Prove of Constant Radius Model

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Abstract

This is the first generation of the model. It's a draft. Who would even care to read the abstract when it's not done anyways?

Keywords: None 1, None 2, None 3

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Rewrite note

This proof will be organized using the following guideline

1. reviewing existing definitions and theorem
2. proving certain property of existing mathematical objects
3. defining the conditions of the model
4. formulating of the model
5. parametrizing the model
6. asserting defined property of the model

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Prelimitary

All citation must goes here, Authors must copy statement they wanted to cite according to the AMS guideline, Any theorem lemma definitions and others must be in theorem environment if possible (see amsthm user-guide for more details), Authors then refer to the statement using \label{} and \cref{}.

Uncited prelimitary

$$C = AB \iff C^i{}_j = \sum_k A^i{}_k B^k{}_j \tag{1}$$

Definition 1. KleinGeometry

Example 1.1. KleinGeometryExamples (I)

Objective

Objective. The objective is to construct

1. a set M

2. an n-dimensional C^{∞} differential structure on set M with chart $\varphi \in M \to \mathbb{R}^n$ (which give it a manifold structure)

3. an algebralic structure (group) on M with operation $\otimes$ (since identity, invertibility, and associativity apply to distance-preserving (curvature-preserving) transformations in the geometry)

4. an metric space on M with metric $d \in M \times M \to \mathbb{R}$
(to be determined) such that

- all of its two-dimensional linear subspaces have sectional curvature of κ
- $d\left(\varphi^{-1}(x), \varphi^{-1}(y)\right)$ is infinitely differentiable by κ for all x and y in $\mathbb{R}^n$

given a natural number n and a real number κ.

Algorithm. (see Figure 1)

Definition 2. *Set of points* $\mathbb{P}_M$ is defined as set of equivalence classes of elements of M with the relation r, where $r := \{ (x, y) \in M \times M \mid d(x, y) = 0 \}$ (Generally speaking, distance between those two elements are zero).

Definition 3. *Set of transformations* $\mathbb{T}_M$ is defined as
$\mathbb{T}_M := \{ f \in M \to M \mid \exists y \in M, f(x) = y \otimes x \}$.

Definition 4. *Principal group* and *subgroup* of the model is defined as

- Let $\mathbb{G}_M$ be a group on set M together with binary operation $\otimes$.
- Let $\mathbb{H}_M$ be a group on set $P \in \mathbb{P}_M$ together with binary operation $\otimes$.

respectively.

Conjecture 1. *If parameters* (κ, n) *is associated with Klein geometry* (G, H) *then* $\mathbb{G}_M \cong G$ *and* $\mathbb{H}_M \cong H$.

From example 1.1,

- *For* $\kappa > 0$, $\mathbb{G}_M \cong \mathrm{O}(n+1)$ *and* $\mathbb{H}_M \cong \mathrm{O}(n)$.
- *For* $\kappa = 0$, $\mathbb{G}_M \cong \mathrm{Euc}(n)$ *and* $\mathbb{H}_M \cong \mathrm{O}(n)$.
- *For* $\kappa < 0$, $\mathbb{G}_M \cong \mathrm{O}^{+}(1, n)$ *and* $\mathbb{H}_M \cong \mathrm{O}(n)$.

Model Foundation

Construction of the set M...

Trigonometry

Definition 5. *Generalized trigonometric functions* $f_k : \mathbb{R} \to \mathbb{R}$ are defined as

$$f_k(\theta) := \begin{cases} g(k\theta) & \text{if } k \geq 0, \\ h(k\theta) & \text{otherwise,} \end{cases}$$

where g (resp. h) are the associated trigonometric (resp. hyperbolic) function.

Example 5.1. *Generalized sine function* (see Figure 3) is defined as

$$\sin_k \theta := \begin{cases} \sin(k\theta) & \text{if } k \geq 0, \\ \sinh(k\theta) & \text{otherwise.} \end{cases}$$

Example 5.2. *Generalized cosine function* (see Figure 4) is defined as

$$\cos_k \theta := \begin{cases} \cos(k\theta) & \text{if } k \geq 0, \\ \cosh(k\theta) & \text{otherwise.} \end{cases}$$

Example 5.3. *Generalized tangent function* (see Figure 5) is defined as

$$\tan_k \theta := \begin{cases} \tan(k\theta) & \text{if } k \geq 0, \\ \tanh(k\theta) & \text{otherwise.} \end{cases}$$

Corollary 1. *Generalized sine and cosine function are infinitely differentiable.*

Proof of corollary 1. The proof is left as an exercise to the other author.

By using mathematical induction, it can be proved that

$$\sin_k^{(4n)} \theta = \theta^{4n} \sin_k \theta$$

$$\cos_k^{(4n)} \theta = \theta^{4n} \cos_k \theta$$

$$\sin_k^{(4n+1)} \theta = \theta^{4n+1} \cos_k \theta$$

$$\cos_k^{(4n+1)} \theta = -\operatorname{sgn}(k) \theta^{4n+1} \sin_k \theta$$

$$\sin_k^{(4n+2)} \theta = -\operatorname{sgn}(k) \theta^{4n+2} \sin_k \theta$$

$$\cos_k^{(4n+2)} \theta = -\operatorname{sgn}(k) \theta^{4n+2} \cos_k \theta$$

$$\sin_k^{(4n+3)} \theta = -\operatorname{sgn}(k) \theta^{4n+3} \cos_k \theta$$

$$\cos_k^{(4n+3)} \theta = \theta^{4n+3} \sin_k \theta$$

Hence, generalized sine and cosine function can be infinitely differentiable. □

Theorem 1 (Pythagorean's identity equivalence)**.**

$$\begin{aligned}
\cos_k^2 \theta + \operatorname{sgn} k \sin_k^2 \theta &= 1 \\
1 + \operatorname{sgn} k \tan_k^2 \theta &= \sec_k^2 \theta \\
\cot_k^2 \theta + \operatorname{sgn} k &= \csc_k^2 \theta
\end{aligned}$$

Proof of theorem 1. The proof is left as an exercise to the other author.

Proof by exhaustion. (Proof by cases.) □

Matrices

Definition 6. *Generalized rotation matrix* is defined as

$$R_k(\theta) := \begin{bmatrix} \cos_k \theta & \sin_k \theta \\ -\sin_k \theta & \cos_k \theta \end{bmatrix},$$

where $\theta \in \mathbb{R}$.

Definition 7. *Position matrix* is defined recursively as

$$P_{k,n}\left(\left\{\theta^1, \ldots, \theta^n\right\}\right) := \begin{bmatrix} P_{k,n-1}\left(\{\theta^1, \ldots, \theta^{n-1}\}\right) & 0 \\ 0 & 1 \end{bmatrix} T_\pi \begin{bmatrix} R_k(\theta^n) & 0 \\ 0 & I \end{bmatrix} T_\pi,$$

$$P_{k,0} := I_1,$$

where $\pi = \begin{pmatrix} 1 & 2 & 3 & \ldots & n & n+1 \\ 1 & n+1 & 3 & \ldots & n & 2 \end{pmatrix}$ and $\theta = \{\theta^i\} \in \mathbb{R}^n$for $i \in \{\, 1..n \,\}$.

Definition 8. *Orientation matrix* is defined as

$$O_n^\pm(\phi_{n-1}, \phi_{n-2}, \ldots, \phi_1) := \begin{bmatrix} 1 & 0 \\ 0 & X_{+1,n-1}^\pm(\phi_{n-1}, \phi_{n-2}, \ldots, \phi_1) \end{bmatrix},$$

$$O_0^\pm := \pm I_1,$$

where $\phi_m \in \mathbb{R}^m$for $m \in \{\, 1..n-1 \,\}$.

Definition 9. *Point matrix* is defined as

$$X^{\pm}_{k,n}\left(\theta,\phi_{n-1},\phi_{n-2},\ldots,\phi_1\right) := P_{k,n}\left(\theta\right)O^{\pm}_{n}\left(\phi_{n-1},\phi_{n-2},\ldots,\phi_1\right),$$

where $\theta\in\mathbb{R}^n$ and $\phi_m\in\mathbb{R}^m$for $m\in\{\,1..n-1\,\}$.

Group structure

Proof of validity of $\mathbb{G}_M$ and $\mathbb{H}_M$ (validity of $\otimes$, proof of the matrices to be $\mathbb{H}_M$ will later be discussed)...

Lemma 1. *Orientation matrix with multiplication is isomorphic to orthogonal group.*

Proof of lemma 1. Let

- the group of orientation matrix $O^{\pm}_n$ with multiplication is $\left(P\left(n\right),\cdot\right)$,
- the group of point matrix $X^{\pm}_{+1,n-1}$ at $k=+1$ with multiplication is $\left(M\left(n\right),\cdot\right)$.

To shows that $P\left(n\right)\cong M\left(n-1\right)$. Consider the mapping

$$\begin{aligned} f: X^{\pm}_{+1,n-1}\left(\phi,\ldots\right)&\mapsto O^{\pm}_n\left(\phi,\ldots\right) && \in M\left(n-1\right)\to P\left(n\right),\\ f^{-1}: O^{\pm}_n\left(\phi,\ldots\right)&\mapsto X^{\pm}_{+1,n-1}\left(\phi,\ldots\right) && \in P\left(n\right)\to M\left(n-1\right),\end{aligned}$$

which by definition 8 is equivalent to

$$f: X\mapsto\begin{bmatrix}1&0\\0&X\end{bmatrix}. \tag{2}$$

For any point matrices $X_1,X_2\in M\left(n-1\right)$, it can be shown that

$$\begin{aligned} f\left(X_1\right)\cdot f\left(X_2\right)&=\begin{bmatrix}1&0\\0&X_1\end{bmatrix}\cdot\begin{bmatrix}1&0\\0&X_2\end{bmatrix} && \text{(substitute Equation (2))}\\ &=\begin{bmatrix}1&0\\0&X_1\cdot X_2\end{bmatrix} && \text{(simplify)}\\ f\left(X_1\right)\cdot f\left(X_2\right)&=f\left(X_1\cdot X_2\right). && (3)\end{aligned}$$

Therefore, $P\left(n\right)\cong M\left(n-1\right)$.

Hence, it is sufficient to shows that $M\left(n-1\right)\cong O\left(n\right)$.

To show that the isomorphism is the identity mapping (to shows that $M\left(n-1\right)=O\left(n\right)$) is sufficient to shows that

1. Every points matrices $X \in M(n-1)$ is an element of orthogonal group $\mathrm{O}(n)$.
2. Every orthogonal matrices $A \in \mathrm{O}(n)$ is an point matrix in $M(n-1)$.

To show that

$$\begin{aligned}&\forall n \in \mathbb{N}, \forall X \in M(n-1), X \in \mathrm{O}(n)\\ &\qquad\iff X \in \{\, A \in \mathrm{GL}(n) \mid AA^T = A^T A = I \,\}\\ &\qquad\iff X \in \mathrm{GL}(n) \wedge AA^T = A^T A = I.\end{aligned}$$

Proof. It is obvious that X is an n-by-n square matrix.

And if X is orthogonal ($XX^T = X^T X = I$), then there exist an n-by-n square matrix $Y = X^T$ such that $XY = YX = T$ (X is invertible).

Hence, it is sufficient to shows that X is orthogonal.

To show that X is orthogonal.

The proof is left as an exercise to the other author.

Use closed property of orthogonal group.

It can be proved that position matrix with such parameter (or its factor) is orthogonal.

It can also be prove inductively that orthogonal matrix is orthogonal by using its block form and base case of point matrix. □

To show that $\forall n \in \mathbb{N}, \forall A \in \mathrm{O}(n-1), \exists X \in M(n-1), A = X$.

Proof. The proof is left as an exercise to the other author.

No idea on how to prove yet. □

□

Lemma 2. *Point matrix with multiplication is isomorphic to principal group of the associated Klein geometry.*

Proof of lemma 2. The proof is left as an exercise to the other author.

Spherical case is proved successfully with lemma 1.

Euclidean case should use limits and first degree taylor series.

hyperbolic case is similar to lemma 1 such that it use closed property. □

Lemma 3. *Position matrix is isomorphic to quotient group of the associated Klein geometry.*

Proof of lemma 3. The proof is left as an exercise to the other author.

It can be seen that $Q \cong M/P \iff M \cong Q \ltimes P$, point matrix is isomorphic to the principal group, orientation matrix is isomorphic to the subgroup, and point matrix is product of position and orientation matrix.

They may be useful for proving that is not yet to be known.

Another way is to seek for general form of the matrix element of the group of the Klein geometry. □

Model Parametrization

Construction of the charts φ...

Definition 10. For point matrix $X^{\pm}_{k,n}(\theta, \phi_1, \phi_2, \ldots, \phi_n)$, n-dimensional vector θ is defined as *position parameter*.

Definition 11. For point matrix X, $(n+1)$-dimensional column vector $p := X \cdot e^1 = X_{1}$ is defined as *position vector*.

This will provide a hint on why the matrices are $\mathbb{H}_M$......

Lemma 4. *For point matrix $X = PO$ where P and O are position and orientation matrix respectively, $p = X_{1} = P_{1}$.*

Proof of definition 7. From definition 8, it is obvious that

$$O^i{}_1 = \begin{cases} 1 & \text{if } i = 1, \\ 1 & \text{otherwise.} \end{cases} \tag{4}$$

$$\begin{aligned} p^i &= X^i{}_1 && \text{From definition 11} \\ &= \sum_j P^i{}_j O^j{}_1 && \text{From Equation (1)} \\ &= P^i{}_1 && \text{From Equation (4)} \end{aligned}$$

$$
\begin{aligned}
p &= X_1 \\
&= P_1
\end{aligned}
$$
□

Lemma 5. *Given position parameter θ, position vector can be evaluated as the following.*

$$
p = \begin{pmatrix} \prod_{j \in \{\,1..n\,\}} \cos_k \theta^j \\ \sin_k \theta^{i-1} \prod_{j \in \{\,i..n\,\}} \cos_k \theta^j \\ \sin_k \theta^n \end{pmatrix}.
$$

Proof of lemma 5. The proof is left as an exercise to the other author.

Use lemma 4 and recursively compute the vector. □

Lemma 6. *Given position vector p, position parameter can be calculated as the following.*

$$
\theta = \begin{pmatrix} \operatorname{arcsin}_k \frac{p^{i+1}}{\prod_{j \in \{\,i+1..n\,\}} \cos_k \theta^j} \\ \operatorname{arcsin}_k p^{n+1} \end{pmatrix}.
$$

Proof of lemma 6. The proof is left as an exercise to the other author.

Use lemma 5 to compute the inverse mapping. □

Trying to explain how domain of the chart can be seperated in such a way that there is a bijective mapping between p and θ...

Coordinate Singularity. It can be seen that at $\cos_k \theta^j = 0$, θ^i is undefined for $i < j$. Such case only occur in spherical geometry at $\theta^j = \left(2m \pm \frac{1}{2}\right) \frac{\pi}{k}$, which is the pole where coordinate lose its degree of freedom (i.e. gimbal's lock) and technically is the boundary of the unique points of single-elliptic geometry.

Parameter Signature. In case of $k > 0$, to assure that the mapping between p and θ, all parameter θ^i is bound to range of $\left(-\frac{1}{2}\frac{\pi}{k}, +\frac{1}{2}\frac{\pi}{k}\right)$ for $i \neq 1$. But given the signature of $\prod_{j \in \{\,1..n\,\}} \cos_k \theta^j$ and the fact that $\cos_k \theta^j > 0$ when $k \leq 0$ or $\theta^j \in \left(\left(2m - \frac{1}{2}\right)\frac{\pi}{k}, \left(2m + \frac{1}{2}\right)\frac{\pi}{k}\right)$, one position parameter θ^1 can be extended to range of $\left(-1\frac{\pi}{k}, +1\frac{\pi}{k}\right)$.

Coordinate Mapping

Proof of validity of ϕ...

Corollary 2. *Position parameter, position vector, and position matrix are equivalence in such ways that provided any of them, the others can be calculated.*

Geometric properties

Construction of the metric d...

Embedding

Definition 12. *Scaled position vector* is defined as $x = \frac{p}{k}$.

Definition 13. Let $M = (P, g)$ be a n-dimension Riemannian manifold on position parameter space with such metric that the map $\phi : P \to P'$ from position parameter space $P \subset \mathbb{R}^n$ to scaled position vector space $P' \subset \mathbb{R}^{n+1}$ is an isometric embedding to $(n+1)$-Euclidean manifold.

Lemma 7. *The metric tensor of M is*

$$g_{ij} = \begin{cases} \prod_{a>i} \cos_k^2 \theta^a & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Proof of lemma 7. The proof is left as an exercise to the other author. □

Locus of scaled position vector

Lemma 8. *For $k > 0$, locus of ϕ is a $(n+1)$-sphere of radius k^{-1}.*

Proof of lemma 8. The proof is left as an exercise to the other author.

Use theorem 1 and et cetera. □

Lemma 9. *For $k < 0$, locus of ϕ is a forward sheet of a two-sheeted $(n+1)$-hyperboloid of radius k^{-1}.*

Proof of lemma 9. The proof is left as an exercise to the other author.

Use theorem 1 and et cetera. □

Lemma 10. *For $k \to 0$, locus of ϕ is a n-Euclidean manifold at infinity.*

Proof of lemma 10. The proof is left as an exercise to the other author.

Use theorem 1 and et cetera. □

Curvature

Proof of validity of the metric d...

Curvature (Method I)

This method may be easier to generalize to Model II where each direction can have partially independent curvature (or even Model III where extrinsic curvature become a thing). But it may be challenging to define Gauss map properly.

Lemma 3.1. *The Gauss map $\nu : \mathbb{P}_M^I \to \mathbb{S}^n$ of $\mathbb{M}_M$ embedded in $\mathbb{R}^{n+1}$ by ϕ is*

$$\nu\left(p\right) =$$

Lemma 3.2. *SecondFundamentalForm*

Lemma 3.3. *PrincipalCurvature*

Curvature (Method II)

This method may be easier to be done (despite the fact that it never finished). But it raises problems when trying to generalize e.g. dealing with extrinsic curvature (which may be introduced in Model III if not to mess with other basis geometries).

Lemma 4.1. *ChristoffelSymbol*

Lemma 4.2. *RiemannCurvatureTensor*

Curvature (Conclusion)

Lemma 5. *SectionalCurvature*

Trace back on why k is used all along instead of κ...

Remark 5.1. It can be seen that $\sec\left(p\right) = \kappa = \operatorname{sgn}\left(k\right)k^2$, hence provided κ, k can be determined and used to evaluate the model.

The Model

Merge all element back to the model...

Definition 6.1. For any parameter $M = \Psi(\kappa, n)$,

$$M := (\{ X_{k,n} \}, \cdot)$$
$$\mathbb{P}_M \equiv \{ \theta \} \equiv \{ p \} \equiv \{ P_{k,n} \}$$
$$\mathbb{T}_M \equiv \{ X_{k,n} \}$$

for injective smooth function $K : \kappa \mapsto k = \operatorname{sgn} \kappa \sqrt{\operatorname{abs} \kappa} \in \mathbb{R} \to \mathbb{R}$.

Definition 6.2.

Assertion 7.1.

Assertion 7.2.

Future plan

Model II

It is very simple to be able to model composite geometries e.g. $S^2 \times E$ by tensor product of the existing model. But to be able to merge them as smooth model may be challenging since not all combination of basis curvature have their own intrinsic geometry. So it may be to find independent variable for each basis or to introduce extrinsic curvature (Model A).

Model B

It is known that E emerged at $n \geq 1$ while S and H emerged at $n \geq 2$ and there's more complex pure geometries than these that emerged in higher dimension. It is interesting and challenging to explore such geometries and prove whether the curvature still works as indicator in such geometries or are there any patterns for their symmetries.

Model A

This model is based on curvature and mostly just 3 basis geometries and extrinsic curvature which seems to be interesting despite some critical result in some combination e.g. $S^1 \times S^1$ vs S^2. It can be even more challenging to have variable curvature with respect to other intrinsic position.

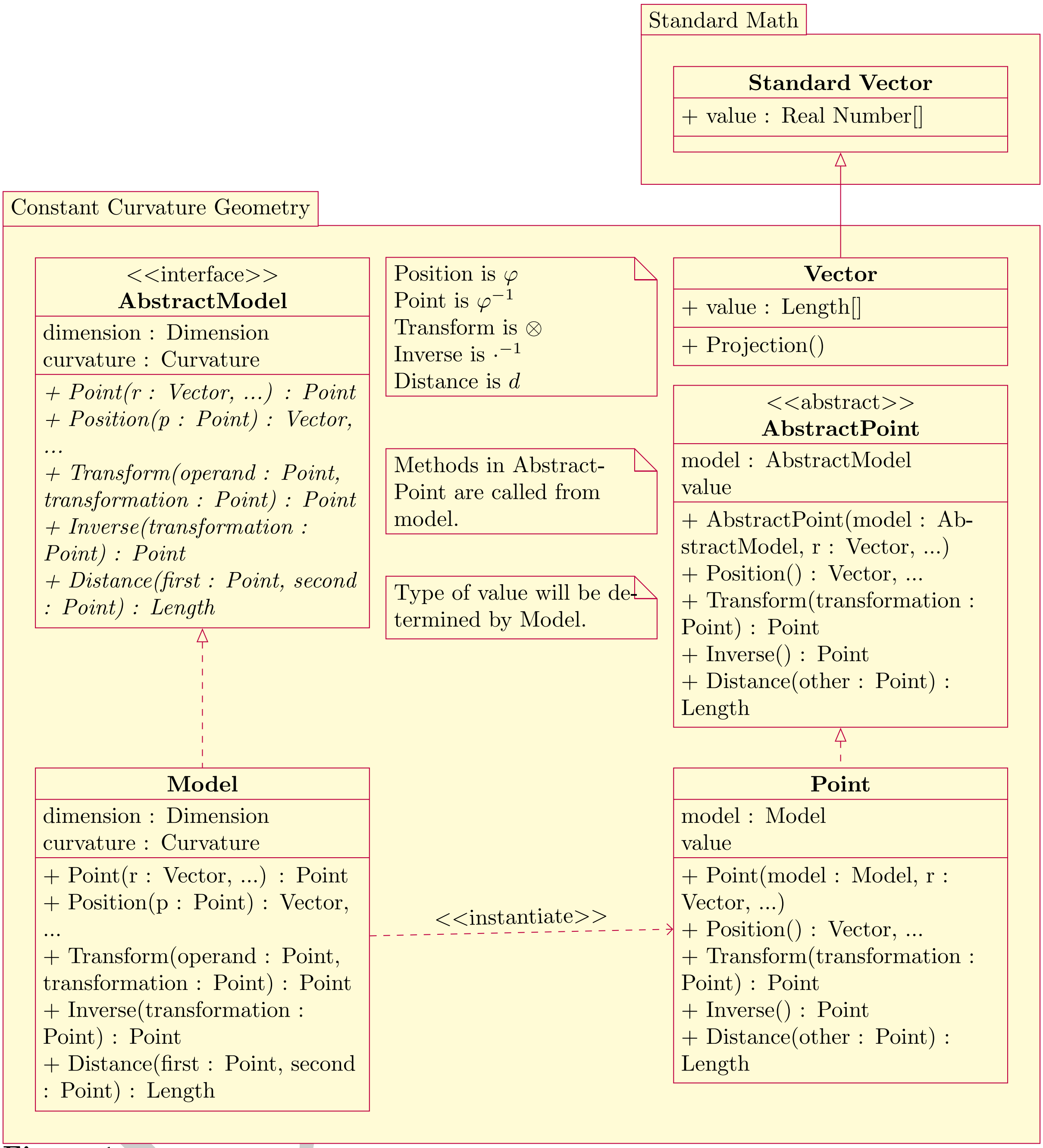

Figure 1

UML class diagram of the model

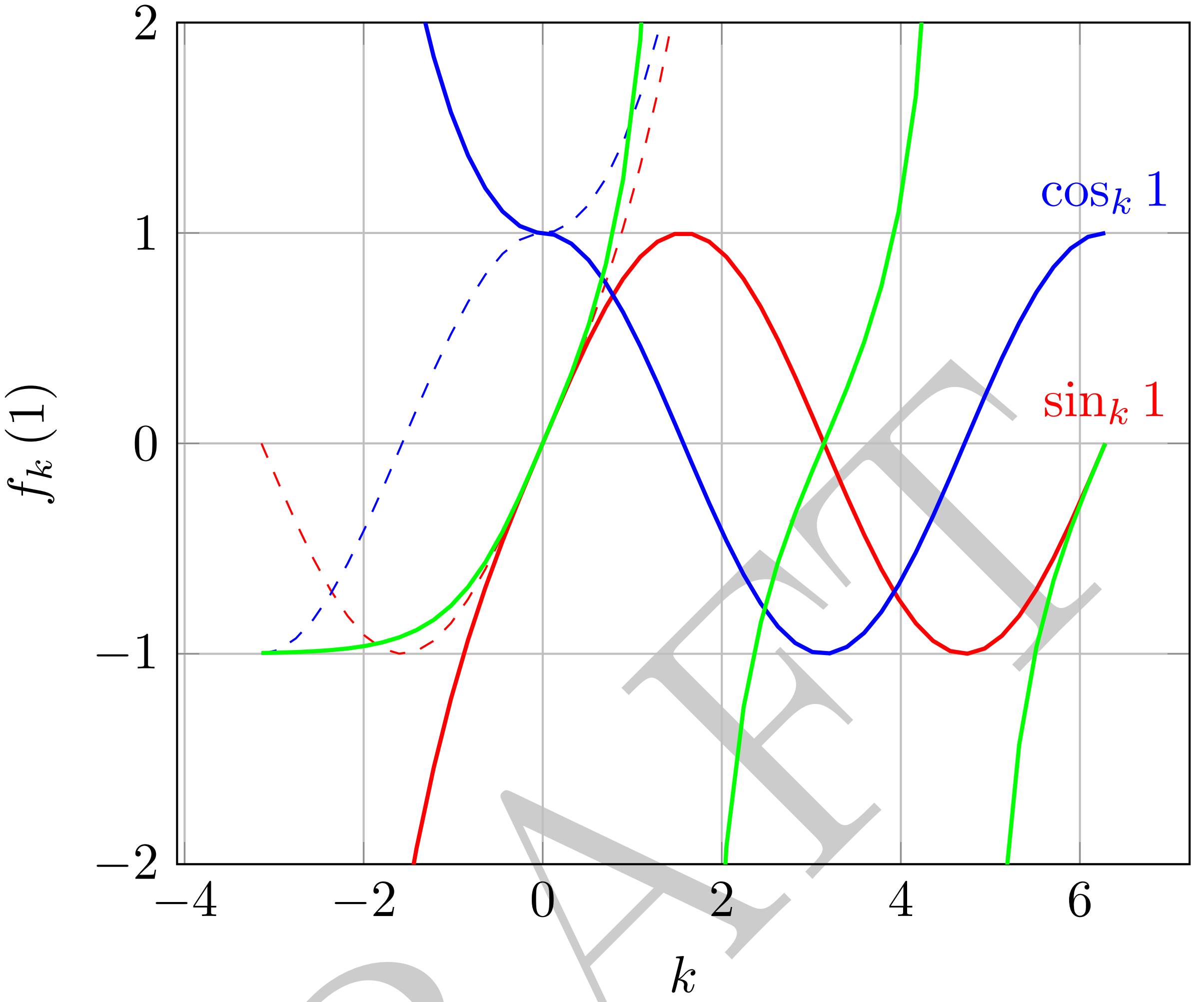

Figure 2

Generalized trigonometric functions as function of k

Note. This graph shows the value of generalized trigonometric functions as solid line and trigonometric and hyperbolic functions in the unused domain as dashed line.

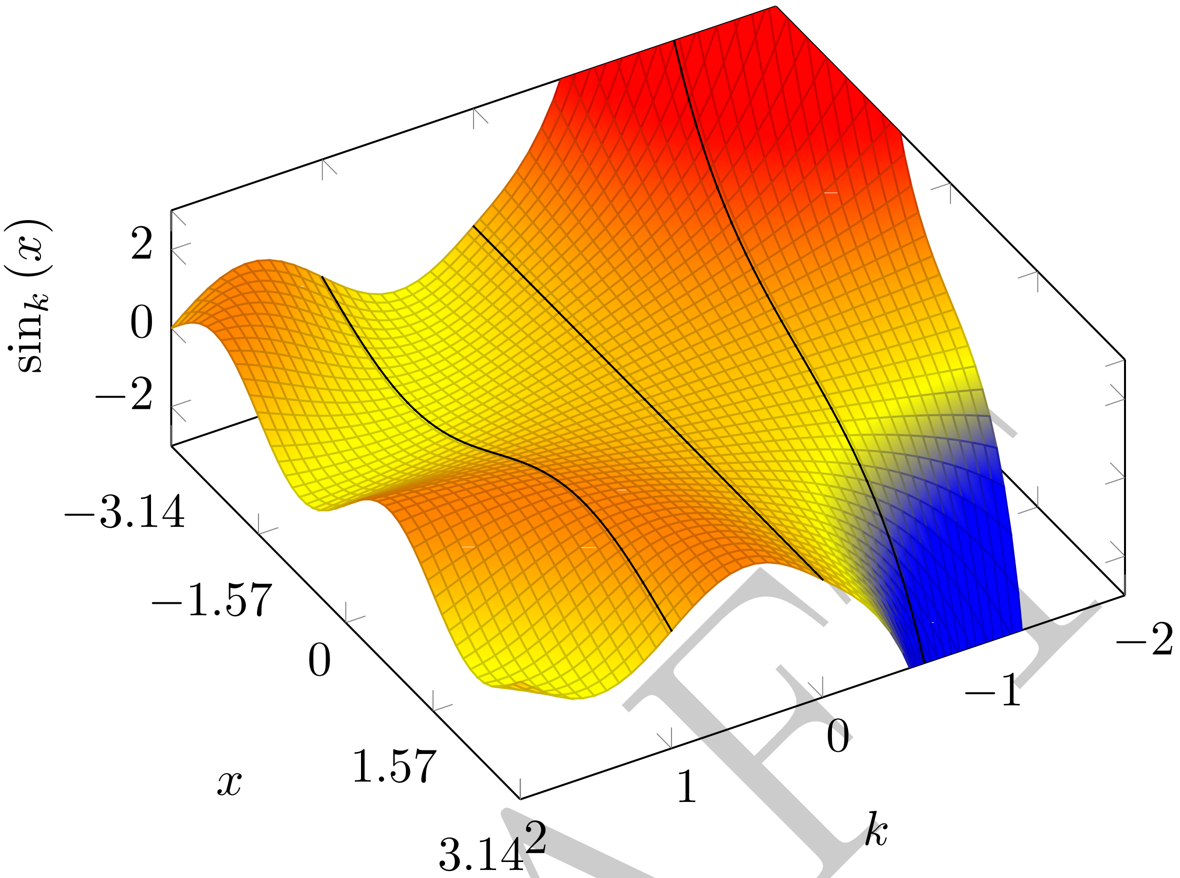

Figure 3

Generalized sine function

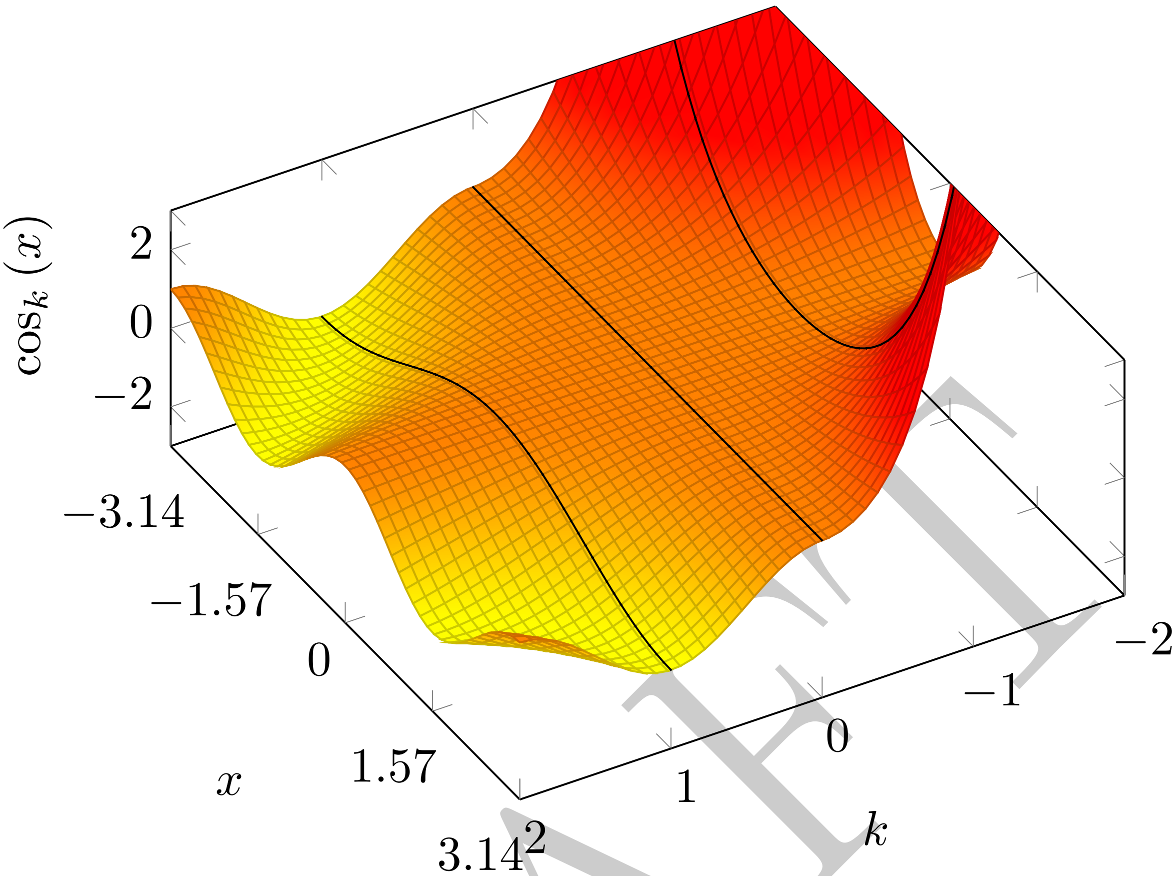

Figure 4

Generalized cosine function

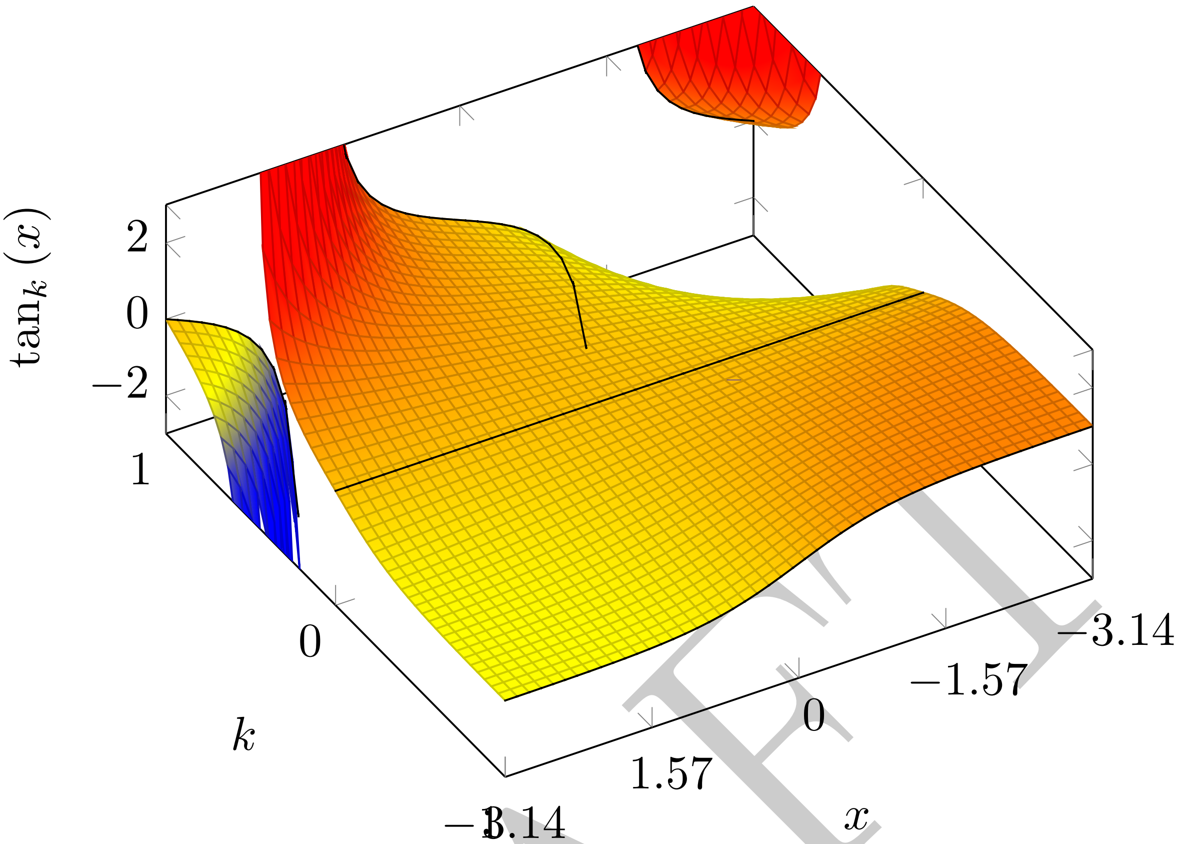

Figure 5

Generalized tangent function