Prove of Constant Radius Model

Hanchai Nonprasart and Sakepisit Maysamat

Mahidol Wittayanusorn School

SCI 30196: Science Project 1

Amornsri Amornvatcharapong, Teerapong Suksumran

February 10, 2022

Abstract

This is the first generation of the model. It's a draft. Who would even care to read the abstract when it's not done anyways?

Keywords: None 1, None 2, None 3

Prove of Constant Radius Model

Contents

Rewrite note

This proof will be organized using the following guideline

1. reviewing existing definitions and theorem
2. proving certain property of the existing mathematical objects
3. defining the conditions of the model
4. formulating of the model
5. parameterize the model
6. asserting defined property of the model

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Preliminary

Tensor

(1) $$(A+B)^{i_1\cdots i_n}{}_{j_1\cdots j_m} = A^{i_1\cdots i_n}{}_{j_1\cdots j_m} + B^{i_1\cdots i_n}{}_{j_1\cdots j_m}$$

(2) $$(\alpha A)^{i_1\cdots i_n}{}_{j_1\cdots j_m} = \alpha A^{i_1\cdots i_n}{}_{j_1\cdots j_m}$$

(3) $$(A\otimes B)^{i_1\cdots i_l i_{l+1}\cdots i_{l+n}}{}_{j_1\cdots j_k j_{k+1}\cdots j_{k+m}} = A^{i_1\cdots i_l}{}_{j_1\cdots j_k} B^{i_{l+1}\cdots i_{l+n}}{}_{j_{k+1}\cdots j_{k+m}}$$

(4) $$(\operatorname{contr} T)^{i_1\cdots i_n}{}_{j_1\cdots j_m} = \sum_a T^{i_1\cdots i_n a}{}_{a j_1\cdots j_m}$$

(5) $$(AB)^i{}_j = \operatorname{contr}(A\otimes B) = \sum_k A^i{}_k B^k{}_j$$

(6) $$\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}\right)_{ij} = \sum_k A_{ik}B_{kj}$$

(7) $$IA = A = AI$$

$$I_n = \operatorname{diag}(1, 1, \ldots, 1) \tag{8}$$

$$I_{a+b} = \begin{bmatrix} I_a & 0_{a\times b} \\ 0_{b\times a} & I_b \end{bmatrix} \tag{9}$$

$$T_{a,b}T_{a,b} = I \tag{10}$$

Differential

$$J_x F = \left(\left. \frac{\partial F_i}{\partial x_j} \right|_x \right)_{i,j} \tag{11}$$

$$Df|_x : T_x\mathbb{R}^k \to T_{f(x)}\mathbb{R}^n = (x, v) \mapsto (f(x), J_x f(v)) \tag{12}$$

Group

Definition 1 (Lie group (Lee, 2013, Chapter 7))**.** A *Lie group* is a smooth manifold G (without boundary) that is also a group in the algebraic sense, with the property that the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$, given by

$$m(g, h) = gh, \qquad i(g) = g^{-1}$$

are both smooth.

Preposition 1 (Lie group (Lee, 2013, Chapter 7))**.** *If G is a smooth manifold with a group structure such that the map $G \times G \to G$ given by $(g, h) \mapsto gh^{-1}$ is smooth, then G is a Lie group.*

Definition 2 (Semidirect product)**.** Suppose H and N are groups, and $\theta : H \times N \to N$ is a smooth left action of H on N. It is said to be an *action by automorphisms* if for each $h \in H$, the map $\theta_h : N \to N$ is a group automorphism of N (i.e., an isomorphism from N to itself). Given such action, we define a new group $N \rtimes_\theta H$, called a *semidirect product* of H and N, as follows. $N \rtimes_\theta H$ is just the Cartesian product $N \times H$; but the group multiplication is defined by

$$(n, h)(n', h') = (n\theta_h(n'), hh').$$

Definition 3. A *Klein geometry* is a pair (G, H) where G is a Lie group and H is a closed Lie subgroup of G such that the (left) coset space

$$X := G/H$$

is connected.

Example 3.1. KleinGeometryExamples (I)

Manifold

Definition 4 (Abstract differentiable manifold (Wolfgang, 2006, Chapter 5A))**.** A *k-dimensional differentiable manifold* (briefly: a k-manifold) is a set M together with a family $(M_i)_{i\in I}$ of subsets such that

1. $M = \bigcup_{i\in I} M_i$ (union),
2. for every $i \in I$ there is an injective map $\varphi_i : M_i \to \mathbb{R}^k$ so that $\phi_i(M_i)$ is open in $\mathbb{R}^k$, and
3. for $M_i \cap M_j \neq \emptyset$, $\varphi_i(M_i \cap M_j)$ is open in $\mathbb{R}^k$.

Definition 5 (Structures on a manifold (Wolfgang, 2006, Chapter 5A))**.** Given a k-dimensional differentiable manifold, one gets additional structure by replacing aditional requirements on the transformation functions $\varphi_j \circ \varphi_i^{-1}$, which belong to the atlas of the manifold; if all $\varphi_j \circ \varphi_i^{-1}$ are (left-hand side), then one speaks of (right-hand side) as follows:

continuous	$\leftrightarrow$	topological manifold
differentiable	$\leftrightarrow$	differentiable manifold
C^1-differentiable	$\leftrightarrow$	C^1-manifold
C^r-differentiable	$\leftrightarrow$	C^r-manifold
C^∞-differentiable	$\leftrightarrow$	C^∞-manifold
real analytic	$\leftrightarrow$	real analytic manifold
complex analytic	$\leftrightarrow$	complex analytic manifold of dimension $\frac{k}{2}$
affine	$\leftrightarrow$	affine manifold
projective	$\leftrightarrow$	projective manifold
conformal	$\leftrightarrow$	manifold with a conformal structure
orienatation-preserving	$\leftrightarrow$	orientable manifold

Definition 6 (Tangent vector (Wolfgang, 2006, Chapter 5B))**.** A *tangent vector* X at p is a derivation (derivative operator) defined on the set of *germs of functions*

$$\mathcal{F}_p(M) := \{\, f : M \to \mathbb{R} \mid f \text{ differentiable} \,\} / \sim ,$$

where the equivalence relation $\sim$ is defined by declaring $f \sim f^*$ if and only if f and f^* coincide in a neighborhood of p. The value $X(f)$ is also referred to as the *directional derivative* of f in the direction X.

This definition means more precisely the following. X is a map $X : \mathcal{F}_p(M) \to \mathbb{R}$ with the two following properties:

1. $X(\alpha f + \beta g) = \alpha X(f) + \beta(g)$, $f, g \in \mathcal{F}_p(M)$ ($\mathbb{R}$-*linearity*);
2. $X(f \cdot g) = X(f) \cdot g(p) + f(p) \cdot X(g)$ for $f, g \in \mathcal{F}_p(M)$ (*product rule*).

(For this to make sense, both f and g have to be defined in a neighborhood of p.)

Briefly: *tangent vectors are derivations acting on scalar functions.*

Corollary 1 (Tangent space (Wolfgang, 2006, Chapter 5B))**.** *The tangent space T_pM of M at p is defined in all cases as the set of all tangent vectors at the point p. By definition T_pM and T_qM are disjoint if $p \neq q$.*

Definition 7 (Derivative (Wolfgang, 2006, Chapter 5B))**.** Let $F : M \to N$ be a differentiable map, and let p, q be two fixed point with $F(p) = q$. Then the *derivative* or the *differential* of F at p is defined as the map

$$DF|_p : T_pM \to T_qN$$

whose value at $X \in T_pM$ is given by $\left(DF|_p(X)\right)(f) = X(f \circ F)$ for every $f \in \mathcal{F}_q(N)$ (which automatically implies the relation $f \circ F \in \mathcal{F}_p(M)$).

Lemma 1 (Chain rule (Wolfgang, 2006, Chapter 5B))**.** *For the derivative as defined in this manner, one has the chain rule in the form*

$$D(G \circ F)|_p = DG|_{F(p)} \circ DF|_p$$

for every composition $M \xrightarrow{F} N \xrightarrow{G} Q$ *of maps, or, more briefly,* $D(G \circ F) = DG \circ DF$*.*

Definition 8 (Riemannian metric (Wolfgang, 2006, Chapter 5C))**.** A *Riemannian metric* g on M is an association $p \mapsto g_p \in L^2(T_pM; \mathbb{R})$ such that the following conditins are satisfied:

1. $g_p(X, Y) = g_p(Y, X)$ for all X, Y, (*symmetry*)
2. $g_p(X, X) > 0$ for all $X \neq 0$, (*positive definiteness*)
3. The coefficient g_{ij} in every local representation (i.e., in every chart)

$$g_p = \sum_{i,j} g_{ij}(p) \cdot dx^i\Big|_p \otimes dx^j\Big|_p$$

are differentiable functions. (*differentiability*)

Remark 1. The pair (M, g) is then called *Riemannian manifold.* One also refers to the Riemannian metric as the *metric tensor.* In local coordinates the metric tensor is given by the matrix (g_{ij}) of functions. In Ricci calculus this is simply written as g_{ij}. (Wolfgang, 2006, Chapter 5C)

Remark 2. A Riemannian metric g defines at every point p an *inner product* g_p on the tangent space T_pM, and therefore the notation $\langle X, Y \rangle$ instead of $g_p(X, Y)$ is also used. The notions of angles and lengths are determined by this inner product, just as these notions are determined by the first fundamental form on surface elements. The length or norm of vector X is given by $||X|| := \sqrt{g(X, X)}$, and the angle β between two tangent vectors X and Y can be defined by the validity of the equation
$\cos\beta \cdot ||X|| \cdot ||Y|| = g(X, Y)$. (Wolfgang, 2006, Chapter 5C)

Curvature

Definition 9 (Christoffel symbols (Wolfgang, 2006, Chapter 4A))**.** 1. The quantities $\partial_k\Gamma_{ij}$ defined by the expressions

$$\partial_k\Gamma_{ij} := \left\langle \nabla_{\frac{\partial f}{\partial u^i}} \frac{\partial f}{\partial u^j}, \frac{\partial f}{\partial u^k} \right\rangle$$

are called the *Christoffel symbols of the first kind.*

2. The quantities ${\Gamma^k}_{ij}$ defined by the expressions

$$\nabla_{\frac{\partial f}{\partial u^i}} \frac{\partial f}{\partial u^j} = \sum_k {\Gamma^k}_{ij} \frac{\partial f}{\partial u^k}$$

are called the *Christoffel symbols of the second kind.*

3. By definition one has $\partial_k\Gamma_{ij} = \partial_k\Gamma_{ji}$, ${\Gamma^k}_{ij} = {\Gamma^k}_{ji}$ as well as $\partial_k\Gamma_{ij} = \sum_m {\Gamma^m}_{ij} g_{mk}$.

Definition 10 (Gauss map (Wolfgang, 2006, Chapter 3B))**.** For a surface element $f : U \to \mathbb{R}^3$, the *Gauss map*

$$\nu : U \to S^2$$

is defined by the formula

$$\nu(u_1, u_2) := \frac{\frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2}}{\left\| \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \right\|}.$$

Definition 11 (Weingarten map, shape operator (Wolfgang, 2006, Chapter 3B))**.** Let $f : U \to \mathbb{R}^3$ be a surface element with Gauss map $\nu : U \to S^2 \subset \mathbb{R}^3$.

1. For every $u \in U$ the image plane of the bilinear map

$$D\nu|_u : T_uU \to T_{\nu(u)}\mathbb{R}^3$$

is parallel to the tangent plane T_uf. By canonically identifying $T_{\nu(u)}\mathbb{R}^3 \cong \mathbb{R}^3 \cong T_{f(u)}\mathbb{R}^3$ we may therefore view $D\nu$ at every point as the map

$$D\nu|_u : T_uU \to T_uf.$$

Moreover, by restricting to the image, we may view the map $Df|)u$ as a linear isomorphism

$$Df|_u : T_uU \to T_uf.$$

In this sense the inverse mapping $(Df|_u)^{-1}$ is well-defined and is also an isomorphism

2. The map $L := -D\nu \circ (Df)^{-1}$ defined pointwise by

$$L_u := -\left(D\nu|_u\right) \circ \left(Df|_u\right)^{-1} : T_u f \to T_u f$$

is called the *Weingarten map* or the *shape operator* of f. Obviously, for every parameter u this is a linear endomorphism of the tangent plane at the corresponding point $f(u)$.

3. L is independent of the parameterization f (up to the choice of the sign of the unit normal vector ν), and it is self-adjoint with respect to the first fundamental form I.

Definition 12 (Hypersurface element (Wolfgang, 2006, Chapter 3F))**.** $f : U \to \mathbb{R}^{n+1}$ is called a *regular hypersurface element*, if $U \subset \mathbb{R}^n$ is open and f is a (C^2-) immersion. The parameter $u = (u_1, \ldots, u_n)$ is associated with the point $f(u)$ with $n+1$ coordinates $f(u) = (f_1(u), \ldots, f_{n+1}(u))$. The *tangent hyperplane* $T_u f$ is the is defined to be the image of $T_u U$ under the map $Df|_u$. Similarly, one defines

- the *Gauss map* $\nu : U \to S^n$ by the unit normal vector $\nu(u)$, which is perpendicular to $T_u f$ (but note: in $\mathbb{R}^{n+1}$ for $n \geq 3$ there is no bilinear vector product of tangent vectors; still one can formally define ν as an n-linear vector product),
- the *Weingarten map* $L = -D\nu \circ (Df)^{-1}$,
- the first, second, and third fundamental forms

$$\begin{aligned}
I = \left(g_{ij}\right)_{i,j=1,\ldots,n} &= \left(\left\langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle\right), \\
II = \left(h_{ij}\right)_{i,j=1,\ldots,n} &= \left(\left\langle \frac{\partial^2 f}{\partial u_i \partial u_j}, \nu \right\rangle\right), \\
III = \left(e_{ij}\right)_{i,j=1,\ldots,n} &= \left(\left\langle \frac{\partial \nu}{\partial u_i}, \frac{\partial \nu}{\partial u_j} \right\rangle\right).
\end{aligned}$$

Definition 13 (Curvature tensor (Wolfgang, 2006, Chapter 4C))**.**

$$R(X, Y)\, Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is a tensor field, which is called the *curvature tensor* of the surface.

Remark 3 (Curvature tensor (Wolfgang, 2006, Chapter 4C)). The left-hand side of the Gauss equation is called the *curvature tensor* and is in general expressed in the form

$$R^s{}_{ikj} := \frac{\partial}{\partial u^k} \Gamma^s{}_{ij} - \frac{\partial}{\partial u^j} \Gamma^s{}_{ik} + \sum_r \left(\Gamma^r{}_{ij} \Gamma^s{}_{rk} - \Gamma^r{}_{ik} \Gamma^s{}_{rj}\right).$$

Example 13.1.

$$S(v) = \pm\nabla_v n$$

Example 13.2. Eigenvalue of second fundamental form (or shape operator)

Objective

Objective. The objective is that given a natural number n and a real number κ, one can construct

1. a set M
2. a Lie group M with operation $\otimes_M$,
3. an n-dimensional C^∞-manifold M with chart $\varphi_i \subset M \to \mathbb{R}^n$,
4. a Riemannian manifold M with inner product $g_p \in T_pM \times T_pM \to \mathbb{R}$

(to be determined) such that

- group action is distance preserved.
- model is continuous with respect to κ (and smooth with respect to each basis).
- for all two-dimensional linear subspaces of the manifold, the sectional

curvature is κ.

(cannot figure formal definition out yet.)

Conjecture 1. *If the parameters (κ, n) is associated with Klein geometry (G, H) then $(M, \otimes_M) \cong G/H$.*

That is, from example 3.1,

- *For $\kappa > 0$, $G \cong \mathrm{O}(n+1)$ and $H \cong \mathrm{O}(n)$.*
- *For $\kappa = 0$, $G \cong \mathrm{Euc}(n)$ and $H \cong \mathrm{O}(n)$.*
- *For $\kappa < 0$, $G \cong \mathrm{O}^+(1, n)$ and $H \cong \mathrm{O}(n)$.*

Model Foundation

Trigonometry

Definition 14. *Generalized trigonometric functions $f_k : \mathbb{R} \to \mathbb{R}$* and $f_k^* : \mathbb{R} \to \mathbb{R}$ are defined as

$$f_k(\theta) := \begin{cases} g(k\theta) & \text{if } k \geq 0, \\ h(k\theta) & \text{otherwise,} \end{cases}$$

$$f_k^*(\theta) := \begin{cases} g(k\theta) & \text{if } k \geq 0, \\ h(-k\theta) & \text{otherwise,} \end{cases}$$

where g (resp. h) are the associated trigonometric (resp. hyperbolic) function.

Example 14.1 (Generalized sine functions)**.**

$$\sin_k \theta := \begin{cases} \sin(k\theta) & \text{if } k \geq 0, \\ \sinh(k\theta) & \text{otherwise.} \end{cases}$$

$$\sin_k^* \theta := \begin{cases} \sin(k\theta) & \text{if } k \geq 0, \\ \sinh(-k\theta) & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \sin(k\theta) & \text{if } k \geq 0, \\ -\sinh(k\theta) & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \sin(|k|\theta) & \text{if } k \geq 0, \\ \sinh(|k|\theta) & \text{otherwise.} \end{cases}$$

(see Figure 2)

Example 14.2 (Generalized cosine functions)**.**

$$\cos_k \theta := \begin{cases} \cos(k\theta) & \text{if } k \geq 0, \\ \cosh(k\theta) & \text{otherwise.} \end{cases}$$

$$\cos_k^* \theta := \begin{cases} \cos(k\theta) & \text{if } k \geq 0, \\ \cosh(-k\theta) & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \cos(k\theta) & \text{if } k \geq 0, \\ \cosh(k\theta) & \text{otherwise.} \end{cases}$$

$$= \cos_k \theta$$

(see Figure 4)

Theorem 1 (Pythagorean's identity equivalence)**.**

$$\cos_k^2 \theta + \operatorname{sgn} k \sin_k^2 \theta = 1$$

Proof of theorem 1. Proof by exhaustion. □

Preposition 2 (Generalized trigonometric functions of sum of arguments)**.**

$$\begin{aligned}
\sin_k(\theta + \phi) &= \sin_k \theta \cos_k \phi + \cos_k \theta \sin_k \phi \\
\sin_k^*(\theta + \phi) &= \sin_k^* \theta \cos_k \phi + \cos_k \theta \sin_k^* \phi \\
\cos_k(\theta + \phi) &= \cos_k \theta \cos_k \phi - \operatorname{sgn} k \sin_k \theta \sin_k \phi \\
&= \cos_k \theta \cos_k \phi - \sin_k^* \theta \sin_k \phi \\
&= \cos_k \theta \cos_k \phi - \sin_k \theta \sin_k^* \phi
\end{aligned}$$

Proof of proposition 2. Proof by exhaustion. □

Preposition 3 (Derivative of generalized trigonometric functions)**.**

$$\begin{aligned}
{\sin_k}'\theta &= k \cos_k \theta \\
{\sin_k^*}'\theta &= |k| \cos_k \theta \\
{\cos_k}'\theta &= -k \sin_k^* \theta
\end{aligned}$$

Proof of proposition 3. Proof by exhaustion. □

Matrices

Definition 15. *Generalized rotation matrix* is defined as

$$R_k(\theta) := \begin{bmatrix} \cos_k \theta & -\sin_k^* \theta \\ \sin_k \theta & \cos_k \theta \end{bmatrix},$$

where $\theta \in \mathbb{R}$.

Corollary 2 (Generalized rotation matrix at zero)**.**

$$R_k(0) = I_2$$

Proof of corollary 2. Obvious □

Corollary 3 (Generalized rotation matrix of sum of arguments)**.**

$$R_k(\theta)\,R_k(\phi) = R_k(\theta+\phi)$$

Proof of corollary 3.

$$\begin{aligned}
R_k(\theta)\,R_k(\phi) &= \begin{bmatrix} \cos_k\theta & -\sin_k^*\theta \\ \sin_k\theta & \cos_k\theta \end{bmatrix} \begin{bmatrix} \cos_k\phi & -\sin_k^*\phi \\ \sin_k\phi & \cos_k\phi \end{bmatrix} && \text{(definition 15)} \\
&= \begin{bmatrix} \cos_k\theta\cos_k\phi + (-\sin_k^*\theta)\sin_k\phi & \cos_k\theta\,(-\sin_k^*\phi) + (-\sin_k^*\theta)\cos_k\phi \\ \sin_k\theta\cos_k\phi + \cos_k\theta\sin_k\phi & \sin_k\theta\,(-\sin_k^*\phi) + \cos_k\theta\cos_k\phi \end{bmatrix} && \text{(Equation (5))} \\
&= \begin{bmatrix} \cos_k\theta\cos_k\phi - \sin_k^*\theta\sin_k\phi & -(\sin_k^*\theta\cos_k\phi + \cos_k\theta\sin_k^*\phi) \\ \sin_k\theta\cos_k\phi + \cos_k\theta\sin_k\phi & \cos_k\theta\cos_k\phi - \sin_k\theta\sin_k^*\phi \end{bmatrix} && \text{(simplify)} \\
&= \begin{bmatrix} \cos_k\theta+\phi & -\sin_k^*\theta+\phi \\ \sin_k\theta+\phi & \cos_k\theta+\phi \end{bmatrix} && \text{(proposition 2)} \\
&= R_k(\theta+\phi) && \text{(definition 15)}
\end{aligned}$$

$$R_k(\theta)\,R_k(\phi) = R_k(\theta+\phi)$$ □

Corollary 4 (Inverse of generalized rotation matrix)**.**

$$R_k(\theta)^{-1} = R_k(-\theta)$$

Proof of corollary 4.

$$\begin{aligned}
R_k(\theta)\,R_k(-\theta) &= R_k(0) && \text{(corollary 3)} \\
&= I_2 && \text{(corollary 2)} \\
R_k(-\theta)\,R_k(\theta) &= R_k(0) && \text{(corollary 3)} \\
&= I_2 && \text{(corollary 2)}
\end{aligned}$$

$$R_k(\theta)\,R_k(-\theta) = R_k(-\theta)\,R_k(\theta) = I_2$$

$$R_k(\theta)^{-1} = R_k(-\theta)$$ □

Definition 16. *Position matrix* is defined recursively as

$$P_{k,n}\left(\left\{\theta^1,\ldots,\theta^n\right\}\right) := \begin{bmatrix} P_{k,n-1}\left(\{\theta^1{}_n,\ldots,\theta^{n-1}\}\right) & 0_{n\times 1} \\ 0_{1\times n} & 1 \end{bmatrix} T_{2,n+1} \begin{bmatrix} R_k(\theta^n) & 0_{2\times n-1} \\ 0_{n-1\times 2} & I_{n-1} \end{bmatrix} T_{2,n+1},$$

$$P_{k,0} := I_1,$$

where $\theta = \{\theta^i\} \in \mathbb{R}^n$ for $i \in \{\,1..n\,\}$.

Definition 17. Let $P\,(n,k)$ be set of position matrices.

Corollary 5 (Position matrix at zero)**.**

$$P_{k,n}\,(0_n) = I_{n+1}$$

Proof of corollary 5. Prove by mathematical induction on n, Let

$$P_{k,n-1}\,(0_{n-1}) = I_n \tag{13}$$

$$\begin{aligned}
P_{k,n}\,(0_n) &= \begin{bmatrix} P_{k,n-1}\,(0_{n-1}) & 0_{n\times 1} \\ 0_{1\times n} & 1 \end{bmatrix} T_{2,n+1} \begin{bmatrix} R_k\,(0) & 0_{2\times n-1} \\ 0_{n-1\times 2} & I_{n-1} \end{bmatrix} T_{2,n+1} && \text{(definition 16)} \\
&= \begin{bmatrix} I_n & 0_{n\times 1} \\ 0_{1\times n} & 1 \end{bmatrix} T_{2,n+1} \begin{bmatrix} R_k\,(0) & 0_{2\times n-1} \\ 0_{n-1\times 2} & I_{n-1} \end{bmatrix} T_{2,n+1} && \text{(Equation (13))} \\
&= \begin{bmatrix} I_n & 0_{n\times 1} \\ 0_{1\times n} & 1 \end{bmatrix} T_{2,n+1} \begin{bmatrix} I_2 & 0_{2\times n-1} \\ 0_{n-1\times 2} & I_{n-1} \end{bmatrix} T_{2,n+1} && \text{(corollary 2)} \\
&= I_{n+1} T_{2,n+1} I_{n+1} T_{2,n+1} && \text{(Equation (9))} \\
&= T_{2,n+1} T_{2,n+1} && \text{(Equation (7))} \\
&= I_{n+1} && \text{(Equation (10))}
\end{aligned}$$

$$P_{k,n}\,(0_n) = I_{n+1}$$

□

Definition 18. *Orientation matrix* is defined as

$$Q_n^{\pm}\,(\phi_{n-1}, \phi_{n-2}, \ldots, \phi_1) := \begin{bmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & X^{\pm}_{+1,n-1}\,(\phi_{n-1}, \phi_{n-2}, \ldots, \phi_1) \end{bmatrix},$$

$$Q_0^{\pm} := \pm I_1,$$

where $\phi_m \in \mathbb{R}^m$for $m \in \{\,1..n-1\,\}$.

Definition 19. Let $Q\,(n)$ be set of orientation matrices.

Corollary 6 (Orientation matrix at zero)**.**

$$Q_n^+ \left(0_{n-1}, 0_{n-2}, \ldots\right) = I_{n+1}$$

Proof of corollary 6. Prove by mathematical induction on n, Let

$$Q_{n-1}^+ \left(0_{n-2}, 0_{n-3}, \ldots\right) = I_n \tag{14}$$

$$\begin{aligned}
Q_n^+ \left(0_{n-1}, 0_{n-2}, \ldots\right) &= \begin{bmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & X_{+1,n-1}^{\pm} \left(0_{n-1}, 0_{n-2}, \ldots\right) \end{bmatrix} && \text{(definition 18)} \\
&= \begin{bmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & P_{+1,n-1} \left(0_{n-1}\right) Q_{n-1}^+ \left(0_{n-2}, 0_{n-3}, \ldots\right) \end{bmatrix} && \text{(definition 20)} \\
&= \begin{bmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & P_{+1,n-1} \left(0_{n-1}\right) I_n \end{bmatrix} && \text{(Equation (14))} \\
&= \begin{bmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & P_{+1,n-1} \left(0_{n-1}\right) \end{bmatrix} && \text{(Equation (7))} \\
&= \begin{bmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & I_n \end{bmatrix} && \text{(corollary 5)} \\
&= I_{n+1} && \text{(Equation (9))}
\end{aligned}$$

$$Q_n^+ \left(0_{n-1}, 0_{n-2}, \ldots\right) = I_{n+1}$$ □

Definition 20. *Point matrix* is defined as

$$X_{k,n}^{\pm} \left(\theta, \phi_{n-1}, \phi_{n-2}, \ldots, \phi_1\right) := P_{k,n} \left(\theta\right) Q_n^{\pm} \left(\phi_{n-1}, \phi_{n-2}, \ldots, \phi_1\right),$$

where $\theta \in \mathbb{R}^n$ and $\phi_m \in \mathbb{R}^m$for $m \in \{ 1..n-1 \}$.

Definition 21. Let $X\left(n,k\right)$ be set of point matrices.

Corollary 7 (Position matrix as subset of point matrix)**.**

$$P\left(k,n\right) \subset X\left(k,n\right)$$

Proof of corollary 7.

$$\begin{aligned} \forall P \in P(k,n) \forall Q \in Q(n), PQ \in X(k,n) && \text{(definition 20)} \\ \implies PI \in X(k,n) && \text{(corollary 6)} \\ \implies P \in X(k,n) && \text{(Equation (7))} \end{aligned}$$

$$P(k,n) \subset X(k,n)$$ □

Corollary 8 (Point matrix at zero)**.**

$$X^{+}_{k,n}(0_n, 0_{n-1}, \dots) = I_{n+1}$$

Proof of corollary 8. It can be implied from corollaries 5 and 6. □

Model Parametrization

Definition 22. For any point matrix $X^{\pm}_{k,n}(\theta, \phi_1, \phi_2, \dots, \phi_n)$, n-dimensional vector θ is defined as *position parameter*.

Definition 23. For point matrix X, $(n+1)$-dimensional column vector $p := \frac{1}{k} X \cdot e^1 = \frac{1}{k} X_1$ is defined as *position vector*.

Definition 24. $P^*(k,n)$ is a set of position vectors.

Lemma 2. *For point matrix $X = PO$ where P and O are position and orientation matrix respectively, $p = \frac{1}{k} X_1 = \frac{1}{k} P_1$.*

Proof of lemma 2.

$$\begin{aligned} p^i &= \frac{1}{k} X^i{}_1 && \text{definition 23} \\ &= \frac{1}{k} \sum_j P^i{}_j O^j{}_1 && \text{Equation (5)} \\ &= \frac{1}{k} P^i{}_1 && \text{definition 18} \end{aligned}$$

$$\begin{aligned} p \\ &= \frac{1}{k} P_1 \end{aligned}$$ □

Lemma 3. *Given position parameter θ, position vector can be evaluated as the following.*

$$\psi_0^{-1} : \theta \mapsto p = \frac{1}{k} \begin{pmatrix} \prod_{j \in \{ 1..n \}} \cos_k \theta^j \\ \sin_k \theta^{i-1} \prod_{j \in \{ i..n \}} \cos_k \theta^j \\ \sin_k \theta^n \end{pmatrix}.$$

Proof of lemma 3. Simplify lemma 2 and definition 16. □

Lemma 4. *Given position vector p, position parameter can be calculated as the following.*

$$\psi_0 : p \mapsto \theta = \begin{pmatrix} \arcsin_k^{\operatorname{sgn} p^1} \frac{kp^2}{\prod_{j \in \{ 2..n \}} \cos_k \theta^j} \\ \arcsin_k \frac{kp^{i+1}}{\prod_{j \in \{ i+1..n \}} \cos_k \theta^j} \\ \arcsin_k kp^{n+1} \end{pmatrix}$$

$$\in \begin{cases} P \to \left(-\frac{\pi}{k}, \frac{\pi}{k}\right] \times \left[-\frac{1}{2}\frac{\pi}{k}, \frac{1}{2}\frac{\pi}{k}\right]^{n-1} & \text{if } k > 0 \\ P \to \mathbb{R}^n & \text{if } k \leq 0 \end{cases}$$

where $\cos_k \left(\arcsin_k^{\pm}(x)\right) = \pm \cos_k (\arcsin_k (x))$.

Proof of lemma 4. From lemma 3,

$$kp = \begin{pmatrix} \prod_{j \in \{ 1..n \}} \cos_k \theta^j \\ \sin_k \theta^{i-1} \prod_{j \in \{ i..n \}} \cos_k \theta^j \\ \sin_k \theta^n \end{pmatrix}$$

$$\sin_k \theta^n = k\theta^{n+1}$$

$$\theta^n = \arcsin_k k\theta^{n+1}$$

$$\sin_k \theta^{i-1} \prod_{j \in \{ i..n \}} \cos_k \theta^j = k\theta^i$$

$$\sin_k \theta^{i-1} = \frac{k\theta^i}{\prod_{j \in \{ i..n \}} \cos_k \theta^j}$$

$$\theta^{i-1} = \arcsin_k \frac{k\theta^i}{\prod_{j \in \{ i..n \}} \cos_k \theta^j}$$

$$\theta^i = \arcsin_k \frac{k\theta^{i+1}}{\prod_{j \in \{ i+1..n \}} \cos_k \theta^j}$$

$$p^1 = \prod_{j \in \{ 1..n \}} \cos_k \theta^j$$

$$
\begin{aligned}
p^1 &= \cos_k \theta^1 \prod_{j \in \{\,2..n\,\}} \cos_k \theta^j \\
\operatorname{sgn} p^1 &= \operatorname{sgn} \cos_k \theta^1 \prod_{j \in \{\,2..n\,\}} \operatorname{sgn} \cos_k \theta^j \\
\operatorname{sgn} p^1 &= \operatorname{sgn} \cos_k \theta^1 \prod_{j \in \{\,2..n\,\}} 1 \\
\operatorname{sgn} p^1 &= \operatorname{sgn} \cos_k \theta^1 \\
\theta &= \begin{pmatrix} \arcsin_k^{\operatorname{sgn} p^1} \frac{kp^2}{\prod_{j \in \{\,2..n\,\}} \cos_k \theta^j} \\ \arcsin_k \frac{kp^{i+1}}{\prod_{j \in \{\,i+1..n\,\}} \cos_k \theta^j} \\ \arcsin_k kp^{n+1} \end{pmatrix}
\end{aligned}
$$

□

Lemma 5.

$$\Psi = \{\, \psi \mid \psi^{-1} \in S^n \to P : \theta \mapsto P_{k,n}(\theta + x) \ \textit{for}\ x \in \mathbb{R}^n \,\}$$

is a coordinate chart of a C^∞ differential structure on P for

$$S = \begin{cases} \left(-\frac{1}{2}\frac{\pi}{k}, +\frac{1}{2}\frac{\pi}{k}\right) & k > 0 \\ \mathbb{R} & k \leq 0 \end{cases}$$

Proof of lemma 5. From definition 4, It is sufficient to shows that

1. R_ψ is an open subset of real vector space (defined),
2. $\bigcup_{\psi \in \Psi} D_\psi = P$ (obvious),
3. transition map is in differentability class C^∞.

To show that $\bigcup_{\psi \in \Psi} D_\psi = P$.

$$
\begin{aligned}
M \in P &\implies \exists \theta_0, M = P_{k,n}(\theta_0) \\
&\implies \exists \theta_0, M = P_{k,n}(0 + \theta_0) \\
&\implies M \in R_{\psi^{-1}} \\
&\implies M \in D_\psi \\
&\implies M \in \bigcup_{\psi \in \Psi} D_\psi
\end{aligned}
$$

$$P \subset \bigcup_{\psi\in\Psi} D_\psi$$

$$\begin{aligned}
M \in \bigcup_{\psi\in\Psi} D_\psi &\implies \exists\psi \in \Psi, M \in D_\psi \\
&\implies \exists\psi \in \Psi, M \in R_{\psi^{-1}} \\
&\implies \exists x_0 \exists\theta \in S^n, M = P_{k,n}\left(\theta + x_0\right) \\
&\implies \exists x_0, M = P_{k,n}\left(0 + x_0\right) \\
&\implies \exists x_0, M = P_{k,n}\left(x_0\right) \\
&\implies M \in P
\end{aligned}$$

$$\bigcup_{\psi\in\Psi} D_\psi \subset P$$

$$\bigcup_{\psi\in\Psi} D_\psi = P$$

To show that every transition map is in differentability class C^∞.

Consider $\psi_1, \psi_2 \in \Psi$ and $x_1, x_2 \in \mathbb{R}^n$ where

$$\psi_i^{-1} \in S^n \to R : \theta \mapsto P\left(\theta + x_i\right).$$

If $\psi_1^{-1}\left(\theta_1\right) = \psi_2^{-1}\left(\theta_2\right)$, let $\phi_i = \theta_i + x_i$.

$$\frac{1}{k}\begin{pmatrix} \prod_{j\in\{\,1..n\,\}} \cos_k {\phi_1}^j \\ \sin_k {\phi_1}^{i-1} \prod_{j\in\{\,i..n\,\}} \cos_k {\phi_1}^j \\ \sin_k {\phi_1}^n \end{pmatrix} = \frac{1}{k}\begin{pmatrix} \prod_{j\in\{\,1..n\,\}} \cos_k {\phi_2}^j \\ \sin_k {\phi_2}^{i-1} \prod_{j\in\{\,i..n\,\}} \cos_k {\phi_2}^j \\ \sin_k {\phi_2}^n \end{pmatrix} \qquad \text{lemma 3}$$

$$\begin{pmatrix} \prod_{j\in\{\,1..n\,\}} \cos_k {\phi_1}^j \\ \sin_k {\phi_1}^{i-1} \prod_{j\in\{\,i..n\,\}} \cos_k {\phi_1}^j \\ \sin_k {\phi_1}^n \end{pmatrix} = \begin{pmatrix} \prod_{j\in\{\,1..n\,\}} \cos_k {\phi_2}^j \\ \sin_k {\phi_2}^{i-1} \prod_{j\in\{\,i..n\,\}} \cos_k {\phi_2}^j \\ \sin_k {\phi_2}^n \end{pmatrix}$$

If $k \geq 0$, by mathematical induction, $\phi_1 = m\frac{\pi}{k} \pm \phi_2$.

Otherwise, by mathematical induction, $\phi_1 = \phi_2$.

Hence, the transition map $\tau_{1,2} = \psi_2 \circ \psi_1^{-1}$ is in the form of $\theta \mapsto c \pm \theta$ and is in differentability class C^∞.

$\square$

Locus of position vector

Lemma 6. *For $k > 0$, P^* is a $(n+1)$-sphere of radius k^{-1}.*

Proof of lemma 6. Simplify lemma 3 using theorem 1. □

Lemma 7. *For $k < 0$, P^* is a forward sheet of a two-sheeted $(n+1)$-hyperboloid of radius k^{-1}.*

Proof of lemma 7. Simplify lemma 3 using theorem 1. □

Lemma 8. *For $k \to 0$, P^* is a n-Euclidean manifold at infinity.*

Proof of lemma 8. Using limits. □

Geometric properties

Embedding

Definition 25. Let $N = (Q, g)$ be a n-dimension Riemannian manifold on position parameter space with such inner product g that the map $\cdot \mapsto \frac{1}{k} \cdot e^1 \in P \to P^*$ is an isometric embedding to $(n+1)$-Euclidean manifold.

Lemma 9. *Position parameter θ^i is associated to the following vector in position vector space.*

$$\frac{\partial}{\partial \theta^i} = \begin{pmatrix} -k \tan_k^* \theta^i p^j \\ k \frac{1}{\tan_k \theta^i} p^{i+1} \\ 0 \end{pmatrix} \frac{\partial}{\partial p^j}$$

Proof of lemma 9.

$$\begin{aligned}
\frac{\partial}{\partial \theta^i} &= \frac{\partial p^j}{\partial \theta^i} \frac{\partial}{\partial p^j} \\
&= \frac{\partial}{\partial \theta^i} \left[\frac{1}{k} \begin{pmatrix} \prod_{l \in \{\,1..n\,\}} \cos_k \theta^l \\ \sin_k \theta^{j-1} \prod_{l \in \{\,j..n\,\}} \cos_k \theta^l \\ \sin_k \theta^n \end{pmatrix} \right] \frac{\partial}{\partial p^j} && \text{lemma 3} \\
&= \frac{1}{k} \frac{\partial}{\partial \theta^i} \left[\begin{pmatrix} \prod_{l \in \{\,1..n\,\}} \cos_k \theta^l \\ \sin_k \theta^{j-1} \prod_{l \in \{\,j..n\,\}} \cos_k \theta^l \\ \sin_k \theta^n \end{pmatrix} \right] \frac{\partial}{\partial p^j}
\end{aligned}$$

$$= \frac{1}{k}\left[\begin{pmatrix} -k\sin_k^*\theta^i\prod_{l\in\{1..n\}/\{i\}}\cos_k\theta^l \\ -k\sin_k^*\theta^i\sin_k\theta^{j-1}\prod_{l\in\{j..n\}/\{i\}}\cos_k\theta^l \\ k\cos_k\theta^i\prod_{l\in\{i+1..n\}}\cos_k\theta^l \\ 0 \end{pmatrix}\right]\frac{\partial}{\partial p^j} \qquad \text{proposition 3}$$

$$= \begin{pmatrix} -\tan_k^*\theta^i\prod_{l\in\{1..n\}}\cos_k\theta^l \\ -\tan_k^*\theta^i\sin_k\theta^{j-1}\prod_{l\in\{j..n\}}\cos_k\theta^l \\ \frac{1}{\tan_k\theta^i}\prod_{l\in\{i+1..n\}}\cos_k\theta^l \\ 0 \end{pmatrix}\frac{\partial}{\partial p^j}$$

$$= \begin{pmatrix} -k\tan_k^*\theta^i p^j \\ k\frac{1}{\tan_k\theta^i}p^{i+1} \\ 0 \end{pmatrix}\frac{\partial}{\partial p^j} \qquad \text{lemma 3}$$

□

Lemma 10. *The metric tensor of N is*

$$g_{ij} = \begin{cases} (1-\operatorname{sgn} k)\tan_k^2(\theta^a)\prod_{1\le j}\cos_k^2\theta^j + \prod_{a<j}\cos_k^2\theta^j & \textit{if } i=j, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof of lemma 10.

$$\begin{aligned} g_{ab}\left[\frac{\partial}{\partial\theta^i}\right] &= \sum_{l,m=1}^{n+1}\frac{\partial p^l}{\partial\theta^a}g_{lm}\left[\frac{\partial}{\partial p^i}\right]\frac{\partial p^m}{\partial\theta^b} \\ &= \sum_{l=1}^{n+1}\frac{\partial p^l}{\partial\theta^a}\frac{\partial p^l}{\partial\theta^b} \end{aligned}$$

If $a<b$,

$$g_{ab} = \begin{bmatrix} -k\tan_k^*(\theta^a)p^j \\ k\frac{1}{\tan_k(\theta^a)}p^{a+1} \\ 0 \end{bmatrix}\cdot\begin{bmatrix} -k\tan_k^*\left(\theta^b\right)p^j \\ k\frac{1}{\tan_k\left(\theta^b\right)}p^{b+1} \\ 0 \end{bmatrix}$$

$$= \sum k^2\tan_k^*(\theta^a)\tan_k^*\left(\theta^b\right){p^j}^2 - k^2\frac{\tan_k^*\left(\theta^b\right)}{\tan_k(\theta^a)}{p^{a+1}}^2$$

$$= \tan_k\left(\theta^b\right)\tan_k\left(\theta^a\right)\prod_{a\le j\le n+1}\cos_k^2\theta^j\left(\prod_{1\le j<a}\cos_k^2\theta^j + \sum\sin_k^2\theta^i\prod_{i<j<a}\cos_k^2\theta^j - \operatorname{sgn}k\right)$$

$$= \tan_k\left(\theta^b\right)\tan_k\left(\theta^a\right)\prod_{a\le j\le n+1}\cos_k^2\theta^j\left(\prod_{1\le j<a}\cos_k^2\theta^j + \operatorname{sgn}k\left(1-\prod_{1\le j<a}\cos_k^2\theta^j\right) - \operatorname{sgn}k\right)$$

$$g_{ab} = 0.$$

If $a > b$, $g_{ab} = g_{ba} = 0$.

If $a = b$,

$$g_{aa} = \begin{pmatrix} -k\tan_k^*\theta^a p^j \\ k\frac{1}{\tan_k\theta^a}p^{a+1} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -k\tan_k^*\theta^a p^j \\ k\frac{1}{\tan_k\theta^a}p^{a+1} \\ 0 \end{pmatrix}$$

$$= \sum_{j=1}^{a}\left(kp^j\tan_k^*\theta^a\right)^2 + \left(kp^{a+1}\cot_k\theta^a\right)^2$$

$$= \sum_{j=1}^{a}\left(kp^j\tan_k\theta^a\right)^2 + \left(kp^{a+1}\cot_k\theta^a\right)^2$$

$$= \sin_k^2\left(\theta^a\right)\prod_{a<j}\cos_k^2\theta^j\left(\prod_{1\le j<a}\cos_k^2\theta^j + \sum_{1\le i<a}\sin_k^2\theta^i\prod_{i<j<a}\cos_k^2\theta^j + \cot_k^2\left(\theta^a\right)\right)$$

$$= \sin_k^2\left(\theta^a\right)\prod_{a<j}\cos_k^2\theta^j\left(\prod_{1\le j<a}\cos_k^2\theta^j + \sum_{1\le i<a}\sin_k^2\theta^i\prod_{i<j<a}\cos_k^2\theta^j + \cot_k^2\left(\theta^a\right)\right)$$

$$= \sin_k^2\left(\theta^a\right)\prod_{a<j}\cos_k^2\theta^j\left(\prod_{1\le j<a}\cos_k^2\theta^j - \operatorname{sgn}k\prod_{1\le j<a}\cos_k^2\theta^j + \operatorname{sgn}k\prod_{1\le j<a}\cos_k^2\theta^j + \sum_{1\le i<a}\sin_k^2\theta^i\prod_{i<j<a}\cos_k^2\right.$$

$$= \sin_k^2\left(\theta^a\right)\prod_{a<j}\cos_k^2\theta^j\left(\prod_{1\le j<a}\cos_k^2\theta^j - \operatorname{sgn}k\prod_{1\le j<a}\cos_k^2\theta^j + \operatorname{sgn}k + \cot_k^2\left(\theta^a\right)\right)$$

$$= \sin_k^2\left(\theta^a\right)\prod_{a<j}\cos_k^2\theta^j\left(\prod_{1\le j<a}\cos_k^2\theta^j - \operatorname{sgn}k\prod_{1\le j<a}\cos_k^2\theta^j + \operatorname{sgn}k + \csc_k^2\left(\theta^a - \operatorname{sgn}k\right)\right)$$

$$= \sin_k^2\left(\theta^a\right)\prod_{a<j}\cos_k^2\theta^j\left(\left(1-\operatorname{sgn}k\right)\prod_{1\le j<a}\cos_k^2\theta^j + \csc_k^2\left(\theta^a\right)\right)$$

$$= \left(1-\operatorname{sgn}k\right)\tan_k^2\left(\theta^a\right)\prod_{j}\cos_k^2\theta^j + \prod_{a<j}\cos_k^2\theta^j$$

□

Curvature

Curvature (Method I)

This method may be easier to generalize to Model II where each direction can have partially independent curvature (or even Model III where extrinsic curvature become a thing). But it may be challenging to define Gauss map properly.

Lemma 11. *Given a position parameter θ, the tangent vector in position vector space can be calculated as follows*

$$\nu(p) = \begin{cases} \begin{bmatrix} kp^1 \\ +kp^i \end{bmatrix} & k > 0 \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} & k = 0 \\ \frac{1}{\sqrt{-1+2(kp^1)^2}} \begin{bmatrix} kp^1 \\ -kp^i \end{bmatrix} & k < 0 \end{cases}.$$

Proof of lemma 11. From lemmas 6 to 8,

$$p \in P \iff \begin{cases} \sum_i {p^i}^2 = k^{-2} & k > 0, \\ p^1 = k^{-1} & k \to 0, \\ {p^1}^2 - \sum_{1<i} {p^i}^2 = k^{-2} \wedge p^1 > 0 & k < 0. \end{cases}$$

Let

$$F(p) = \begin{cases} \sum_i {p^i}^2 - k^{-2} & k > 0, \\ p^1 - k^{-1} & k \to 0, \\ {p^1}^2 - \sum_{1<i} {p^i}^2 - k^{-2} & k < 0. \end{cases}$$

$$\begin{aligned} n &= l \nabla F(p) \\ &= l \begin{cases} \nabla \sum_i {p^i}^2 - k^{-2} & k > 0, \\ \nabla p^1 - k^{-1} & k \to 0, \\ \nabla {p^1}^2 - \sum_{1<i} {p^i}^2 - k^{-2} & k < 0. \end{cases} \end{aligned}$$

$$= l\begin{cases}\begin{pmatrix}2p^i\end{pmatrix} & k>0,\\ \begin{pmatrix}1\\0\end{pmatrix} & k\to 0,\\ \begin{pmatrix}-2p^1\\2p^i\end{pmatrix} & k<0.\end{cases}$$

$$\hat{n} = \pm\begin{cases}\frac{1}{||p||}\begin{pmatrix}p^i\end{pmatrix} & k>0,\\ \begin{pmatrix}1\\0\end{pmatrix} & k\to 0,\\ \frac{1}{||p||}\begin{pmatrix}-p^1\\p^i\end{pmatrix} & k<0.\end{cases}$$

$$= \pm\begin{cases}\frac{1}{k^{-1}}\begin{pmatrix}p^i\end{pmatrix} & k>0,\\ \begin{pmatrix}1\\0\end{pmatrix} & k\to 0,\\ \frac{1}{\sqrt{-1+2(kp^1)^2}}\begin{pmatrix}-p^1\\p^i\end{pmatrix} & k<0.\end{cases}$$

$$\nu = \hat{n} := \begin{cases}\begin{pmatrix}kp^i\end{pmatrix} & k>0,\\ \begin{pmatrix}1\\0\end{pmatrix} & k\to 0,\\ \frac{1}{\sqrt{-1+2(kp^1)^2}}\begin{pmatrix}p^1\\-p^i\end{pmatrix} & k<0.\end{cases} \qquad \square$$

Lemma 12.

$$L(p) = \begin{cases} -kI & k > 0, \\ 0 & k \to 0, \\ \frac{1}{||p||}\left(\frac{1}{||p||^2} pp^T \operatorname{diag}(-1, 1 \ldots, 1) - I\right) & k < 0. \end{cases}$$

Proof of lemma 12.

$$\begin{aligned} \frac{\partial}{\partial p^j}\frac{1}{||p||} &= \frac{\partial}{\partial ||p||}\frac{1}{||p||}\frac{\partial}{\partial p^j}||p|| \\ &= -\frac{1}{||p||^2}\frac{\partial}{\partial p^j}\sqrt{\sum_i {p^i}^2} \\ &= -\frac{1}{||p||^2}\frac{\partial}{\partial \sum_i {p^i}^2}\sqrt{\sum_i {p^i}^2}\frac{\partial}{\partial p^j}\sum_i {p^i}^2 \\ &= -\frac{1}{||p||^2}\frac{1}{2\sqrt{\sum_i {p^i}^2}}2p^j \\ &= -\frac{1}{||p||^2}\frac{1}{2\,||p||}2p^j \\ &= -\frac{p^j}{||p||^3} \end{aligned}$$

From definition 12,

$$\begin{aligned} L(p) &= -(D\nu \circ (Df)^{-1})(p) \\ &= -(D\nu)(p) \\ &= -\left(\frac{\partial}{\partial p^j}\nu^i\bigg|_p\right)_{i,j} \end{aligned}$$

$$(L(p))_{i,j} = \begin{cases} -\frac{\partial}{\partial p^j} kp^i & k > 0, \\ -\frac{\partial}{\partial p^j} 1 & k \to 0 \text{ and } i = 1, \\ -\frac{\partial}{\partial p^j} 0 & k \to 0 \text{ and } i \neq 1, \\ -\frac{\partial}{\partial p^j}\left(\frac{1}{||p||}p^1\right) & k < 0 \text{ and } i = 1, \\ -\frac{\partial}{\partial p^j}\left(-\frac{1}{||p||}p^i\right) & k < 0 \text{ and } i \neq 1, \end{cases}$$

$$= \begin{cases} -k & k > 0 \text{ and } i = j, \\ 0 & k > 0 \text{ and } i \neq j, \\ 0 & k \to 0, \\ -\left(\frac{1}{||p||}\frac{\partial}{\partial p^j}p^1 + p^1\frac{\partial}{\partial p^j}\frac{1}{||p||}\right) & k < 0 \text{ and } i = 1, \\ \left(\frac{1}{||p||}\frac{\partial}{\partial p^j}p^i + p^i\frac{\partial}{\partial p^j}\frac{1}{||p||}\right) & k < 0 \text{ and } i \neq 1, \end{cases}$$

$$= \begin{cases} -k & k > 0 \text{ and } i = j, \\ 0 & k > 0 \text{ and } i \neq j, \\ 0 & k \to 0, \\ -\frac{1}{||p||} + \frac{{p^1}^2}{||p||^3} & k < 0 \text{ and } i = j = 1, \\ \frac{p^1p^j}{||p||^3} & k < 0 \text{ and } j \neq i = 1, \\ \frac{1}{||p||} - \frac{{p^i}^2}{||p||^3} & k < 0 \text{ and } i = j \neq 1, \\ -\frac{p^ip^j}{||p||^3} & k < 0,\ i \neq j \text{ and } i \neq 1, \end{cases}$$

For $k > 0$, $L(p) = -kI$.

For $k \to 0$, $L(p) = 0$.

For $k < 0$, $L(p) = \frac{1}{||p||}\left(\frac{1}{||p||^2}pp^T \operatorname{diag}(-1, 1 \ldots, 1) - I\right)$. □

Lemma 13.

Proof of lemma 13. The proof is left as an exercise to the other author.

For $k > 0$,

$$\begin{aligned} 0 &= \det L(p) - \lambda I \\ &= \det -kI - \lambda I \\ &= \det(-k - \lambda)I \\ &= (-k - \lambda)^{n+1} \det I \\ &= (-k - \lambda)^{n+1} \\ \lambda &\in \{ -k, \ldots, -k \} \end{aligned}$$

For $k < 0$,

$$
\begin{aligned}
0 &= \det L(p) - \lambda I \\
&= \det \frac{1}{||p||} \left(\frac{1}{||p||^2} pp^T \operatorname{diag}(-1, 1 \dots, 1) - I \right) - \lambda I \\
&= \det \frac{1}{||p||} \left(\frac{1}{||p||^2} pp^T \operatorname{diag}(-1, 1 \dots, 1) - I \right) - \left(\lambda' - \frac{1}{||p||} \right) I \qquad \lambda' := \lambda + \frac{1}{||p||} \\
&= \det \frac{1}{||p||^3} pp^T \operatorname{diag}(-1, 1 \dots, 1) - \frac{1}{||p||} I - \left(\lambda' - \frac{1}{||p||} \right) I \\
&= \det \frac{1}{||p||^3} pp^T \operatorname{diag}(-1, 1 \dots, 1) - \lambda' I \\
&= \det \frac{1}{||p||^3} pp^T \operatorname{diag}(-1, 1 \dots, 1) - \frac{1}{||p||^3} \lambda'' I \qquad \lambda'' := ||p||^3 \lambda' \\
&= \frac{1}{||p||^{3(n+1)}} \det pp^T \operatorname{diag}(-1, 1 \dots, 1) - \lambda'' I \\
&= \det pp^T \operatorname{diag}(-1, 1 \dots, 1) - \lambda'' I \\
&= \lambda''^n \left(\lambda'' + p^{12} - \sum_{i \neq 1} p^{i2} \right) \\
0 &= \lambda''^n \left(\lambda'' + \frac{1}{k} \right) \\
\lambda'' &\in \left\{ -\frac{1}{k}, 0, \dots, 0 \right\} \\
\lambda' &\in \left\{ -\frac{1}{k\,||p||^3}, 0, \dots, 0 \right\} \\
\lambda &\in \left\{ -\frac{1}{k\,||p||^3} - \frac{1}{||p||}, -\frac{1}{||p||}, \dots, -\frac{1}{||p||} \right\} \\
&= \left\{ -\frac{1 + ||p||^2}{k\,||p||^3}, -\frac{1}{||p||}, \dots, -\frac{1}{||p||} \right\} \\
&= \left\{ -\frac{2k^2 p^{1^2}}{k\,||p||^3}, -\frac{1}{||p||}, \dots, -\frac{1}{||p||} \right\} \\
&= \left\{ -\frac{2k p^{1^2}}{||p||^3}, -\frac{1}{||p||}, \dots, -\frac{1}{||p||} \right\} \\
&= \left\{ k^2\,||p||\, \frac{-2p^{1^2}}{k\,||p||^4}, -\frac{1}{||p||}, \dots, -\frac{1}{||p||} \right\} \\
&= \left\{ k^2\,||p||\, \frac{-2p^{1^2}}{k\,||p||^4}, -\frac{1}{||p||}, \dots, -\frac{1}{||p||} \right\}
\end{aligned}
$$

$\square$

Curvature (Method II)

This method may be easier to be done (despite the fact that it never finished). But it raises problems when trying to generalize e.g. dealing with extrinsic curvature (which may be introduced in Model III if not to mess with other basis geometries).

Lemma 14.

Lemma 15.

Curvature (Conclusion)

Lemma 16.

CurvatureParameter. It can be seen that $\sec(p) = \kappa = \operatorname{sgn}(k)k^2$. Hence, when provided κ, k can be determined and used to evaluate the model.

The Model

Definition 1.1. For any parameter κ, n,

$$\mathbb{M} := M$$

$$\otimes_{\mathbb{M}} := \cdot$$

$$\varphi_{\mathbb{M}} := X \mapsto \theta$$

$$g_{\mathbb{M}} := g$$

$$\mathbb{P}_{\mathbb{M}} \equiv R$$

$$\mathbb{T}_{\mathbb{M}} \equiv M$$

for injective smooth function $K : \kappa \mapsto k = \operatorname{sgn} \kappa \sqrt{|\kappa|} \in \mathbb{R} \to \mathbb{R}$.

Assertion 1.1.

Assertion 1.2.

Future plan

Model II

It is very simple to be able to model composite geometries e.g. $S^2 \times E$ by tensor product of the existing model. But to be able to merge them as smooth model may be challenging since not all combination of basis curvature have their own intrinsic geometry. So it may be to find independent variable for each basis or to introduce extrinsic curvature (Model A).

Model B

It is known that E emerged at $n \geq 1$ while S and H emerged at $n \geq 2$ and there's more complex pure geometries than these that emerged in higher dimension. It is interesting and challenging to explore such geometries and prove whether the curvature still works as indicator in such geometries or are there any patterns for their symmetries.

Model A

This model is based on curvature and mostly just 3 basis geometries and extrinsic curvature which seems to be interesting despite some critical result in some combination e.g. $S^1 \times S^1$ vs S^2. It can be even more challenging to have variable curvature with respect to other intrinsic position.

References

Lee, J. M. (2013). *Introduction to smooth manifolds* (2nd ed., Vol. 218). Springer.

Wolfgang, K. (2006). *Differential geometry: Curves - surfaces - manifolds* (B. Hunt, Trans.; 2nd ed., Vol. 16). American Mathematical Society.

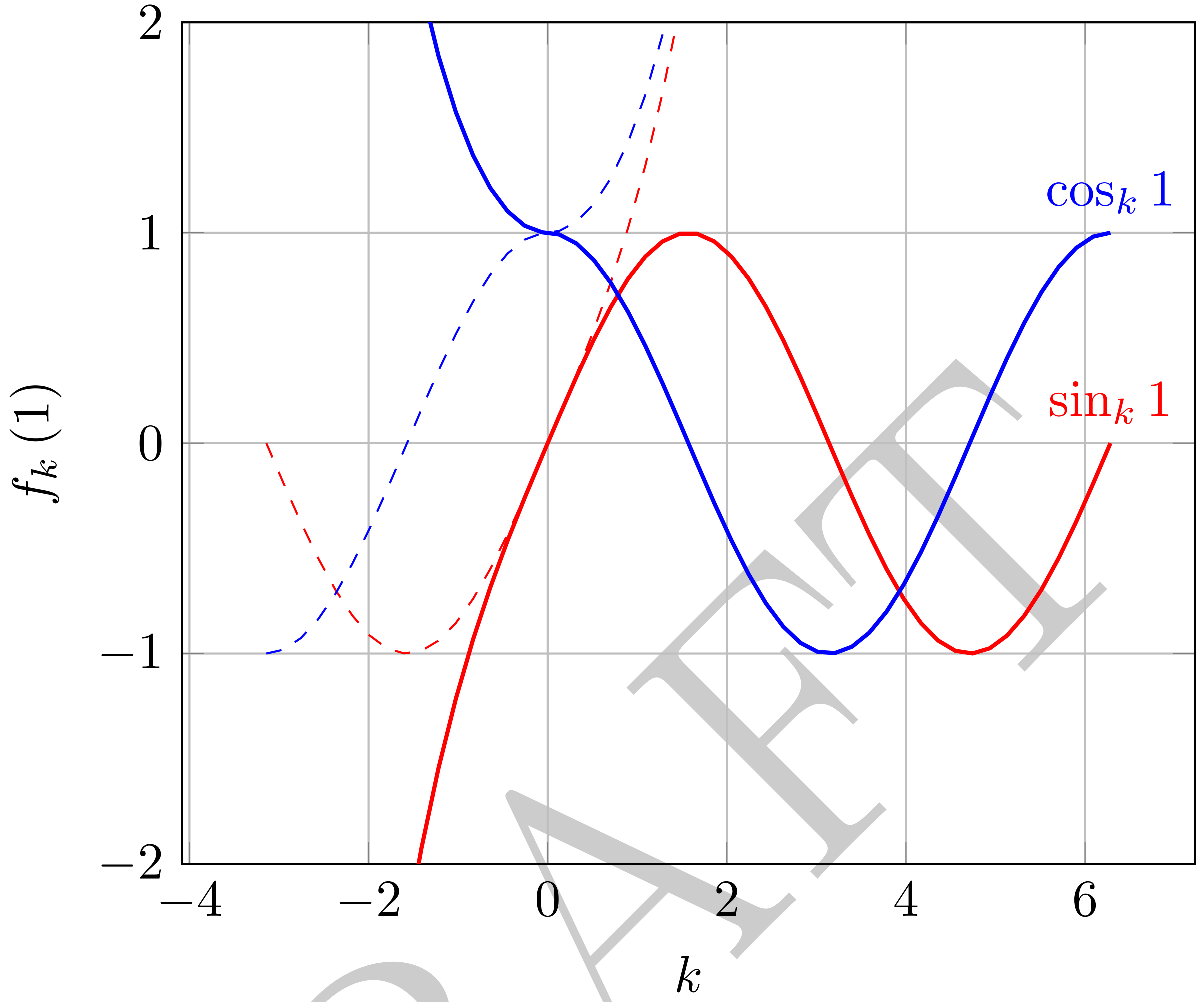

Figure 1

Generalized trigonometric functions as function of k

Note. This graph shows the value of generalized trigonometric functions as solid line and trigonometric and hyperbolic functions in the unused domain as dashed line.

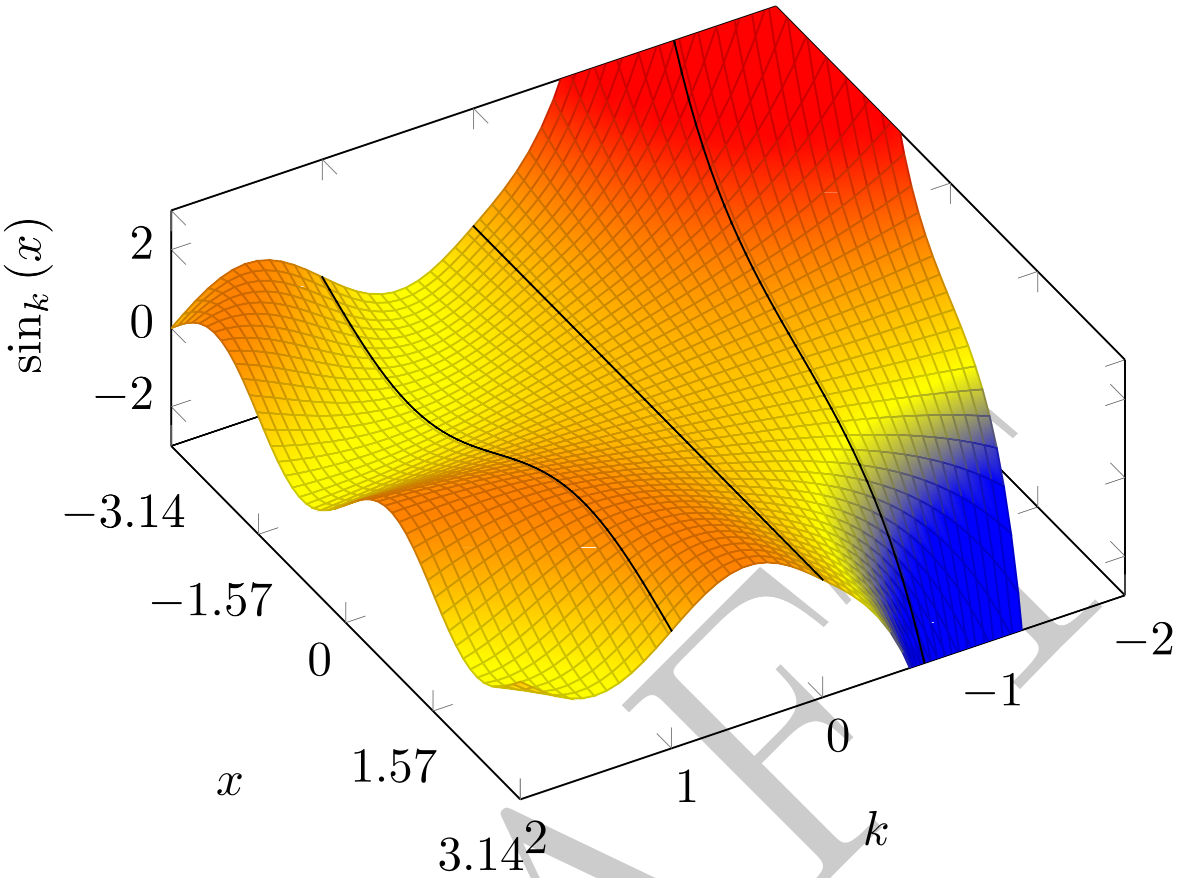

Figure 2

Generalized sine function

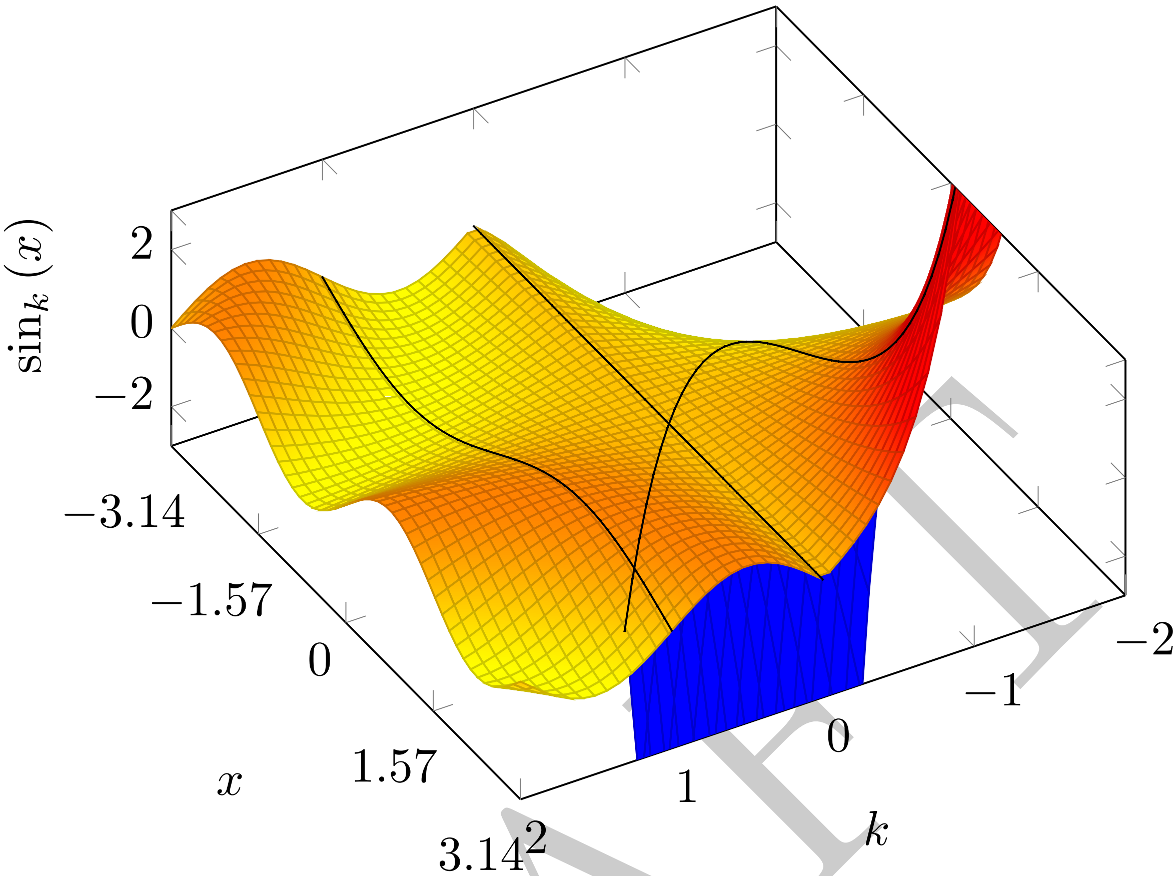

Figure 3

Generalized sine function variant

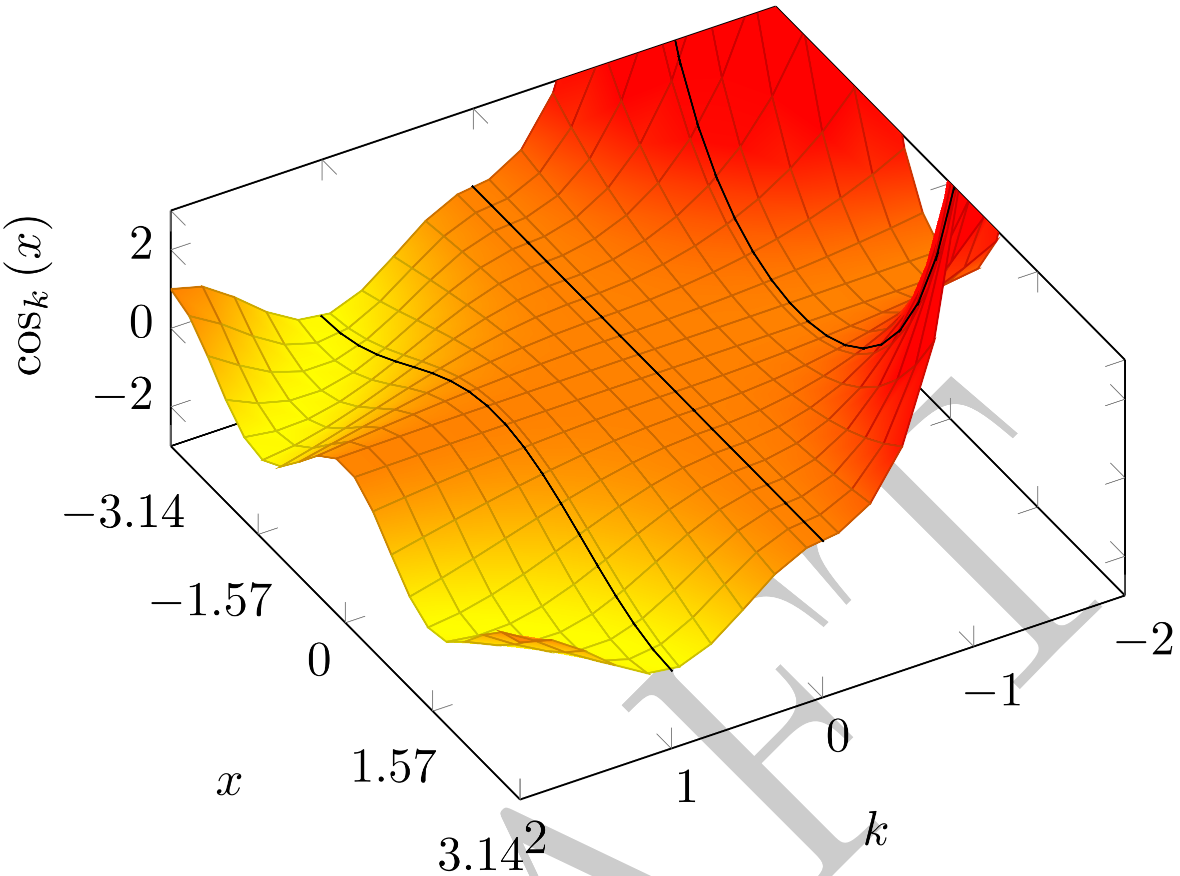

Figure 4

Generalized cosine function