Geometric properties

Embedding

Definition 1. Let $N = (Q, g)$ be a n-dimension Riemannian manifold on position parameter space with such inner product g that the map $\cdot \mapsto \frac{1}{k} \cdot e^1 \in P \to P^*$ is an isometric embedding to $(n+1)$-Euclidean manifold.

Remark 1. It have been proved that there is no isometric embedding from H^n to E^{n+1}. Hence, this section is based on a faulty assumption and would not provide a useful information.

Lemma 1. *Position parameter θ^i is associated to the following vector in position vector space.*

$$\frac{\partial}{\partial \theta^i} = \begin{pmatrix} -k \tan_k^* \theta^i p^j \\ k \frac{1}{\tan_k \theta^i} p^{i+1} \\ 0 \end{pmatrix} \frac{\partial}{\partial p^j}$$

Proof of lemma 1.

$$\begin{aligned}
\frac{\partial}{\partial \theta^i} &= \frac{\partial p^j}{\partial \theta^i} \frac{\partial}{\partial p^j} \\
&= \frac{\partial}{\partial \theta^i} \left[\frac{1}{k} \begin{pmatrix} \prod_{l \in \{1..n\}} \cos_k \theta^l \\ \sin_k \theta^{j-1} \prod_{l \in \{j..n\}} \cos_k \theta^l \\ \sin_k \theta^n \end{pmatrix} \right] \frac{\partial}{\partial p^j} && ?? \\
&= \frac{1}{k} \frac{\partial}{\partial \theta^i} \left[\begin{pmatrix} \prod_{l \in \{1..n\}} \cos_k \theta^l \\ \sin_k \theta^{j-1} \prod_{l \in \{j..n\}} \cos_k \theta^l \\ \sin_k \theta^n \end{pmatrix} \right] \frac{\partial}{\partial p^j} \\
&= \frac{1}{k} \left[\begin{pmatrix} -k \sin_k^* \theta^i \prod_{l \in \{1..n\}/\{i\}} \cos_k \theta^l \\ -k \sin_k^* \theta^i \sin_k \theta^{j-1} \prod_{l \in \{j..n\}/\{i\}} \cos_k \theta^l \\ k \cos_k \theta^i \prod_{l \in \{i+1..n\}} \cos_k \theta^l \\ 0 \end{pmatrix} \right] \frac{\partial}{\partial p^j} && ??
\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} -\tan_k^* \theta^i \prod_{l\in\{\,1..n\,\}} \cos_k \theta^l \\ -\tan_k^* \theta^i \sin_k \theta^{j-1} \prod_{l\in\{\,j..n\,\}} \cos_k \theta^l \\ \frac{1}{\tan_k \theta^i} \prod_{l\in\{\,i+1..n\,\}} \cos_k \theta^l \\ 0 \end{pmatrix} \frac{\partial}{\partial p^j} \\
&= \begin{pmatrix} -k \tan_k^* \theta^i p^j \\ k \frac{1}{\tan_k \theta^i} p^{i+1} \\ 0 \end{pmatrix} \frac{\partial}{\partial p^j} && ??
\end{aligned}
$$

□

Lemma 2. *The metric tensor of N is*

$$
g_{ij} = \begin{cases} (1-\operatorname{sgn} k)\tan_k^2(\theta^a)\prod_{1\leq j}\cos_k^2\theta^j + \prod_{a<j}\cos_k^2\theta^j & \textit{if } i=j, \\ 0 & \textit{otherwise.} \end{cases}
$$

Proof of lemma 2.

$$
\begin{aligned}
g_{ab}\left[\frac{\partial}{\partial \theta^i}\right] &= \sum_{l,m=1}^{n+1} \frac{\partial p^l}{\partial \theta^a} g_{lm}\left[\frac{\partial}{\partial p^i}\right]\frac{\partial p^m}{\partial \theta^b} \\
&= \sum_{l=1}^{n+1} \frac{\partial p^l}{\partial \theta^a}\frac{\partial p^l}{\partial \theta^b}
\end{aligned}
$$

If $a<b$,

$$
\begin{aligned}
g_{ab} &= \begin{bmatrix} -k\tan_k^*(\theta^a)\,p^j \\ k\frac{1}{\tan_k(\theta^a)}p^{a+1} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -k\tan_k^*\left(\theta^b\right)p^j \\ k\frac{1}{\tan_k\left(\theta^b\right)}p^{b+1} \\ 0 \end{bmatrix} \\
&= \sum k^2 \tan_k^*(\theta^a)\tan_k^*\left(\theta^b\right){p^j}^2 - k^2\frac{\tan_k^*\left(\theta^b\right)}{\tan_k(\theta^a)}{p^{a+1}}^2 \\
&= \tan_k\left(\theta^b\right)\tan_k(\theta^a)\prod_{a\leq j\leq n+1}\cos_k^2\theta^j\left(\prod_{1\leq j<a}\cos_k^2\theta^j + \sum \sin_k^2\theta^i\prod_{i<j<a}\cos_k^2\theta^j - \operatorname{sgn} k\right) \\
&= \tan_k\left(\theta^b\right)\tan_k(\theta^a)\prod_{a\leq j\leq n+1}\cos_k^2\theta^j\left(\prod_{1\leq j<a}\cos_k^2\theta^j + \operatorname{sgn} k\left(1-\prod_{1\leq j<a}\cos_k^2\theta^j\right) - \operatorname{sgn} k\right)
\end{aligned}
$$

$g_{ab} = 0.$

If $a > b$, $g_{ab} = g_{ba} = 0$.

If $a = b$,

$$
\begin{aligned}
g_{aa} &= \begin{pmatrix} -k \tan^*_k \theta^a p^j \\ k \frac{1}{\tan_k \theta^a} p^{a+1} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -k \tan^*_k \theta^a p^j \\ k \frac{1}{\tan_k \theta^a} p^{a+1} \\ 0 \end{pmatrix} \\
&= \sum_{j=1}^{a} \left(kp^j \tan^*_k \theta^a\right)^2 + \left(kp^{a+1} \cot_k \theta^a\right)^2 \\
&= \sum_{j=1}^{a} \left(kp^j \tan_k \theta^a\right)^2 + \left(kp^{a+1} \cot_k \theta^a\right)^2 \\
&= \sin^2_k (\theta^a) \prod_{a<j} \cos^2_k \theta^j \left(\prod_{1\le j<a} \cos^2_k \theta^j + \sum_{1\le i<a} \sin^2_k \theta^i \prod_{i<j<a} \cos^2_k \theta^j + \cot^2_k (\theta^a) \right) \\
&= \sin^2_k (\theta^a) \prod_{a<j} \cos^2_k \theta^j \left(\prod_{1\le j<a} \cos^2_k \theta^j + \sum_{1\le i<a} \sin^2_k \theta^i \prod_{i<j<a} \cos^2_k \theta^j + \cot^2_k (\theta^a) \right) \\
&= \sin^2_k (\theta^a) \prod_{a<j} \cos^2_k \theta^j \left(\prod_{1\le j<a} \cos^2_k \theta^j - \operatorname{sgn} k \prod_{1\le j<a} \cos^2_k \theta^j + \operatorname{sgn} k \prod_{1\le j<a} \cos^2_k \theta^j + \sum_{1\le i<a} \sin^2_k \theta^i \prod_{i<j<a} \cos^2_k \right. \\
&= \sin^2_k (\theta^a) \prod_{a<j} \cos^2_k \theta^j \left(\prod_{1\le j<a} \cos^2_k \theta^j - \operatorname{sgn} k \prod_{1\le j<a} \cos^2_k \theta^j + \operatorname{sgn} k + \cot^2_k (\theta^a) \right) \\
&= \sin^2_k (\theta^a) \prod_{a<j} \cos^2_k \theta^j \left(\prod_{1\le j<a} \cos^2_k \theta^j - \operatorname{sgn} k \prod_{1\le j<a} \cos^2_k \theta^j + \operatorname{sgn} k + \csc^2_k (\theta^a - \operatorname{sgn} k) \right) \\
&= \sin^2_k (\theta^a) \prod_{a<j} \cos^2_k \theta^j \left((1 - \operatorname{sgn} k) \prod_{1\le j<a} \cos^2_k \theta^j + \csc^2_k (\theta^a) \right) \\
&= (1 - \operatorname{sgn} k) \tan^2_k (\theta^a) \prod_{j} \cos^2_k \theta^j + \prod_{a<j} \cos^2_k \theta^j
\end{aligned}
$$

□

Curvature

Curvature (Method I)

This method may be easier to generalize to Model II where each direction can have partially independent curvature (or even Model III where extrinsic curvature become a thing). But it may be challenging to define Gauss map properly.

Lemma 3. *Given a position parameter θ, the tangent vector in position vector space* *can be calculated as follows*

$$\nu(p) = \begin{cases} \begin{bmatrix} kp^1 \\ \\ +kp^i \end{bmatrix} & k > 0 \\ \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} & k = 0 \\ \frac{1}{\sqrt{-1+2(kp^1)^2}} \begin{bmatrix} kp^1 \\ \\ -kp^i \end{bmatrix} & k < 0 \end{cases} .$$

Proof of lemma 3. From **??????**,

$$p \in P \iff \begin{cases} \sum_i {p^i}^2 = k^{-2} & k > 0, \\ p^1 = k^{-1} & k \to 0, \\ {p^1}^2 - \sum_{1<i} {p^i}^2 = k^{-2} \wedge p^1 > 0 & k < 0. \end{cases}$$

Let

$$F(p) = \begin{cases} \sum_i {p^i}^2 - k^{-2} & k > 0, \\ p^1 - k^{-1} & k \to 0, \\ {p^1}^2 - \sum_{1<i} {p^i}^2 - k^{-2} & k < 0. \end{cases}$$

$$\begin{aligned} n &= l \nabla F(p) \\ &= l \begin{cases} \nabla \sum_i {p^i}^2 - k^{-2} & k > 0, \\ \nabla p^1 - k^{-1} & k \to 0, \\ \nabla {p^1}^2 - \sum_{1<i} {p^i}^2 - k^{-2} & k < 0. \end{cases} \\ &= l \begin{cases} \begin{pmatrix} 2p^i \end{pmatrix} & k > 0, \\ \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix} & k \to 0, \\ \begin{pmatrix} -2p^1 \\ \\ 2p^i \end{pmatrix} & k < 0. \end{cases} \end{aligned}$$

$$\hat{n} = \pm \begin{cases} \frac{1}{||p||}\begin{pmatrix} p^i \end{pmatrix} & k > 0, \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & k \to 0, \\ \frac{1}{||p||}\begin{pmatrix} -p^1 \\ p^i \end{pmatrix} & k < 0. \end{cases}$$

$$= \pm \begin{cases} \frac{1}{k^{-1}}\begin{pmatrix} p^i \end{pmatrix} & k > 0, \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & k \to 0, \\ \frac{1}{\sqrt{-1+2(kp^1)^2}}\begin{pmatrix} -p^1 \\ p^i \end{pmatrix} & k < 0. \end{cases}$$

$$\nu = \hat{n} := \begin{cases} \begin{pmatrix} kp^i \end{pmatrix} & k > 0, \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & k \to 0, \\ \frac{1}{\sqrt{-1+2(kp^1)^2}}\begin{pmatrix} p^1 \\ -p^i \end{pmatrix} & k < 0. \end{cases}$$

$\square$

Lemma 4.

$$L(p) = \begin{cases} -kI & k > 0, \\ 0 & k \to 0, \\ \frac{1}{||p||}\left(\frac{1}{||p||^2}pp^T \operatorname{diag}(-1, 1 \ldots, 1) - I\right) & k < 0. \end{cases}$$

Proof of lemma 4.

$$\begin{aligned} \frac{\partial}{\partial p^j}\frac{1}{||p||} &= \frac{\partial}{\partial\,||p||}\frac{1}{||p||}\frac{\partial}{\partial p^j}||p|| \\ &= -\frac{1}{||p||^2}\frac{\partial}{\partial p^j}\sqrt{\sum_i {p^i}^2} \\ &= -\frac{1}{||p||^2}\frac{\partial}{\partial \sum_i {p^i}^2}\sqrt{\sum_i {p^i}^2}\frac{\partial}{\partial p^j}\sum_i {p^i}^2 \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{||p||^2}\frac{1}{2\sqrt{\sum_i {p^i}^2}}2p^j \\
&= -\frac{1}{||p||^2}\frac{1}{2\,||p||}2p^j \\
&= -\frac{p^j}{||p||^3}
\end{aligned}
$$

From ??,

$$
\begin{aligned}
L(p) &= -(D\nu \circ (Df)^{-1})(p) \\
&= -(D\nu)(p) \\
&= -\left(\frac{\partial}{\partial p^j}\nu^i\bigg|_p\right)_{i,j}
\end{aligned}
$$

$$
\begin{aligned}
(L(p))_{i,j} &= \begin{cases}
-\frac{\partial}{\partial p^j}kp^i & k > 0, \\
-\frac{\partial}{\partial p^j}1 & k \to 0 \text{ and } i = 1, \\
-\frac{\partial}{\partial p^j}0 & k \to 0 \text{ and } i \neq 1, \\
-\frac{\partial}{\partial p^j}\left(\frac{1}{||p||}p^1\right) & k < 0 \text{ and } i = 1, \\
-\frac{\partial}{\partial p^j}\left(-\frac{1}{||p||}p^i\right) & k < 0 \text{ and } i \neq 1,
\end{cases} \\
&= \begin{cases}
-k & k > 0 \text{ and } i = j, \\
0 & k > 0 \text{ and } i \neq j, \\
0 & k \to 0, \\
-\left(\frac{1}{||p||}\frac{\partial}{\partial p^j}p^1 + p^1\frac{\partial}{\partial p^j}\frac{1}{||p||}\right) & k < 0 \text{ and } i = 1, \\
\left(\frac{1}{||p||}\frac{\partial}{\partial p^j}p^i + p^i\frac{\partial}{\partial p^j}\frac{1}{||p||}\right) & k < 0 \text{ and } i \neq 1,
\end{cases}
\end{aligned}
$$

$$= \begin{cases} -k & k > 0 \text{ and } i = j, \\ 0 & k > 0 \text{ and } i \neq j, \\ 0 & k \to 0, \\ -\frac{1}{||p||} + \frac{{p^1}^2}{||p||^3} & k < 0 \text{ and } i = j = 1, \\ \frac{p^1 p^j}{||p||^3} & k < 0 \text{ and } j \neq i = 1, \\ \frac{1}{||p||} - \frac{{p^i}^2}{||p||^3} & k < 0 \text{ and } i = j \neq 1, \\ -\frac{p^i p^j}{||p||^3} & k < 0,\ i \neq j \text{ and } i \neq 1, \end{cases}$$

For $k > 0,\ L(p) = -kI$.

For $k \to 0,\ L(p) = 0$.

For $k < 0,\ L(p) = \frac{1}{||p||}\left(\frac{1}{||p||^2} pp^T \operatorname{diag}(-1, 1 \dots, 1) - I\right)$. □

Lemma 5.

Proof of lemma 5. The proof is left as an exercise to the other author.

For $k > 0$,

$$\begin{aligned} 0 &= \det L(p) - \lambda I \\ &= \det -kI - \lambda I \\ &= \det\,(-k - \lambda) I \\ &= (-k - \lambda)^{n+1} \det I \\ &= (-k - \lambda)^{n+1} \\ \lambda &\in \{\, -k, \dots, -k \,\} \end{aligned}$$

For $k < 0$,

$$\begin{aligned} 0 &= \det L(p) - \lambda I \\ &= \det \frac{1}{||p||}\left(\frac{1}{||p||^2} pp^T \operatorname{diag}(-1, 1 \dots, 1) - I\right) - \lambda I \\ &= \det \frac{1}{||p||}\left(\frac{1}{||p||^2} pp^T \operatorname{diag}(-1, 1 \dots, 1) - I\right) - \left(\lambda' - \frac{1}{||p||}\right) I \qquad \lambda' := \lambda + \frac{1}{||p||} \\ &= \det \frac{1}{||p||^3} pp^T \operatorname{diag}(-1, 1 \dots, 1) - \frac{1}{||p||} I - \left(\lambda' - \frac{1}{||p||}\right) I \end{aligned}$$

$$
\begin{aligned}
&= \det \frac{1}{||p||^3} pp^T \operatorname{diag}(-1, 1\ldots, 1) - \lambda' I \\
&= \det \frac{1}{||p||^3} pp^T \operatorname{diag}(-1, 1\ldots, 1) - \frac{1}{||p||^3} \lambda'' I && \lambda'' := ||p||^3 \lambda' \\
&= \frac{1}{||p||^{3(n+1)}} \det pp^T \operatorname{diag}(-1, 1\ldots, 1) - \lambda'' I \\
&= \det pp^T \operatorname{diag}(-1, 1\ldots, 1) - \lambda'' I \\
&= \lambda''^n \left(\lambda'' + p^{12} - \sum_{i \neq 1} p^{i2} \right) \\
0 &= \lambda''^n \left(\lambda'' + \frac{1}{k} \right) \\
\lambda'' &\in \left\{ -\frac{1}{k}, 0, \ldots, 0 \right\} \\
\lambda' &\in \left\{ -\frac{1}{k\, ||p||^3}, 0, \ldots, 0 \right\} \\
\lambda &\in \left\{ -\frac{1}{k\, ||p||^3} - \frac{1}{||p||}, -\frac{1}{||p||}, \ldots, -\frac{1}{||p||} \right\} \\
&= \left\{ -\frac{1 + ||p||^2}{k\, ||p||^3}, -\frac{1}{||p||}, \ldots, -\frac{1}{||p||} \right\} \\
&= \left\{ -\frac{2k^2 p^{12}}{k\, ||p||^3}, -\frac{1}{||p||}, \ldots, -\frac{1}{||p||} \right\} \\
&= \left\{ -\frac{2k p^{12}}{||p||^3}, -\frac{1}{||p||}, \ldots, -\frac{1}{||p||} \right\} \\
&= \left\{ k^2 \, ||p|| \, \frac{-2p^{12}}{k\, ||p||^4}, -\frac{1}{||p||}, \ldots, -\frac{1}{||p||} \right\} \\
&= \left\{ k^2 \, ||p|| \, \frac{-2p^{12}}{k\, ||p||^4}, -\frac{1}{||p||}, \ldots, -\frac{1}{||p||} \right\}
\end{aligned}
$$

□

Curvature (Method II)

This method may be easier to be done (despite the fact that it never finished). But it raises problems when trying to generalize e.g. dealing with extrinsic curvature (which may be introduced in Model III if not to mess with other basis geometries).

Lemma 6.

Lemma 7.

Curvature (Conclusion)

Lemma 8.

CurvatureParameter. It can be seen that $\sec(p) = \kappa = \operatorname{sgn}(k)k^2$. Hence, when provided κ, k can be determined and used to evaluate the model.