Abdullah MEMİŞOĞLU 171024001

HWTC #04

Q1 $X = \{\text{Telefonda geçirilen zaman}\}$

$A = \{X > 5 \text{ dakika}\}$ $\quad B = \{X > 15 \text{ dakika}\}$

$P\{B|A\} = ?$

$P\{B|A\} = P\{X > 15 | X > 5\} = P\{X > 10\} \rightarrow C = \{X > 10\}$

$C^c = \{X \leq 10\} = cdf_X(x)$

$P\{X > 10\} = P\{X \leq 10\}^c = 1 - \underbrace{P\{X \leq 10\}}_{f_1} = \boxed{1 - f_1 = P\{X > 10\}}$

$f_1 = cdf_X(10) = 1 - exp(-\lambda . x) = 1 - exp(-0.1 . 10) = \boxed{1 - e^{-1}}$

$P\{B|A\} \stackrel{?}{=} P\{X > 10\} = 1 - f_1 = 1 - (1 - e^{-1}) = \frac{1}{e}$ //

Q2 $X \sim N(0, \sigma^2)$ ise $\mu = 0$ $Var(x) = \sigma^2$

buradan $pdf_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)}{2\sigma^2}\right)$

Is $pdf_X(x)$ above an even function of x?

$$pdf_X(x) \overset{?}{=} pdf_X(-x)$$

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(+x-\mu)^2}{2\sigma^2}\right)\Bigg|_{\mu=0} \overset{?}{=} \frac{1}{\sigma\sqrt{2\pi}}\left(\frac{-(-x-\mu)^2}{2\sigma^2}\right)\Bigg|_{\mu=0}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right) = \frac{1}{\sigma\sqrt{2\pi}}\left(\frac{-x^2}{2\sigma^2}\right) \checkmark$$

And also,

$$P\{X \geq a\} = \int_a^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-y^2}{2\sigma^2}\right) dy$$

$$\boxed{\begin{array}{l} g = -y, \ dg = -dy \\ \text{int. sınırlar} \\ \cancel{x} = a \rightarrow g = -a \\ \cancel{x} = +\infty \rightarrow g = -\infty \end{array}}$$

$$P\{X \geq a\} = \int_{-a}^{-\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-g^2}{2\sigma^2}\right)(-dg)$$

$$= -\int_{-a}^{-\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-g^2}{2\sigma^2}\right) dg = \boxed{\int_{-\infty}^{-a} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-g^2}{2\sigma^2}\right)}$$

$$\int_a^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right) dy = \int_{-\infty}^{-a} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-g^2}{2\sigma^2}\right)$$ ise $P\{X \geq a\} = P\{X \leq -a\}$

$$P\{X > a\} = P\{X \geq a\} = P\{X < -a\} = P\{X \leq -a\}$$

$$\boxed{P\{X < -a\} + P\{X > -a\} = 1}$$

$$\boxed{P\{X \leq -a\} = 1 - P\{X > -a\}}$$

$$\boxed{P\{X > a\} = P\{X < -a\} = 1 - P\{X \geq -a\}}$$

$Q_3 \to$ Her biri bağımsız 5 deneme için binomial R.V

$$mgf_Z(t) = E[\exp(tZ)] = \sum_{k=0}^{n} \exp(tk)\binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \cdot \underbrace{[p.\exp(t)]}_{x}^k \underbrace{q}_{y}^{n-k} = \boxed{[q + p\exp(t)]^n = mgf_Z(t)}$$

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} \cdot x^k \cdot y^{n-k} = (x+y)^n}$$

$\boxed{n \text{ deneme sayısı} = 5}$
$\boxed{p = 0.6, \quad q = 0.4}$

$$\boxed{[0.4 + 0.6\exp(t)]^5 = mgf_Z(t)}$$

Deneme Oyunu için Bernoulli RV (success p, failure q)

$$mgf_Y(t) = \sum_{i=0}^{1} \exp(ta_i)\, pmf_X(a_i) = \exp(t.0) \cdot q + \exp(t.1) \cdot p$$

$$= q + p.\exp(t) = \boxed{0.4 + 0.6\exp(t) = mgf_Y(t)}$$

Buradan görüyoruz ki

$$mgf_Y(t) = \prod_{i=1}^{5} mgf_{Z_i}(t) = (mgf_{Z_1}(t))^5$$

5 taneden 1 tane seç

$\forall i \in [1,2,3,4,5]$ için

X_i iid

... in them properly

Q4 channel 1 : game with 3 penalties
channel 2 : " " 5 "
channel 3 : " " 7 "
channel 4 : " " 9 "

cevap

$$\boxed{\frac{1}{4}\left(2p_5 q_5 + 4p_7 q_7^3 + 6p_9 q_9^5\right)}$$

X_3 = number of successful shots out of 3 on ch1
X_5 = " " " " " " 5 " ch2
X_7 = " " " " " " 7 " ch3
X_9 = " " " " " " 9 " ch4

$A = \{$I witness a goal after missing the first 3 shots on TV$\}$

$P\{A\} = ?$

$B_1 = \{$I turn on ch1 and stay there$\}$ B_i for $i = 1,2,3,4$
$B_2 = \{$ " " " ch2 " " " $\}$ $B_i \cap B_j = 0$ $i \neq j$ mut. excl.
$B_3 = \{$ " " " ch3 " " " $\}$ $B_1 \cup B_2 \ldots \cup B_4 = E$ all inclusive
$B_4 = \{$ " " " ch4 " " " $\}$

$$P\{A\} = P\{\bigcup_{i=1}^{4} \{A \cap B_i\}\} \overset{ax\ 3}{=\!=\!=} \sum_{i=1}^{4} P\{A \cap B_i\} \underset{\text{cond. prob}}{\overset{df\ of}{=\!=\!=}} \sum_{i=1}^{4} P\{A|B_i\} \cdot P\{B_i\}$$

$$\boxed{\sum_{i=1}^{4} P\{A|B_i\} \cdot P\{B_i\}}$$

$P\{B_i\}$ for $i \in \{1,2,3,4\}$ $P\{B_i\} = P\{B_j\}$ $i \neq j$ so

$$\boxed{P\{B_1\} = P\{B_2\} = P\{B_3\} = P\{B_4\} = \frac{1}{4}}$$

ch1 $\underbrace{\bullet\bullet\bullet}$ $= \binom{0}{1} \cdot p^1 \cdot q^{-1} \longrightarrow$ is not possible $\rightarrow 0$

ch2 $\underbrace{\bullet\bullet\bullet}\bullet\bullet$ $= \binom{2}{1} \cdot p_5^1 \cdot q_5^1 = 2p_5 \cdot q_5$

ch3 $\underbrace{\bullet\bullet\bullet}\bullet\bullet\bullet\bullet$ $= \binom{4}{1} \cdot p_7^1 \cdot q_7^3 = 4p_7 q_7^3$

ch4 $\underbrace{\bullet\bullet\bullet}\bullet\bullet\bullet\bullet\bullet\bullet$ $= \binom{6}{1} \cdot p_9^1 \cdot q_9^6 = 6 \cdot p_9 \cdot q_9^5$

$$\boxed{P\{A\} = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 2p_5 q_5 + \frac{1}{4} \cdot 4p_7 q_7^3 + \frac{1}{4} \cdot 6p_9 q_9^5} \; *$$

HW TC Q4 – Question 2 – Bonus HW

Is $pdf_X(x)$ an even function of x?

$$pdf_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \qquad X \sim N(\mu, \sigma^2)$$

for an even function $pdf_X(x) \overset{?}{=} pdf_X(-x)$

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \overset{?}{=} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(-x-\mu)^2}{2\sigma^2}\right)$$

$pdf_X(x) \neq pdf_X(-x)$ $\quad \mu \neq 0$ iken

Is $pdf_Y(y)$ an even function of y? $\quad Y \sim N(0, \sigma^2)$

$pdf_Y(y) \overset{?}{=} pdf_Y(-y)$

$$pdf_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{y^2}{2\sigma^2}\right) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(-y)^2}{2\sigma^2}\right) = pdf_Y(-y)$$

so $pdf_Y(y)$ is an even function.

HWBONUS #08

Q1- Look up the English words below and learn to use them properly

Convenient: Suitable or agreeable to the needs or purpose

The House is convenient to all transportation

Convenience: the quality of being convenient, suitability

Convenience utensils that can be discarded after use

for convenience: easiness, accomodation

Maybe he keeps that wagon out there all the time for convenience.

Verbose: Containing more words than necessary

*An example of verbose is someone who can talk for five minutes without pausing for the other person to speak.

have continuous measurement

HW Bonus #08 $\boxed{Q_2}$

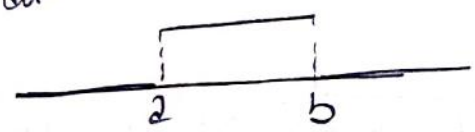
: What is a Median? What is a Mean? This is in our context of probability. the mean and the median of uniform random variable are the same?

Mean: Ortalama değer (Average Value of a random variable)

Median: In Probability theory, the median is the value separating the higher half from the lower half of a data sample, a population. For a dataset it may be thought as the Middle value.

★ "Uniform Random Variable" göz önüne alındığında

Belirli aralıkta sabit olasılıksal değer alan sürekli "random variable" türüdür.



[a,b] aralığında sabit(k) gibi bir değeri var.

Eşitlik vardır.

tanıma göre Mean → $\boxed{\frac{b+a}{2}}$

tanıma göre Median → $\underbrace{\frac{b-a}{2}}_{\text{a ve b'nin orta noktaya uzaklığı}} + \underbrace{a}_{\text{başlangıç noktası}} = \frac{a+b}{2}$

Q3 - Matlab / Octave has a "rand" function that generates uniform random variables obeying the distribution described by $U(0,1)$ in the interval $[0,1]$. What should we do in order to obtain uniform random variable realizations in the following intervals?

(a) Generate relizations obeying $U(-0.5, +0.5)$

(b) Generate relizations obeying $U(+5, +12)$

(a) rand(n) fonksiyonu n×n boyutunda matrise [0,1] aralığında sayılar dizmektedir. Ve MATLAB'da bir matris ile sayı toplama - çıkarma işlemi matrisin tüm elemanlarına o sayıyı eklemek ya da çıkarmaktır. Böylece; $X_1 = rand(n)$ ve $X_1 = X_1 - 0.5$ komutu ile [0,1] aralığındaki değerler (−0.5, 0.5) aralığına getirilebilir.

(b) $rand(n) = X_1$ olsun $U(+5, +12)$

$$0 \le X_1 \le 1 \xrightarrow{*7} 0 \le 7X_1 \le 7 \xrightarrow{+5} \boxed{5 \le 7X_1 + 5 \le 12}$$

$X_1 = 7 * X_1 + 5$ → matrisin her elemanı 7 ile çarpılıp 5 eklenir.

7 ile çarpacağımızı 0'ın yutan eleman özelliğini kullanarak düşündük.

formülize edersek $U(a,b)$ ise $\boxed{X_1 = a + (b-a) * rand(n)}$

Q4 Plot the cdf of the generated uniformly distributed random numbe realizations

(a)

```
a=-0.5; b=0.5;
X=a+(b-a).*rand(100,1)
V=[a:(b-a)/99:b];
plot(v,X);
```

(b)

```
a=5; b=12;
X=a+(b-a).*rand(100,1)
V=[a:(b-a)/99:b];
plot(v,x);
```

Q5 What is ~~##~~ a histogram in a probabilistic context? What do we use it for

☆ We use histograms when we have continous measurement and want to understand the distribution of values and look for outliers.

These graphs take your continous measurement and place them into ranges of values known as bins.

Q6: Check if the CDF in eq3 is a proper one through
$\lim_{x \to -\infty} cdf_x(x) = 0$ $\lim_{x \to \infty} cdf_x(x) = 1$

$\lim_{x \to -\infty} cdf_x(x) = 1 - \exp(-\lambda x) \Big|_{x=-\infty}$ (support(x) = [0, +∞), old. −∞ değeri alamaz. x = 0 için 0)
$\lim_{x \to -\infty} cdf_x(x) = 0$

$\lim_{x \to +\infty} cdf_x(x) = \Big|_{x=+\infty} 1 - \exp(-\lambda x) = 1 - \underbrace{\exp(-\infty)}_{0} = \underline{1}$

Q7: Check that $\boxed{-x \cdot \exp(-\lambda x) \Big|_{x=0}^{x \to +\infty} = 0}$

$-x \cdot \exp(-\lambda x) \Big|_{x=0}^{+\infty} = \underbrace{-\infty \cdot 0 - 0}_{\text{belirsizlik}} = -\infty \cdot 0$ (belirsizlik)

$\lim_{x \to \infty} \dfrac{-x}{\dfrac{1}{\exp(-\lambda x)}} = \dfrac{\infty}{\infty}$ belirsizlik l'hespital $= -\lim_{x \to \infty} \dfrac{1}{\dfrac{+\lambda \exp(-\lambda x)}{(\exp(-\lambda x))^2}}$

$\boxed{-\lim_{x \to \infty} \dfrac{(\exp(-\lambda x))}{\lambda} = 0}$

$-x \exp(-\lambda x) \Big|_{x=0}^{+\infty} = \underbrace{\lim_{x \to +\infty} (-x \exp(-\lambda x))}_{0} - \underbrace{(-x \exp(-\lambda x)) \Big|_{x=0}}_{0}$

$0 - 0 = \underline{\underline{0}}$

Q_8: Show that the first term on the RHS of eq. 11 is again zero

$$-x^2 \cdot \exp(-\lambda x) \Big|_{x=0}^{+\infty} \overset{?}{=} 0$$

$$-\lim_{x \to +\infty} \frac{x^2}{\frac{1}{\exp(-\lambda x)}} = \frac{\infty}{\infty} \text{ l'hospital} = \frac{2x}{\frac{\exp(-\lambda x) \cdot \lambda}{(\exp(-\lambda x))^2}} = \frac{2x \cdot \exp(-\lambda x)}{\lambda}$$

$$\lim_{x \to +\infty} \frac{2}{\lambda} \cdot x \cdot \exp(-\lambda x) = \underbrace{\frac{2}{\lambda}}_{0} \underbrace{\lim_{x \to +\infty} x \exp(-\lambda x)}_{0} = \underline{\underline{0}}$$

buldук

Q_9 $P\{X > t+s \mid X > t\}$ x for $t, s > 0$

$X > t+s \subset X > t$ bu yüzden kesişimleri $\{X > t+s\}$

conditional probability

$$\frac{P\{X > t+s, X > t\}}{P\{X > t\}}$$

$$= \frac{P\{X > t+s\}}{P\{X > t\}} = \frac{P\{\{X \le t+s\}^c\}}{P\{\{X \le t\}^c\}} \xrightarrow{A + A^c = 1} \frac{1 - P\{X \le t+s\}}{1 - P\{X \le t\}}$$

$P\{X \le t+s\} = \mathrm{cdf}_X(t+s) = \exp(-\lambda(t+s))$, $P\{X \le t\} = \mathrm{cdf}_X(t) = \exp(-\lambda t)$

$$= \frac{1 - (1 - \exp{-\lambda}(t+s))}{1 - (1 - \exp{-\lambda}(t))} = \frac{\exp{-\lambda}(t+s)}{\exp{-\lambda} t} = \frac{\exp(-\lambda t) \cdot \exp(-\lambda s)}{\exp{-\lambda} t}$$

$$= \boxed{\exp(-\lambda s)}$$

cdf'e benzetmek için $\rightarrow 1 - (\underbrace{1 - \exp(-\lambda s)}_{\mathrm{cdf}_X(s)})$

$$= 1 - P\underbrace{\{X \le s\}}_{\mathrm{cdf}_X(s)} \xrightarrow{A^c + A = E} \underline{P\{X > s\}}$$

CamScanner ile tarandı

... that eq. 6 is true

Q10 How does one generate samples of an exponentially distributed random variable through a random number generator?

see $y = 1 - \exp(-\lambda x) \rightarrow \exp(-\lambda x) = 1 - y$, $-\lambda x = \ln|1-y|$ $x = \frac{-1}{\lambda} \ln|1-y|$ eq.19

What is the significance of eq.19.

---

* generate samples of an exponentially distribution

1) Compute the cdf, for exponential distribution the cdf is $F(x) = 1 - e^{-\lambda x}$

2) $y = cdf_x(x)$ on the range of X, $y = 1 - e^{-\lambda x}$ $x \geq 0$

3) $y = 1 - e^{-\lambda x} \rightarrow 1 - y = e^{-\lambda x} \rightarrow \ln|1-y| = -\lambda x \rightarrow \boxed{\frac{-1}{\lambda} \ln|1-y| = x}$ eq1

4) Generate uniform random numbers $y_1, y_2 \ldots$

we know that for $i = 1, 2 \ldots$ where y_i is a uniformly distributed random number on $(0,1)$

5) $y_1, y_2 \ldots$ uniformly distributed random numbers so, $(1-y_1), (1-y_2)(1-y_3) \ldots$ " " " " too.

6) so the calculation can be simplified as $\boxed{x_i = \frac{-1}{\lambda} \ln y_i}$ eq2

7) Before we simplify the $\boxed{x_i = \frac{-1}{\lambda} \ln y_i}$, we said y is a uniform random numbers on $(0,1)$ so, we can generate all x in support when y_i for $i = 1, 2, 3 \ldots$ numbers on $(0,1)$

CamScanner ile tarandı

... of the result in eq. 15 to show that eq. 6 is true

Q11 In terms of stochastic process jargon: What is the relation between a Poisson process and an exponentially distributed sojourn time What is "sojourn"

Relation Between the Poisson and Exponential Distribution

* If we expect λ events on average for each unit of time, then the average waiting time between events is Exponentially distributed, with parameter λ (thus average wait time is $1/\lambda$), and the number of events counted in each unit of time is Poisson distributed with parameter λ.

sojourn time: As we say above, the average wait time in exponentially distribution is sojourn time.

Q12 $\lim_{x \to -\infty} cdf_X(x) \stackrel{?}{=} 0$

$$cdf_X(x) = P\{X \leq x\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{(\varrho-\mu)^2}{2\sigma^2}\right) d\varrho$$

$$\lim_{x \to -\infty} cdf_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{-\infty} \exp\left(-\frac{(\varrho-\mu)^2}{2\sigma^2}\right) d\varrho \Rightarrow$$

Integral sınırları eşitse altında alan kalmayacağından

$$\int_a^a f(x)\,dx = 0$$

$$\boxed{\lim_{x \to -\infty} cdf_X(x) = 0}$$

Q13: Make use of the result in eq.15 to show that eq.6 is true

eq.15 $\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = 1$ eq.6 $\lim_{x \to +\infty} cdf_X(x) = 1$

$$\lim_{x \to +\infty} cdf_X(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \quad X \sim N(\mu, \sigma^2)$$

$$eq(15) = 1 = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy = \lim_{y \to +\infty} cdf_Y(y) \quad \text{when } Y \sim N(0,1)$$

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy = 1 = \boxed{\lim_{y \to +\infty} cdf_Y(y) = 1} \quad Y = N \sim (0,1)$$

Ancak bizim $X \sim N(\mu, \sigma^2)$ için bulmamız gerekli.

$$\lim_{x \to +\infty} cdf_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$ burada $\rho = \frac{x-\mu}{\sigma}$

dönüşümü yaparsak $\boxed{d\rho = \frac{dx}{\sigma}}$ $x \to +\infty$, $\rho \to +\infty$ $\boxed{dx = \sigma \cdot d\rho}$

$$\lim_{\rho \to +\infty} cdf_X(\rho) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{\rho^2}{2}\right) \cdot \sigma \cdot d\rho$$

$$= \frac{1}{\sigma} \cdot \sigma \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\rho^2}{2}\right) d\rho}_{1\,(eq.15)} = \frac{1}{\sigma} \cdot \sigma = 1$$

böylece $\lim_{x \to +\infty} cdf_X(x) = 1$ bulunur. $X \sim N(\mu, \sigma^2)$

CamScanner ile tarandı

Q14: The first term on the RHS of (eq.16) is much easier to deal with if we recall $\int_{-\infty}^{+\infty} g(x).h(x)\,dx = 0$ if one of $\{g(x), h(x)\}$ is even the other one is odd function of x. Show how to make use of eq.19 to prove eq.18

Suppose that $f(x) = g(x).h(x)$

= h(x) and g(x) both even, f(x) e
" " " " " odd, f(x) ev
" one of them even the other one odd,

$\boxed{f(x) \text{ odd}}$

$f(x) = g(x).h(x)$ odd func.

for odd functions, $\boxed{\int_{-a}^{a} F(x)\,dx = 0}$

So $\boxed{\int_{-\infty}^{+\infty} g(x)h(x)\,dx = \int_{-\infty}^{+\infty} f(x)\,dx = 0}$

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu).\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx = \int_{-\infty}^{\infty} \overset{\text{odd function}}{y.}\underbrace{\exp\left(\frac{-y^2}{2\sigma^2}\right)}_{\text{even function we prove it in HWTC\#04}}$$

$\boxed{\begin{array}{l} x-\mu = y \\ dx = dy \end{array}}$

$$\boxed{\text{so } \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu).\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx = 0}$$

$f(y) = \exp\left(\frac{-y^2}{2\sigma^2}\right)$ $\qquad f(y) \overset{?}{=} f(-y)$ $\qquad f(y) = \exp\left(-\frac{y^2}{2\sigma^2}\right)$

$f(-y) = \exp\left(-\frac{(-y)^2}{2\sigma^2}\right)$

$f(y) = f(-y)$ so $\boxed{f(y) \text{ even function}}$

CamScanner ile tarandı

Q: 14.2 Show that $E[x-\mu] = E[x-E[x]] = 0$

$$E[x-\mu] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x-\mu) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$x-\mu = \rho$ $\quad dx = d\rho \longrightarrow \dfrac{1}{\sigma\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \rho \cdot \underbrace{\exp\left(-\frac{\rho^2}{2\sigma^2}\right)}_{f(\rho)}$

$f(\rho)$ is even function

ρ is odd function

$f(\rho).\rho \longrightarrow$ odd function $\displaystyle\int_{-a}^{a} f(\rho).\rho \, d\rho = 0$ (odd function)

So $\dfrac{1}{\sigma\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \rho . \exp\left(-\frac{\rho^2}{2\sigma^2}\right) = \underline{E[x-\mu] = 0}$

-2

$\dfrac{-y)^2}{2\sigma^2}$

Q15: Explain why the derivation in (Eq.28) is correct it might help to think in terms of a random variable. $y \sim N(0, \sigma^2)$

$$E[(x-E[x])^2] \overset{E[x]=\mu}{=\!=} E[(x-\mu)^2] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x-\mu)^2 \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$E[x] \Rightarrow x = x-\mu$

$y = x-\mu \quad dy = dx$

$$\mathrm{var}(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} y^2 \exp\left(-\frac{y^2}{2\sigma^2}\right) dy \qquad u = y \quad dv = y \cdot \exp\left(\frac{-y^2}{2\sigma^2}\right) dy$$

$$du = dy \quad v = -\sigma^2 \cdot \exp\left(\frac{-y^2}{2\sigma^2}\right)$$

$u.v - \int v\,du$

$$\left[\underbrace{-y.\sigma^2 \cdot \exp\left(\frac{-y^2}{2\sigma^2}\right)\Big|_{y\to-\infty}^{\infty}} - \int_{-\infty}^{\infty} -\sigma^2 \exp\left(\frac{-y^2}{2\sigma^2}\right)..\right] \cdot \frac{1}{\sigma\sqrt{2\pi}}$$

$$\lim_{x\to+\infty} \frac{-y\sigma^2}{\frac{1}{\exp\left(-\frac{y^2}{2\sigma^2}\right)}} \overset{l'hasp.}{=} \frac{-\sigma^2}{\frac{+2y.\cancel{\exp\left(\frac{-y^2}{2\sigma^2}\right)}}{\left(\exp\left(\frac{-y^2}{2\sigma^2}\right)\right)^{\cancel{2}}}} = \frac{-\sigma^2 \cdot \exp\left(-\frac{y^2}{2\sigma^2}\right)}{2} = 0$$

$$\frac{1}{\sigma\sqrt{2\pi}}\left[+\int_{-\infty}^{\infty} \sigma^2 \exp\left(\frac{-y^2}{2\sigma^2}\right)\right] = \sigma^2 \underbrace{\int_{-\infty}^{\infty} \exp\left(\frac{-y^2}{2\sigma^2}\right) \cdot \frac{1}{\sigma\sqrt{2\pi}}}$$

$y = x-\mu$ ise $\underbrace{\lim_{x\to+\infty} \mathrm{cdf}_x(x)}_{1 \text{ bulduk}}$

$\mathrm{var}(x) = \sigma^2$ bulunur.

$$\int_{-\infty}^{\infty} \exp\left(\frac{-y^2}{2\sigma^2}\right) \cdot \frac{1}{\sigma\sqrt{2\pi}} = \mathrm{cdf}_y(y)$$

$Y \sim N(0, \sigma^2)$

$\boxed{\mu = 0 \text{ durumu}}$

Q16 Look up the "Q-function" associated with the standart Gaussian random Variable $Z \sim N(0,1)$. What is the Q function used for?

Q-function is complementary of CDF function

$Q = P\{X \geq x\}$ so

$$P\{X \geq x\} = \int_x^\infty pdf_X(x)\,dx = \int_x^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$$y = \frac{x-\mu}{\sigma}$$

$$P\left\{y > \frac{x-\mu}{\sigma}\right\} = \int_{\frac{x-\mu}{\sigma}}^\infty \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy}_{B}$$ we see that,

B part is actually $pdf_X(x)$ for $X \sim N(0,1)$

so, $$Q(z) = \int_z^\infty \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)\,dy$$

$$Q(z) = Q\left(\frac{x-\mu}{\sigma}\right) = P\left\{y > \frac{x-\mu}{\sigma}\right\}$$ so $Z \sim N(0,1)$

because of $P\{y > x\}$

Thus Q-function gives the area of the shaded curve with the transformation $y = \frac{x-\mu}{\sigma}$ applied to the Gaussian pdf.

Q-function evaluates the tail probability of normal distribution.

Q17: How do we generate samples of a Normally distributed random variable Look up the Box-Muller Algorithm for example.

Generating samples of a Normally distributed random variables has two ways like Central Limit theorem, Box Muller transform

A very common thing to do with a probability distribution is to sample from it. In other words, we want to randomly generate numbers such that the values of x are in proportion to the pdf. So for the standart Normal Distri. $N \sim (0,1)$ most of values would fall close to somewhere around $X=0$.

Sampling Using Box Muller Transform.

Box Muller transform is a neat little "trick" that allows us to sample from a pair of normally distributed variables using a source of only uniformly distributed variables. Given two independent uniformly distributed random variables U_1, U_2 on the interval $(0,1)$ we define two new random variables, R and θ, that intuitively representing polar coordinates as such:

$$R = \sqrt{-2\ln U_1}, \quad \theta = 2\pi U_2$$

$$X = R\cos\theta = \sqrt{-2\ln U_1}\cos(2\pi U_2)$$

$$Y = R\sin\theta = \sqrt{-2\ln U_1}\sin(2\pi U_2)$$

CamScanner ile tarandı

Q18 : What is the function that accepts as argument $X \sim N(\mu, \sigma^2)$ and yields $Z \sim N(0,1)$?

+ Q function $X \sim N(\mu, \sigma^2)$ and $Z \sim N(0, 1)$

$Q = P\{X > x\}$ $\quad z = \frac{x - \mu}{\sigma}$ $\quad P\{z > \frac{x-\mu}{\sigma}\} = \int_{z}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-y^2}{2}\right) dy$

$\sigma = 1$ $\quad \mu = 0$

Q19: Show that for the mgf in (eq.1) $\frac{d^k}{dt^k}\left[mgf_X(t)\right]\Big|_{t=0} = E[X^k]$

$$\boxed{\int_{X \in support}^{S} \exp(tx) \cdot Pdf_X(t)\, dt = mgf_X(t)}$$

Integral ve türevin lineerliğinden

$$\frac{d}{dt} \int_{X \in support} \exp(tx) \cdot Pdf_X(x)\, dt \xlongequal{\text{Integral}} \int \frac{d}{dt} \exp(tx) \cdot Pdf_X(x)\, dt$$

$$= \int x \cdot \exp(tx) \cdot Pdf_X(x)\, dt \Big|_{t=0} = \int x \cdot Pdf(x)\, dt = E[x]$$

$$\frac{d^k}{dt^k} \int_{X \in support} x \cdot \exp(tx)\, Pdf_X\, dt \xlongequal{\text{lineerlik}} \int_{X \in sup.} \frac{d^k}{dt^k} \exp(tx)\, Pdf_X(x)\, dt$$

$$= \int_{X \in sup} x^k \cdot \exp(tx)\, Pdf_X(x)\, dt = E[X^k]$$

Q20: Why $\int_{x \in support} [\dots] dx = \int_{-\infty}^{\infty} [\dots] dx$ That is, why are the two integral expressions equivalent, when $[\dots]$ contains "$pdf_X(x)$" as a term

Because,

$pdf_X(x) = P\{X = x\}$ then suppose that $support(x) = x_i, x_{i+1} \dots x_j$

we know that $P\{X = x_a\} = 0$ for $\{a < i\} \cup \{a > j\}$ then we know

$$\int_{-\infty}^{\infty} [\dots] dx = \underbrace{\int_{-\infty}^{x_i} [\dots] dx}_{0} + \int_{x_i}^{x_j} [\dots] dx + \underbrace{\int_{x_j}^{\infty} [\dots] dx}_{0}$$

$$\int_{-\infty}^{\infty} [\dots] dx = \int_{x_i}^{x_j} [\dots] dx = \int_{x \in support(x)} [\dots] dx$$

Q21: Define another Gaussian RV Y with $Y \sim N(t, 1)$ and write down $pdf_Y(z)$ / $\mu = t$ $\sigma^2 = 1$

$$pdf_Y(z) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(z-t)^2}{2}\right)$$

$$\lim_{z \to \infty} cdf_Y(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z-t)^2}{2}\right) = 1$$

$\lim_{z \to \infty} cdf_Y(z) = 1$ olduğu kanıtlandı burada $\mu = t$ $\sigma^2 = 1$ özel durumu incelenmektedir.

Q22: Show that $E[X]=\mu$ $var(X)=\sigma^2$ Assuming now that X is also a Gaussian RV, then we have as $X \sim N(\mu,\sigma^2)$

$$E[X] = \int_{x \in support(X)} x \cdot pdf_X(x)\,dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx$$

$$X \to X-\mu+\mu$$

$$\left[\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx\right] + \left[\frac{\mu}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx\right]$$

$$x-\mu=y \quad \underbrace{\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} y \exp\left(-\frac{y^2}{2\sigma^2}\right)dx}_{\text{Bu terimin benzerinin sıfır olduğunu ödev içerisinde bulduk}}$$

$$+ \mu \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx}_{\lim_{x \to +\infty} cdf_X(x) = 1 \text{ bulmuştuk}} \qquad \boxed{E[X]=\mu} \text{ bulunur}$$

$$var(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x-\mu)^2 \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx \qquad \boxed{x-\mu=y}$$

$$var(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 \exp\left(-\frac{y^2}{2\sigma^2}\right)dy \qquad u=y \quad dV = y\exp\left(\frac{-y^2}{2\sigma^2}\right)dy$$

$$du=dy \quad V = -\sigma^2 \exp\left(\frac{-y^2}{2\sigma^2}\right)$$

$$= \sigma^2 \left[\underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-y^2}{2\sigma^2}\right)}_{\underbrace{\lim_{x \to +\infty} cdf_X(y)}_{1}}\right] \qquad X \sim N(0,\sigma^2)$$

$$var(x) = \sigma^2$$

CamScanner ile tarandı

Q23 : with $Z \sim N(0,1)$ and X defined as in (eq. 26) how would you show that X is also Gaussian RV

Burada lineerlik özelliği kullanılacak.

$X = \sigma z + \mu$ ise $\boxed{z = \frac{x-\mu}{\sigma}}$

$cdf_z(z)$ lineer z $(0,1)$ aralığındayken

$cdf_z(z) = P\{z \leq \frac{x-\mu}{\sigma}\}$ $cdf_z(z)$ z, $(0,1)$ aralığında lineer ise

$cdf_z(z) = cdf_x(\frac{x-\mu}{\sigma})$ $cdf_x(\frac{x-\mu}{\sigma})$ 'de $(\mu, \mu+\sigma)$ aralığında lineerdir

böylece X uniformly distributed

CamScanner ile tarandı

Q24. Can you derive a compact formula for all of the Moments of $Z \sim N(0,1)$ through its mgf as given in (eq.25)

$$mgf_Z(t) = \exp\left(+\frac{t^2}{2}\right)$$

$$\frac{d^k}{dt^k}\left(mgf_Z(t)\right)\Big|_{t=0} = E[x^k]$$

$E[t] = t.\exp\left(\frac{t^2}{2}\right)\Big|_{t=0} = 0$

$E[t^2] = 1.\exp\left(\frac{t^2}{2}\right) + t^2\exp(t^2/2) \quad (t^2+1)\exp(t^2/2)$

$E[t^3] = t.\exp\left(\frac{t^2}{2}\right) + 2t.\exp(t^2/2) + t^3\exp(t^2/2) = (t^3+3t)\exp(t^2/2)$

$E[t^4] = \exp\left(\frac{t^2}{2}\right) + t^2\exp(t^2/2) + 2\exp(t^2/2) + 2t^2\ldots + 3t^2\ldots + t^4\ldots\Big|_{t=0} = 3$

$E[t^5] = t\ldots + 2t\ldots + t^3\ldots + 2t\ldots + 4t\ldots + 2t^3\ldots + 6t\ldots + 3t^3\ldots + 4t^3\ldots + t^5 = 0$

$E[t^5] = (t^5 + 10t^3 + 15t)\, e^{t^2/2}$

(Türev kısmı uzun sürdüğünden işlemler ödev kısmına geçirilmedi.)

$E[t^6] = 15$

$E[t] = t.\exp\left(\frac{t^2}{2}\right)\Big|_{t=0} = 0$

$E[t^2] = (t^2+1)\exp\left(\frac{t^2}{2}\right)\Big|_{t=0} = 1$

$E[t^3] = (t^3+3t)\exp\left(\frac{t^2}{2}\right)\Big|_{t=0} = 0$

$E[t^4] = (t^4+6t^2+3)\exp\left(\frac{t^2}{2}\right)\Big|_{t=0} = 3$

$E[t^5] = (t^5 + 10t^3 + 15t)\exp(t^2/2)\Big|_{t=0} = 0$

$E[t^6] = (t^6 + 15t^4 + 45t^2 + 15)\exp(t^2/2) = 15$

$E[t^7] = (t^7 + 21t^5 + 105t^3 + 105t)\exp(t^2/2)$

$E[t^8] = (t^8 + 28t^6 + 210t^4 + 420t^2 + 105)\exp(t^2/2)$

$E[t^9] = (t^9 +$

k	$E[t^k]$
1	0
2	1
3	0
4	3
5	0
6	15
7	0
8	105

$E[t^k] = f(k)$ olsun

$$f(k) = \begin{cases} 0, & \text{if } k = \text{odd} \\ (k-1).(k-3).\ldots.1, & \text{if } k = \text{even} \end{cases}$$