Q1 HWTC #13

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$A \sim Uniform(15, 30)$

$R = 40lt$ deterministic value

$S_1 \sim Bernoulli(P_1 = 1)$
$S_2 \sim Bernoulli(P_2 = 0.6)$

Toplam gaz = g

$P\{g > 40\}$ $g = A_1 + A_2$

$P\{A_1 + A_2 \geq 40\} = 1 - P\{A_1 + A_2 \leq 40\}$

$P\{\ldots\} \ cdf_{A_1 + A_2}(40) = 1 - \ldots$

$Q2: \; 0 < x < 2 \quad 0 < y < 1$ Abdullah

$pdf_{X,Y}(x,y) = cxy$

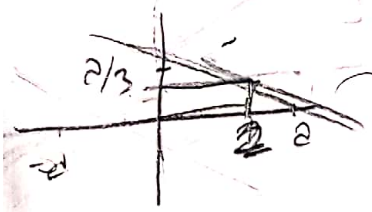
$\iint_{\substack{support(x)\\support(y)}} pdf_{X,Y}(x,y)\,dx\,dy = 1$, $\int_0^1\int_0^2 \underbrace{pdf_{X,Y}(x,y)}_{cxy}\,dx\,dy = 1$

$\int_0^1\int_0^2 cxy\,dx\,dy = \int_0^1 \Big|_0^2 cy\frac{x^2}{2}\,dy = \int_0^1 2cy\,dy = cy^2\Big|_0^1 = 1$

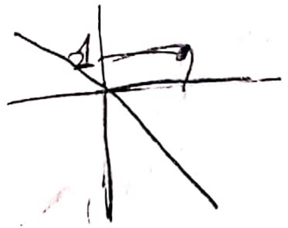
$\boxed{c = 1}$

$P\{X>3Y\}$ ise $P\{3Y-X \le 0\}$ $Z = 3Y-X$

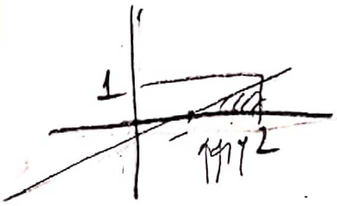
$\boxed{3y-x=a}$ $y = \frac{x+a}{3}$



$a > 3$ için $cdf_Z(a) = 1$



$a < 0$ için $cdf_Z(a) = 0$



$-1 < a < 1$ için $cdf_Z(a) =$

$P\{X>3Y\} = \iint_{X>3Y} xy\,dx\,dy = \int_0^{2/3}\int_{3y}^{2} xy\,dx\,dy$

$X > 3Y$ ise

$\int_0^{2/3} \Big|_{3y}^{2} \frac{x^2}{2} y\,dy$ $\left(\frac{4-9y^2}{2}\right)y$ $\Big|_0^1$ $\frac{4}{9} - \frac{9y^4}{8} = \frac{2}{9}$

$\frac{\cancel{8}.16}{\cancel{8}.81}\,\frac{2}{9}$

Q3 X and Y are two iid RV $\lambda = 1$ $Z = X + Y$

$P\{Z > 2 \mid X < 1\} = ?$

$P\{X + Y > 2 \mid X < 1\} \rightarrow P\{Y > 1\}$

$1 - P\{Y \leq 1\} = 1 - cdf_Y(1) = 1 - (1 - exp(-\lambda y))$

$exp(-1) = 0.367$

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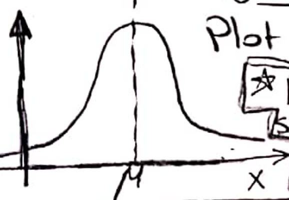
$\boxed{\text{Gaussian RV}} \rightarrow X \sim N(\mu, \sigma^2) \rightarrow \text{support}(X) = (-\infty, +\infty)$

$\text{Pdf}_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$, $\text{Cdf}_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{(\varrho-\mu)^2}{2\sigma^2}\right) d\varrho$

$X \sim N(\mu, \sigma^2)$ o.ü $E[X] = \mu$, $E[X^2] = E[(X-\mu)^2] + 2\mu E[X-\mu] + \mu^2$, $Var(X) = \sigma^2$



Plot for $\text{pdf}_X(x)$

☆ $\text{Pdf}_X(x)$ is symetric around μ ☆

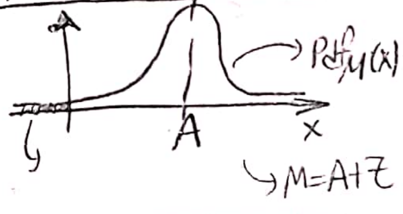


fig1

$\boxed{\text{Gaussian RV Example}}$

Amplitude $A > 0$, Noise signal $Z \sim N(0, \frac{N_0}{2})$

M : the mesage signal as a randon variable

$P\{\text{bit 0 received} \mid \text{bit 1 sent}\} = P\{M \leq 0\}$

$\text{Pdf}_M(x) = P\{Z + A \leq x\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty} \exp\left(-\frac{\varrho^2}{2\sigma^2}\right) d\varrho$

$\text{Pdf}_M(x) = \frac{1}{\sigma\sqrt{2\pi}} * \exp\left(-\frac{(x-A)^2}{2\sigma^2}\right)$ eq.1

$\boxed{\text{eq.1 also says } M \sim N(A, N_0/2)}$

☆ Express in terms of Q-function of a standart Randon Variable

$P\{M \leq 0\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{(\alpha-A)^2}{2\sigma^2}\right) d\alpha$, $\sigma = \sqrt{\frac{N_0}{2}}$, $y = \frac{\alpha - A}{\sigma}$, $dy = \frac{d\alpha}{\sigma}$

$P\{M \leq 0\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{A}{\sigma}} \exp(-y^2/2)\, dy = P\{Y \leq \frac{-A}{\sigma}\}$, $Y \sim N(0,1)$, $P\{\text{interpreted as bit 0} \mid \text{bit 1 is sent}\} = Q\left(A\sqrt{\frac{2}{N_0}}\right)$

$-A + Z = R$, $P\{R \geq 0\} = P\{\text{misinterpreted as bit 1} \mid \text{bit 0 is sent}\}$, $P\{\text{Error}\} = P\{\text{misinterpreted as bit 0} \mid \text{bit 1 is sent}\} . P\{\text{bit 1 is sent}\} + P\{1|0\}P\{0\}$

$\boxed{\text{Sums of RV}}$ ☆ $\text{support}(X) = \text{support}(Y) = (-\infty, +\infty)$,

☆ ☆ ☆ $\text{Pdf}_{X,Y}(x,y) = \text{Pdf}_{X|Y}(x|y) . \text{Pdf}_Y(y) = \text{Pdf}_{Y|X}(y|x) . \text{Pdf}_X(x)$

$\text{Pdf}_X(x) = \int_{-\infty}^{+\infty} \text{Pdf}_{X,Y}(x,y)\, dy$

$\text{Pdf}_Y(y) = \int_{-\infty}^{+\infty} \text{Pdf}_{X,Y}(x,y)\, dx$

$Z = X + Y$ o.ü, $\text{cdf}_Z(a) = P\{Z \leq a\} = P\{X + Y \leq a\}$

$P\{X + Y \leq a\} = \iint_{x+y \leq a} \text{Pdf}_{X|Y}(x|y)\, \text{Pdf}_Y(y)\, dx\, dy \rightarrow \int_{-\infty}^{+\infty} \int_{-\infty}^{a-y} \text{Pdf}_{X|Y}(x|y)\, \text{Pdf}_Y(y)\, dx\, dy = \int_{-\infty}^{+\infty} \text{cdf}_{X|Y}(a-y|y)\, \text{Pdf}_Y(y)\, dy$

$= \text{cdf}_{X|Y}(a-y|y) = \int_{-\infty}^{a-y} \text{Pdf}_{X|Y}(x|y)\, dx = P\{X \leq a-y \mid Y = y\}$

$\text{Pdf}_Z(a) = \frac{d}{da}[\text{cdf}_Z(a)] = \int_{-\infty}^{+\infty} \text{Pdf}_{X|Y}(a-y|y)\, \text{Pdf}_Y(y)\, dy = \int_{-\infty}^{+\infty} \text{Pdf}_{X,Y}(a-y, y)\, dy$

$\text{Pdf}_Z(a) = \int_{-\infty}^{+\infty} \text{Pdf}_X(a-y) . \text{Pdf}_Y(y)\, dy = \text{Pdf}_Z(a) = \text{Pdf}_X(a) * \text{Pdf}_Y(a)$

$P\{A\} = E[E[\mathbb{I}_{\{A\}} | Y]] = E[P\{A|Y\}]$, $E[X|Y] \rightarrow RV$, $P\{A|Y\} \rightarrow RV$

$\text{mgf}_Z(t) = \text{mgf}_X(t) . \text{mgf}_Y(t) \Rightarrow$ X and Y are two independent RV's

$\text{mgf}_U(t) = [\text{mgf}_{V_1}(t)]^n$ $V_i \sim$ Bernoulli → iid for all i

$U \sim$ Binomial

Sums of Independent RV , X and Y → two independent RV, $Z=X+Y$ mgf of $Z = ?$

$$mgf_Z(t) = E[\exp(tZ)] = \sum_{i=-\infty}^{+\infty} \exp(t z_i)\, pmf_Z(z_i) \quad , \quad pmf_{X,Y}(x,y) = pmf_X(x) \cdot pmf_Y(y)$$

$$mgf_Z(t) = E[\exp(t(x+y))] = \sum_i \sum_j [\exp(t(x_i+y_j))] \cdot pmf_{XY}(x_i, y_j) \quad \begin{pmatrix} -\infty < i < +\infty \\ -\infty < j < +\infty \end{pmatrix}$$

$$\sum_{i,j}\sum [\exp tx] \cdot [\exp ty] \cdot [pmf_X(x_i)] \cdot [pmf_Y(y_j)]$$

$Mgf_X(t)$ $\quad$ $Mgf_Y(t)$ → X ve Y bağımsız olluğundan çarpımlar olarak yazılır.

independency $= Mgf_X(t) \cdot Mgf_Y(t)$

A binomial RV → parameter (n) → Bernoulli trials (P) → sum of n iid

U → binomial RV ; $\{V_1, V_2, V_3 \dots V_n\}$ → embedded bernoulli RVs.

$\boxed{U = V_1 + V_2 + \dots V_n}$ then $mgf_U(t) = \prod_{i=1}^{n} mgf_{V_i}(t) \xRightarrow{iid} mgf_U(t) = [mgf_{V_1}(t)]^n$ → iid olduğu için her mgf_{V_i} aynı değerdedir. Oy ♡ eder ü ♡ $\hat{a}$ldıl

$Mgf_U(t) = [q + \exp(t) \cdot P]^n$

$mgf_{V_1}(t) = [q + \exp(t) P]$

Sum of Continous RV , X, Y: Continous RV , $Z = X+Y$

$pdf_{X,Y}(x,y) = cxy \quad 0 \le x \le 2 \quad 0 \le y \le 1$ $\quad \int_0^1\int_0^2 pdf_{X,Y}(x,y)\,dx\,dy = 1$ olmalı.

→ bu sayede C katsayısı bulunur.

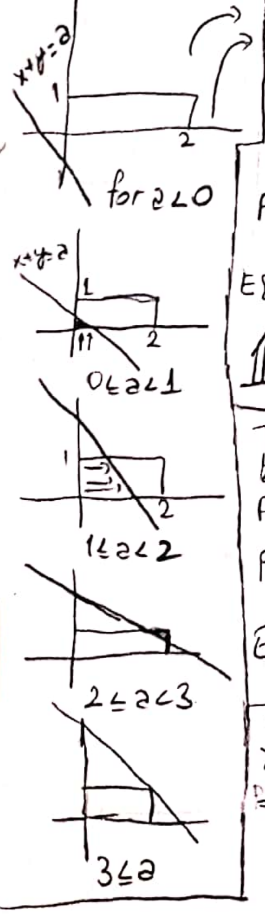
c katsayısı bulunduktan sonra: $pdf_X(x) = \int_0^1 pdf_{X,Y}(x,y)\,dy$ $\quad$ $\int_0^2 pdf_{Y,X}(y,x)\,dx$

independency check: $pdf_{X,Y}(x,y) \overset{?}{=} pdf_X(x) \cdot pdf_Y(y)$ bakılır.

$cdf_Z(a) = P\{Z \le a\}$

$0 \le a \le 1$ için $\int_0^a \int_0^{a-x} cxy\, dy\, dx$, $1 \le a \le 2$ için $\int_0^1 \int_0^{a-y} cxy\, dx\, dy$

$2 \le a \le 3$ için $P\{Z \le a\} = 1 - P\{Z > a\}$ $\quad P\{Z > a\} = \int_{a-1}^{2} \int_{a-x}^{1} cxy\, dy\, dx$



Conditional Expectation $\quad E[X] = \sum_{k=-\infty}^{+\infty} x_k\, pmf_X(x_k)$ $\quad support(Y) = \bigcup_{k=-\infty}^{+\infty} \{y_k\}$ $\{Y = y_k\}$ $k \in \mathbb{Z}$ (integer)

$\{X = x_k\} = \{\{X = x_k\} \cap E\} = \{\{X = x_k\} \cap \{\bigcup_{m=-\infty}^{+\infty} \{Y = y_m\}\}\} = \{\bigcup_{m=-\infty}^{+\infty} \{X = x_k, Y = y_m\}\}$ → mutually ex. all inclusive

$P\{X = x_k\} = \sum_{m=-\infty}^{+\infty} P\{X = x_k, Y = y_m\} = \sum_{m=-\infty}^{+\infty} P\{X = x_k \mid Y = y_m\} P\{Y = y_m\}$ ✶ $E[X] = \sum_{k=-\infty}^{+\infty} x_k \cdot \left[\sum_{m=-\infty}^{+\infty} P\{X = x_k \mid Y = y_m\} P\{Y = y_m\}\right]$

$E[X] = \sum_{m=-\infty}^{\infty} P\{Y = y_m\} E[X \mid Y = y_m]$ $\Rightarrow E[X] = E[E[X \mid Y]]$ ✶✶

$\mathbb{1}_{\{A\}} = \begin{cases} 1, & \text{if A occurs} \\ 0, & \text{otherwise} \end{cases}$, $P\{A\} = E[\mathbb{1}_{\{A\}}]$ → $P\{A\} = E[E[\underbrace{\underbrace{\mathbb{1}_{\{A\}}}_{RV} \mid \underbrace{Y}_{RV}}_{RV}]]$, $P\{A\} = E[P\{\underbrace{A}_{event} \mid \underbrace{Y}_{RV}\}]$

deterministic value $\quad$ det. Value

Cond. Exp. EXAMPLE

Thelma postane soyacak olursa \$100 ile \$500 arasında kazanacak. Başarılı olma olasılığı 0.3, \$400'den fazla kazanma olasılığı nedir?

$A \sim Uniform(a_L = \$100, a_H = \$500)$, $S \sim Bernoulli(P = 0.3)$, $\mathbb{1}_{\{S=1\}} = 1 - \mathbb{1}_{\{S=0\}}$, $R = \$400$ det. value

$P\{A.S > R\} = P\{A \cdot \mathbb{1}_{\{S=1\}} > R\} = E[\mathbb{1}_{\{A.S > R\}}] = E[E[\mathbb{1}_{\{A.S > R\}} \mid S]] = E[P\{A.S > R \mid S\}]$

$E[P\{A.S > R \mid S\}] = P\{A.0 > R \mid S = 0\} \cdot P\{S = 0\} + P\{A.1 > R\} \cdot P\{S = 1\}$ → independency

sum of cont. RV Example

X ve Y toplamlarının cdf ve pdf'ini belirle: $Z = X+Y$ $\quad cdf_Z(a) = \iint_{x+y \le a} pdf_{X|Y}(x|y)\, pdf_Y(y)\, dx\, dy$

$= \int_{-\infty}^{+\infty}\int_{-\infty}^{a-y} pdf_{X|Y}(x|y)\, pdf_Y(y)\, dx\, dy = \boxed{\int_{-\infty}^{+\infty} cdf_{X|Y}(a-y|y)\, pdf_Y(y)\, dy = cdf_Z(a)}$

$pdf_Z(a) = \frac{d}{da} cdf_Z(a) = \int_{-\infty}^{+\infty} pdf_{X|Y}(a-y|y)\, pdf_Y(y)\, dy = \int_{-\infty}^{+\infty} pdf_{X,Y}(a-y, y)\, dy = pdf_Z(a)$

→ $pdf_Z(a) = pdf_X(a) * pdf_Y(a)$