$\mu = E(X)$

Functions of Random Variables and Their Expected Values

Similar to the univariate case, we can also find the **expected value (average or mean)** of functions of multiple random variables.

Expected value of a function of two random variables (Bivariate Expectations)

Suppose $g(x,y)$ is a real-valued function. If X and Y are random variables with joint pmf of $p_{X,Y}(x,y)$ or pdf of $f_{X,Y}(x,y)$, then $\boldsymbol{E}[\,\boldsymbol{g}(\boldsymbol{X},\ \boldsymbol{Y})\,]$ denotes the expected value of $\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y})$ and is computed as

$$\boxed{\boldsymbol{E}[\,\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y})\,]} = \sum_{all\ x}\sum_{all\ y}[\,\boldsymbol{g}(\boldsymbol{x},\ \boldsymbol{y}) \times \boldsymbol{p}_{X,Y}(\boldsymbol{x},\boldsymbol{y})\,] \qquad \text{; if } X \text{ and } Y \text{ are DISCRETE}$$

Notation

Recall: $E(X) = \sum_{all\ x} x\, p(x)$

$$\boldsymbol{E}[\,\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y})\,] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}) \times \boldsymbol{f}_{X,Y}(\boldsymbol{x},\boldsymbol{y})\ \boldsymbol{dx}\,\boldsymbol{dy} \qquad \text{; if } X \text{ and } Y \text{ are CONTINUOUS}$$

Recall: $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

Ex 1: The input (X) and output (Y) of a system subject to random perturbations are described probabilistically by the joint pmf given in the table below:

		Output (Y)				
		1	**2**	**3**	**4**	**5**
Input (X)	**1**	7/71	2/71	8/71	5/71	4/71
	2	4/71	2/71	5/71	5/71	9/71
	3	2/71	4/71	8/71	5/71	1/71

(a) Compute the expected value of absolute difference between input and output of the system.

Here, $g(x,y) = |x-y|$. Find $E(|X-Y|)$

$$E[|X-Y|] = \sum_{all\ x}\sum_{all\ y} |x-y|\ p(x,y)$$

$$= \left[|1-1| \times \frac{7}{71}\right] + \left[|1-2| \times \frac{2}{71}\right] + \cdots + \left[|3-5| \times \frac{1}{71}\right]$$

$$= \frac{110}{71} = 1.549$$

This is the typical (or average) value of absolute difference between input and output.

(b) Compute the E(XY). Here, $g(X,Y) = XY$.

$$E(XY) = \sum_{\text{all } x}\sum_{\text{all } y} xy\, p(x,y)$$

$$= \left(1\times1\times\frac{7}{71}\right) + \left(1\times2\times\frac{2}{71}\right) + \cdots + \left(3\times5\times\frac{1}{71}\right)$$

$$= \frac{428}{71} = 6.025$$

(c) Compute the expected value of ratio between input and output. Here, $g(X,Y) = \frac{X}{Y}$

$$E\left(\frac{X}{Y}\right) = \sum_{\text{all } x}\sum_{\text{all } y} \frac{x}{y}\, p(x,y)$$

$p(1,2)$ $p(3,5)$

$$= \left(\frac{1}{1}\times\frac{7}{71}\right) + \left(\frac{1}{2}\times\frac{2}{71}\right) + \cdots + \left(\frac{3}{5}\times\frac{1}{71}\right) = 0.795775$$

(d) Compute the E[(X-1)/Y]. Here, $g(X,Y) = \frac{X-1}{Y}$

$$E\left(\frac{X-1}{Y}\right) = \sum_{\text{all } x}\sum_{\text{all } y} \left(\frac{x-1}{y}\right) p(x,y)$$

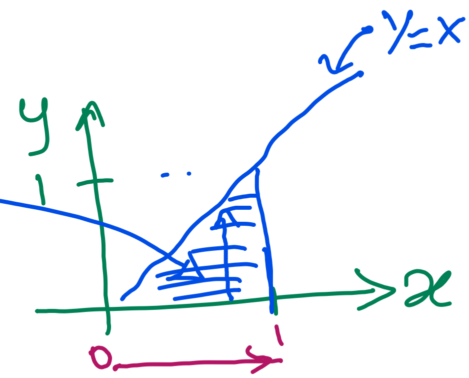
$p(2,1)$ $p(3,5)$

$$= \left[\left(\frac{1-1}{1}\right)\times\frac{7}{71}\right] + \left[\left(\frac{2-1}{1}\right)\times\frac{4}{71}\right] + \cdots + \left[\left(\frac{3-1}{5}\right)\times\frac{1}{71}\right]$$

$$= \frac{25.95}{71} = 0.365493$$

Ex 2: Consider the following function.

$$f_{X,Y}(x,y) = \begin{cases} \frac{5}{3}(1 + x^2y) \ ; & 0 < y \le x \le 1 \\ 0 \ ; \ \text{Elsewhere} \end{cases}$$



(a) Compute E(XY).

Here, $g(X,Y) = XY$.

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

$0 < \boxed{y} < x < 1$

$$= \int_0^1 \int_0^x xy\, \frac{5}{3}(1+x^2y)\, dy\, dx$$

$0 < y < \boxed{x} < 1$

$$\text{OR} \quad \int_0^1 \int_y^1 xy\, \frac{5}{3}(1+x^2y)\, dx\, dy$$

$$= \frac{145}{504} \approx 0.2877$$

(b) Compute the expected value of ratio between Y and X. $g(X,Y) = \frac{Y}{X}$.

$$E\left(\frac{Y}{X}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{y}{x} f(x,y)\, dx\, dy$$

$$= \int_0^1\int_y^1 \frac{y}{x}\, \frac{5}{3}(1+x^2 y)\, dx\, dy$$

OR $\int_0^1\int_0^x \frac{y}{x}\, \frac{5}{3}(1+x^2 y)\, dy\, dx$ $= \underline{\underline{0.52778}}$

$f(x,y) = f(x)f(y)$ or $p(x,y) = p(x)p(y)$

Independence in probability **can make our lives easier in lots of ways!**

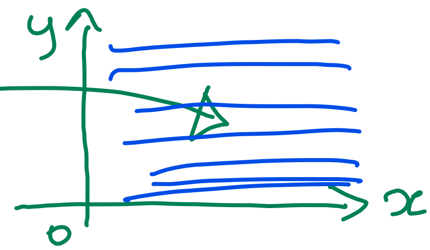
Let $g(X)$ is a function of X, and $h(Y)$ is a function of Y. **If** two random variables X **and** Y **are independent**, then

Product of function of X and function of Y

$$E[\,g(X) \times h(Y)\,] = E[\,g(X)\,] \times E[\,h(Y)\,]$$

Product of Expected values

Ex 3: The amplitudes of two signals X and Y have joint pdf:



$$f_{X,Y}(x,y) = \begin{cases} e^{-x/2}\, y\, e^{-y^2} & ; \quad x > 0,\ y > 0 \\ 0 ; & \text{Otherwise} \end{cases}$$

(a) Are X, Y independent? Yes, because support of X and support of Y are NOT related, and the joint pdf can be written as a product of a non negative function of X and a non negative function of Y.

(b) If $E(X) = 2$ and $E(Y) = \sqrt{\pi}/2$, then compute the $E(XY)$. Here, $g(X,Y) = XY$

$g(x)$ $h(y)$

Since we found X, Y are independent,

$$E(XY) = E(X) * E(Y)$$

$$= 2 * \frac{\sqrt{\pi}}{2}$$

$$= \sqrt{\pi} = \underline{\underline{1.772}}$$