HW TC #15

Q1

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$$S = [S_1 \; S_2]^T = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

$$S^T = \{[S_1 \; S_2]^T\}^T = [S_1 \; S_2]$$

$$E[S S^T] = E\Big[\underbrace{\underset{2\times1}{\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}} \underset{1\times2}{[S_1 \; S_2]}}_{\text{matris çarpımı 2x2 matris verir}} \Big] = E\begin{bmatrix} S_1 . S_1 & S_1 . S_2 \\ S_1 . S_2 & S_2 . S_2 \end{bmatrix}$$

linearity of expectation and matrix $\longrightarrow$ $= \begin{bmatrix} E[S_1 S_1] & E[S_1 S_2] \\ E[S_1 S_2] & E[S_2 S_2] \end{bmatrix}$

$S_2 = S_{1new} \Rightarrow E[S_{1new}] = \frac{1}{p}$ from lecture derivations based on geometric RV set-up

$S_1 \rightarrow E[S_1] = \frac{1}{p}$ for geometric RV

So $E[S S^T] = \begin{bmatrix} \frac{1+q}{p^2} & \\ & \end{bmatrix}$

$\begin{bmatrix} E' & E[S_1 . S_2] \end{bmatrix}$

$$E[S_1^2] = \frac{1+q}{p^2}$$

$$E[S_1 . S_2] = E[$$

Q2 $A \sim N(\mu = 0, \sigma^2 = 25 cm^2)$

$X \sim N(\mu = 0, \sigma^2 = 1)$

$$P\left\{ \left| \frac{A-\mu}{\sigma} \right| \leq 5 \,\middle|\, \left| \frac{A-\mu}{\sigma} \right| \leq 10 \right\}$$

$$= \frac{P\{|X| \leq 5, |X| \leq 10\}}{P\{|X| \leq 10\}} \quad \text{eq.1}$$

↳ Baye's Rule

10 cm'den düşük old. bilindiğine göre 5cm'den yakın olma olasılığı

$$\Rightarrow P\{|X| \leq 5, |X| \leq 10\} = P\{X \leq 5\}$$

$\{X \leq 5\} \xrightarrow{\text{implies}} \{|X| \leq 10\}$

then eq.1 → $\dfrac{P\{|X| \leq 5\}}{P\{|X| \leq 10\}} =$

cdf of x is even function

$$cdf_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{s^2}{2\sigma^2}\right) \to Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{+\infty} \exp\left(-\frac{s^2}{2\sigma^2}\right)$$

$s \to -x$ $cdf_X(x) = Q(x)$ for changing dummy so,

$$P\{|X| \leq 5\} = P\{-5 \leq x \leq 5\} = 2 cdf_X(5) - 1 = \underline{2Q(-5) - 1}$$

$$P\{|X| \leq 10\} = P\{-10 \leq x \leq 10\} = 2 cdf_X(10) - 1 = 2Q(-10) - 1$$

$$P\left\{ \left| \frac{A-\mu}{\sigma} \right| \leq 5 \,\middle|\, \left| \frac{A-\mu}{\sigma} \right| \leq 10 \right\} = \boxed{\frac{2Q(-5) - 1}{2Q(-10) - 1}}$$

Q4:

$A_1 \sim \text{Uniform}(\$10, \$25)$

$A_2 \sim \text{Uniform}(\$10, \$25)$

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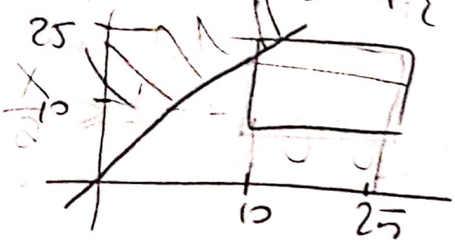
$P\{ \underbrace{\{A_1 \geq 2A_2\}}_{X} \cup \underbrace{\{A_2 \geq 2A_1\}}_{Y} \}$ X and Y are mutually ex.

so $P\{ \{X\} \cup \{Y\} \} \overset{\text{Axiom 3}}{=\!=} \overbrace{P\{A_1 \geq 2A_2\}}^{P_1} + \overbrace{P\{A_2 \geq 2A_1\}}^{P_2}$

for $10 \leq A_1 \leq 25$, $10 \leq A_2 \leq 25$

$P\{A_1 \geq 2A_2\} = P\{ \frac{A_1}{A_2} \geq 2 \} \longrightarrow \frac{A_1}{A_2}$



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Gaussian RV $\rightarrow X \sim N(\mu, \sigma^2)$ support(x) = $(-\infty, +\infty)$

$$\boxed{pdf_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)} \quad \boxed{cdf_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{(\vartheta-\mu)^2}{2\sigma^2}\right)}$$

Log-Normal Distribution $\Rightarrow X \sim N(0,1)$, $Y = \exp(X)$

$$cdf_Y(y) = P\{Y \le y\} = P\{\exp(X) \le y\} = P\{X \le \ln(y)\} \rightarrow cdf_Y(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\ln y} \exp\left(-\frac{x^2}{2}\right) dx$$

$$pdf_Y(y) = \frac{d}{dy} cdf_Y(y) = \frac{1}{y\sqrt{2\pi}} \exp\left(-\frac{[\ln(y)]^2}{2}\right)$$

$\boxed{\text{Question from HWTC\#13}} \Rightarrow$ öğrenci kaç puan alması gerektiğini hesaplayacak, Not $\rightarrow G \sim N(\mu = 55, \sigma^2 = 100)$ Eğer not μ'nün 2σ kadar yakınında değilse ve not çok düşükse sikıntı öğrencinin sıkıntısız çıkma ihtimalini $Z \sim N(0,1)$ şeklinde ifade edilebilecek cdf fonk. ile elde et.

$$P\{|G-\mu| \le 2\sigma\} = ? \rightarrow P\left\{\left|\frac{G-\mu}{\sigma}\right| < 2\right\} = P\left\{-2 < \frac{G-\mu}{\sigma} < 2\right\} \quad G \sim N(\mu, \sigma^2), \; Z \sim N(0,1)$$

$$P\{-2 < Z \le 2\} = \boxed{2 cdf_Z(2) - 1} = \quad Z = \frac{G-\mu}{\sigma}$$

$\boxed{\text{Question From HWTC \#04}} \Rightarrow$ A certain Gaussian RV, X is known to be such that $X \sim N(0, \sigma^2)$ $a > 0$ ise $P\{X > a\} = ?$ in terms of $P\{X > -a\}$

for $X \sim N(0, \sigma^2) \rightarrow pdf_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$ $\boxed{\text{pdf is an even funct. of } x}$

then $P\{Y \ge a\} = P\{Y > a\} = P\{Y < -a\} = P\{Y \le -a\}$

$$P\{Y \ge a\} = \int_a^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \left(-\frac{y^2}{2\sigma^2}\right) dy \quad \vartheta = -y \quad d\vartheta = -dy \quad \{y = a\} \cdot \{\vartheta = -a\}, \{y \to +\infty\} \cdot \{\vartheta \to -\infty\}$$

$\boxed{\text{Conditional Expectation Example}}$ (Miner Trapped in Mine)

A miner is trapped in a mine. There are 3 gates at his location for which at each any occasion selection of one is as likely as the others. One of the gates takes the miner out of the mine in 3 hours. He is to reach the same original location after arduous journey of 5 hours. The last gate again is to conduct him to his location at the outset after 7 hours. What is the expected amount of time for the miner to reach out of the mine.

SOL D: Duration of the journey until the miner makes it out of the mine

$E[D] = ?$ S: Reports the index of the gate that the miner selects with the pmf for S as in $P\{S=1\} = P\{S=2\} = P\{S=3\} = 1/3$ $\quad E[D] = E[E[D|S]]$

$$= \sum_{k=1}^{3} E[D|S=k] P\{S=k\} \rightarrow \boxed{E[D|S=1] = 3hrs} \quad E[D|S=2] = 5 + E[D], \; E[D|S=3] = 7 + E[D]$$

$$E[D] = 3 \cdot \frac{1}{3} + [5 + E[D]] \cdot \frac{1}{3} + [7 + E[D]] \cdot \frac{1}{3} = 1 + \frac{2E[D]}{3} + 4 = E[D]$$

$$5 = \frac{E[D]}{3} \rightarrow E[D] = 15hrs$$

Cond. Expect. Example (Earned sum on trials Ending in Success)

We are taking shots on goal with our trials accounted for by iid Bernoulli RV with probability p. We are paid a sum of money for each successful shot on goal. Each sum of money thus paid is RV itself independent of other such payments and the number of shots taken.

$N \sim Binomial(n, p)$, $A_k \sim Uniform(a_1, a_2)$, R: Deterministic value $\quad P\left\{\sum_{k=1}^{N} A_k > R\right\}$

$S = \left\{\sum_{k=1}^{N} A_k > R\right\} \rightarrow P\{S\} = E[\mathbb{1}_{\{S\}}]$ indicator RV for events $\quad P\{S\} = E[\mathbb{1}_{\{S\}}] = E[E[\mathbb{1}_{\{S\}}|N]] = P\{S\} = E[P\{S|N\}]$

$$P\{S\} = \sum_{m=0}^{n} P\{S|N=m\} P\{N=m\} \rightarrow P\{N=m\} = \binom{n}{m} p^m (1-p)^{n-m} \rightarrow P\{S|N=m\} = P\left\{\sum_{k=1}^{m} A_k > R | N=m\right\}$$

$pdf_A(x) = pdf_{A_1}(x) \cdots pdf_{A_m}(x)$ $\quad \boxed{P\{A_1 + A_2 \le b\} = \int_{-\infty}^{+\infty} P\{A_1 + A_2 \le b | A_2 = a\} \cdot pdf_{A_2}(a)} = P\left\{\sum_{k=1}^{m} A_k > R\right\}$

$$= \int_{-\infty}^{+\infty} A_1 \le b - a | A_2 = a\} \, pdf_{A_2}(a) \, da = \boxed{\int_{-\infty}^{+\infty} cdf_A(b-a) \cdot pdf(a) da}$$

$\operatorname{Pdf}_{A_1+A_2}(b) = \int_{-\infty}^{+\infty} \operatorname{pdf}_A(b-a)\, \operatorname{pdf}_A(a)\,da = \boxed{\operatorname{pdf}_A(b) * \operatorname{pdf}_A(a)}$

$P\{A_1+A_2+A_3 \leq b\} = \operatorname{cdf}_{A_1+A_2+A_3}(b) = \int_{-\infty}^{+\infty} P\{A_1+A_2+A_3 \leq b \mid A_3 = a\} \operatorname{pdf}_{A_3}(a)\,da$

$= \int_{-\infty}^{+\infty} P\{A_1+A_2 \leq b-a \mid A_3 = a\} \operatorname{pdf}_{A_3}(a)\,da \rightarrow \operatorname{pdf}_{A_1+A_2+A_3} = \frac{d}{db} \operatorname{cdf}_{A_1+A_2+A_3}(b)$

$= \int_{-\infty}^{+\infty} \operatorname{pdf}_{A_1+A_2}(b-a) \operatorname{pdf}_{A_3}(a)\,da \Rightarrow \operatorname{pdf}_{A_1+A_2+A_3}(b) = \underbrace{\operatorname{pdf}_{A_1+A_2}(b)}_{\operatorname{pdf}_A(b) * \operatorname{pdf}_A(b)} * \operatorname{pdf}_{A_3}(a)\,da$

$= \operatorname{pdf}_A(b) * \operatorname{pdf}_A(b) * \operatorname{pdf}_A(b) \rightarrow P\{S \mid N = m\} = \int_{R} \operatorname{pdf}_A^{*3}(\xi)\,d\xi = P\{S \mid N=0\} = P\{\sum_{r=1}^{0} A_r > R\}$

Markov – Chebyshev inequalities

for X as a random variable with only nonnegative values $\operatorname{support}(X) \cap (-\infty, 0) = \emptyset$

$P\{X \geq a\} = E[\mathbb{1}_{\{X \geq a\}}]$ if $a > 0 \rightarrow P\{X \geq a\} = 1$, $\mathbb{1}_{\{X \geq a\}} \leq \frac{X}{a}$ $\quad \underbrace{\begin{bmatrix} \mathbb{1}_{\{X \geq a\}} = 0 \\ \frac{X}{a} \geq 0 \end{bmatrix}}_{\text{if } X < a} \underbrace{\begin{bmatrix} \mathbb{1}_{\{X \geq a\}} = 1 \\ \frac{X}{a} \geq 1 \end{bmatrix}}_{\text{if } X \geq a}$

$E[\mathbb{1}_{\{X \geq a\}}] = P\{X \geq a\} \leq E[\frac{X}{a}] = \frac{E[X]}{a}$

$\boxed{P\{X \geq a\} \leq \frac{E[X]}{a}}$ — Chebyshev Inequality

for any RV X, $\operatorname{support}((X - E[X])^2) \cap (-\infty, 0) = \emptyset$

any real number k (nonzero) $k^2 > 0$ then for k^2 and $(X - E[X])^2$ $\quad P\{(X - E[X])^2 > a\} = 1$ for any $a < 0$

$\boxed{P\{(X - E[X])^2 \geq k^2\} \leq \frac{E[(X - E[X])^2]}{k^2}} \Rightarrow \underbrace{\{(X - E[X])^2 \geq k^2\}}_{|X - E[X]| \geq k}$

$P\{|X - E[X]| \geq k\} \leq \frac{\operatorname{Var}(X)}{k^2}$ $\quad \boxed{\operatorname{Var}(X) = E[(X - E[X])^2]}$

Autocorrelation

The autocorrelation function for the indicated process $X(t)$ is def. as

$\tilde{R}_{XX}(t_1, t_2) = E[X(t_1) X(t_2)]$

- The stochastic process $X(t)$ is assumed to be scalar. $\tilde{R}_{XX}(\cdot,\cdot)$ function (this is a function not a stochastic process) has two arguments, both of which are continous time entries.

WSS process için $\tilde{R}_{XX}(t_1, t_2) = R_{XX}(t_1 - t_2)$'dir.

$R_{XX}(t_1 - t_2) = R_{XX}(\tau) \rightarrow \tau = t_1 - t_2 \rightarrow E[X(t+\tau) X(t)]$ for any t

$= E[X(t) \cdot X(t+\tau)] = R_{XX}((t) - (t+\tau)) = R_{XX}(\tau) = R_{XX}(-\tau)$

$\Rightarrow E[(X(t+\tau) \pm X(t))^2] \geq 0 \rightarrow E[X^2(t+\tau) \pm 2X(t+\tau)X(t) + X^2(t)] = E[X(t+\tau) X(t+\tau)] \pm 2E[X(t+\tau) X(t)]$

$+ E[X(t) X(t)] = R_{XX}(0) \pm 2R_{XX}(\tau) + R_{XX}(0) = \boxed{2R_{XX}(0) \mp 2R_{XX}(\tau) \geq 0}$

$R_{XX}(0) \geq R_{XX}(\tau)$, $-R_{XX}(0) \leq R_{XX}(\tau) \longrightarrow |R_{XX}(\tau)| \leq R_{XX}(0) \rightarrow$ PSD $\Rightarrow S_{XX}(\omega)$ as follows

$S_{XX}(\omega) = \mathcal{F}\{R_{XX}(\tau)\} = \int_{-\infty}^{+\infty} R_{XX}(\tau) \exp(-j\omega\tau)\,d\tau$, $S_{XX}(\omega)$ is the CTFT of $R_{XX}(\tau)$

$\boxed{R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) \exp(j\omega\tau)\,d\omega}$ $\quad f = \frac{\omega}{2\pi}$, $\boxed{\begin{aligned} \tilde{S}_{XX}(f) &= \int_{-\infty}^{+\infty} R_{XX}(\tau) \exp(-j2\pi f\tau)\,d\tau \\ R_{XX}(\tau) &= \int_{-\infty}^{+\infty} \tilde{S}_{XX}(f) \exp(j2\pi f\tau)\,df \end{aligned}}$

$\boxed{R_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega)\,d\omega = \int_{-\infty}^{+\infty} S_{XX}(f)\,df}$ $\quad \boxed{R_{XX}(0) = E[X^2(t)]}$ for any t $\quad S_{XX}(\omega) = \tilde{S}_{XX}(f)$

$S_{XX}(\omega)$ is real, even in ω, $S_{XX}(\omega) \geq 0$ for all ω