HWTC #08

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Q1: X and Y are two contınous RV

$support(X) = (-\infty, +\infty)$
$support(Y) = (-\infty, +\infty)$

Joint pdf of X and Y $\rightarrow pdf_{X,Y}(x,y)$

$$\boxed{pdf_X(x) = \int_{-\infty}^{+\infty} pdf_{X,Y}(x,y)\,dy} \qquad \boxed{pdf_Y(y) = \int_{-\infty}^{+\infty} pdf_{X,Y}(x,y)\,dx}$$

$Z = X + Y$

$pdf_{X,Y}(x,y) = pdf_{X|Y}(x|y)\,pdf_Y(y)$
$= pdf_{Y|X}(y|x)\,pdf_X(x)$

$cdf_Z(a) = P\{Z \le a\} = P\{X+Y \le a\}$

$= \iint_{x+y \le a} pdf_{X|Y}(x|y)\,pdf_Y(y)\,dx\,dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{a-y} pdf_{X|Y}(x|y)\,pdf_Y(y)\,dx\,dy =$

$= \int_{-\infty}^{+\infty} cdf_{X|Y}(a-y|y)\,pdf_Y(y)\,dy$

$\boxed{cdf_{X|Y}(a-y|y) = P\{X \le a-y \mid Y = y\}}$

$\boxed{\text{if X and Y are independent}}$

$$\boxed{\begin{aligned} pdf_{X|Y}(x|y) &= pdf_X(x) \\ pdf_{Y|X}(y|x) &= pdf_Y(y) \end{aligned}}$$

$$\boxed{\begin{aligned} pdf_Z(a) &= \frac{d}{da}\left[cdf_Z(a)\right] = \\ &= \int_{-\infty}^{+\infty} pdf_{X|Y}(a-y|y)\,pdf_Y(y)\,dy \end{aligned}}$$

then $\boxed{pdf_Z(a) = pdf_X(a) * pdf_Y(a)}$ $(* \rightarrow$ konvolüsyon$)$

X and Y contınous RV $\quad Z = X + Y$

$0 \le x \le a, \quad 0 \le y \le b$

$\boxed{pdf_{X,Y}(x,y) = cxy}$

$pdf_X(x) = \int_0^b \frac{4}{a^2b^2} \cdot xy \cdot dy = \frac{4}{a^2b^2} \cdot x \cdot \frac{b^2}{2}$

$\int_0^b\int_0^a cxy\,dx\,dy = 1$ durdu

$\boxed{pdf_X(x) = \frac{2x}{a^2}}$ $\qquad \boxed{pdf_Y(y) = \int_0^a \frac{4}{a^2b^2} xy\,dx}$

$\frac{a^2}{2} c \cdot \frac{b^2}{2} = 1$

$\boxed{c = \frac{4}{a^2b^2}}$

$\boxed{pdf_Y(y) = \frac{2y}{b^2}}$

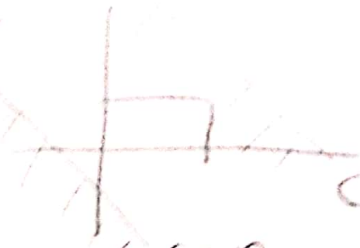
$pdf_{X,Y}(x,y) = pdf_X(x) \cdot pdf_Y(y)$ ise independent

$\frac{4}{a^2b^2} \cdot xy = \frac{2x}{a^2} \cdot \frac{2y}{b^2}$ ✓ X and Y independent

1.2 $Z = X+Y$ $\quad P\{Z \le c\} = \iint_{\substack{0 \le x \le a \\ 0 \le y \le b \\ x+y \le c}} pdf_{X,Y}(x,y)\,dx\,dy$ for $a \ge b$

if $c \le 0 \rightarrow cdf_Z(a) = 0$

if $c \ge a+b \rightarrow cdf_Z(a) = 1$



$c > a+b$

$c \le 0$

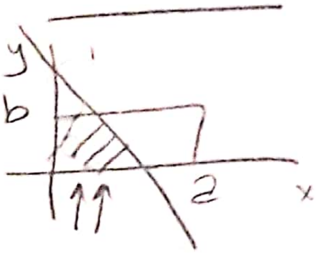
$0 \le c \le b$

$$P\{Z \le a\} = \int_0^c \int_0^{c-x} xy\,dy\,dx$$

$$= \int_0^c \frac{(c-x)^2}{2} \cdot x\,dx = \frac{1}{2}\int_0^c \frac{x^3 - 2cx^2 + c^2x}{2} = \frac{1}{2}\Big|_0^c \frac{x^4}{8} - \frac{cx^3}{3} + \frac{c^2x^2}{4}$$

$$\frac{c^4}{8} - \frac{c^4}{3} + \frac{c^4}{4} = \boxed{\frac{c^4}{24}} \qquad pdf_Z(a) = \frac{c^3}{6} = \frac{d}{dc}(pdf_Z(c))$$

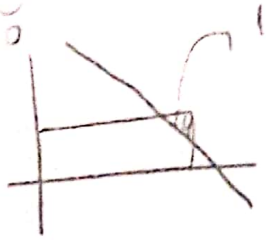
$b \le c \le a$



$$\int_0^1 \int_0^{c-y} xy\,dx\,dy = \int_0^1 \frac{1}{2} y(z-y)^2\,dy = \frac{1}{2}\int_0^1 (y^3 - 2cy^2 + c^2y)$$

$$= \frac{1}{2}\Big|_{y=0}^{1} \left[\frac{y^4}{4} - \frac{2c}{3}y^3 + \frac{c^2y^2}{2}\right] = \frac{1}{2}\left[\frac{1}{4} - \frac{2c}{3} + \frac{c^2}{2}\right]$$

$$pdf_Z(c) = \frac{d}{dc}(cdf_Z(c)) = \boxed{-\frac{2}{3} + c}$$



$1 - cdf_Z(a)$

$$\longrightarrow \int_{c-b}^{a} \int_{c-x}^{b} xy\,dy\,dx = \int_{c-b}^{a} \left(\frac{b^2 - (c-x)^2}{2}\right) x\,dx$$

$c \ge a+b$ ise $cdf_Z(c) = 1$

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Q2: continous RV X $Y = X^2 + 2$

$$cdf_Y(y) \leq P\{Y \leq y\} = P\{X^2 + 2 \leq y\}$$

$$= P\{|X| \leq \sqrt{y-2}\} \Rightarrow P\{-\sqrt{y-2} \leq X \leq \sqrt{y-2}\}$$

$$= P\{X \leq \sqrt{y-2}\} - P\{X \leq -\sqrt{y-2}\} \rightarrow \text{simetriden}$$

$-\sqrt{y-2}, \sqrt{y-2}$

$$= \boxed{cdf_X(\sqrt{y-2}) - cdf_X(-\sqrt{y-2})} = cdf_Y(y)$$

$$\frac{d}{dy} cdf_Y(y) = Pdf_Y(y)$$

$$Pdf_Y(y) = \frac{d}{d(\sqrt{y-2})} \cdot cdf_X(\sqrt{y-2}) \cdot \frac{d\sqrt{y-2}}{dy} - \frac{d}{d(-\sqrt{y-2})} \cdot cdf_X(-\sqrt{y-2})$$

$$Pdf_Y(y) = \frac{1}{2\sqrt{y-2}} \left(Pdf_X(\sqrt{y-2}) + Pdf_X(-\sqrt{y-2}) \right)$$

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Q3: S_k: Index of the week at which the kth successful sales attempt occurs

$S \sim$ Geometric(p) $A_k \sim$ Uniform

kovaryans için expected değeri bulalım.

$E\left[\sum_{k=1}^{\cdot} \exp(-rw S_k) A_k\right]$ $E[\exp(-rw S_k)]$

$S_k = \underbrace{V_1 + V_2 \ldots V_k}_{V_k \sim \text{Geometric}(p),\ \text{iid}}$ V_i's are independent so

V_i's are equal each other so; $\prod_{i=1}^{k} E[\exp(-rw V_i)]$

$[E[\exp(-rw V_1)]]^k \longrightarrow \boxed{E[\exp(-rw V_1)] = Mgf_{V_1}(-rw)}$

$\operatorname{cov}(\underbrace{S_k}_{\text{index}}, \exp(-rw \underbrace{S_{k-1}}_{\text{bir önceki haftanın indirimi}})) = E[S_k \cdot \exp(-rw S_{k-1})] - E[S_k] \cdot E[\exp(-rw S_{k-1})]$

$E[\exp(-rw \cdot S_k)] = E[\exp(-rw S_1)]^k, \quad E[S_k] = E[S_1]^k = \left(\frac{1}{\lambda}\right)^k$

$\operatorname{cov}(S_k, \exp(-rw \cdot S_k)) = E[S_1 \cdot \exp(-rw S_1)]^k - \frac{1}{\lambda} \cdot E[\exp(-rw S_1)]^k$