Probability Rules

Introduction

Experiment is an activity from which an outcome (or result) is obtained.

Outcome of an experiment is subjected to uncertainty (i.e. exact result is unknown until the experiment is performed). However, possible results are known before conducting the experiment. Such a study is called a **random experiment.**

- **Ex 1**: Toss a coin once and observe the top face. All possible outcomes are:

- **Ex 2**: Roll a pair of regular dice. Suppose the total on the dice is of interest. All possible outcomes are:

- **Ex 3 (Computer memories)**: Suppose we store an n-bit word consisting of all 0s at a particular location. When we read it back, we may not get all 0s. In fact, any n-bit word may be read out if the memory location is faulty. The set of all possible n-bit words are: $\{(b_1, \ldots, b_n) \mid b_i = 0 \text{ or } 1\}$.
  If n = 2, then possible outcomes are:
- **Ex 4 (Optical communication systems)**: Since the output of a photodetector is a random number of photoelectrons. The logical possible outcomes are nonnegative integers. That is, $\{0,1,2, \ldots\}$. Notice that we include 0 to account for the possibility that no photoelectrons are observed.
- **Ex 5 (Wireless communication systems)**: Noncoherent receivers measure the energy of the incoming waveform. Since energy is a nonnegative quantity, the possible outcomes are nonnegative real numbers. That is,

Sample Space is a set of all possible outcomes (or results) of a random experiment and denoted by **Ω**.

- **Ex 1**: Ω = { _____ ,_____ }

- **Ex 2**: Ω = { _____ , _____, . . . , ______ }

- **Ex 3**: Ω = { ________ , _______ , ________ , _______ }

- **Ex 4**: Ω = { _______ , ______ , ______ , ______ }

- **Ex 5**: Ω = { }

Mathematical Notation for listing possible outcomes:

{ x | membership rule}

Ex: $\left\{x \;\middle|\; \frac{x}{2} \in \mathbb{Z}\right\}$

which is read as "all the x values given (or knowing) that x/2 is an element of $\mathbb{Z}$" where $\mathbb{Z}$ indicates integers. That is, {

Event is a subset (small collection of outcomes) of sample space and denoted by a capitol letter.
Event and set are interchangeable.

Ex 6: If the sample space is the set of all triples (b_1,b_2,b_3), where the b_i are 0 or 1, then any particular triple, say (0,0,0) or (1,0,1) would be an outcome.

Event W would be a subset such as the set of all triples with exactly one 1; that is, W = {

Ex 7: Which of following are possible events when you toss a coin once and observe the top face?

Sample space = Ω = { _____ , _____ }

A = {H is observed } = {   }

B = {either H or T is observed} = { _______ , _______ }

Null Set (Empty Event) is an event with no elements and denoted by Ø

In example 7, C = {neither H nor T is observed}

Definitions of Set Operations:

Consider two events A, B. Then

- **Complement of $\boldsymbol{A}$** is defined as a set of all elements in Ω that are not in A. It is denoted by $\boldsymbol{A'}$, $\boldsymbol{A^c}$, **or** $\boldsymbol{\overline{A}}$.

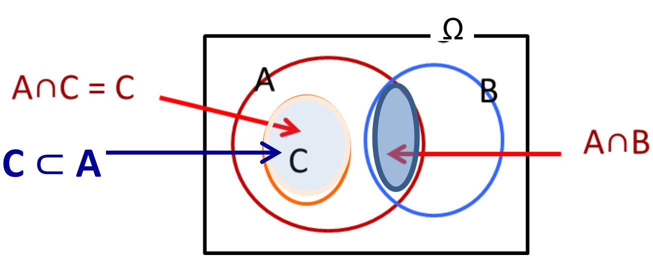
- **Intersection** of two events A, B is defined as a set of all elements common to events A, B. It is denoted by $\boldsymbol{A \cap B}$. It can also denoted by $\boldsymbol{A\ and\ B}$.

- **Union** of events A, B is defined as a set of all elements in A or B or both. It is denoted by $\boldsymbol{A \cup B}$. It can also denoted by $\boldsymbol{A\ or\ B}$.

- Event B is a **subset** of event A if every element in B is also in A. It is denoted by $\boldsymbol{B \subset A}$. This is also said as B is contained in A.

❖ **The null set is a subset of every event.**

Venn Diagrams are useful to see set operations. Events are represented by regions.



Events A, B are **disjoint (mutually exclusive)** if and only if intersection of events A and B is empty. Disjoint events **do not have common elements** and they **do not occur at the same time**.

$$\textbf{Events A, B are Disjoint} \quad \Leftrightarrow \quad \mathbf{A \cap B = \emptyset}$$

Examples of mutually exclusive (disjoint) events:

- Rising the Sun and rising the Moon cannot occur in the same location at the same time.
- It is impossible to observe both an odd number and an even number when you roll a regular die once.
- A student cannot take both STAT 351 and MTH 252 in a given quarter because MTH 252 is a prerequisite for STAT 351.
- The output of a binary machine number cannot be both 0 and 1 at the same time.
- “It is raining outside this building at this moment" and "It is not raining outside this building at this moment"

Exhaustive events

Events A, B are exhaustive if and only if union of events A and B is the sample space. Combined elements of exhaustive events are the elements of the sample space.

$$\textbf{Events A, B are Exhaustive} \quad \Leftrightarrow \quad \mathbf{A \cup B = \Omega}$$

- $A_1, A_2, \dots, A_n$ are **mutually exclusive (or disjoint)** events if they are **pairwise mutually exclusive (or disjoint)** events

- $A_1, A_2, \dots, A_n$ are **pairwise mutually exclusive (or disjoint)** events if and only if $A_i \cap A_j = \emptyset$ for all $i \neq j$

- $A_1, A_2, \dots, A_n$ are **exhaustive** events if and only if $A_1 \cup A_2 \cup \dots \cup A_n = \Omega$

- $A_1, A_2, \dots, A_n$ are **exhaustive and pairwise mutually exclusive** events if and only if $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $A_1 \cup A_2 \cup \dots \cup A_n = \Omega$

Partition

A family of events is a partition if and only if those events are exhaustive and pairwise mutually exclusive.

$$\textbf{Partition of sample space} \quad \Leftrightarrow \quad \bigcup_{i=1}^{n} A_i = \Omega \quad \textbf{and} \quad \textbf{for all } i \neq j, A_i \cap A_j = \emptyset$$

Ex 8: A fair coin is tossed twice, so Ω =

Let A = {observing no H} =

B = {observing one H} =

C = {observing H after T} =

Write the outcomes for A^c =

B^c =

C^c =

Write the outcomes for $A \cap B$ =

$A \cap C$ =

$B \cap C$ =

$A \cap B \cap C$ =

Are events A, B mutually exclusive?

Are events A, C mutually exclusive?

Are events C, B mutually exclusive?

Are events A, B, C mutually exclusive?

Write the outcomes for $A \cup B$ = {Observing no H or one H} =

$A \cup C$ = {Observing no H or H after T} =

$B \cup C$ = {Observing one H or H after T} =

$A \cup B \cup C$ = {Observing no H or one H or H after T} =

Basic Properties (algebra of sets):

$A \cap \emptyset =$ ___ , $A \cup \emptyset =$ ___

$A \cap A^c =$ ___ , $A \cup A^c =$ ___

$\Omega^c =$ ___ , $(A^c)^c =$ ___

DeMorgan's Law:

$(A \cap B)^c = A^c \cup B^c$

$(A \cup B)^c = A^c \cap B^c$

Distributive Law:

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

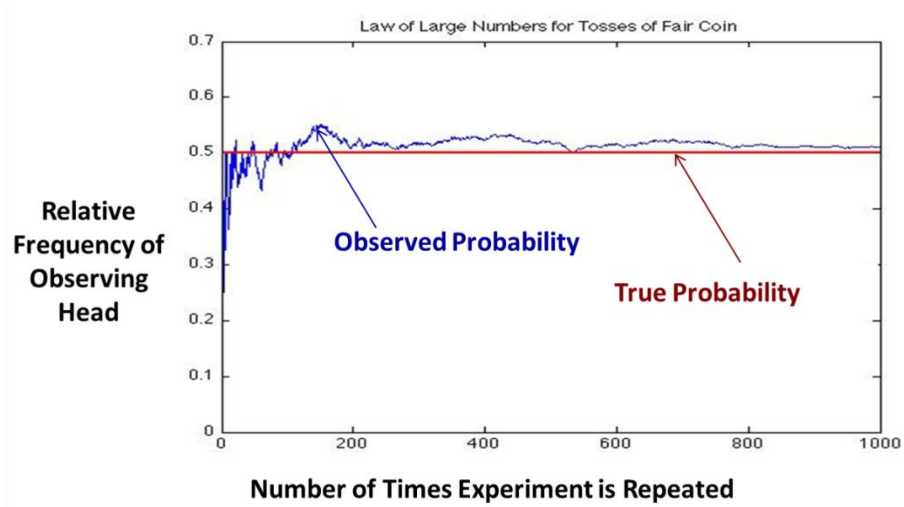
Probability

Probability of any outcome of a random phenomenon is the proportion of times the outcome would occur in a very long series of repetitions.

All outcomes are equally likely in a random phenomenon.

For example, if you consider a tossing fair coin, there are two possible sides to observe (head or tail) each with probability of 0.5. Does that mean when you toss a fair coin you observe head once and tail once?

- ❖ $P(A)$ is the relative frequency of occurring event A in the long run if the experiment is repeated several times (500 or 1000 or more).



The **probability of event A**, denoted by $P(A)$, is a numerical measure that shows likelihood or chance of occurring event A.

The **cardinality** of an event A is denoted by $|\boldsymbol{A}|$ and is the number of elements in the event A. The cardinality of Ω can be countably finite, countably infinite or uncountably infinite.

Ex 9: Roll two fair regular dice and observe the top faces. $\Omega = \{(1,1),\ (1,2),\ (1,3), \dots, (6,6)\}$ has 36 elements and finite. (**Countably finite)**

Ex 10: Toss a fair coin until 1st H is observed. $\Omega = \{H,\ TH,\ TTH,\ TTTH, \dots\}$ has infinite countable elements. (**Countably infinite**)

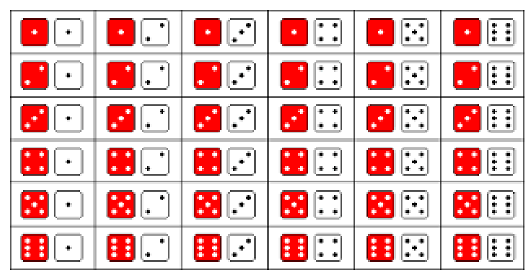
Ex 11: Noncoherent receivers measure the energy of the incoming waveform. Since energy is a nonnegative quantity, the sample space consists of nonnegative real numbers. Ω is [0,∞). (**Uncountably infinite**)

Let $|\Omega| = \boldsymbol{N} < \infty$, where all N outcomes of an experiment are equally likely. Then

$$\boldsymbol{P(A)} = \frac{|\boldsymbol{A}|}{\boldsymbol{N}}$$

$P(\Omega) =$ $\qquad$ $P(\emptyset) =$

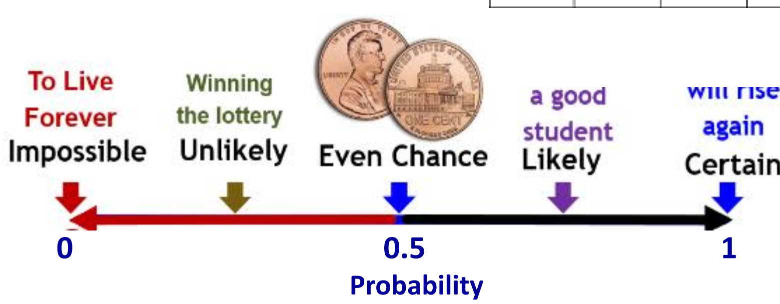
Ex 12: What is the probability of getting a total of 11 when two dice are tossed?



$|\Omega| = N =$

A = {Total is 11} =

$P(A) = \dfrac{|A|}{N} =$



Axioms of Probability

Axiom 1 **:** $\boldsymbol{P(A) \geq 0}$

Axiom 2 : $\boldsymbol{P(\Omega) = 1}$

Axiom 3 : If $A_1, A_2, \dots$ are a sequence of **pairwise mutually exclusive** (or **disjoint**) events in Ω (i.e. $A_i \cap A_j = \emptyset$ for all $i \neq j$), then

$$\boldsymbol{P(A_1 \cup A_2 \cup \dots) = \sum_{i=1}^{\infty} P(A_i)}$$

If $A_1, A_2, \dots, A_n$ are **pairwise mutually exclusive** and by taking $A_{n+1} = A_{n+2} = \cdots = \emptyset$, then

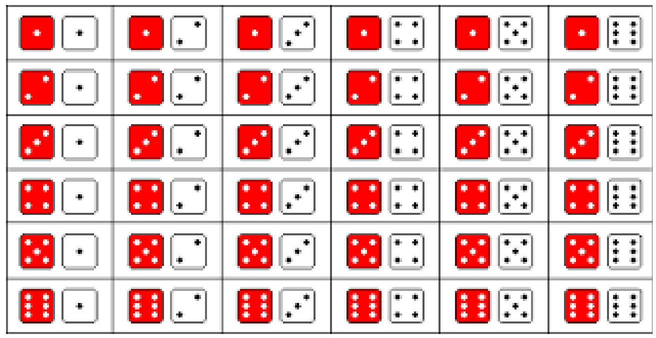
$$\boldsymbol{P(A_1 \cup A_2 \cup \dots \cup A_n) = \sum_{i=1}^{n} P(A_i)}$$

For each event A,

$$\boldsymbol{P(A) + P(A^c) = 1} \quad \textbf{and} \quad \boldsymbol{P(A) = 1 - P(A^c)} \quad \textbf{and} \quad \boldsymbol{P(A^c) = 1 - P(A)}$$

If $\boldsymbol{A \subset B}$, then $\boldsymbol{P(A) \leq P(B)}$

Ex 13: What is the probability of getting a total of 7 or 11 when two dice are tossed?



$|\Omega| = N =$

$A = \{\text{Total is } 11\} =$

$B = \{\text{Total is } 7\} =$

$P(B) =$

A and $B = A \cap B = \{\text{Total is 7 and } 11\} =$

A or $B = A \cup B = \{\text{Total is 7 or } 11\} =$

$P(A \cup B) =$

What is the probability of getting a total of at least 4 when two dice are tossed?

$C = \{\text{Total is at least } 4\} =$

$C^c =$

$P(C^c) =$

- $P(A \cup B) = P(\ A \text{ or } B) \ = \ P(\text{At least one of the two events occur})$

The Additive Law of Probability

If $A,\ B$ are any events, then

$$P(A \cup B) = \ P(A) + \ P(B) - P(A \cap B)$$

If $A,\ B$, and C are any events, then

$$P(A \cup B \cup C) = \ P(A) + \ P(B) + \ P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) \ + \ P(A \cap B \cap C)$$

If A, B are any events, then $P(A \cup B) \leq \ P(A) + \ P(B)$

Ex 14: Past records show that 4 of 135 parts are defective in length, 3 of 141 are defective in width, and 2 of 347 are defective in both. Use these figures to estimate probabilities of the individual events assuming that defects occur independently in length and width.

a) What is the probability that a part produced under the same conditions will have at least one of the two defects (that is, defective in length or defective in width)?

b) What is the probability that a part will have neither defect?

c) What is the probability that a part will have defect in length but no defect in width?

If $A_1,\ A_2,\ \dots\ ,\ A_n$ are any events, then

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P\left(\bigcup_{i=1}^{n} A_i\right) =$$

$$\sum_{i=1}^{n} P(A_i) - \sum \sum_{1 \leq i,j \leq n} P\left(A_i \cap A_j\right) + \sum \sum_{1 \leq i,j,k \leq n} \sum P\left(A_i \cap A_j \cap A_k\right) - \cdots + (-1)^{n-1} P(A_1 \cap A_2 \dots \cap A_n)$$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$