HW BONUS #16

Solutions of HWTC#12

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Q1: $h(t) \xleftrightarrow{F} H(\omega)$, $F\{h(t)\} = F\{\delta(t) - \omega_0 \exp(-\omega_0 t) u(t)\}$

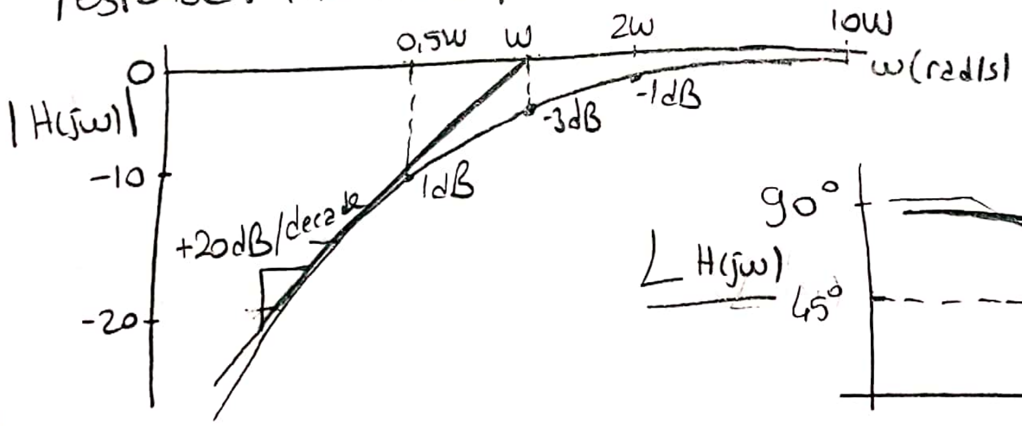
$\Rightarrow F\{\delta(t)\} = 1$, $F\{\omega_0 \exp(-\omega_0 t)\} = \dfrac{\omega_0}{j\omega + \omega_0}$ linearity of fourier,

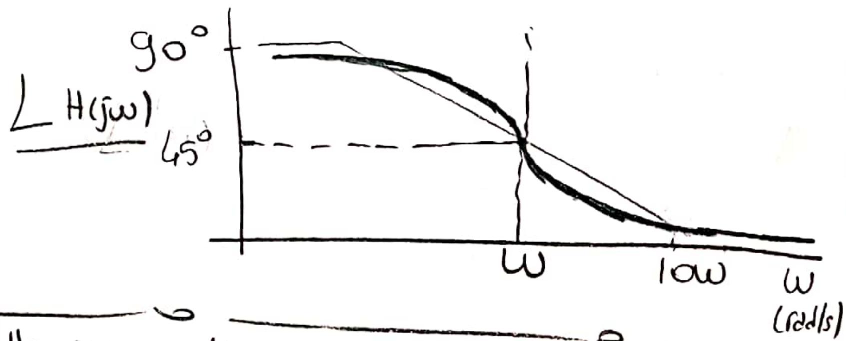
$$F\{h(t)\} = 1 - \frac{\omega_0}{j\omega + \omega_0} = \boxed{\frac{j\omega}{j\omega + \omega_0}}$$

$$F\{h(t)\} = H(\omega) = \frac{j\omega/\omega_0}{1 + j\omega/\omega_0} \rightarrow \lim_{\omega \to +\infty} \frac{j\omega/\omega_0}{1 + j\omega/\omega_0} = \frac{\infty}{\infty} \text{ so l'hospital}$$

then $\displaystyle\lim_{\omega \to +\infty} \frac{j/\omega_0}{j/\omega_0} = 1 \rightarrow \boxed{\lim_{\omega \to +\infty} H(\omega) = 1}$

So W_0 serves as a low pass cut off frequency, $H(\omega)$ has a high pass response. The Bode plots are as follows.





Now, we are interested in the following system.

$N(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$

(zero-mean white noise process with constant PSD $N_0/2$)

proved $Y(t)$ is WSS. Therefore, the following quantities are relevant.

$$E[Y(t)] = \mu_Y, \quad R_{YY}(\tau) = E[Y(t) \cdot Y(t+\tau)]$$

$$S_{YY}(\omega) = F\{R_{YY}(\tau)\} = |H(\omega)|^2 S_{NN}(\omega)$$

$$S_{YY}(\omega) = \left| \frac{\frac{j\omega}{\omega_0}}{1 + \frac{j\omega}{\omega_0}} \right|^2 \cdot \frac{N_0}{2}$$

$$10 \log_{10}(S_{YY}(\omega)) = 20 \log_{10} \left| \frac{\frac{j\omega}{\omega_0}}{1 + \frac{j\omega}{\omega_0}} \cdot \frac{N_0}{2} \right|$$

HW Bonus 1. Compute the final forms of the expressions in (eq.3).(eq.4) and (eq.5c)

$\boxed{\text{eq.3}}$ $E[Y(t)] = \mu_y$

$$Y(t) = \int_{-\infty}^{+\infty} h(t-s)N(s)\,ds \Rightarrow E[Y(t)] = E\left[\int_{-\infty}^{+\infty} h(t-s)N(s)\,ds\right]$$

$$\Rightarrow E[Y(t)] = E\left[\int_{-\infty}^{+\infty} h(t-s).N(s)\,ds\right] = \int_{-\infty}^{+\infty} E[h(t-s)N(s)\,ds]$$

linearity of expectation $\Rightarrow$ LTI system is deterministic so $h(t-s)$ comes out $\int_{-\infty}^{+\infty} h(t-s).E[N(s)]ds$

$N(s)$ is WSS so $E[X(s)] \rightarrow$ constant μ_N

$$E\{Y(t)\} = \mu_N.\int_{-\infty}^{+\infty} h(t-s)\,ds, \quad v = t-s \quad s = t-v \quad dv = -ds$$

$$= \mu_N.\int_{+\infty}^{-\infty} h(v)[-dv] = \underbrace{\left[\int_{-\infty}^{+\infty} h(v)\,dv\right]}_{A}\mu_N$$

$$A \quad F\{h(t)\} = H(w) = \int_{-\infty}^{+\infty} h(t).\exp(-jwt)\,dt$$

$$A = H(0) = \int_{-\infty}^{+\infty} h(t)\exp(-j.0.t)\,dt = \int_{-\infty}^{+\infty} h(t)\,dt$$

$$E[Y(t)] = \mu_N.A = \underbrace{\mu_N}_{\text{constant}}.\underbrace{H(0)}_{\text{constant}} = \mu_Y$$

$$\boxed{E[Y(t)] = \mu_Y}$$

eq.4 $R_{YY}(\tau) = E[Y(t).Y(t+\tau)]$

$Y(t) = \int_{-\infty}^{+\infty} h(s)\, N(t-s)\, ds$, $\quad Y(t+\tau) = \int_{-\infty}^{+\infty} h(r)\, N(t+\tau-r)\, dr$ so

$$\underbrace{E[Y(t).\,Y(t+\tau)]}_{\boxed{eq.1}} = \tilde{R}_{YY}(t, t+\tau) = E\left[\int_{-\infty}^{+\infty} dr\, h(r).\int_{-\infty}^{+\infty} ds\, h(s)\, N(t-s).\, N(t+\tau-r)\right]$$

linearity of the integral and expectation

$$\int_{-\infty}^{+\infty} dr\, h(r).\underbrace{\int_{-\infty}^{+\infty} ds\, h(s).\, E[\underbrace{N(t-s).N(t+\tau-r)}_{R_{NN}(\tau-r+s)}]}_{A}$$

eq.2

$v = -s$, $dv = -ds$

$$A = \int_{+\infty}^{-\infty} [-dv]\, \underbrace{h(-v)}_{\tilde{h}(v)}\, R_{NN}(\tau-r-v)$$

$$= \int_{-\infty}^{+\infty} \tilde{h}(v)\, dv\, R_{NN}(\tau-r-v) = \underbrace{R_{NN}(\tau-r) * \tilde{h}(\tau-r)}_{G(\tau-r)}$$

using eq.1 and eq.2

$$\tilde{R}_{YY}(t+t+\tau) = \int_{-\infty}^{+\infty} dr\, h(r).\, G(\tau-r) = h(\tau) * \underbrace{G(\tau)}_{R_{NN}(\tau) * \underbrace{\tilde{h}(\tau)}_{h(-\tau)}}$$

$$\tilde{R}_{YY}(t, t+\tau) = \underbrace{h(\tau) * R_{NN}(\tau) * h(-\tau)}_{R_{YY}(\tau)}$$

so;

$$\tilde{R}_{YY}(t, t+\tau) = \boxed{R_{YY}(\tau) = E[Y(t).Y(t+\tau)]}$$

$\boxed{\text{eq. 5c}}$ $$S_{YY}(w) = \left| \frac{\frac{jw}{w_0}}{1+\frac{jw}{w_0}} \right|^2 \cdot \frac{N_0}{2}$$

$$\mathcal{F}\{h(t)\} = H(w) = \int_{-\infty}^{+\infty} h(\tau)\, \exp(-jw\tau)\, d\tau \quad \} \; v = -\tau$$

$$= \int_{+\infty}^{-\infty} h(-v)\, \exp(+jwv)\, [-dv] \rightarrow \boxed{H(w) = \int_{-\infty}^{+\infty} h(-v) \cdot \exp(+jwv)\, dv}$$

$$H^*(w) = \int_{-\infty}^{+\infty} h(-v)\, \exp(-jwv)\, dv = \mathcal{F}\{h(-v)\}$$ for the PSD

$$S_{YY}(w) = \mathcal{F}\{R_{YY}(\tau)\} = \mathcal{F}\{h(\tau) * R_{NN}(\tau) * h(-\tau)\} =$$

$$= H(w) \cdot H^*(w) \cdot S_{NN}(w) \longrightarrow$$ property of Fourier transform about convolution.

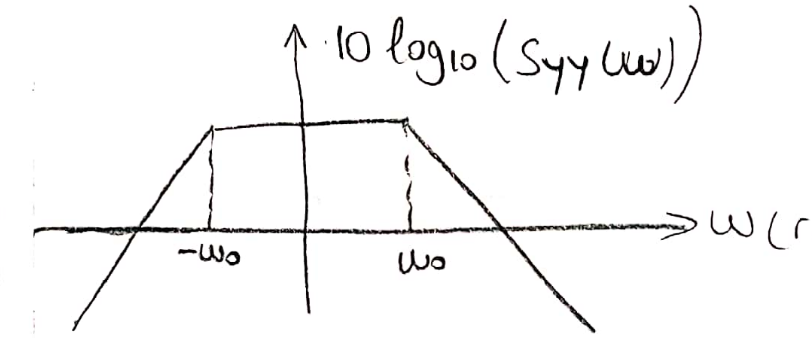
$$S_{YY}(w) = |H(w)|^2 \cdot S_{NN}(w)$$

we know $\longrightarrow$ $H(w) = \dfrac{\frac{jw}{w_0}}{1+\frac{jw}{w_0}}$ and $S_{NN}(w) = \dfrac{N_0}{2}$ so,

$$\boxed{S_{YY}(w) = \left| \frac{\frac{jw}{w_0}}{1+\frac{jw}{w_0}} \right|^2 \frac{N_0}{2}}$$

Conditional expectation formula

HW Bonus 2 Plot the expression in (eq.6) on the ω-axis.



$$\text{for } S_{YY}(\omega) = \left| \frac{\frac{j\omega}{\omega_0}}{1+\frac{j\omega}{\omega_0}} \right|^2 \frac{N_0}{2}$$

from lecture "Response of Low Pass Filter to white noise"

HWTC# 12 Q2:

S_k: Index of the kth success event in a sequence of indefinite length of iid Bernoulli trials with success probability p.

$S_1 \sim \text{Geometric}(P)$ pmf $f_{S_1}(k) = q^{k-1}p$

a,b : Deterministic indices with the following values

$a=5$, $b=10$ then $P\{S_1 < 1+a, S_2 > b\} = ?$

$$= P\{S_1 \le a, S_2 > b\} = \sum_{k=1}^{a} P\{S_2 > b \mid S_1 = k\} \cdot P\{S_1 = k\}$$

for iid RV's S_1 and $\hat{S}_1$, we may express the following

$P\{S_2 > b \mid S_1 = k\}$ $\quad S_2 = S_1 + \hat{S}_1$

HW Bonus. 3 Explain each step of (eq.11)

$S_2 = S_1 + \hat{S}_1$ we know S_1 and $S_2 \to S_k \to$ a sequence of indefinite length of iid Bernoulli trials

so def of S_k and iid the formula can be written as $S_2 = S_1 + \hat{S}_1$

$$P\{S_2 > b \mid S_1 = k\} = P\{S_1 + \hat{S}_1 > b \mid S_1 = k\} = P\{\hat{S}_1 > b-k \mid S_1 = k\}$$

Used "given that $S_1 = k$" then $\quad$ Given that $S_1 = k$ than LHS of "|" $S_1 \to k$

$$\underbrace{P\{\hat{S}_1 > b-k \mid S_1 = k\}}_{\text{independent } \hat{S}_1 \text{ and } S_1} = P\{\tilde{S}_1 > b-k\} = P\{S_1 > b-k\}$$

$\tilde{S}_1$ and S_1 are iid

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HW bonus 4. Through (eq. 11) compute the explicit expression for (eq. 10)

$$P\{S_1 \le a, S_2 > b\} = \sum_{k=1}^{a} \underbrace{P\{S_2 > b \mid S_1 = k\}}_{P\{S_1 > b-k\} = 1 - P\{S_1 \le b-k\}} . P\{S_1 = k\}$$

$$\boxed{\sum_{k=1}^{a} \left(1 - cdf_{S_1}(b-k)\right) . pmf_{S_1}(k)}$$

$b = 10,\ a = 5$ $\quad \sum_{k=1}^{5} \left(1 - cdf_{S_1}(10-k)\right) . \underbrace{pmf_{S_1}(k)}_{q^{k-1} . p}$

$$\boxed{cdf_{S_1}(10-k) = 1 - (1-p)^{10-k}}$$

$$\sum_{k=1}^{5} \left(1 - \left(1 - (1-p)^{10-k}\right)\right) . p . (1-p)^{k-1} = \sum_{k=1}^{5} (1-p)^{10-k} . p . (1-p)^{k-1}$$

$$= \sum_{k=1}^{5} (1-p)^{9} . p = \boxed{5p . q^{9}} \quad \boxed{q = 1-p}$$

Q2 (part b) $P\{S_1 < a+1, S_2 < b+1\} = P\{S_1 \le a, S_2 \le b\}$

$$P\{S_1 \le a, S_2 \le b\} = \underbrace{P\{S_1 \le a\}}_{cdf_{S_1}(a) = cdf_{S_1}(5)} - \underbrace{P\{S_1 \le a, S_2 > b\}}_{5pq^{9}}$$

$$(1 - q^{a}) = (1 - q^{5})$$

$$\boxed{P\{S_1 \le a, S_2 \le b\} = (1 - q^{5}) - 5pq^{9}}$$

Q2 (Part c)

$$P\{S_2 < b+1\} = P\{S_2 \le b\} = \lim_{a \to b^-} \underbrace{P\{S_1 \le a, S_2 \le b\}}_{(1-q^{a}) - 5pq^{9}}$$

$$= (1 - q^{b}) - 5pq^{9} = \boxed{(1 - q^{10}) - 5pq^{9}}$$

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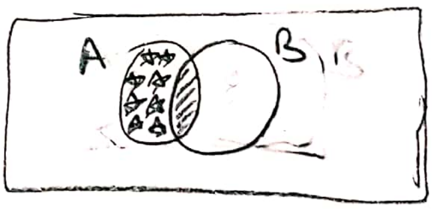
Conditional expectation formula.

Q2 – Hw bonus 5 – Verify (eq.13)

$$P\{S_1 \le a, S_2 \le b\} = P\{S_1 \le a\} - P\{S_1 \le a, S_2 > b\}$$

A ve B evrensel küme içinde iki olay olsun.

$P\{A, B\} + P\{A, B^c\} = P\{A\}$ olmalıdır. Eğer



Taralı bölge $P\{A,B\}$'yi,
işaretli bölge $P\{A,B^c\}$'yi temsil eder.

böylece $P\{A,B\} + P\{A,B^c\} = P\{A\}$ olduğu görüldü.

$$P\{\underbrace{S_1 \le a}_{A}, \underbrace{S_2 \le b}_{B}\} + P\{\underbrace{S_1 \le a}_{A}, \underbrace{S_2 > b}_{B^c}\} = P\{\underbrace{S_1 \le a}_{A}\}$$

$P\{S_1 \le a, S_2 \le b\} = P\{S_1 \le a\} - P\{S_1 \le a, S_2 > b\}$ bulunur.

Hw bonus 6 – Compute an explicit expression for the probability in (eq.13)

$$P\{S_1 \le a, S_2 \le b\} = \underbrace{P\{S_1 \le a\}}_{cdf_{S_1}(a)} - \underbrace{P\{S_1 \le a, S_2 > b\}}_{5pq^9}$$

$5pq^9$ } Hw bonus 4'te bulundu

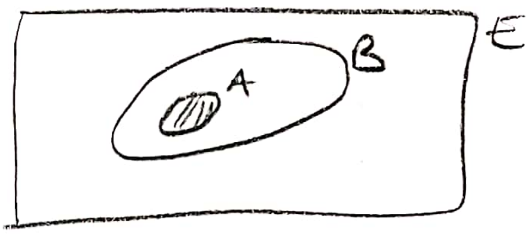
$$\boxed{P\{S_1 \le a, S_2 \le b\} = (1-q^5) - 5pq^9}$$

Hw bonus 7. Go through this computation and compute an explicit expression for (eq. 14)

$$\{S_1 \le b-1\} \xrightarrow[?]{\text{implies}} \{S_2 \le b\}$$

$$\{S_2 \le b\} \xrightarrow[?]{\text{implies}} \{S_1 \le b-1\}$$

$$\underbrace{\{S_2 \le b\}}_{A} \xrightarrow{\text{implies}} \underbrace{\{S_1 \le b-1\}}_{B} \equiv$$



$$\boxed{P\{S_2 \le b\} = P\{S_2 \le b, P\{S_1 \le b-1\}}$$

$\hookrightarrow$ eq. 15

$P\{A, B\} = P\{A\}$

$= P\{S_2 \le b, P\{S_1 \le b-1\} = \lim_{a \to b^+} P\{S_1 \le a, S_2 \le b\}$ olduğu görülür.

Part d. ★ Bu Maddede doğruluk kontrolü yapılması isteniyor. Bu kontrol Hw bonus 7. kapsamında yapılmıştır.

HWTC#12 $\boxed{Q3}$.

S_k: Arrival time of the first attack in minutes counting from 14:00. These are RV s. But $S_0 = 0$ min deterministically.

T_k: Interarrival times for the attacks. These are iid RV.

$T_k = S_k - S_{k-1}$, $T_k \sim \text{Exponential}(\lambda = \frac{1}{5} \text{min})$ $T_1 = S_1 - S_0 = S_1 - 0 = S_1$

R: Deterministic time for the money transfer to be concluded without any attacks interupting. $R = 10$ min

L: Random variable reporting the total amount of time for the money transfer to be concluded in the presense of attacks including the deterministic duration R.

Looking forward to $E[L] = E[E[L|S_1]] = ?$

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Conditional expectation formula.

$$E[L] = E\big[E[L|S_1]\big] = \int_0^{+\infty} E[L|S_1]\, pdf_{S_1}(x)\,dx$$

split the Integral two parts.

$$E[L] = \int_0^R \underbrace{E[L|S_1=x]}_{E[x+L']}\, \underbrace{pdf_{S_1}(x)}_{\lambda\exp(-\lambda x)}\,dx + \int_R^{+\infty} \underbrace{E[L|S_1=x]}_{R}\, \underbrace{pdf_{S_1}(x)}_{\lambda(\exp(-\lambda x))}\,dx$$

$$E[L] = \frac{1}{\lambda}\left[\exp(\lambda R) - 1\right]$$

Hw bonus 8. Explain how we come to interpret the parametric expectation values in the integral expression of (eq. 21).

$$E[L] = \int_0^R \underbrace{E[L|S_1=x]} pdf_{S_1}(x)\,dx\,dx + \int_R^{+\infty} E[L|S_1=x]\, pdf_{S_1}(x)\,dx$$

$$\int_0^R \left[E[x] + E[L']\right] \cdot \lambda \cdot \exp(-\lambda x)\,dx + \int_R^{+\infty} R \cdot \lambda \cdot \exp(-\lambda x)$$

$$\int_0^R \frac{1}{\lambda} \cdot \lambda\exp(-\lambda x) + \int_0^R E[L']\,\lambda \cdot \exp(-\lambda x)\,dx + \int_R^{+\infty} R \cdot \lambda \exp(-\lambda x)\,dx$$

$E[x] = \frac{1}{\lambda}$ for exponential RV

$$\Big|_0^R -\frac{1}{\lambda}\exp(-\lambda x) + \Big|_0^R -E[L'] \cdot \exp(-\lambda x) + \Big|_R^{+\infty} -R\exp(-\lambda x) = E[L]$$

$$\left(-\frac{1}{\lambda}\exp(-R\lambda) + \frac{1}{\lambda}\right) + \left(-E[L'] \cdot \exp(-\lambda R) + E[L']\right) + \left(0 + R \cdot \exp(-\lambda R)\right) = E[L]$$

$E[L]$ and $E[L']$ identically distributed.

$$\frac{1}{\lambda} - \left(\frac{1}{\lambda} + R\right)\exp(-\lambda R) = E[L] \cdot \exp(-\lambda R) \quad \frac{\exp(-\lambda R)}{\text{böl.}}$$

$$E[L] = \frac{\exp(\lambda R)}{\lambda} - \left(\frac{1}{\lambda} + R\right) \neq \frac{1}{\lambda}\left(\exp(\lambda R) - 1 + R\lambda\right)$$

HwTc # 12

Q4:

L: The random variable reporting the time it takes for the money transfer to be concluded, including the deterministic time for the actual transfer, which is denoted by R.

$$P\{L \le t\} = cdf_L(t)$$

$L_{raw} = L - R$, $P\{L_{raw} \le t\} = cdf_{L_{raw}}(t)$

we are going to formulate an equation for the survival function of L_{raw}, which is $f(t) = P\{L_{raw} > t\} = 1 - cdf_{L_{raw}}(t)$

Recall:

S_k: Denotes the arrival times of the attacks, with k (index) corresponding to the kth attack.

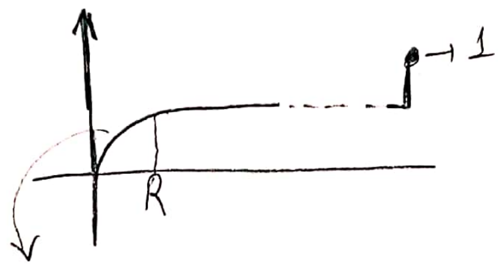
we might suppose that an attack arrives just at the begining (14:00), and we denote by S_0 as the time, in minutes away from 14:00 of the 0th attack, and in this formulation $S_0 = 0$

$$T_k = S_k - S_{k-1} \sim Exp(\lambda) \longrightarrow \boxed{T_1 = S_1 - S_0 = S_1 - 0 = S_1}$$

$$F(t) = cdf_{T_1, row}(t) = \begin{cases} cdf_{T_1}(t) & t \le R \\ cdf_{T_1}(R) & t > R \end{cases}$$

$T_{1,row}$ is the RV used to account for the transient stochastic process with arrival times. $T_{k,row} \Rightarrow$ for $k = 1, 2, \dots$ are iid

$S_{0,row} = 0$, $T_{k,row} = S_{k,row} - S_{k-1,row}$



this jump is given by $1 - F(\infty)$
$= 1 - cdf_{T_1}(R)$

This portion between 0 and R is given by $cdf_{T_1}(t)$

$$f(t) = P\{L_{raw} > t\} = E\left[P\{L_{raw} > t \mid S_{1raw}\}\right]$$

$$= \int_t^{+\infty} P\{L_{raw} > t \mid S_{1,row} = x\} \cdot d\left[cdf_{S_{1,row}}(x)\right] +$$

$$+ \int_0^t \underbrace{P\{L_{raw} > t \mid S_{1,raw} = x\}}_{P\{\cdots\} = 1} d\left[cdf_{S_{1,raw}}(x)\right]$$

$$\boxed{cdf_{S_{1,row}} = cdf_{T_1,row} = F(x)}$$

Replace $\rightarrow L_{row} \Longleftrightarrow S_{1,row} + \hat{L}_{row}$, L_{row} and $\hat{L}_{row}$ identically distributed.

$$f(t) = P\{L_{row} > t\} = \int_t^{+\infty} dF(x) + \int_0^t \underbrace{P\{S_{1,row} + L_{row} > t \mid S_{1row} = x\}}_{P\{L_{1row} > t \mid S_{1,row} = x\} = P\{x + \hat{L}_{row} > t\}} \cdot d[F(x)]$$

$$= \left[F(\infty) - F(t)\right] + \int_0^t \underbrace{P\{\hat{L}_{raw} > t - x\}}_{P\{L_{raw} > t - x\} \cdots} d[F(x)]$$

L_{raw} and $\hat{L}_{raw}$ are identically distributed.

$$f(t) = \left[F(\infty) - F(t)\right] + \int_0^t f(t-x) \cdot d[F(x)]$$

$$\Rightarrow f(t) = \left[F(\infty) - f(t)\right] + \underbrace{F * f(t)}_{\text{convolution}}$$

Let us suppose there exists a function with following property

$$U_{raw} * F(t) = U_{raw}(t) - 1$$

the Reneval function $\rightarrow S_k$ row for $k = 0, 1, \ldots$

$$U_{raw} * f(t) = U_{raw} * [F(\infty) - F(t)] + U_{raw} * F * f(t)$$

$$= F(\infty) . U_{raw}(t) - [U_{raw}(t) - 1] + U_{raw} * f(t) - f(t)$$

$$\Rightarrow f(t) = 1 + [F(\infty) - 1] . U_{raw}(t) \Rightarrow 1 - [1 - F(\infty)] U_{raw}(t)$$

$$= \boxed{P\{L_{aw} > t\} = f(t)}$$

$$\Rightarrow 1 - f(t) = cdf_{L_{raw}}(t) = P\{L_{raw} < t\}$$

$$= [1 - F(\infty)] U_{raw}(t) \quad , \quad U_{raw}(t) < \frac{1}{1 - F(\infty)}$$

$$\lim_{t \to +\infty} U_{raw}(t) = \frac{1}{1 - F(\infty)} \rightarrow \text{figure from pg-10}$$

$$1 - F(\infty) = 1 - cdf_{T_1}(R) = 1 - [1 - \exp(-\lambda R)] = \exp(-\lambda R)$$

Hw bonus 9. You might want to verify all steps of (eq.13) according to the identity in (eq.12) and the convolution definition used in this write up.

we know $f(t) = [F(\infty) - F(t)] + F * f(t)$ and $U_{raw} * f(t) = U_{raw}(t) - 1$

then $U_{raw} * f(t) = U_{raw} * [[F(\infty) - F(t)] + F * f(t)]$ konvolüsyonun dağılma özelliği

$$U_{raw} * f(t) = U_{raw} * [F(\infty) - f(t)] + U_{raw} * F * f(t)$$

$$U_{raw} * f(t) = U_{raw} * F(\infty) - \underbrace{U_{raw} * f(t)}_{[U_{raw}(t) - 1]} + U_{raw} * F * f(t)$$

$$U_{raw} * f(t) = F(\infty) U_{raw} - [U_{raw}(t) - 1] + U_{raw} * F * f(t)$$

$$U_{raw} * f(t) = 1 - U_{raw}(t) + U_{raw} F(\infty) + U_{raw} * F * f(t)$$

$$U_{raw} * f(t) = 1 - [1 - F(\infty)] . U_{raw}(t) + \underbrace{U_{raw} * F * f(t)}_{U_{raw}(t) * f(t) - f(t)}$$

$$\cancel{U_{raw} * f(t)} = 1 - [1 - F(\infty)] \cdot U_{raw}(t) + \cancel{U_{raw}(t) * f(t)} - f(t)$$

$$f(t) = 1 - [1 - F(\infty)]\, U_{raw}(t) =$$

$$\boxed{\text{Eq. 13g} \rightarrow 1 - f(t) = [1 - F(\infty)]\, U_{raw}(t)}$$