HW Bonus #18 - HW TC #14

Q1: For the following joint pdf

$Pdf_{X,Y}(x,y) = \begin{cases} cxy & \text{for } 0<x<1,\ 0<y<1 \end{cases}$ $\quad P\{\frac{X}{Y}>2 \mid X<\frac{1}{2}\}$

(a) $\int_0^1\int_0^1 Pdf_{X,Y}(x,y)\,dy\,dx = 1$ için $\rightarrow \int_0^1\int_0^1 cxy\,dy\,dx = 1 = \frac{c}{4}$, $\boxed{c=4}$

(b)

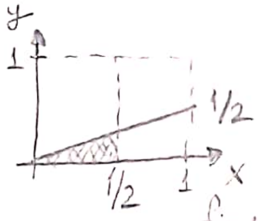

fig 1.1a

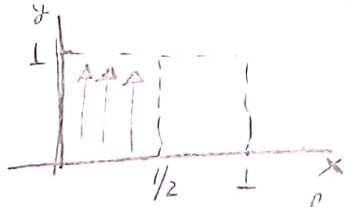

fig 1.1b

fig 1.1a $\rightarrow P\{\frac{X}{Y}>2, X<\frac{1}{2}\}$, fig 1.1b $P\{X<\frac{1}{2}\}$

HW bonus$_1$: Express the 2x2=4 integral expressions indicated and solve for the result of each.

$P\{\frac{X}{Y}>2, X<\frac{1}{2}\} = \int_0^{1/2}\int_0^{x/2} 4xy\,dy\,dx = \int_0^{1/2} \frac{4x}{2}\left(\frac{x}{2}\right)^2 = \int_0^{1/2}\frac{2x\cdot x^2}{4} = \int_0^{1/2}\frac{x^3}{2} = \Big|_0^{1/2}\frac{x^4}{8} = \frac{1}{16}\cdot\frac{1}{8} = \frac{1}{128}$

$P\{X<\frac{1}{2}\} = \int_0^{1/2}\int_0^1 4xy\,dy\,dx = \int_0^{1/2}\Big|_0^1 \frac{4xy^2}{2} = \int_0^{1/2} 2x\,dx = \Big|_0^{1/2} x^2 = \frac{1}{4}$

end of bonus

HW bonus$_2$ Confirm the end result.

$P\{\frac{X}{Y}>2 \mid X<\frac{1}{2}\} = \frac{P\{\frac{X}{Y}>2, X<\frac{1}{2}\}}{P\{X<\frac{1}{2}\}} = \frac{\frac{1}{128}}{\frac{1}{4}} = \frac{1}{32}$ end of bonus.

Q_2: With $X(t)$ and $Y(t)$ as WSS processes what about $Z(t) = X(t) + Y(t)$, two checks are necessary

$$E[Z(t)] = E[X(t) + Y(t)] = E[X(t)] + E[Y(t)]$$

Hw. bonus$_3$: Perform the check.

$$Z(t) = \int_{-\infty}^{+\infty} h(t-s)\, Z(s)\, ds$$

$$E[Z(t)] = E[X(t)] + E[Y(t)] = E\left[\int_{-\infty}^{+\infty} h(t-s).\, X(s)\right] + E\left[\int_{-\infty}^{+\infty} h(t-s).\, Y(s)\right]$$

$$\xrightarrow[\text{of expected and integral}]{\text{linearity}} \int_{-\infty}^{+\infty} h(t-s)\, \underbrace{E[X(s)]}_{M_x}\, ds + \int_{-\infty}^{+\infty} h(t-s)\, \underbrace{Y(s)}_{M_y}\, ds$$

M_x and M_y are constant because $X(t)$ and $Y(t)$ are WSS process

$$= \int_{-\infty}^{+\infty} h(t-s)\, M_x\, ds + \int_{-\infty}^{+\infty} h(t-s)\, M_y\, ds \qquad t-s = v \quad -ds = dv$$

$$= M_x \int_{+\infty}^{-\infty} h(v)(-dv) + M_y \int_{+\infty}^{-\infty} h(v)(-dv) = M_x \int_{-\infty}^{+\infty} \underbrace{h(v)dv}_{A} + M_y \int_{-\infty}^{+\infty} \underbrace{h(v)dv}_{A}$$

$$A \Rightarrow \mathcal{F}\{h(t)\} = H(\omega) = \int_{-\infty}^{+\infty} h(t) \exp(-j\omega t) dt$$

$$A = H(0)$$

$$E[Z(t)] = \underbrace{M_x . H(0) + M_y . H(0)}_{\text{constant in time.}} \checkmark$$

ilk kontrol sağlandı

end of bonus.

Check 2. $\tilde{R}_{ZZ}(t_1, t_2)$

$$E[Z(t_1).Z(t_2)] = E[[X(t_1) + Y(t_1)].[X(t_2) + Y(t_2)]]$$

$$= \underbrace{E[X(t_1)\,X(t_2)]}_{\tilde{R}_{XX}(t_1,t_2)} + E[Y(t_1)\,Y(t_2)] + E[X(t_1)\,Y(t_2)] + E[Y(t_1)\,X(t_2)]$$

$$\tilde{R}_{XX}(t_1,t_2) + \tilde{R}_{YY}(t_1,t_2) + E[X(t_1)].E[Y(t_2)] + E[X(t_2)].E[Y(t_1)]$$

eq. 2.3d

Q2.2

Hw bonus. Answer these questions

from eq.2.3d

$$\tilde{R}_{ZZ}(t_1,t_2) = \tilde{R}_{XX}(t_1,t_2) + \tilde{R}_{YY}(t_1,t_2) + \underbrace{E[X(t_1)].E[Y(t_2)] + E[X(t_2)]E[Y(t_1)]}_{0}$$

X ve Y'nin WSS olması argümanı ne olursa olsun expected değeri sıfırdır

$$\tilde{R}_{ZZ}(t_1,t_2) = \tilde{R}_{XX}(t_1,t_2) + \tilde{R}_{YY}(t_1,t_2)$$

X ve Y WSS process olduğu sürece

$$\tilde{R}_{XX}(t_1,t_2) = R_{XX}(t_1-t_2)$$ olduğu 11.2.3 ve 11.2.4 dersi kapsamında görülmektedir.

$$\tilde{R}_{YY}(t_1,t_2) = R_{YY}(t_1-t_2)$$

Bu durumda $\tilde{R}_{ZZ}(t_1,t_2) = R_{XX}(t_1-t_2) + R_{YY}(t_1-t_2)$ bulunur

$\tilde{R}_{ZZ}(t_1,t_2)$, t_1-t_2 argümanlı fonksiyonlar ile ifade edilebilmektedir.

check 2✓

end of bonus.

Q3: Define the random variables.

A: Daily number of cases from distributor A, $A \sim \text{Poisson}(\lambda_A)$

B: " " " " " " B, $B \sim \text{Poisson}(\lambda_B)$

N: Total number of cases received today (Deterministic)

N_A: Total number of cases received today from distributor A (Deterministic)

Bulmamız gereken olasılık

$$P\{A = N_A \mid A+B = N\}$$

$$\text{eq. 3.1} \rightarrow \binom{N}{N_a}\left(\frac{\lambda_A}{\lambda_A+\lambda_B}\right)^{N_A} \cdot \left(\frac{\lambda_B}{\lambda_A+\lambda_B}\right)^{N-N_a}$$

Hw bonus.4: Review the Poisson Decomposition Theorem and confirm the result in eq. 3.1

Poisson Decomposition Theorem

$A \sim Poisson(\lambda_A)$, $B \sim Poisson(\lambda_B)$ bir C RV tanımlayalım.

$C \sim Poisson(\lambda = \lambda_A + \lambda_B)$ A ve B'nin independent olduğu bilinmektedir.

$X = \{A$ ve B independent, $C = A+B\}$
$Y = \{C$ bir Poisson RV$\}$ $\Big\}\ X \Longleftrightarrow Y$

$$P\{A=i, B=j\} \xrightarrow{\text{independency}} \underbrace{P\{A=i\}}_{Pdf_A(i)} . \underbrace{P\{B=j\}}_{Pdf_B(j)} = \exp(-\lambda_A)\frac{(\lambda_A)^i}{i!} . \exp(-\lambda_B)\frac{(\lambda_B)^j}{j!}$$

$$= \exp(-(\lambda_A+\lambda_B)) . \frac{\lambda_A^i . \lambda_B^j}{i!\ j!} \left[\frac{(\lambda_A+\lambda_B)^{i+j}}{(\lambda_A+\lambda_B)^{i+j}} . \frac{(i+j)!}{(i+j)!}\right]$$

Parantez içi düzenleme için eklenmektedir. Sonucu değiştirmeyecektir.

$$= \exp(-(\lambda_A+\lambda_B)) . \left(\frac{(\lambda_A+\lambda_B)^{i+j}}{(i+j)!}\right) \underbrace{\frac{(i+j)!}{i!\ j!}}_{C\binom{i+j}{i}} . \left(\frac{\lambda_A}{\lambda_A+\lambda_B}\right)^i \left(\frac{\lambda_B}{\lambda_A+\lambda_B}\right)^j$$

$$P\{A=i, B=j\} = P\{A+B=i+j, A=i\} = P\{C=k, A=i\} =$$

$$= \exp(-(\lambda_A+\lambda_B)) \frac{(\lambda_A+\lambda_B)^k}{k!} . \binom{k}{i} \left(\frac{\lambda_B}{\lambda_A+\lambda_B}\right)^i \left(\frac{\lambda_B}{\lambda_A+\lambda_B}\right)^{k-i}$$

$$\sum_{i=0}^{k} P\{C=k, A=i\} = \boxed{P\{C=k\} = \exp(-(\lambda_A+\lambda_B)) \frac{(\lambda_A+\lambda_B)^k}{k!}}$$

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then $$P\{A=i \mid A+B=i+j\} = \frac{P\{A=i, A+B=i+j\}}{P\{A+B=i+j\}} =$$

$$= \frac{\cancel{\exp(-(\lambda_A+\lambda_B))}\ \cancel{\frac{(\lambda_A+\lambda_B)^{i+j}}{(i+j)!}} \binom{i+j}{i} \left(\frac{\lambda_A}{\lambda_A+\lambda_B}\right)^i . \left(\frac{\lambda_B}{\lambda_A+\lambda_B}\right)^j}{\cancel{\exp(-(\lambda_A+\lambda_B))}\ \cancel{\frac{(\lambda_A+\lambda_B)^{i+j}}{(i+j)!}}}$$

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$$P\{A=i \mid A+B=i+j\} = \binom{i+j}{i} \cdot \left(\frac{\lambda_A}{\lambda_A+\lambda_B}\right)^i \cdot \left(\frac{\lambda_B}{\lambda_A+\lambda_B}\right)^j$$ bulunur.

end of bonus.

Böylece eq.3.1 elde edilir.

$$P\{A=N_A \mid A+B=N\} \longrightarrow A\to A,\ N_A \to i,\ A+B \to C,\ N \to i+j$$

formül uygulanabilir.

$$\hookrightarrow = \binom{N}{N_A} \cdot \left(\frac{\lambda_A}{\lambda_A+\lambda_B}\right)^{N_A} \cdot \left(\frac{\lambda_B}{\lambda_A+\lambda_B}\right)^{N-N_A}$$

$\lambda_A = 10$, $\lambda_B = 15$, $N = 20$ $N_A = 12$

$$= \binom{20}{12}\left(\frac{10}{25}\right)^{12}\left(\frac{15}{25}\right)^{8} = \underbrace{\frac{20!}{12! \cdot 8!}}_{125970} \cdot \left(\frac{2}{5}\right)^{12} \cdot \left(\frac{3}{5}\right)^{8} = 0.0354$$

Q4: Define RV's

N: Number of steps taken $N \sim \text{Poisson}(\lambda)$

V_k: The direction taken in the horizontal orientation at each step indexed by k, where $k \in \{1, \dots N\}$, with N as the random variable defined above

☆ V_k's are iid with distribution $\left.\begin{array}{l} P\{V_k = +1\} = P_R \\ P\{V_k = -1\} = P_L \end{array}\right\} P_R + P_L = 1$

N and V_k $k \in \{1, \dots, N\}$ are independent.

$$g_n(R) = P\left\{\left|\sum_{k=1}^{n} V_k\right| < R\right\} \rightarrow$$ bu olasılık aşağıdaki olasılık hesabı için bulunmalı.

$$\boxed{P\left\{N > N_s \;\middle|\; \left|\sum_{k=1}^{n} V_k\right| < R\right\}} \quad \text{eq.1}$$

Q4.2

eq.1 olasılığı Baye's rule ile aşağıdaki gibi ifade edilebilir

$$P\{N>N_s \mid |\sum_{k=1}^{N} V_k| < R\} = \frac{P\{|\sum_{k=1}^{N} V_k| < R \mid N>N_s\}.P\{N>N_s\}}{P\{|\sum_{k=1}^{N} V_k| < R \mid N>N_s\} P\{N>N_s\} + P\{|\sum_{k=1}^{N} V_k| < R \mid N\leq N_s\}. P\{N\leq N_s\}}$$

eq.4.5.a ←

Hw bonus.5 Explain each step of (eq.4.5)

eq.4.5.a → Klasik Baye's rule kullanılır.

$$P\{B_j|A\} = \frac{P\{A|B_j\}.P\{B_j\}}{\sum_{k=1}^{N} P\{A|B_k\}.P\{B_k\}}$$ formül benzetimi uyarlanır.

eq.4.5.b →

$$\frac{\sum_{\ell=N_s+1}^{+\infty} P\{|\sum_{k=1}^{N} V_k|<R \mid N=\ell\}.P\{N=\ell\}}{\sum_{\ell=0}^{+\infty} P\{|\sum_{k=1}^{N} V_k|<R \mid N=\ell\}.P\{N=\ell\}}$$

☆ Bayes rule kuralları gereği pay kısmı spesifik değer belirtirken payda olası tüm durumları inceler burada $N>N_s$ şartı sigma sınırları ile sağlanır. Paydada N'in tüm durumları olacağından sigma sınırları $0 \to +\infty$ olarak belirlenir.

eq.4.5.c →

$$1 - \frac{\sum_{\ell=0}^{N_s} P\{|\sum_{k=1}^{N} V_k|<R \mid N=\ell\} P\{N=\ell\}}{\sum_{\ell=0}^{N_s} P\{|\sum_{k=1}^{N} V_k|<R \mid N=\ell\} P\{N=\ell\}}$$

eq.4.5.c İç terimleri eşit olan iki sigma içerir eq.4.5.b ile eşdeğerliği şöyle gösterilir (Burada sigma içi terimlerin eş olmasından yararlanılır.)

$$1 - \frac{\sum_{\ell=0}^{N} \cdots}{\sum_{\ell=0}^{+\infty}} = \frac{\sum_{\ell=0}^{+\infty} - \sum_{\ell=0}^{N} \cdots}{\sum_{\ell=0}^{+\infty}} = \frac{\sum_{\ell=N_s+1}^{+\infty}}{\sum_{\ell=0}^{+\infty} \cdots}$$

böylece eq.4.5.b ile eq.4.5.c eşdeğerdir.

Q4.3

$$1-\frac{\sum_{\ell=0}^{N_s} P\{|\sum_{k=1}^{N} V_k| < R \mid N=\ell\} P\{N=\ell\}}{\sum_{\ell=0}^{+\infty} P\{|\sum_{k=1}^{N} V_k| < R \mid N=\ell\} P\{N=\ell\}} = A$$

$$1-\frac{\sum_{\ell=0}^{N_s} P\{|\sum_{k=1}^{\ell} V_k| < R \mid N=\ell\} P\{N=\ell\}}{\sum_{\ell=0}^{+\infty} P\{|\sum_{k=1}^{\ell} V_k| < R \mid N=\ell\} P\{N=\ell\}} = B$$

Burada A ve B eşdeğer olasılık belirtir çünkü $N=\ell$ bilgisi "given that" bölgesinde yer alır sigma üst sınırı $N \to \ell$ olabilir. end of bonus.

Öyleyse B olasılığı aranacaktır.

Bunun için not: $\underline{P\{|\sum_{k=1}^{\ell} V_k| < R\}}\Big|_{\ell=0} = P\{|\sum_{k=1}^{0} V_k| < R\} = \underbrace{P\{0 < R\}}_{1}$

$$\boxed{P\{N=\ell\} = \exp(-\lambda)\cdot\frac{\lambda^\ell}{\ell!}}$$ $N \sim \text{Poisson}(\lambda)$ old. için

$g_n(\ell)$ fonk.'nunda $n \to \ell$ değişimi uygulanırsa

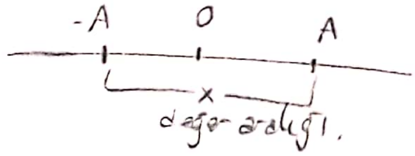
$$g_\ell(R) = P\{|\sum_{k=1}^{\ell} V_k| < R\} = P\{-R < \sum_{k=1}^{\ell} V_k < R\} = P\{-R < \sum_{k=1}^{\ell} V_k \leq R-1\}$$

$$\underbrace{P\{\sum_{k=1}^{\ell} V_k \leq R-1\}}_{h_\ell(R-1)} - \underbrace{P\{\sum_{k=1}^{\ell} V_k \leq -R\}}_{h_\ell(-R)}$$

Hw bonus 6: Explain each step of (eq. 4.8) Q4.4

$$g_\ell(R) = P\left\{\left|\sum_{k=1}^{\ell} V_k\right| < R\right\}$$ → n olan dummy ℓ ile değişti

$$= P\left\{-R < \sum_{k=1}^{\ell} V_k < R\right\}$$ → Mutlak değer özelliği $|x| < A$ ise, $-A < x < A$ yazılır.



$$= P\left\{-R < \sum_{k=1}^{\ell} V_k \le R-1\right\}$$ → $\sum_{k=1}^{\ell} V_k$ terimi discrete olduğundan sınır 1 br azaltılarak "$<$" → "$\le$" değişimi yapılır.

$$= P\left\{\left\{-R < \sum_{k=1}^{\ell} V_k\right\} \cup \left\{\sum_{k=1}^{\ell} V_k \le R-1\right\}\right\} \xrightarrow{\text{Axiom 3}}$$

$$= \underbrace{P\left\{-R < \sum_{k=1}^{\ell} V_k\right\}}_{\underbrace{-P\left\{\sum_{k=1}^{\ell} V_k \le -R\right\}}_{h_\ell(-R)}} + \underbrace{P\left\{\sum_{k=1}^{\ell} V_k \le R-1\right\}}_{h_\ell(R-1)} = \boxed{h_\ell(R-1) - h_\ell(R)}$$

end of bonus.

$$h_\ell(R-1) = 1, \quad h_\ell(-R) = 0, \quad \boxed{g_\ell(R) = 1}$$ all for $\ell < R$

$$h_\ell(R-1) < 1, \quad h_\ell(-R) > 0, \quad g_\ell(R) < 1$$ for $\ell \ge R$

for $\ell = R$ → $h_\ell(R-1) = 1 - P_R^R$, $h_\ell(-R) = P_L^R$

$$\boxed{g_R(R) = 1 - P_R^R - P_L^R}$$

Q4.5 Can you come up with a Matrix–vector product-based formulation to compute probabilities that will help you reach $g_l(R)$ for a given N_s?
Hw bonus 7. Check how this can be done: (Markov Chains, Probability transition Matrix)

A Markov chain is mathematical system that experiences transitions from one state to another according to certain probabilistic rules. The probability of transitioning to any particular state is dependent solely on the current state and time elapsed.

Markov chains may be modeled by finite state machines (FSM) and RANDOM WALKS provide a prolific example of their usefulness in Mathematics.

our situation. $g_l(R) = h_l(l-1) - h_l(R)$

if $l < R$ $g_l(R) = 1$, $l = R \cdot g_l(l) = 1 - P_R^R - P_L^R$, $l \geq R$ $g_l(R) < 1$

$$P^1 = \begin{array}{c} \\ N_0 \\ N_1 \\ \vdots \\ N_s \end{array} \begin{array}{c} \begin{array}{cccc} N_0 & N_1 & \cdots & N_s \end{array} \\ \begin{pmatrix} P_{N_0N_0} & P_{N_0N_1} & \cdots & P_{N_0N_s} \\ P_{N_1N_0} & P_{N_1N_1} & \cdots & P_{N_1N_s} \\ \vdots & & & \\ P_{N_sN_0} & \cdots & & P_{N_sN_s} \end{pmatrix} \end{array}$$

$N_s < R$ ise

P^2
P^3
$\vdots$
P^N

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