CS 624
Stirling's Formula

1 Stirling's formula

How fast does $n!$ grow? Well, we can find some simple bounds as follows:

lower bound. We have

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$$

and all except the first factor on the right-hand side is at least 2, so certainly

$$2^{n-1} \leq n!$$

1.1 Exercise *Prove in fact that for any $n \geq 4$,*

$$2^n \leq n!$$

upper bound.

$$\begin{aligned} n! &= 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n \\ &\leq n \cdot n \cdot n \cdot \cdots \cdot n \end{aligned}$$

so

$$n! \leq n^n$$

and so we have for all $n \geq 4$, that

$$2^n \leq n! \leq n^n$$

Taking logarithms, we see that

$$n \leq \log_2 n! \leq n \log_2 n$$

1.2 Exercise *Show that this is really true; that is, show that*

$$\log_2 n^n = n \log_2 n$$

This is nice, but we really want a tighter approximation. And the upper bound is pretty much as good as we need in that direction, but the lower bound can be improved. Here's one way that can be done:

$$\begin{aligned} n! &= \Pi_{j=1}^{n} j \\ &\geq \Pi_{j=\lceil \frac{n}{2} \rceil}^{n} \frac{n}{2} \\ &= \left(\frac{n}{2}\right)^{\lfloor \frac{n}{2} \rfloor} \\ &\geq \frac{2}{n}\left(\frac{n}{2}\right)^{\frac{n}{2}} \end{aligned}$$

and so

$$\frac{n}{2}\log_2 \frac{n}{2} - \log_2 \frac{n}{2} \leq \log_2 n!$$

which simplifies to

$$\frac{1}{2} n \log_2 n - \frac{n}{2} - \log_2 n + 1 \leq \log_2 n!$$

which shows that

$$\log_2 n! = \Omega(n \log_2 n)$$

and so putting this all together, we get

$$\log_2 n! = \Theta(n \log_2 n) \tag{1}$$

and in fact, we now know that we could use any base for the logarithm.

1.3 Exercise *You should be able to justify all the computations above. In doing this, you will need to make use of the identity*

$$n = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil$$

which you should prove as well.

Equation (1) will be very useful to us. In fact, it is a simple form of what is called Stirling's formula. A slightly more precise form of Stirling's formula is this:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right) \tag{2}$$

In fact, one can even extend this:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \Theta\left(\frac{1}{n^3}\right)\right) \tag{3}$$

and one can even go farther—there is a formula for the coefficients in the series on the right. And when we do that, we get the final version of Stirling's formula. For our purposes in this course, however, we only need equation (1). The more refined forms do come up, however, and if you go on in this area (or others) you will likely see them.