HWTC #15 HW BONUS # 19

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Q1: S_k: the index of the kth success event in an indefinitely long succession of iid. Bernoulli trials with the common success probability "p" $S_1 \sim Geometric(p)$

Note that S_2 can be interpreted as

$$S_2 = S_1 + \hat{S}_1 \quad \text{where} \quad S_1, \hat{S}_1 \sim Geometric(p)$$

$S_1, \hat{S}_1$ are the indices of the first success events for non overlapping successions of Bernoulli trials

S_1 and $\hat{S}_1$ are iid

$$SS^T = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \begin{bmatrix} S_1 & S_2 \end{bmatrix} = \begin{bmatrix} S_1^2 & S_1 S_2 \\ S_1 S_2 & S_2^2 \end{bmatrix}$$

$$E[SS^T], \quad E[S_k^2] = Var(S_k) + [E[S_k]]^2$$

HW Bonus: What is an outer vector product or a dyadic product in linear algebra? Definition of outer vector product

$$u = (u_1, u_2 \dots u_m), \quad v = (v_1, v_2, v_3 \dots v_n)$$

$$uv^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \begin{bmatrix} v_1, v_2 \dots v_n \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \dots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \dots & u_2 v_n \\ \vdots & & & \\ u_m v_1 & \dots & \dots & u_m v_n \end{bmatrix} \quad eq.1$$

outer product $u \otimes v$ is defined as the $m \times n$ matrix A obtained by multiplying each element of u by each element of v.

$$u \otimes v = A = uv^T$$

Dyadic product is the same. ("farkı Dyadic product işlemi vektör kullanırken. outer product aynı işlemi Matris ile yapar.)

end of bonus.

Hw bonus2. Is the matrix m (eq.14b) symetric? If so, how would you prove that.

simetrik matris transpose'si kendisine eşdeğer matris belirten yapılardır. Kontrol edecek olursak

$$\begin{bmatrix} S_1^2 & S_1S_2 \\ S_1S_2 & S_2^2 \end{bmatrix}^T = \begin{bmatrix} S_1^2 & S_1S_2 \\ S_1S_2 & S_2^2 \end{bmatrix}$$

Transpose'sinin kendisine eşdeğer olduğu görülüyor. Bu durumda simetriktir.

end of bonus.

$$E[S_k^2] = Var(S_k) + [E[S_k]^2]$$

eq.2

Computation of eq.15 is easy since S_1 is a geometric RV. for S_2, one should better make use of (eq.11)

One has to recall that
var (sum of independent RV)
= sum of the variances of the indicated independent RV.S, in order to deal with eq.2 for S_2.

Hw bonus3 Therefore compute $E[S_2^2]$

$S_2 = S_1 + \hat{S}_1$ olduğunu söylemiştik bu durumda eq.2 denklemi

$$E[S_2^2] = Var(S_2) + [E[S_2]^2] = Var(S_1+\hat{S}_1) + [E[S_1+\hat{S}_1]^2]$$

şu açıklamaya göre: var (sum of ind. RV) = sum of the variances of the indicated independent R.V.
ve expectation lineerliğine göre

$$E[S_2^2] = \underbrace{Var(S_1)}_{\frac{1-p}{p^2}} + \underbrace{Var(\hat{S}_1)}_{\frac{1-p}{p^2}} + E[S_1]^2 + E[\hat{S}_1]^2$$

$$\frac{1-p}{p^2} + \frac{1-p}{p^2} + \left(\frac{1}{p}\right)^2 + \left(\frac{1}{p}\right)^2$$

for geometric RV

$$Var(S_1) = \frac{1-p}{p^2} = Var(\hat{S}_1)$$

$$E[S_1] = \frac{1}{p} = E[\hat{S}_1]$$

$$E[S_2^2] = \frac{1-p+1-p+1+1}{p^2} = \frac{4-2p}{p^2}$$

end of bonus.

Hw bonus 4. Explain each step of eq. 1.6

$$E[S_1 S_2] = E[S_1[S_2 - S_1 + S_1]] = E[S_1[\hat{S_1} + S_1]]$$

$S_1[S_2 - S_1 + S_1] = S_1 . S_2$ olduğundan $\quad \hat{S_1} = S_2 - S_1$ olduğundan

$$E[S_1[\hat{S_1} + S_1]] = E[S_1 . \hat{S_1}] + E[S_1^2] = E[S_1] . E[\hat{S_1}] + E[S_1^2]$$

dağılma özelliği

Independency

$$E[S_1 S_2] = \frac{1}{p} \cdot \frac{1}{p} + \frac{2-p}{p^2}$$

$$E[S_1^2] = Var(S_1) + E[S_1]^2$$

$$\boxed{E[S_1 S_2] = \frac{3-p}{p^2}}$$ ☆ Hw bonus. compute the explicit expression for $E[S_1, S_2]$

$$\frac{1-p}{p^2} + \frac{1}{p^2} = \frac{2-p}{p^2}$$

end of bonus.

$$\boxed{E[S_2^2] = \frac{4-2p}{p^2} \quad E[S_1^2] = \frac{2-p}{p^2}} \Rightarrow$$

$$= E\begin{bmatrix} S_1^2 & S_1 S_2 \\ S_1 S_2 & S_2^2 \end{bmatrix} \xrightarrow[\text{and expectation}]{\text{linearity of matrix}} \begin{bmatrix} E[S_1^2] & E[S_1 S_2] \\ E[S_1 S_2] & E[S_2^2] \end{bmatrix} =$$

$$= ☆ \begin{bmatrix} \frac{2-p}{p^2} & \frac{3-p}{p^2} \\ \frac{3-p}{p^2} & \frac{4-2p}{p^2} \end{bmatrix} ☆$$

Hw bonus 6. Compute an explicit expression for $E[S_m S_n]$.

$m=n$ ise $E[S_m^2] = Var(S_m) + E[S_m]^2$

$m<n$ ise

$E[S_m[S_n + S_{n-1} - S_{n-1} + S_{n-2} + S_{n-2} \cdots + S_1 - S_1]] =$

Q2: Let $X \sim N(\mu=0, \sigma^2 = 25cm^2)$

This question asks the following probability $P\{|X| < 5cm | X < 10cm\}$

Consider the following

$$P\{|X| \le 5cm | X < 10cm\} = \frac{P\{|X| < 5cm, X < 10cm\}}{P\{X < 10cm\}} = \frac{P\{-5 < X < 5, X < 10cm\}}{P\{X < 10cm\}}$$

$$\frac{X-\mu}{\sigma} = Z = \frac{X}{5cm} \qquad Z \sim N(\mu=0, \sigma^2=1)$$

$$P\left\{\frac{X}{\sigma} < \frac{10}{\sigma}\right\} = P\{Z<2\} = 1 - P\{Z \ge 2\} = 1 - Q(2)$$

explain each step of (eq. 2.6)

$\frac{X}{\sigma} = Z$ tanımlaması yapıldıktan sonra $P\{X < 10\}$ terimi $\frac{X}{\sigma}$'ya benzetilir.

$P\left\{\frac{X}{\sigma} < \frac{10}{5}\right\} = P\{Z < 2\}$ elde edilir.

$P\{Z<2\} = 1 - \underbrace{P\{Z \ge 2\}}_{Q(2)}$

Q fonksiyonu gaussian RV için cdf ile çok benzerlik gösterir. Sadece cdf fonksiyonu $-\infty \to x$'e integral alırken Q fonksiyonu $x \to +\infty$'a alır. Bu durumda formatında $Z \ge 2$ olmalı.

Bu durumda

$$P\{X<10\} = 1 - Q(2)$$ olarak elde edilir.

(eq.1)

$$cdf_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{\rho^2}{2\sigma^2}\right)$$

4

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{+\infty} \exp\left(-\frac{\rho^2}{2\sigma^2}\right)$$

$$P\{|x| \le 5\,cm\} = P\{-5 \le x \le 5\,cm\}$$

$$= P\left\{-\frac{5}{\sigma} \le \frac{x}{\sigma} \le \frac{5}{\sigma}\right\} = P\{-1 \le z \le 1\}$$

Q fonksiyonu integral için çift "fonk." konudurunda Q tek fonk.

$$P\{-1 \le z\} = P\{z \ge -1\} = Q(-1) = -Q(1) \rightarrow \text{tek fonk.}$$

$$P\{z \le 1\} = 1 - P\{z \ge 1\} = 1 - Q(1)$$

$$P\{\{-1 \le z\} \cup \{z \le 1\}\} \xrightarrow{Ax.3} \underbrace{P\{-1 \le z\}}_{-Q(1)} + \underbrace{P\{z \le 1\}}_{1-Q(1)} = 1 - 2Q(1) \quad eq.2$$

$$\boxed{\frac{P\{|x| \le 5, |x \le 10\}}{P\{x \le 10\}} = \frac{P\{|x| \le 5\}}{P\{x \le 10\}} =}$$

How bows. Derive this final answer $\{|x| \le 5\} \xrightarrow{\text{IMPLIES}} \{x \le 10\}$

$$\frac{P\{|x| \le 5\}}{P\{x \le 10\}} = \frac{1 - 2Q(1)}{1 - Q(2)} \rightarrow \text{from eq.1 and eq.2}$$

Q3: We will have to make use of Chebyshev's inequality which reads:

$P\{|x - E[x]| \ge k\} \le \frac{var(x)}{k^2}$ according to Q2: $E[x] = 0$, $var(x) = 25\,cm^2$

Aranılan cevap $P\{|x| \le 5\,cm\}$

$$P\{|x - 0| \ge k = 5\} \le \frac{25}{5^2} \rightarrow P\{|x| \ge 5\} \le 1$$

$$P\{|x| \ge 5\} = 1 - P\{|x| \le 5\} \le 1 \qquad 0 \le P\{|x| \le 5\}$$

$$P\{|x| \le 5\} \ge 0$$

Hw bonus Show that the indicated bound will be lower bound, and also tell us if the inequality in (eq.3.4) says much or not?

$$P\{|X-0|\geq k=5\} \leq \frac{25}{5^2} \rightarrow \boxed{P\{|X|\geq 5\}\leq 1}$$

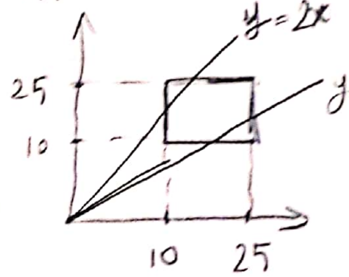
$$P\{|X|\geq 5\} = 1-P\{|X|<5\}\leq 1 \rightarrow 0\leq P\{|X|<5\}$$

$P\{|X|\leq 5\}\geq 0 \rightarrow$ Buradan aranılan olasılık değerinin her halükarda 0'dan büyük olması beklendiği görüldü. Lower bound sağlandı.
$P\{|X|<5\}$'in olası tüm dataları sıfırdan büyük olmalı.
Bu eşitsizlik ekstra bir bilgi (bu soru için) katmamaktadır.
Bir olasılık değerinin sıfırdan büyük olacağı zaten bilinmektedir.

Q4: Let us define our random variables.
X, Y: the amounts of money that the brothers each raise. X and Y are iid. and X, Y ~ Uniform($10, $25) And we are asked

$$P\{X>2Y\}+P\{Y>2X\} = 2P\{Y>2X\} = 2E[P\{Y>2X|X\}]$$

In dealing with the probability in (eq.4.1b), the following graphical approach will be much more preferred.



$$2E[P\{Y>2X|X\}] = 2\int_{10}^{12.5}\int_{2x}^{25} Pdf_{X,Y}(x,y)\,dy\,dx$$

Hw bonus Explain each step of (eq.4.1.b)

$$P\{X>2Y\}+P\{Y>2X\} = 2P\{Y>2X\}$$

☆ Burada X ve Y random variable ların iid olmasından dolayı $P\{Y>2X\} = P\{X>2Y\}$ diyebiliriz.

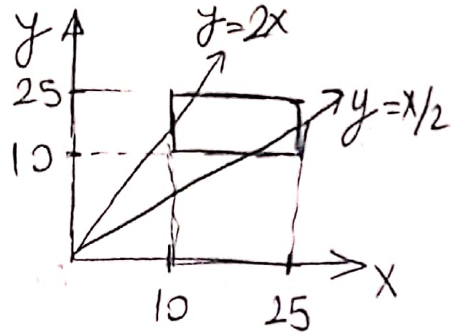
$$= 2E[P\{Y>2X|X\}]$$

→ conditional expectation açılımı

$P\{Y>2X\} = E[P\{Y>2X|X\}]$ olarak yazılabilir

end of bonus. 6

Hw Bonus. How else can we express this integral?



$$2\int_{10}^{12.5}\int_{2x}^{25} pdf_{X,Y}(x,y)\,dy\,dx$$

Burada y'sınırı sayı sınırı kabul edilmiştir x sınırı kabul edilerek veya $y=\frac{x}{2}$ kullanılarak integralin 3 alternatifi bulunur

x sayı sınırı

$$2\int_{20}^{25}\int_{10}^{y/2} pdf_{X,Y}(x,y)\,dy\,dx \quad \text{end of bonus}$$

Hw Bonus. How do you express this joint pdf?

$pdf_{X,Y}(x,y) = pdf_X(x) . pdf_Y(y) \longrightarrow$ X ve Y independent olduğu sürece joint pdf bu şekilde yazılabilir.

$$pdf_X(x) = pdf_Y(y) = \frac{1}{\underset{b-a}{25-10}} = \frac{1}{15}$$

$$\boxed{pdf_{X,Y}(x,y) = \left(\frac{1}{15}\right)^2}$$

end of bonus.

$$2\int_{10}^{12.5}\int_{2x}^{25}\frac{1}{225}\,dy\,dx = 2\int_{10}^{12.5}\frac{(25-2x)}{225}\,dx = 2\int_{10}^{12.5}\frac{1}{9}-\frac{2x}{225}\,dx$$

$$= \frac{2.(2.5)}{9} - 2.\Big|_{10}^{12.5}\frac{x^2}{225} = \frac{5}{9} - \frac{2}{225}(12.5-10).(12.5+10)$$

$$= \frac{5}{9} - \frac{\cancel{2}}{\underset{45}{\cancel{225}}}.\cancel{2.5}.\underset{\frac{1}{2}}{\cancel{22.5}} = \frac{5}{9}-\frac{1}{2} = \frac{1}{18}$$

[HW Bonus. What is $P\{Y>2X \mid X=x\}$ in eq. 4.3 ?]

$$P\{Y>2X \mid X=x\} = P\{Y>2x\} = 1-P\{Y\leq 2x\} = 1-\text{cdf}_Y(2x)$$

then $2\int_{10}^{12.5} P\{Y>2X \mid X=x\}\,\text{pdf}_X(x)\,dx$ sonucu $\frac{1}{18}$ vermelidir

$$2\int_{10}^{12.5}(1-\text{cdf}_Y(2x))\cdot\frac{1}{15}\,dx = \frac{1}{18} = \frac{2}{15}\int_{10}^{12.5}(1-\text{cdf}_Y(2x))\,dx$$

$\frac{1}{b-a}$

$$\boxed{\frac{5}{12} = \int_{10}^{12.5} 1-\left(\frac{2x-10}{15}\right)dx}$$

$\frac{x-a}{b-a}$

$$\int_{10}^{12.5}\frac{25}{15}-\frac{2x}{15}\,dx = \int_{10}^{12.5}\frac{25}{15}\,dx - 2\int_{10}^{15}\frac{2x}{15}\,dx$$

$$\Big|_{10}^{12.5}\frac{25}{15}x - \Big|_{10}^{12.5}\frac{x^2}{15}$$

$$\frac{25\cdot(2.5)}{15} - \frac{(12.5)^2-10^2}{15} = \frac{25}{6} - \frac{(22.5)\cdot(2.5)}{15\,6} = \frac{2.5}{6} = \frac{5}{12}$$

Olasılık değerinin doğru seçildiği 2 yoldan da sonucun $\frac{1}{18}$ çıkması gerektiği bilgisi ile teyit edildi.

8