GEBZE TECHNICAL UNIVERSITY

ENGINEERING FACULTY

ELECTRONICS ENGINEERING

ELECTRONICS 218

PROBABILITY AND RANDOMNESS

BONUS HW 07

Name - Surname	Abdullah Memişoğlu
Student ID	171024001

Bonus HW #07 Abdullah MEMİŞOĞLU 171024001

Q1 → Show that the Taylor expansion of the function exp(x) around $x_0=0$ is given by $\exp(x)=\sum_{k=0}^{+\infty}\frac{x^k}{k!}$ (eq.16)

Taylor expansion

$$f(x)=\sum_{n=0}^{+\infty}\frac{f^{(n)}(a)}{n!}\cdot(x-a)^n = f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2\cdots$$

$a=0$ özel durumu için ve $f(x)=\exp(x)$ kabul edersek bu açılım

$$\exp(x)=\sum_{n=0}^{+\infty} f(0)+\frac{f'(0)}{1!}(x-0)+\frac{f''(0)}{2!}(x-0)^2\cdots$$

$$f(x)=f'(x)=f''(x)=\cdots=\exp(x) \quad \exp(0)=1=f(0)=f'(0)\cdots$$

$$\boxed{\exp(x)=\sum_{n=0}^{+\infty} 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\cdots}$$

Bu bilindiğine göre

$\boxed{\lim_{i\to\infty} cdfy(i)=1}$ olduğu kanıtlanabilir

$$\lim_{i\to+\infty} cdfy(i)=\sum_{k=-\infty}^{i}\Big[\underbrace{\exp(-\lambda)\cdot\frac{\lambda^k}{k!}}_{Pmfy(k)}\Big]=\exp(-\lambda)\cdot\Big[\underbrace{1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}\cdots}_{\exp(\lambda)}\Big]$$

$$\boxed{\lim_{i\to+\infty} cdfy(i)=\exp(-\lambda)\cdot\exp(\lambda)=1}$$

$Q_2 \rightarrow$ Explain the derivation in (eq.23) by going through the steps one by one (eq.23 = $E[Y^2] = \lambda + \lambda^2$)

$$E[Y^2] = \sum_{k=0}^{+\infty} k^2 \, pmf_Y(k) = \underbrace{\sum_{k=0}^{+\infty} (k^2 - k + k) \, pmf_Y(k) = \sum_{k=0}^{+\infty} k.(k-1) \, pmf_Y(k) + \sum_{k=0}^{+\infty} k \, pmf_Y(k)}$$

Bu adım k(k-1) elde edip k! ile sadeleştirip Taylor expansion uygulayabilmek için yapıldı

$$= \underbrace{\sum_{k=0}^{+\infty} k.(k-1).pmf_Y(k)}_{A} + \underbrace{\sum_{k=0}^{+\infty} k.pmf_Y(k)}_{E[Y]=\lambda}$$

$$A \xrightarrow[\text{açılır}]{pmf_Y(k)} \sum_{k=2}^{+\infty} k.(k-1).\exp(-\lambda).\frac{\lambda^k}{k!} \xrightarrow{k! = k(k-1).(k-2)!} \sum_{k=2}^{+\infty} \exp(-\lambda).\frac{\lambda^k}{(k-2)!}$$

k=0 için 0

k=1 için 0

Biliyoruz ki $\boxed{\sum_{k=2}^{+\infty} \frac{\lambda^{k-2}}{(k-2)!} = \sum_{j=0}^{+\infty} \frac{\lambda^j}{j!} = \exp(\lambda) \quad \boxed{eq.1}}$ } Bunu elde etmek için;

$$\sum_{k=2}^{+\infty} \exp(-\lambda).\frac{\lambda^k}{(k-2)!} = \lambda^2 \exp(-\lambda).\underbrace{\sum_{k=2}^{+\infty} \frac{\lambda^{k-2}}{(k-2)!}}_{\exp(\lambda).(eq.1)} = \boxed{\lambda^2 = A}$$

$E[Y^2] = A + E[Y] = \underline{\underline{\lambda^2 + \lambda}}$ bulunur

Q3: The moment generating function is known to be related to the Laplace transform of the pmf. Search for and recall the Laplace transform definition. And as a related concept, look up what the "characteristic function" as applies to again random variables is all about.

* The moment-generating function of a random variable X is

$$\boxed{w_X(s) = E[e^{sX}] = \int_0^{+\infty} p(x)e^{sx}dx} \quad \text{eq.1}$$

$s \in \mathbb{R}$ için, eğer $p \rightarrow$ probability density function (negatif olmayan durumda) Laplace transform şu şekilde işler

$$\mathcal{L}\{p\}(s) = E[e^{-sX}] = \int_0^{\infty} p(x)e^{-sx}dx$$

$w_X(s) = \mathcal{L}\{p\}(-s)$ tam olarak doğru bir ifade değil negatif değerlerde sıfır değeri aldığından bir dirac delta fonksiyonu ilave edilmeli.

$$w^*(s) = E[e^{sX}] = p_0 e^{s \times 0} + \int_0^{+\infty} p(x)e^{sx}dx = \boxed{p_0 + \mathcal{L}\{p\}(-s)}$$

characteristic function: # of a random variable is a variation on the moment generating function rather than use the expected value of tX, it uses the expected value of $eitX$. This means the characteristic function of a random variable is the fourier transform of its density function.

---

from here, it's not that you want me to do. But I want to add this to HW.

MGF → # is literally the function that generates the moments
$- E[X], E[X^2], E[X^3] \dots E[X^n]$

$$MGF_X(t) = E[e^{tX}] \longrightarrow E[X^n] = \frac{d^n}{dt^n}\left(MGF_X(t)\right)\Big|_{t=0}$$

$$E[X] = \frac{d}{dt} MGF_X(t)\Big|_{t=0} = MGF_X'(0)$$

$$E[X^2] = \frac{d^2}{dt^2} MGF_X(t)\Big|_{t=0} = MGF_X''(0)$$

Prove with taylor series

① $e^{tX} = 1 + tX + \frac{(tX)^2}{2!} \dots$

② $E[e^{tX}] = E\left(1 + tX + \frac{(tX)^2}{2} \dots\right)$

③ $E[e^{tX}] = E[1] + E[tX] \dots$

④ $\frac{d}{dt}E[e^{tX}] = \frac{d}{dt}\left(E[1] + tE[X] \dots\right)$

$t=0$ için
$= 0 + E[X] + 0 + 0 \dots = E[X]$

böylece 1, 2, 3 … n türevleri sırasıyla $E[X], E[X^2] \dots E[X^n]$'i verecektir.

Q4 You may now show yourself that for $k \geq 1$ we have

$$\frac{d^k}{dt^k}\Big[E[\exp(tx)]\Big] = E[X^k]$$

★ $\exp(tx)$ fonksiyonu kullanılma sebebi Taylor açılımında her türev alındığında sabit olan elemanın bir birim ötelenmesi ve her türevinde sadece bir adet sabit bulunmasıdır

Taylor series

$$\rightarrow e^{tx} = 1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} \cdots$$

$$\rightarrow E[e^{tx}] = E\left[1 + tx + \frac{(tx)^2}{2!} \cdots\right]$$

$$\rightarrow E[e^{tx}] = E[1] + t.E[x] + \frac{t^2}{2!}E[x^2] \cdots$$

bu durumda $E[x^2]$ ve $E[x^3]$ bulup görelim

$$E[x^2] = \frac{d^2}{dt^2}\Big(\underbrace{E[1]}_{\frac{d}{dt}\text{'de }(0)} + \underbrace{tE[x]}_{\frac{d^2}{dt^2}\text{ ile } 0} + \frac{t^2}{2!}E[x^2] \underbrace{\cdots}_{t=0 \text{ için her biri sıfır olur}}\Big)$$

$$\frac{d}{dt} \rightarrow \frac{2t}{2!}$$

$$\frac{d^2}{dt^2} = \frac{2}{2!}E[x^2] = E[x^2] ★$$

$$E[x^3] = \frac{d^3}{dt^3}\left[E[1] + tE[x] + \frac{t^2}{2!}E[x^2] + \frac{t^3}{3!}E[x^3] \cdots\right]$$

$$\frac{d}{dt}E[e^{tx}] = \left[E[x] + tE[x^2] + \frac{3t^2}{3!}E[x^3] \cdots\right]$$

$$\frac{d^2}{dt^2}E[e^{tx}] = \left[E[x^2] + t.E[x^3] \cdots\right]$$

$$\frac{d^3}{dt^3}E[e^{tx}] = \Big[E[x^3] \underbrace{\cdots}_{\text{her biri } t \text{ elemanı içerdiği için sıfır } t=0 \text{ old. 'dan}}\Big]$$

$$\frac{d^3}{dt^3}E[e^{tx}] = E[x^3] \longrightarrow \boxed{\frac{d^n}{dt^n}E[e^{tx}] = E[x^n]}$$

Q5: Show that (through (eq.7)) $E[exp(tX)]|_{t=0} = 1$ and

$$\frac{d^k}{dt^k}\left[E[exp(tX)]\right]\Big|_{t=0} = E[X^k] - P \text{ for } k \in [1, +\infty]$$

$$\rightarrow E[exp(tX)] = q.exp(t.0) + p.exp(t.1)$$

$$= q + P(exp(t)) = 1 - p + P(exp(t)) = 1 + P(-1 + exp(t))$$

$$E[exp(tX)]\Big|_{t=0} = 1 + P(\cancel{1} + exp(t))\Big|_{t=0} = 1 + \underbrace{P(-1+1)}_{0} = \underline{\underline{1}}$$

$$\frac{d^k}{dt^k}\left[E[exp(tX)]\right]\Big|_{t=0} = E[X^k]$$

$$= \frac{d}{dt}E[exp(tX)] = \frac{d}{dt}\left(1 - P + P.exp(t)\right) = P.exp(t)$$

$$= \frac{d^2}{dt^2}E[exp(tX)] = \frac{d}{dt}(P.exp(t)) = P.exp(t)$$

$$\vdots$$

$$\frac{d^k}{dt^k}E[exp(tX)] = \frac{d}{dt}(P.exp(t)) = P.exp(t)$$

$$\frac{d^k}{dt^k}E[exp(tX)]\Big|_{t=0} = P.exp(t)\Big|_{t=0} = \underline{\underline{P}}$$

Q6 8 Look up what an indicator function is especially associated with a probabilistic event, denoting it as, eg., $\mathbb{1}_{\{\cdot\}}$ where $\{\cdot\}$ is supposed to denote the event that indicator... ...function operates on. And then show that (with X as our Bernoulli RV in this account) $P\{X=1\} = E\left[\mathbb{1}_{\{X=1\}}\right] = P$

☆ Let E be a sample space and $A \subseteq E$ be an event. The indicator function of the event A, denoted by $\mathbb{1}_E$, is a random variable defined as,

$$\mathbb{1}_{A(\omega)} = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

for example we toss a die, the sample space $\rightarrow E = \{1,2,3,4,5,6\}$

Define an event that described by the sentence "An even number appears face up. The A event $\rightarrow A = \{2,4,6\}$ then,

$$\mathbb{1}_{A(\omega)} = \begin{cases} 0 & \text{if } \omega=1 \\ 1 & \text{if } \omega=2 \\ 0 & \text{if } \omega=3 \\ 1 & \text{if } \omega=4 \\ 0 & \text{if } \omega=5 \\ 1 & \text{if } \omega=6 \end{cases}$$

---

we know that,

$$E[X] = \sum_{i=0}^{1} k \cdot P\{X=k\} = 0 \cdot P\{X=0\} + 1 \cdot P\{X=1\}$$ ancak burada

$E\left[\mathbb{1}_{\{X=1\}}\right] \rightarrow k=1$ durumu $1 \cdot P\{X=1\}$ $\quad \mathbb{1}_{\{X=1\}}$ = Sample space içerisinde $x=1$ olduğu durumlarda $\underline{1}$ diğer durumlarda $\underline{0}$

böylece $E\left[\mathbb{1}_{\{X=1\}}\right] = 1 \cdot P\{X=1\} = \underset{\text{bernoulli RV}}{P}$ bulunur

Q7: Go through the steps in the derivation of eq.10 and explain every step. What do I mean by the condition imposed in just under the result

$$\boxed{\star \text{ eq.10} \rightarrow \frac{p.\exp(t)}{1-q.\exp(t)}}$$

$\star$ condition $\star$ $\left(|q.\exp(t)| < 1\right)$

$mgf_Y(t) = E[\exp(tY)]$

for geometric RV $\rightarrow pmf_Y(k) = q^{k-1}.p$

$$E[\exp(tY)] = \sum_{k=-\infty}^{+\infty} \exp(tk).q^{k-1}p \qquad E[Y] = k.q^{k-1}p \text{ ise}$$

support$(Y) = 1,2,\ldots$ old. için;

$$E[\exp(tY)] = \sum_{k=1}^{+\infty} \exp(tk)q^{k-1}.p$$

$\exp(tk) = e^{tk}$ bunu $(e^t)^k$ olarak yazabiliriz

$$E[\exp(tY)] = \sum_{k=1}^{+\infty} \exp(t)^k.q^{k-1}.p$$

$$E[\exp(tY)] = \sum_{k=1}^{+\infty} (\exp(t).q)^k.\frac{p}{q}$$

$|q.\exp(t)| < 1$ olduğu sürece

$$\boxed{\sum_{k=1}^{+\infty} (\exp(t).q)^k = \frac{1}{1-q.\exp(t)}}$$

olarak yazılabilir

Bu koşulun sağlandığını varsayarak ilerleyelim

$$E[\exp(tY)] = \frac{p}{\cancel{q}}.\cancel{q}.\exp(t).\underbrace{\sum_{k=1}^{+\infty} (\exp(t).q)^{k-1}}_{\frac{1}{1-q\exp(t)}}$$

$$\boxed{E[\exp(tY)] = p.\exp(t).\frac{1}{1-q\exp(t)}}$$

$$\boxed{E[\exp(tY)] = \frac{p.\exp(t)}{1-q\exp(t)}}$$

1. ve 2. momentleri yerine koyup denediğimizde ve karşılaştırdığımızda mgf denklemini doğru bulduğumuzu görüyoruz. Yaptığımız varsayımın böylece doğru olduğunu gördük.

$$\boxed{\text{Provided} \rightarrow |q.\exp(t)| < 1}$$

Q88 Go through the steps of the derivation in (Eq. 11), explaining each step. You might find the following identity useful

$$\left(\frac{a}{b}\right)' = \frac{a'b - b'a}{b^2}$$

$mgf_Y(t) \rightarrow$ her bir RV için, kimliksel olmak üzere her bir moment değeri için expected value veren fonksiyondur.

$$E[Y] = \frac{d}{dt}\left[mgf_Y(t)\right] = \frac{d}{dt}\left[\frac{p.\exp(t)}{1-q.\exp(t)}\right] \qquad \frac{a'b - b'a}{b^2}$$

$$\left.\frac{\overbrace{p.\exp(t)}^{a'}.\overbrace{(1-q\exp(t))}^{b} - \overbrace{(-q.\exp(t))}^{b'}.\overbrace{(p.\exp(t))}^{a}}{[1-q\exp(t)]^2}\right|_{t=0}$$

$t=0$ için $$\frac{(p.1).(1-q.1)-(-q.1).(p.1)}{(1-q.1)^2} = \frac{p.(1-q)-(-q).p}{(1-q)^2}$$

$$= \underbrace{\frac{p-\cancel{pq}+\cancel{pq}}{(1-q)^2}}_{\left.\frac{p.\exp(t)}{[1-q.\exp(t)]^2}\right|_{t=0}} \quad \frac{p=1-q}{\quad} \quad \frac{p}{p^2} = \frac{1}{p}//$$

Q9: Go through the steps of eq.13 and explain each step, again making use of eq.12

$$eq.13 \rightarrow E[Y^2] = \frac{p+2q}{p^2}$$

$$eq\,12 \rightarrow \left(\frac{a}{b}\right)' = \frac{a'b - b'a}{b^2}$$

$$E[Y^2] = \frac{d^2}{dt^2}\left[mgf_Y(t)\right]\Big|_{t=0}$$ } Bir önceki soruda $\underbrace{\frac{d}{dt}\left[mgf_Y(t)\right]\Big|_{t=0}}_{\text{bulmuştuk.}}$

$$E[Y^2] = \frac{d}{dt}\Big[\underbrace{\frac{d}{dt}\,mgf_Y(t)}_{\frac{P.\exp(t)}{[1-q.\exp(t)]^2}}\Big]\Big|_{t=0}$$

$$E[Y^2] = \frac{d}{dt}\left[\frac{P\exp(t) \rightarrow a}{[1-q.\exp(t)]^2 \rightarrow b}\right]$$

$$\frac{\overbrace{P.\exp(t)}^{a'}.\Big[\overbrace{(1-q.\exp(t))^2}^{b}\Big] - \Big[\overbrace{2.(1-q.\exp(t)).(-q)}^{b'}\Big].\overbrace{P.\exp(t)}^{a}}{[1-q.\exp(t)]^4}\Bigg|_{t=0}$$

$$\boxed{t=0 \text{ ise} \rightarrow \frac{P.1.(1-q.1)^2 - (2.(1-q.1).(-q).P.1)}{(1-q)^4}}$$

$$= \frac{P.(1-q)^2 - 2(1-q).(-q).P}{(1-q)^4} \underset{=}{\overset{p=1-q}{}} \frac{P.p^2 - 2P.P.(-q)}{p^4}$$

$$= \frac{p^2(p+2q)}{p^4} = \frac{p+2q}{p^2} \;//$$

Q10 : The computation in eq.15 is carried out through the binomial expansion theorem.

$$mgf_Z(t) = E[\exp(tZ)] = \sum_{k=0}^{n} \exp(t.k)\binom{n}{k}.p^k.q^{n-k}$$

$\boxed{\exp(tk) = \exp(t)^k}$

$$= \sum_{k=0}^{n} \underbrace{[p.\exp(t)]}_{x}^{k}.\binom{n}{k}.\underbrace{q}_{y}^{n-k}$$

Binomial expansion

$$= \underbrace{\sum_{k=0}^{n}\binom{n}{k}.x^k.y^{n-k}}_{[x+y]^n} = [x+y]^n$$

$= (x+y)^n = \underline{\underline{(p.\exp(t)+q)^n}}$ bulunur.

$$E[\exp(tz)] = (p.\exp(t)+q)^n$$

Q11 : Look up the English word "infinitesimal", search for it in a mathematical context and get to know what it is used for in for example continuity definitions

→ infinitesimal : sonsuz küçük değer

A function $f(x)$ is said to be continous when

$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$ in this formula x is not equal exactly a but they are <u>infinitesimally</u> close eachother.

Q12. Look up the technical definition of continuity.

Continuity : if we have $A_1 \subseteq A_2 \subseteq A_3 \ldots$, $A = \bigcup_{n=1}^{\infty} A_n$ and A_i is infinitesimally close to A_{i+1} then $\lim_{n\to\infty} P\{A_n\} = P\{A\}$

and also

f $x \to c$, then $f(x) \to f(c)$ f is continous function

$Q_{13} \rightarrow$ Try to examine in the support of (eq.1) how the parenthesis in the set {"[", "(", ")", "]"} affect my choices for utilizing either of those in the set {"<", "≤"} in eq.2. See if maked a mistake

"[" symbol means initial value of the interval is inclusive and "]" symbol means stop value of the interval is enclusive so if $[a_0, a_1]$ an interval for x value then it can be demonstrated as $\boxed{a_0 \leq x \leq a_1}$, "(" and ")" symbols mean the initial and stop values are exclusive then (a_2, a_3) is interval for y value then,

$\boxed{a_2 < y < a_3}$

$Q_{14} \rightarrow$ Show through the fundamental theorem of calculus that

$$\frac{d}{dx}\left[cdf_X(x)\right] = \frac{d}{dx}\left[\int_{-\infty}^{x} Pdf_X(\rho)\,d\rho\right] = Pdf_X(x)$$

Using the Second Fundamental Theorem

Theorem $\rightarrow \frac{d}{dx}\int_{a}^{x} f(t)\,dt = f(x)$ eğer sınırlar x'in fonksiyonu ise

eğer sınırlar x'in fonksiyonu ise $\left[\frac{d}{dx}\int_{y(x)}^{z(x)} f(t)\,dt = f(z(x)).z'(x) - f(y(x)).y'(x)\right]$

böylece, $\frac{d}{dx}\left[cdf(x)\right] = \frac{d}{dx}\left[\int_{-\infty}^{x} Pdf_X(\rho).d\rho\right] = Pdf_X(x)$

$Q_{15} \rightarrow$ Through (eq.14) show that (eq.15) is correct

$$eq.14 = \int_{-\infty}^{+\infty} \exp(t\rho) \cdot pdf_X(\rho) \cdot d\rho = mgf_X(t)$$

$$eq.15 = E[X^k] = \frac{d^k}{dt^k}\left[mgf_X(t)\right]\Big|_{t=0}$$

$$mgf_X(t) = E[\exp(tx)] = \int_{\substack{support\\(\rho)}} \exp(t\rho) \cdot pdf_X(\rho) d\rho$$

$$\exp(t\rho) = 1 + t\rho + \frac{(t\rho)^2}{2!} + \frac{(t\rho)^3}{3!} \cdots$$

$$mgf_X(t) \int_{-\infty}^{\infty} \left(1 + \frac{t\rho}{1!} + \frac{(t\rho)^2}{2!} \cdots\right) \cdot pdf_X(\rho) d\rho$$

$$mgf_X(t) = \int_{-\infty}^{\infty} \left(1 + \frac{t\rho}{1!} + \frac{(t\rho)^2}{2!} \cdots \frac{(t\rho)^k}{k!} \cdots\right) pdf_X(\rho) d\rho$$

$$\frac{d^k}{dt^k}\left[mgf_X(t)\right] = \int_{-\infty}^{\infty} \left(\cancel{1} \; \cancel{\frac{d^k}{dt}} t\rho \; \cancel{\frac{d^k}{dt^k}} \frac{(t\rho)^2}{2} \cdots \frac{d^k}{dt^k} \frac{(t\rho)^k}{k!} + \frac{(t\rho)^{k+1}}{k+1!}\right) pdf_X$$

k. elemandan önceki terimler türev ile sonraki terimler $t=0$ iken 0 olur türevin ve integralin sıralaması sürekli fonksiyonlarda önemsiz

türev sonucunda k. eleman $\frac{k.(k-1).(k-2)\cdots \rho^k}{k!} = \rho^k$ $\boxed{\text{türevde } pdf_X(\rho) \text{ t'ye göre sabit kabul edilir}}$

böylece $\frac{d^k}{dt^k}\left[mgf_X(t)\right] = \int_{-\infty}^{\infty} \rho^k \cdot pdf_X(\rho) d\rho$ bulunur.

ki bu tanım notlar Paerisinde eq.13 te bulunmaktadır.

$$\star \; E[X^k] = \int_{-\infty}^{+\infty} \rho^k pdf_X(\rho) d\rho = \frac{d^k}{dt^k}\left[mgf_X(t)\right] \; \star$$

Bonus HW TC # 03

Sol 1.

$\boxed{A = \{\text{Ayşe'nin önce teslim etmesi}\}}$

$\boxed{A^c = \{\text{Ali'nin önce teslim etmesi}\}}$

Bu iki olay "all inclusive" ve "mutually exclusive" olduğu için birbirlerinin "complementary"si olduğu söylenebilir

Bu durumda $P\{A\} = 2P\{A^c\}$ ise $P\{A\} + P\{A^c\} = 1$

$2P\{A^c\} + P\{A^c\} = 1$ $\boxed{P\{A\} = \frac{2}{3}}$ $\boxed{P\{A^c\} = \frac{1}{3}}$

$\boxed{B = \{5 \text{ sorunun çözülmesi}\}}$

$\lambda = 3$ Ayşe için

$\lambda = 10$ Ali için

$\{B|A\} = \{\text{Ayşe'nin 5 soruyu önce çözmesi}\} \rightarrow \boxed{P\{B|A\} = \frac{e^{-3} \cdot 3^5}{5!}}$

$\{B|A^c\} = \{\text{Ali'nin 5 soruyu önce çözmesi}\} = P\{B|A^c\}$

$\boxed{P\{B|A^c\} = \frac{e^{-10} \cdot 10^5}{5!}}$

Bize sorulan

$\{\text{5 soru geldiği bilindiğine göre Ayşe'nin getirmesi}\} = P\{A|B\}$

$$P\{A|B\} = \frac{P\{B|A\}\, P\{A\}}{P\{B|A\} \cdot P\{A\} + P\{B|A^c\} \cdot P\{A^c\}} = \boxed{\frac{\frac{e^{-3} \cdot 3^5}{5!} \cdot \frac{2}{3}}{\frac{e^{-3} \cdot 3^5}{5!} \cdot \frac{2}{3} + \frac{e^{-10} \cdot 10^5}{5!} \cdot \frac{1}{3}}}$$

$$= \frac{\frac{2e^{-3} \cdot 3^5}{5!}}{\frac{2e^{-3} 3^5}{5!} + \frac{e^{-10} \cdot 10^5}{5!}} = \frac{2e^{-3} \cdot 3^5}{2e^{-3} 3^5 + e^{-10} \cdot 10^5} = \frac{24.196}{28.736} = \underline{0.842}$$

Sol 2. S_1 ve S_2 "Random variable" olmak üzere

$S_1 = \{$ index of the first succesful shot on goal $\}$

$S_2 = \{$ index of the second succesfull shot on goal $\}$

(a) Oyuncunun başarılı olan ilk denemenin indexi i, ise 2.nin j olması olasılığı.

$P\{S_2 = j \mid S_1 = i\}$ $\qquad P\{S_1 = i\} = q^{i-1}p$

$P\{S_2 = j\} = P\{i < S_2 = j\}$

$$P\{\{S_2 = j\} \mid S_1 = i\} = \frac{P\{\{S_2 = j\}, \{S_1 = i\}\}}{P\{S_1 = i\}} \underset{=}{\text{independence}} \frac{P\{S_2 = j\} \cdot \cancel{P\{S_1 = i\}}}{\cancel{P\{S_1 = i\}}}$$

$= \underline{P\{S_2 = j\}}$

independence koşuluna göre cevap $P\{S_2 = j\}$ ancak bunlar "event" 'e independent denebilmesi için;

$\{S_2 = j\} = X_k$, X_k for $k \in \{S_1+1, S_1+2, \dots\}$ olmak üzere

$\{i < S_2 = j\} = X_k$ $\quad \{i < S_2\} \cup \{S_2 = j\} = \{i < S_2\} + \{S_2 = j\} - \{i < S_2\} \cap \{S_2 = j\}$

$$P\{S_2 < i\} \cup P\{S_2 = j\} \underset{\text{Axiom 3}}{=} \underbrace{P\{S_2 < i\}}_{A} + \underbrace{P\{S_2 = j\}}_{B} \qquad \boxed{B = q^{j-1} \cdot p}$$

$$A = P\{\underbrace{i \le S_2}_{cdf_x(i)}\}^c = cdf_x(i) = 1 - q^i \rightarrow \text{complementary} \rightarrow 1 - (1 - q^i) = q^i$$

$$\boxed{A + B = P\{S_2 = j\} = q^i + q^{j-1} \cdot p}$$

Sol 2.b $P\{S_2=j\}=Pmf_{S_2}(j)=?$

Attempt trial index $\rightarrow$ $\underbrace{1,2,3\ldots\ldots(k-1)}_{\text{burada 1 adet başarı olmalı.}}$ $\overset{\text{2nd success}}{k}$ $\quad j\in\{1,2,\ldots k-1\}$

$$P\{S_2=j\}=\Big[\underbrace{\binom{k-1}{1}p.q^{k-2}}_{\text{first success}}\Big].\underbrace{p}_{\text{2nd success}}=\underline{(k-1).p^2.q^{k-2}}$$

$$\boxed{Pmf_{S_2}(j)=(j-1).p^2.q^{j-2}\quad support(k)=2,3\ldots\ldots j}$$

HW TC #03 2(a) HW BONUS

If we term the described process in the question as a "stohastic process", through X_k, where X_k records the result of the kth Beroulli trial (random variable) for $k\in\{1,2,\ldots\}$ we say that " X_k for $k\in\{S_1+1, S_1+2\ldots\}$ is <u>independent</u> of the past described as X_j for $j\in\{1,2,\ldots S_1\}$ = ?

<u>Sol</u> The definition of independence may be extended from random vectors to a <u>stochastic process</u>. Random variables obtained by sampling the process at any n times $t_1,\ldots,t_n$ are independent random variables for any n. formally, a stochastic process $\{X_t\}_{t\in T}$ is called independent, if and only if for all $n\in N$ and for all $t_1\ldots t_n\in T$

$P\{x_1,x_2,x_3\ldots x_t\}=P\{x_1\}.P\{x_2\}.\ldots P\{x_t\}$

According to definition, we can say $P\{\{S_1=i\},\{S_2=j\}\}=P\{S_1=i\}.P\{S_2=j\}$ if and only if <u>support$(i)=1,2,\ldots S_1$, support$(j)=S_1+1, S_1+2\ldots S_2$</u>

What is a "stopping time", Are S_1 and S_2 stopping time?

In a sense, a <u>stopping time</u> is a random time that does not require that we see into the future. That is, we can tell whether or not $T\leq t$ from our information at time t.

So we can define S_1 and S_2 as stopping times.

HWTC #03 2(a) HW Bonus

Answer to Question 2a also bring about the Concept of "time homogeneity" for a stochastic process? What do we mean by this term

Definition: A stochastic process is said to be homogeneus in space if the (transition probability between any two states values at two given times depends on the difference between those state values.

According to definition we can say our situation and this definition are compatible $[1 \le i \le S_1 \quad S_1+1 \le j \le S_2]$ because of finite number of states)

$$pmf_{S_1}(i) \neq pmf_{S_1}(i+1)$$
$$pmf_{S_2}(j) \neq pmf_{S_2}(j+1)$$

HWTC #03 2(b) Bonus HW

Can you come up with a parametric formula for the following pmf family. $pmf_{S_m}(k) = P\{S_m = k\}$ $S_m = \{$index of the mth successful attempt$\}$

$$P\{S_m = k\} = \underbrace{1, 2, 3, \dots (k-1)}_{(m-1) \text{ times successful attempt}}, \underset{\text{mth successful attempt}}{k}$$

so

$$P\{S_m = k\} = \left[\binom{k-1}{m-1} \cdot p^{m-1} \cdot q^{k-1-(m-1)}\right] \cdot p$$

$$\boxed{P\{S_m = k\} = \binom{k-1}{m-1} \cdot p^m \cdot q^{k-m}}$$

What is the support for S_3 with $pmf_{S_3}(k)$ as in eq.6

eq.6 $\Rightarrow$ $P\{S_3 = k\} = \frac{(k-1) \cdot (k-2)}{2} p^3 \cdot q^{k-3}$

for $k = 1, 2 \rightarrow P\{S_3 = k\} = 0$ so support $(k) = 3, 4, \dots$

Prove that $\sum_{k \in \text{support}} \text{Pmf}_{S_3}(k) = 1$

$$\sum_{k=3}^{+\infty} \text{Pmf}_{S_3}(k) = \sum_{k=+3}^{+\infty} \frac{(k-1).(k-2)}{2} . p^3 . q^{k-3} = \frac{p^3}{2} \sum_{k=+3}^{+\infty} (k-1).(k-2).q^{k-3}$$

$$= \frac{p^3}{2} \sum_{k=3}^{+\infty} \frac{d^2}{dq^2}\left(q^{k-1}\right) = \frac{p^3}{2} . \frac{d^2}{dq^2}\underbrace{\left(\sum_{k=3}^{+\infty} q^{k-1}\right)}_{A}$$

$$A = q^2+q^3+ \cdots\cdots = q^2(1+q+\cdots) = \frac{q^2}{1-q}$$

$$= \frac{p^3}{2} \frac{d^2}{dq^2}\left(\underbrace{\frac{q^2}{1-q}}_{B}\right)$$

$$B' = \frac{2q.(1-q) - (-1).q^2}{(1-q)^2} = \frac{2q-2q^2+q^2}{(1-q)^2} = \frac{2q-q^2}{(1-q)^2}$$

$$B'' = \frac{(2-2q)\overset{q^2-2q+1}{(1-q)^2} - 2.(1-q).(-1).(2q-q^2)}{(1-q)^4} =$$

$$= \cancel{-2q^3} + \cancel{4q^2} - 2q + \cancel{2q^2} - \cancel{4q} + 2 + \underbrace{(2-2q)(2q-q^2)}_{\cancel{4q} - \cancel{4q^2} - \cancel{2q^2} + \cancel{2q^3}}$$

$$= \frac{-2q+2}{(1-q)^4} . \frac{p^3}{2} \overset{(p=1-q)}{=\!=} \frac{2(1-q).p^3}{(1-q)^4 . 2} = \frac{2.p.p^3}{p^4.2}$$

$$= 1$$