < X < y\}}_{A} \cap \underbrace{\{-\infty < X < -y\}}_{B} = \{-\infty < X < y\}$



$\{-y \le X \le +y\} = B^c \cap A$ → then $P\{-y \le X \le y\} = P\{B^c \cap A\}$

$\boxed{A = B \cup B^c \cap A}$ B and $B^c \cap A$ are mutually exclusive $P\{A\} \overset{Ax3}{=} P\{B\} + P\{B^c \cap A\}$ eq.1

From eq.1 $P\{B^c \cap A\} = P\{A\} - P\{B\} = \underbrace{P\{-\infty < X < y\}}_{cdf_X(y)} - \underbrace{P\{-\infty < X < -y\}}_{cdf_X(-y)}$

$\boxed{cdf_Y(y) = cdf_X(y) - cdf_X(-y)}$ → $\frac{d}{dy}[cdf_Y(y)] = pdf_Y(y) = pdf_X(y) - pdf_X(-y)(-1) =$

$\boxed{pdf_Y(y) = pdf_X(y) + pdf_X(-y)}$

Question from HWTC#08
we know cdf and pdf of X then what is $Y = X^2 + 2$

Solution $Y = x^2 + a \to cdf_Y(y) = P\{Y \le y\} = P\{X^2 + a \le y\} = P\{X^2 \le y - a\} = P\{|X| \le \sqrt{y-a}\}$

$P\{-(y-a) \le X \le y-a\}$ $\underbrace{\{-\infty \le X \le -(y-a)\}}_{B}$ $\underbrace{\{-\infty \le X \le y-a\}}_{A}$ $A \cap B = \{-\infty \le X \le y-a\}$

$A = B \cup B^c \cap A \Rightarrow P\{B\} + P\{B^c \cap A\} = P\{A\}$ $P\{B^c \cap A\} = P\{-(y-a) \le X \le y-a\} = P\{A\} - P\{B\}$

$\boxed{\frac{d}{dy}(cdf_Y(y)) = pdf_Y(y) = pdf(y-a) + pdf(-(y-a))}$

$cdf(y-a) - cdf(-(y-a))$

Autocorrelations and PSD of WSS → White Noise,

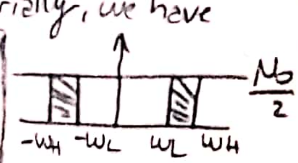
Autocorrelations → $R_{NN}(t_1, t_2) = E[N(t_1) N(t_2)]$

$= R_{NN}(t_1 - t_2) = \frac{N_0}{2}\delta(t_1 - t_2) = \boxed{\frac{N_0}{2}\delta(\tau)}$,

White noise is a stochastic process has a flat (constant nonzero) characteristic for its PSD the white noise process assumed to be $N(t)$ is a WSS process with a zero mean $E[N(t)] = 0$

and then the PSD → $S_{NN}(w) = F\{R_{NN}(\tau)\} = \frac{N_0}{2}$ (constant for all w)

Pictorially, we have



PSD of white noise $N(t)$ and a frequency band of interest $[w_L, w_H]$

Average power in the band

$\frac{1}{2\pi}\left[\int_{w_L}^{w_H} S_{NN}(w)dw + \int_{-w_H}^{-w_L} S_{NN}(w)dw\right]$

$= N_0(f_H - f_L)$

Binomial RV from HWTC#06 → Given that out of 5 independent shots ($p=0,6$) taken, 3 successful attempts exists, what is the probability that only one failure occurs in the last two attempts

$A = \{$5 bağımsız atışın 3ü başarılı$\}$, $B = \{$Son iki denemeden yalnızca 1'i başarılı$\}$

$P\{B|A\} = \frac{P\{B,A\}}{P\{A\}}$ → $P\{B,A\}$ → $\underbrace{\cdot\,\cdot\,\cdot}_{2}$ $\underbrace{\cdot\,\cdot}_{1}$ $\binom{3}{2} \cdot p^2 q \cdot \binom{2}{1} \cdot p \cdot q$

$P\{A\} = \sum_{k=3}^{3} \binom{5}{k} \cdot p^k \cdot q^{5-k} = \binom{5}{3} \cdot p^3 q^2 = 10p^3q^2$ $\boxed{6p^3q^2 = P\{B,A\}}$

$P\{B|A\} = \frac{6p^3q^2}{10p^3q^2} = 0.6$

Binomial RV from HWTC#05
bağımsız 5 atış atan birinin ilk 3'te 1 son 2'de 1 sayı alma olasılığı nedir?

p ve q'lar cinsinden → $\underset{A}{\cdot\,\cdot\,\cdot}$ $\underset{B}{\cdot\,\cdot}$ A and B independent $P\{A,B\} = P\{A\} \cdot P\{B\}$

$P\{A\} = \binom{3}{1} \cdot p \cdot q^2$
$P\{B\} = \binom{2}{1} \cdot p \cdot q$ } $6p^2q^3$