CS 624
Harmonic Numbers

1 The harmonic numbers and logarithms

Before you go any farther, look at the first few pages of my expository paper "Some Early Analytic Number Theory" (up through page 5, anyway) which you can get to from my web site (`http://www.cs.umb.edu/~offner`). (I really mean this. This is important. Stop now and don't read any more here until you have read those pages.)

Here are some graphs that are modified from that paper:

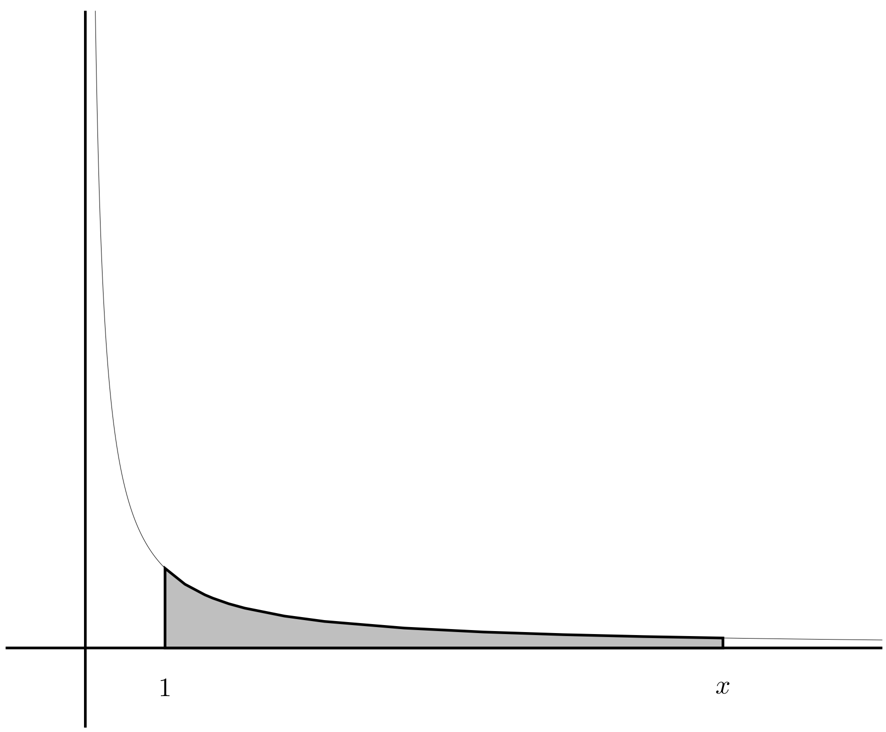

Figure 1: The graph of the function $f(x) = \frac{1}{x}$. The shaded area is $\log x$.

In Figure 2, we show how to build rectangles underneath this curve whose sizes are just the terms of the harmonic series.

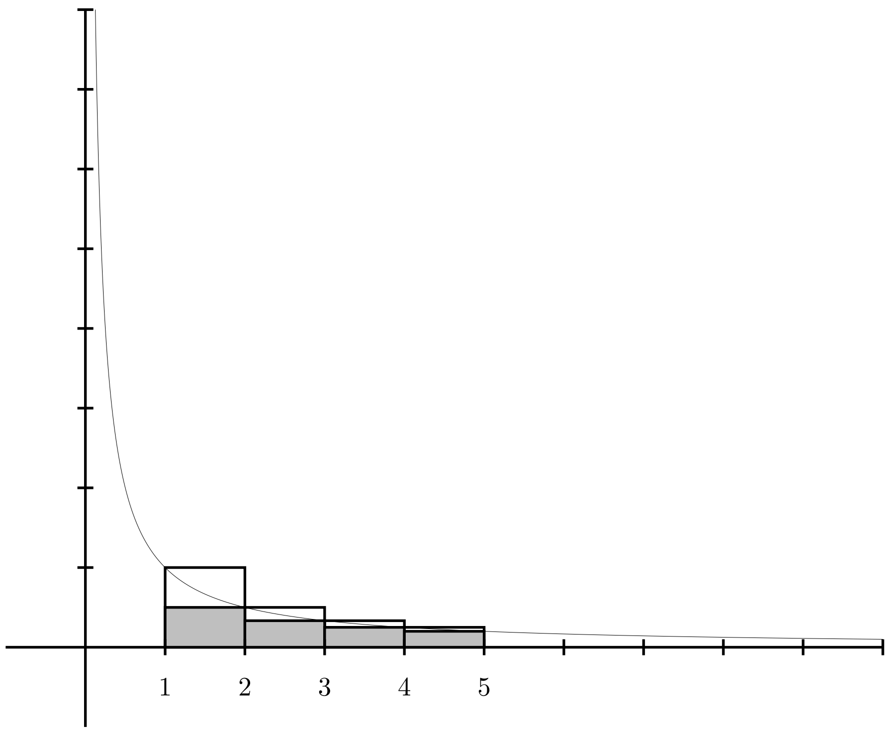

Figure 2: The same graph, with lower and upper approximations to the area. The shaded rectangles have area $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ...

We can see from this that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} < \log n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$$

1.1 Exercise *You have to be able to justify this. Can you show clearly where each side of this inequality comes from? In particular, why does the series end with $\frac{1}{n}$ on one side and $\frac{1}{n-1}$ on the other?*

This is important. Suppose I asked this on a test?

Thus, if we define

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

we have

$$H_n - 1 < \log n < H_{n-1} = H_n - \frac{1}{n}$$

and so

$$\frac{1}{n} < H_n - \log n < 1$$

so $\log n$ is a good approximation to H_n.

The numbers H_n are called *harmonic numbers*, because the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

is known as the *harmonic series.*

1.2 Exercise *Show that the harmonic series diverges. That is, show that the sum tends to infinity as you take more and more terms. To be precise, show that given any $A > 0$ (think of A as a large number), there is a number n_0 such that for $n > n_0$, $\sum_{k=1}^{n} \frac{1}{k} > A$.*

You almost certainly learned how to do this in first-year calculus, if not before. (You don't really need calculus for this at all.) And in fact, the proof is also in the expository article I referred you to at the beginning of this section. But can you write it out clearly without copying it? Could you write it out if you woke up "in the middle of the night", without agonizing about it?

2 Sums of harmonic numbers

You may remember from first year calculus that (provided $0 < a < b$)

$$\begin{aligned} \int_a^b \log t \, dt &= (x \log x - x)\big|_a^b \\ &= (b \log b - b) - (a \log a - a) \end{aligned} \tag{1}$$

2.1 Exercise *Prove this fact, using integration by parts.*

Since we have just seen that the harmonic numbers are well approximated by logarithms, it seems reasonable to guess that something similar to equation (1) might hold for harmonic numbers. And indeed it does. We prove this by writing what we are looking for as a double sum and interchanging the order of summation. In doing this, we have to be careful about the bounds of the sums.

2.2 Theorem

$$\sum_{k=1}^{n} H_k = (n+1)H_n - n$$

Proof.

$$\begin{aligned}
\sum_{k=1}^{n} H_k &= \sum_{k=1}^{n}\sum_{j=1}^{k}\frac{1}{j} \\
&= \sum_{j=1}^{n}\sum_{k=j}^{n}\frac{1}{j} \\
&= \sum_{j=1}^{n}\frac{n-j+1}{j} \\
&= \sum_{j=1}^{n}\Big(\frac{n+1}{j}-\frac{j}{j}\Big) \\
&= (n+1)\sum_{j=1}^{n}\frac{1}{j}-\sum_{j=1}^{n}1 \\
&= (n+1)H_n - n
\end{aligned}$$

□

2.3 Exercise *Let us take a simple form of the result for logarithms that we started with—let's set $a = 1$. We get (remember that $\log 1 = 0$)*

$$\int_1^b \log t\,dt = (b\log b - b) - (1\log 1 - 1) = b\log b - (b-1)$$

See if you can prove this result using the same idea as in the proof of Theorem 2.2. Of course, you won't have sums—you'll have integrals. But the general idea should be quite similar.