Ex 2: Suppose we roll a regular fair die three times and observe top face. Let X be the number of times we observe number 5. Is X a binomial random variable?

The support of $X = \{0, 1, 2, 3\}$

Does the experiment have a fixed number of trials (rolls)? Yes, $n = 3$

Are there only 2 disjoint and exhaustive outcomes of interest for each trial? Yes,

Success: observing 5

Are trials independent? Yes

Does probability of a "success" remain constant across the trials? Yes, $p = \frac{1}{6}$

Yes, X has a Binomial $(n = 3, p = 1/6)$.

In MATLAB: X has B(n,p)

When a is an integer from 0 to n,

Number of successes

Probability of observing at most a successes = P(X ≤ a) = binocdf(a, n, p)

Probability of observing more than a successes = P(X > a) = binocdf(a, n, p, 'upper')

Ex 3: Transmission Error

A binary communications channel introduces a bit error in a transmission with probability of 0.01. Let X be the number of errors in 10 independent transmissions. $p = 0.01$, $n = 10$

Support of x: $\{0, 1, 2, \ldots, 10\}$

(a) Write the pmf of X.

$$p(x) = P(X = x) = \begin{cases} \binom{10}{x}(0.01)^x(1-0.01)^{10-x} & ; x = 0, 1, 2, \ldots, 10 \\ 0 & ; \text{otherwise} \end{cases}$$

(b) Find the probability of 8 errors in 10 transmissions.

$$P(X=8) = \boxed{\binom{10}{8}(0.01)^8(1-0.01)^{10-8}}$$

$$= \frac{10!}{8!(10-8)!} \times (0.01)^8 (0.99)^2 = 4.41 \times 10^{-15}$$

Almost impossible

Almost zero

(c) Find the probability of one or fewer errors in 10 transmissions. $\{\boxed{0, 1}, 2, \ldots, 10\}$

$P(X \leq 1) = 0.9957 = P(X=0) + P(X=1)$

Use $\boxed{\text{binocdf}(1, 10, 0.01)}$ in Matlab.

(d) Find the probability of at least 2 errors in 10 transmissions. $\{0, 1, \boxed{2, 3, 4, \ldots, 10}\}$

$P(X \geq 2) = 0.0043 = 1 - P(X \leq 1)$

Use $\boxed{\text{binocdf}(1, 10, 0.01, \text{'upper'})}$ in Matlab.

$P(X \geq 2) = P(X > 1)$

At most 2 = 2 or less

At least 2 = 2 or higher

Key Questions:

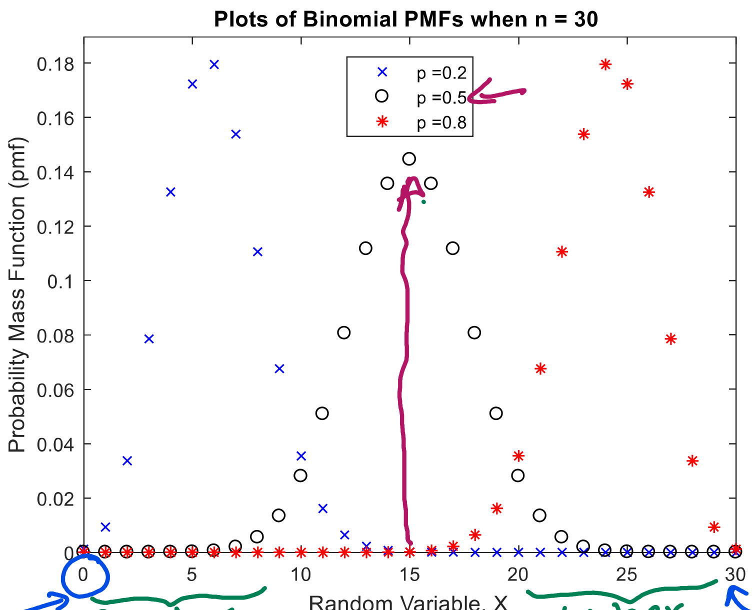
- How is a Binomial random variable similar to a Bernoulli random variable?

  Each trial has disjoint and exhaustive outcomes (success, failure)

- How are they different?

  Bernoulli has one trial.
  Binomial has n number of trials.

mean of Binomial is $n \times p = 30 \times 0.5 = 15$




```
%% Plot of Binomial Distributions
n = 30;
x = 0:n;
p1=0.2; p2=0.5 ; p3= 0.8;
y1 = binopdf(x,n,p1);
y2 = binopdf(x,n,p2);
y3 = binopdf(x,n,p3);
plot(x,y1,'xb',x,y2,'ok', x,y3,'*r')
xlabel('Random Variable, X')
ylabel ('Probability Mass Function (pmf)')
title(['Plots of Binomial PMFs when n = '
num2str(n) ' '])
legend(strcat('p = ',num2str(p1)),
strcat('p = ',num2str(p2)), strcat('p =
',num2str(p3)),'Location','north')
ylim([0,max(max([y1(:),y2(:),y3(:)]))+.01])
```

min   smaller values   higher values   n

- **If p < 0.5, then** smaller number of successes is more likely. (Right-skewed)
- **If p = 0.5, then** medium number of successes is more likely. (symmetric)
- **If p > 0.5, then** higher number of sccesses is more likely. (Left-skewed)

Ex 4: A 10-digit binary string is randomly generated. What is the probability that the ten digits (each of which is either 0 or 1) sum to 7?

Let X be the number of digits with 1 in 10 digits.
$p = P(\text{each digit is } 1) = 0.5$
If the sum is 7, there are seven 1s.

$$P(X=7) = \underbrace{\binom{10}{7}}_{120} (0.5)^{7} (1-0.5)^{10-7} = \underline{\underline{0.1172}}$$

Binomial : Support is $\{0, 1, 2, \ldots, n\}$

Geometric Random Variable and Distribution

A random variable which gives **the number of the Bernoulli trial on which the first success occurs** is a **geometric random variable** and has a **geometric distribution**.

- There is **no fixed number of trials**. It is unknown the maximum times experiment should be repeated.
- There are only **2 outcomes of interest** for each trial (success, failure).
- The trials are independent.
- The **probability of a "success" is a constant** across the trials. The probability of success is denoted by **p**

Parameter is p.

$1 - p = P(\text{Failure})$

$X - 1 =$ Number of failures

$\mathbf{X \sim Geometric\ (p)}$

$\mathbf{pmf} : p(x) = P(X = x) = \mathbf{p(1-p)^{x-1}}$; $x = 1, 2, \ldots$ and $0 < p < 1$

$\mu =$ $\mathbf{Mean} : E(X) = \frac{1}{p}$

$\sigma^2 =$ $\mathbf{Variance} : Var(X) = \frac{1-p}{p^2}$

$q = 1 - p$

Geometric Series for $|r| < 1$: $\sum_{k=0}^{\infty} (ar^k) = \frac{\mathbf{a}}{\mathbf{1 - r}}$

Key Questions:

$\{1, 2, \ldots\}$

- Why does the support of the geometric random variable start at 1 (not zero)?

  Geometric variable counts the number of Bernoulli trials to observe 1 success. Therefore, there must be at least one trial to observe 1 success.

- How is a Binomial random variable similar to a Geometric random variable?

  They both use independent Bernoulli trials

- How are they different?

  Binomial counts number of successes in n trials.
  Geometric counts the total number of trials to observe 1st success.

In MATLAB: $\boxed{Y = X - 1}$ is the number of failures before 1st success

When a is an integer from 0,

at most a

Probability of observing at most a failures before 1st success = P(X ≤ a) = geocdf(a, p)

Probability of observing more than a failures before 1st success = P(X > a) = geocdf(a, p, 'upper')

(*) Note: In Matlab, Y is number of failures before 1st success and therfore, support of Y starts at zero.

$P(X \geq 3) = 1 - P(X \leq 2) = 1 - [P(X=1) + P(X=2)]$
$= 1 - [0.8 \times (1-0.8)^{1-1} + 0.8 \times (1-0.8)^{2-1}]$

Ex5: Message Transmission in a feedback channel

Let X be the number of times a message needs to be transmitted until it arrives correctly at its destination. Support of X is $\{1, 2, 3, 4, \ldots\}$

Let p be the probability a message arrives correctly at its destination.

(a) Write the pmf of X

$$p(x) = P(X=x) = \begin{cases} p(1-p)^{x-1} & \text{for } x = 1, 2, 3, \ldots \\ 0 & \text{for otherwise} \end{cases}$$

(b) If $p = 0.8$, find the probability that a message needs to be transmitted at least 3 times until it arrives correctly at its destination.

$$P(X \geq 3) = \sum_{x=3}^{\infty} [0.8 \times (1-0.8)^{x-1}] = \underline{\underline{0.04}}$$

(*) In Matlab, $Y = X - 1$. Therefore, $P(X \geq 3) = P(Y+1 \geq 3) = P(Y \geq 2)$
$= P(Y > 1)$.

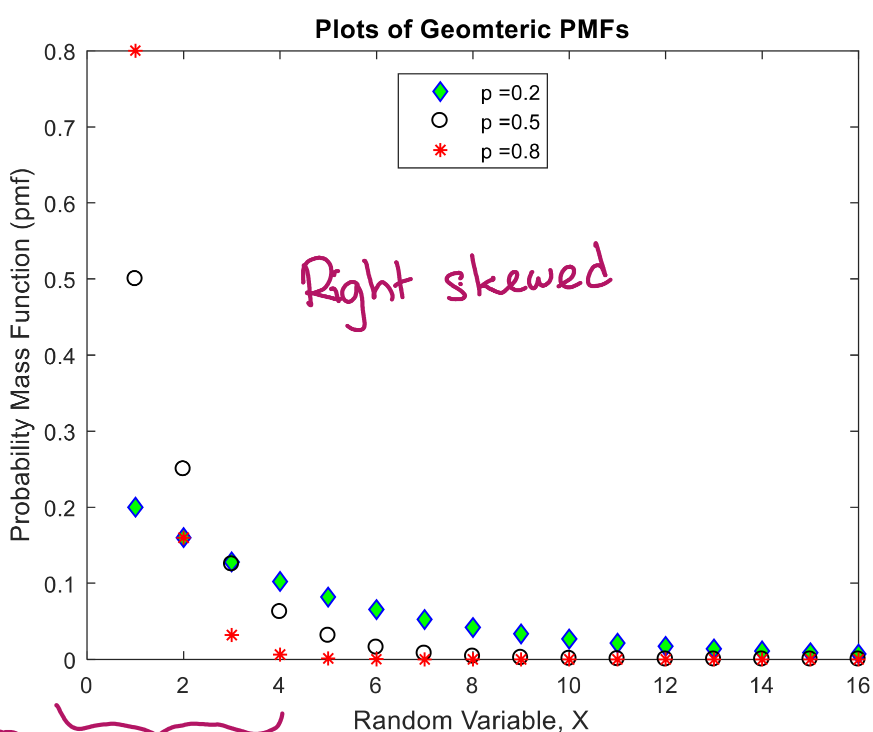
Use geocdf(1, 0.8, 'upper')

(c) If $p = 0.8$, find the expected number of times a message needs to be transmitted until it arrives correctly at its destination.

$$\mu = E(X) = \frac{1}{p} = \frac{1}{0.8} = \underline{\underline{1.25}}$$

(d) If $p = 0.8$, find the variance of number of times a message needs to be transmitted until it arrives correctly at its destination.

$$\sigma^2 = Var(X) = \frac{1-p}{p^2} = \frac{1-0.8}{0.8^2} = \underline{\underline{0.3125}}$$




```
%% Plots of Geometric Distributions
x = 0:15;
p1=0.2; p2=0.5 ; p3= 0.8;
y1 = geopdf(x,p1);
y2 = geopdf(x,p2);
y3 = geopdf(x,p3);
plot(x+1,y1,'db',
'MarkerFaceColor',[0 1 0]) % Specify
fill color
hold on;
plot(x+1,y2,'ok', x+1,y3,'*r')
xlabel('Random Variable, X')
ylabel ('Probability Mass Function
(pmf)')
title('Plots of Geomteric PMFs')
legend(strcat('p = ',num2str(p1)),
strcat('p = ',num2str(p2)), strcat('p
= ',num2str(p3)),'Location','north')
```

(*) Smaller values are more likely.