Generic Threshold Circuit for Schnorr Signatures

We describe a generic counting and threshold comparison procedure for Schnorr signatures (as implemented according to SchnorrSignature.pdf) implemented as circuit over the 'SNARK field' F with modulus bit length denoted by $len|F|$. As fixed circuit it should be able to treat up to

$$N < L = len|F|,$$

public keys,

$$pk_1, pk_2, \ldots, pk_N,$$

arranged in a linearly ordered list, with *Null keys* pk_{NULL} to fill up to the maximum number N, if there a less signer.

Normative notes

For our purpose, we assume $pk_{NULL} \in \mathbb{G}$ to be a *phantom key*, i.e. an arbitrary fixed element from the group $\mathbb{G} = MNT4 - 753 = EC(F)$ (as used by the signature scheme) to which nobody knows the secret key. A simple way to do this in a publicly verifiable manner is by choosing it as hash of some public data, for example

$$pk_{NULL} = H(\text{"magic string"}),$$

where H is any hash-to-curve algorithm and some publicly declared "magic string" (e.g. "Strontium Sr 90").

Notice that the upper bound for N is quite arbitrarily chosen to satisfy

$$d = len(N) < len|F| - 1,$$

as needed for the threshold enforcer described below, and not too large anyway for performance. However, any other N the bit length of which satisfies the above inequality is fine.

The Generic Threshold Gadget

Our gadget $Threshold(m, h)$ is as follows.

- Public input:
  - message m (as single element from F),
  - $h \in F$, the POSEIDON Hash as commitment on the parameters of the threshold scheme (in our case the threshold $t \in F$), and the public keys $pk_1, pk_2, \ldots, pk_N$ allowed to sign (including null keys).
- Private input / witnesses:
  - the threshold $t \in F$,
  - the public keys $pk_1, \ldots, pk_N$ (including possible null keys) in the same order as done for the computation of h,
  - Schnorr signatures $\sigma_1, \ldots, \sigma_N$, including arbitrarily chosen null signatures σ_{NULL} (e.g., $\sigma_{NULL} = (0, 0)$) to fill up to full length,
  - Boolean variables $b_0, \ldots, b_{d-1}$ for the threshold comparison.

It's circuit is based on three components, as depicted below:

1. *Poseidon* Gadget, which enforces the privatly chosen pk_i to hash to the given fingerprint h,
2. the $SchnorrVerdict$ gadget, as described in SchnorrVerdict.pdf, which enforces the Boolean verdicts v_i to reflect a valid/invalid signature, and
3. the *threshold enforcer*, which uses a simple length-restriction argument to force that the number $v = \sum_{i=1}^{N} v_i$ of valid signatures satisfies $v \geq t$.

The Poseidon gadget

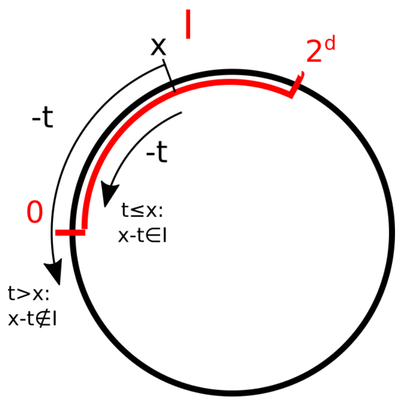
is as described in Poseidon.pdf, extended to the domain of N field elements (the public keys to be hashed).

Threshold enforcer

To guarantee that two integers x, t from $I = [0, 2^d - 1]$ as subset of F satisfy $x \geq t$, we take $x - t$ in F and force it to be in the same interval I simply by demanding an at most d bit integer representation

$$\begin{aligned} x &= \sum_{k=0}^{d-1} b_k \cdot 2^k. \\ 0 &= b_k \cdot (b_k - 1), \qquad k = 0, \ldots, d-1. \end{aligned}$$

Note that d needs to be smaller than the length L of the field modulus (In practice it is much smaller, e.g. $d = 4$), so that $2^{d+1} < |F|$.



Notice that this gadget comes almost for free, demanding only $d + 1$ rank one constraints.

Comment, or why it works although risking modular reduction at any point during the calculation: any integer solution (b_i) of

$$\begin{aligned} x &= b_0 + b_1 \cdot 2 + \ldots + b_d \cdot 2^{d-1} \bmod r, \\ 0 &= b_i \cdot (1 - b_i) \bmod r, \qquad i = 0, \ldots, d-1. \end{aligned}$$

is forced by the Boolean constraints $0 = b_i \cdot (1 - b_i)$ to be of the form

$$b_i = \epsilon_i + k \cdot r, \quad \epsilon_i \in \{0, 1\},$$

where $k \cdot r$ is a (positive or negative) multiple of the modulus r. Hence, by the first equation, x and

$$\sum_{i=0}^{d-1} \epsilon_i \cdot 2^i$$

can still only differ by a multiple of r. But this means that x has a representation (mod r) as an integer from I.