Conditional Probability:

Often, we have partial information about a certain phenomenon and wish to know how this affects the probabilities of outcomes of interest to us, if at all. For example, we might want to know the probability of rain today, given that it rained yesterday.

Ex 15: Toss a fair die. Let A = {less than 4 is rolled} = and B ={Odd number is rolled}=
What is the probability of throwing a number less than 4 given that an odd number was thrown?

Conditional Probability

The probability that A occurs given (or knowing) that B occurred is denoted by $\boldsymbol{P(A|B)}$ and given by

$$\boldsymbol{P(A|B) = \frac{P(A \cap B)}{P(B)}} \quad , \ \textbf{if} \quad \boldsymbol{P(B) > 0}$$

Conditional probability is important when we are interested in the probability structure under certain restrictions.

- MATH 252 is a prerequisite for STAT 351. How likely that you will get an A for STAT351 course if your grade for MATH 252 is B?
- How likely is it that a person has a certain disease given that a medical test was negative?
- How likely is it that a person does not have a certain disease given that a medical test was positive?
- A spot shows up on a radar screen. What is the chance that it is correspond to an aircraft?

Ex 16: Let 10,000 people are tested for a cancer. Among them, 185 are actually cancer-free and tested positive while 1 person actually has the cancer and tested negative.
Actually, only 50 people among 10,000 have the cancer. Fill in the blanks of the following table to summarize these data.

	Negative test for cancer	**Positive test for cancer**	**Actual Total**
Actual Cancer-free			
Actual Cancer			
Total			

a) Find the probability that a randomly chosen person actually has the cancer. (This is the **cancer rate**)

b) Find the probability that a randomly chosen person is actually cancer-free and tests negative.

c) If a randomly chosen person is actually cancer-free, then find the probability that this person tests positive. This is called **false positive rate** of this test.

d) What is the **false negative rate** of this test?

Ex 17: Suppose that in a class of 100 kids 45 like Sprite, 60 like Cola and 20 like both. If a kid is randomly selected and ask the drink they like, estimate the following probabilities.

(a) A randomly selected kid likes at least one of these drinks

(b) If a randomly selected kid like at least one of these drinks, the probability that kid likes Cola

Multiplicative Law of Probability

The probability that intersection of two events A, B is given by

$$\begin{aligned} \boldsymbol{P(A \cap B)} &= \boldsymbol{P(A|B) \times P(B)} \\ &= \boldsymbol{P(B|A) \times P(A)} \end{aligned}$$

If $A_1, A_2, \ldots, A_k$ is a collection of events, then

$$\boldsymbol{P(A_1 \cap A_2 \cap \ldots \cap A_k) = P(A_1)\, P(A_2 \mid A_1)\, P[A_3 \mid (A_1 \cap A_2)] \ldots\, P[A_k \mid (A_1 \cap A_2 \cap \ldots \cap A_{k-1})]}$$

Ex 18: You have 10 computer chips, two of which are known to be defective.

(a) What is the probability that a randomly selected chip is defective?

(b) Your friend randomly picks 3 chips one after the other without replacement. Find the probability that 1st is defective and next two are not defective.

Law of Total Probability

If $B_1, B_2, \ldots, B_k$ is a collection of **mutually exclusive** ($B_i \cap B_j = \emptyset$ for all $i \neq j$) and **exhaustive** ($B_1 \cup B_2 \cup \ldots \cup B_k = \Omega$) events, and if $P(B_i) > 0$ for all i, then for any event A,

$$\boldsymbol{P(A) = \sum_{i=1}^{k} [\, P(A|B_i) \times P(B_i) \,]}$$

Venn Diagram:

Ex 19: Linda, Minnie, and Robin are medical assistants at a local clinic. Linda sees 20% of the patients, while Minnie and Robin each see 40% of the patients. Suppose that 60% of Linda's patients receive flu shots, while 30% of Minnie's patients receive flu shots and 10% of Robin's patients receive flu shots.

(a) What is the probability that a randomly selected patient gets a flu shot?

(b) Given that a patient receives a flu shot, find the conditional probability that Linda gave the shot.

Ex 20: The probability that a cell in a wireless system is overloaded is 1/3. Given that it is overloaded, the probability of a blocked call is 0.3. Given that it is not overloaded, the probability of a blocked call is 0.1. Find the probability that your call is blocked.

Bayes' Rule

If $B_1, B_2, \ldots, B_k$ is a collection of **mutually exclusive** ($B_i \cap B_j = \emptyset$ for all $i \neq j$) and **exhaustive** ($B_1 \cup B_2 \cup \ldots \cup B_k = \Omega$) events, and if $P(B_i) > 0$ for all i, then for any event A,

$$\boldsymbol{P(B_j|A) = \frac{P(A \cap B_j)}{P(A)} = \frac{P(A|B_j) \times P(B_j)}{P(A)} = \frac{P(A|B_j) \times P(B_j)}{\sum_{i=1}^{\infty}[\, P(A|B_i) \times P(B_i)\,]}}$$

Ex 19 Contd.: Linda sees 20% of the patients, while Minnie and Robin each see 40% of the patients. Suppose that 60% of Linda's patients receive flu shots, while 30% of Minnie's patients receive flu shots and 10% of Robin's patients receive flu shots.
Find the conditional probability that the patient was seen by Robin given that patient received a flu shot.

Ex 21: Due to an Internet configuration error, packets sent from New York to Los Angeles are routed through El Paso, Texas with probability 3/4.
Given that a packet is routed through El Paso, suppose it has conditional probability 1/3 of being dropped.
Given that a packet is not routed through El Paso, suppose it has conditional probability 1/4 of being dropped.

(a) Find the probability that a packet is dropped.

(b) Find the probability that a packet is routed through El Paso given that it is not dropped.

Ex 22: A computer virus spreads around the world, all reporting to a master computer. The good people capture the master computer and find that one million out of 1000 million computers have been infected (but don't know which ones). Governments decide to take action!

No one can use the internet until their computer passes the "virus-free" test.

The test gives a positive result 99% of the times given that the computer actually has the virus. But the test has a **false positive rate** of 2%. That is, 2% of the time it says the computer has the virus when it doesn't have the virus.

If a randomly selected computer is tested positive, then how likely that computer actually has the virus? (That is, what percent of computers actually has the virus if those were tested positive?)

Ex 23: A student dormitory in a college consists of the following:

30% are freshman of whom 10% own a car

40% are sophomore of whom 20% own a car

20% are juniors of whom 40% own a car

60% of seniors own a car

Assume that there are no other types of students at this dorm. A student is randomly selected from this dormitory.

(a) What is the probability that this student is a senior?

(b) If the student owns a car, find the probability that this student is a senior.

Independence

Sometimes the occurrence of one event, B, will have no effect on the probability of another event, A. If A and B are unrelated, then intuitively it should be the case that

$$P(A|B) = P(A)$$

Also, it follows that

$$P(A \cap B) = P(A) \times P(B)$$

Statistical Independence

Two events, A and B, are statistically **independent** if and only if

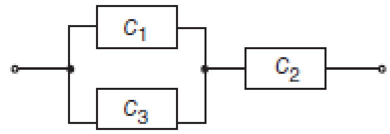
$$\boldsymbol{P(A \cap B) = P(A) \times P(B)}$$

$$\boldsymbol{P(A|B) = P(A)}$$

$$\boldsymbol{P(B|A) = P(B)}$$

Ex 24: Three bits are transmitted across a noisy channel and the number of correct receptions is noted. Find the probability that the number of correctly received bits is two, assuming bit errors are mutually independent and that on each bit transmission the probability of correct reception is λ for some fixed $0 \leq \lambda \leq 1$.

Ex 25: Given three components with respective reliabilities $R_1 = 0.8$, $R_2 = 0.75$, and $R_3 = 0.98$, compute the reliabilities of each system shown in figure below. Assume all components fail independently.



If A and B, are statistically **independent,** then

$$\mathbf{A \text{ and } B^c \text{ are independent, that is } P(A \cap B^c) = P(A) \times P(B^c)}$$

$$\mathbf{A^c \text{ and } B \text{ are independent, that is } P(A^c \cap B) = P(A^c) \times P(B)}$$

$$\mathbf{A^c \text{ and } B^c \text{ are independent, that is } P(A^c \cap B^c) = P(A^c) \times P(B^c)}$$

A collection of events $A_1, A_2, \dots, A_n$ are **pairwise independent** if and only if

$$\boldsymbol{P(A_i \cap A_j) = P(A_i) \times P(A_j) \text{ for all } i \neq j}$$