Bivariate Probability Distributions

Recall: A (univariate) random variable is defined to be a function from a sample space Ω into the real numbers, ℝ. Often we are interested simultaneously in two or more outcomes of a random experiment rather than one. For example, we might be interested in…

- the voltage signals at two points in a circuit at some specific time
- the repeated measurement of a certain quantity such as the repeated measurement ("sampling") of the amplitude of an audio or video signal that varies with time
- Number of customers waiting in two lines at the grocery store
- Daily average temperature and power usage
- Number of hours spent studying and test score
- Dosage of a drug and blood pressure

In each of these examples, there are two random variables, and we are interested in the 2-dimensional random vector (X, Y). The concept of discrete/continuous random variable, independent and pmfs/pdfs can be extended to **bivariate random variables,** as well as n-dimensinoal random variables.

Definition: (Joint PMF)

If X and Y are jointly distributed/bivariate (absolutely) **discrete** random variables, then

$$\boldsymbol{p_{X,Y}(x,y) = P(X = x\ , Y = y) \quad = \quad P(X = x\ \cap\ Y = y)}$$

is a **joint (bivariate) probability <u>mass</u> function (joint pmf)** if it holds the following conditions:

$*\quad \mathbf{0 \le p_{X,Y}(x,y) \le 1 \quad for\ all \quad x\ in\ X\ , \qquad y\ in\ Y}$

$*\quad \displaystyle\sum_{all\ x}\sum_{all\ y} \boldsymbol{p_{X,Y}(x,y)} \ = \mathbf{1}$

Ex 1: In a binary communications channel, let X denote the bit sent by the transmitter and let Y denote the bit received at the other end of the channel.

		Receive (Y)	
		0	**1**
Send (X)	**0**	**0.45**	**?**
	1	**0.03**	**0.47**

(a) What is the probability that channel sends 0 and 1 is received from the other side of channel?

(b) Compute the probability that absolute difference between bit send and bit receive is zero.

(c) Compute the probability that absolute difference between bit send and bit receive is not zero.

Ex 2: The input (X) and output (Y) of a system subject to random perturbations are described probabilistically by the joint pmf given in the table below:

		Output (Y)				
		1	**2**	**3**	**4**	**5**
Input (X)	**1**	**7/71**	**2/71**	**8/71**	**5/71**	**4/71**
	2	**4/71**	**2/71**	**5/71**	**5/71**	**9/71**
	3	**2/71**	**4/71**	**8/71**	**5/71**	**1/71**

(a) What is the probability that input is 1 and output is an odd number?

(b) What is the probability that input is 2 and output is less than 4?

(c) What is the probability that input is 2 or output is 3?

(d) What is the probability that input is 3?

Definition: Relationship between Bivariate PMFs and Univariate PMFs
Let X and Y are jointly distributed/bivariate (absolutely) **discrete** random variables with joint pmf $\boldsymbol{p_{X,Y}(x,y)}$.
Then the **<u>marginal</u> probability <u>mass</u> function of $\boldsymbol{X}$** variable is given by

$$\boldsymbol{p_X(x) = \sum_{all\ y} p_{X,Y}(x\,,y)}$$

The **<u>marginal</u> probability <u>mass</u> function of $\boldsymbol{Y}$** variable is given by

$$\boldsymbol{p_Y(y) = \sum_{all\ x} p_{X,Y}(x\,,y)}$$

Ex 2 Contd.: Compute the marginal pmf of input (X) and marginal pmf of output (Y) of the system.

Ex 3: If the joint probability distribution of X and Y is given as below, compute the marginal pmf of Y.

$$p_{X,Y}(x,y) = \begin{cases} \dfrac{x+y}{30} & ; \ x = 0,1,2,3 \ ; \ y = 0,1,2 \\ 0 & ; \ \text{Otherwise} \end{cases}$$

Definition: (Joint PDF)
If X and Y are jointly distributed/bivariate (absolutely) **continuous** random variables, then $\boldsymbol{f_{X,Y}(x,y)}$ is a **joint (bivariate) probability <u>density</u> function (joint pdf)** if it holds:

* $\boldsymbol{f_{X,Y}(x,y) \geq 0 \ \text{ for } \ -\infty < x < \infty \ , \quad -\infty < y < \infty}$
* $\boldsymbol{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1}$

Ex 4: The amplitudes of two signals X and Y have joint pdf:

$$f_{X,Y}(x,y) = \begin{cases} e^{-x/2}\, y\, e^{-y^2} & ; \quad x > 0\,, \;\; y > 0 \\ 0\,; & \text{Otherwise} \end{cases}$$

(a) Draw the support of this pdf in Cartesian plane.
(Support is the set of possible values).

(b) Prove the given $f_{X,Y}(x,y)$ is a valid joint pdf.

(c) Compute the probability that X>5 and Y<3. That is, find P(X>5, Y<3).

(e) Compute the probability that 2X < Y−10. That is, find P(2X < Y−10).

Definition: Relationship between Bivariate PDFs and Univariate PDFs

Let X and Y are jointly distributed/bivariate (absolutely) **continuous** random variables with joint pdf $\boldsymbol{f_{X,Y}(x,y)}$.

Then the **<u>marginal</u> probability <u>density</u> function of $\boldsymbol{X}$** variable is given by

$$\boldsymbol{f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy}$$

That is, if we **integrate $\boldsymbol{Y}$** continuous variable **out** of the bivariate/joint pdf, we are left with the pdf of the X variable.

The **<u>marginal</u> probability <u>density</u> function of $\boldsymbol{Y}$** variable is obtained by **integrating $\boldsymbol{X}$** continuous variable **out** of the bivariate/joint pdf as shown below.

$$\boldsymbol{f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx}$$

Ex 4 Contd. : Find the marginal pdf of amplitude of each signal. Check whether each is a valid pdf to verify your answer.

Ex 5: Consider a series connection of two components, with respective lifetimes X and Y. The joint pdf of the lifetimes is given by

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{200} & ; \quad (x,y) \in A \\ 0 & ; \text{ Elsewhere} \end{cases}$$

where A is the triangular region in the (x,y) plane with
the vertices (100, 100), (100, 120), and (120, 120).

(a) Draw and shade the support of this function on Cartesian plane.

(b) Compute the marginal pdf of X and marginal pdf of Y.

(c) Compute the probability that the reliability of this connection is more than 105 hours.

Independent Random Variables

Definition: Independence of Random Variables

Two random variables, X and Y, are **independent** if and only if

- **Using pdf:**

$$f_{X,Y}(x,y) = f_X(x) \times f_Y(y)$$

where $f_X(x)$ and $f_Y(y)$ are marginal pdfs.

- **Using pmf:**

$$p_{X,Y}(x,y) = p_X(x) \times p_Y(y)$$

where $p_X(x)$ and $p_Y(y)$ are marginal pmfs.

for every pair of real numbers (x, y). **Be careful with the support!!!**

❖ **If the support is dependent, then variables are NOT independent.**

❖ Further, we **don't need the marginal pmfs/pdfs to show independence**! We can show independence using non-negative functions of the variables themselves.

"How to tell whether X and Y are independent or not?"

Are the **supports of X and Y** dependent? —YES→ **X and Y are dependent**

↓ NO

Is the **joint** PDF or PMF **a product of Marginals?** —YES→ **X and Y are independent**

↓ NO

X and Y are dependent

Ex 6: Explain whether the two variables are independent based on pmf or pdf for the following scenarios:

(a) The input (X) and output (Y) of a system subject to random perturbations are described probabilistically by the joint pmf given in the table below:

		Output (Y)				
		1	**2**	**3**	**4**	**5**
Input (X)	**1**	7/71	2/71	8/71	5/71	4/71
	2	4/71	2/71	5/71	5/71	9/71
	3	2/71	4/71	8/71	5/71	1/71

(b) The amplitudes of two signals X and Y have joint pdf:

$$f_{X,Y}(x, y) = e^{-x/2}\, y\, e^{-y^2} \quad ; \quad x > 0\,,\ \ y > 0$$

(c) Consider the joint probability distribution of X and Y given by

$$p_{X,Y}(x,y) = \begin{cases} \dfrac{xy^2}{13} \end{cases} ; \quad (x,y) = (1,1),(1,2),(2,2) \quad \text{and zero otherwise}$$

(d) Consider a series connection of two components, with respective lifetimes X and Y. The joint pdf of the lifetimes is given by

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{200} & ; \quad (x,y) \in A \\ 0 & ; \text{ Elsewhere} \end{cases}$$

where A is the triangular region in the (x,y) plane with the vertices (100, 100), (100, 120), and (120, 120).

Ex 7: Let the marginal pmf of each of X, Y variables is as given below.

$$\text{pmf of } X \text{ is} \quad p_X(x) = \frac{x}{6} \quad ; \quad x = 1,2,3 \quad \text{and zero otherwise}$$

$$\text{pmf of } Y \text{ is} \quad p_Y(y) = \frac{y^2}{5} \quad ; \quad y = 1,2 \quad \text{and zero otherwise}$$

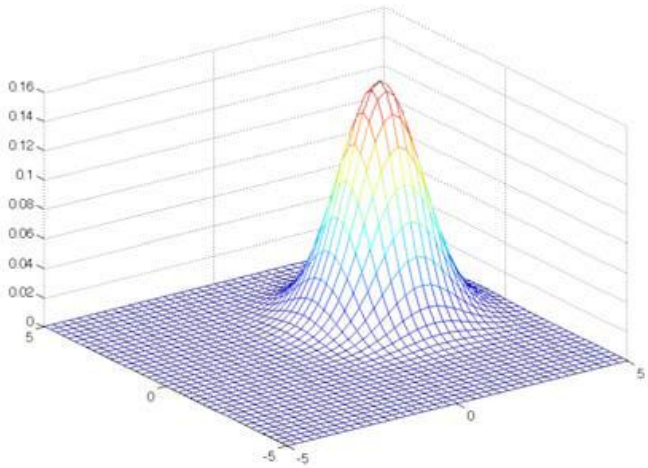
If the two variables are independent, then

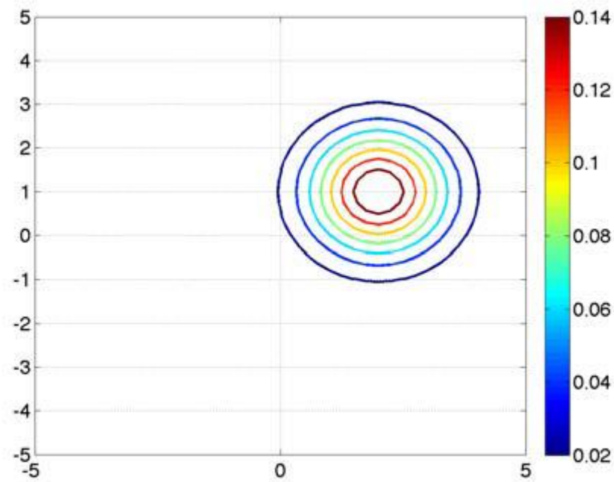
(a) compute the probability that both of X and Y is an odd number.

(b) compute the joint pmf of variables.

Ex 8: Suppose that the lifetimes of two components are independent of one another.
The lifetime of the first component (X_1) has an exponential distribution with mean of 1000 hours.
The lifetime of the 2^{nd} component (X_2) has an exponential distribution with mean of 1200 hours.
Compute the probability that lifetime of first component is less than life of 2^{nd} component.

Ex 9: Let X and Y denote the position of an electron in the 2-dimensional Cartesian plane. Due to the uncertainty principle X and Y can't be measured exactly and are random variables.
You are told that the measurement along the X-axis is independent from the measurement along the Y -axis.
Furthermore, $X \sim N(2,1)$ and $Y \sim N(1,1)$.
Find the joint pdf of X and Y.



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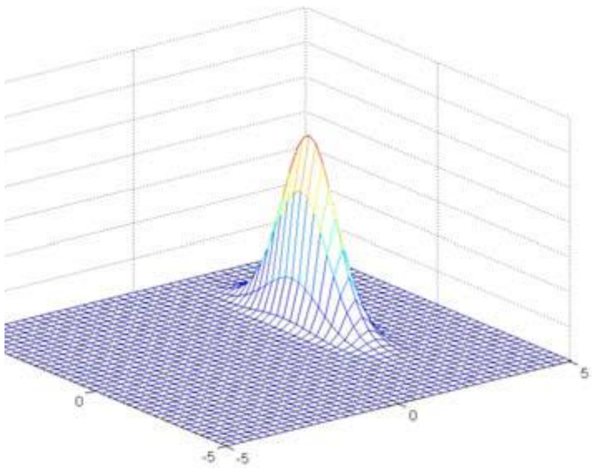
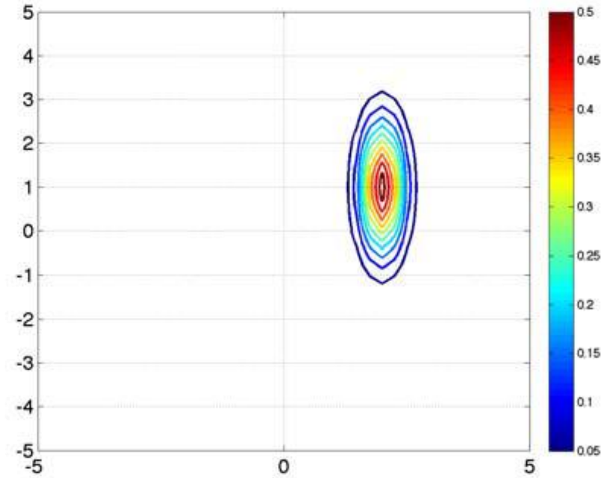
%% Plot of Bivariate Normal (Gaussian) pdf
mu = [2 1]; %Define Means %
sigma = [1 0; 0 1]; %Define the variance-covariance matrix%
% Here, since X, Y are independent, cov(X,Y)=0 %
x = -5:0.2:5; % Define a range of value for X%
y = -5:0.2:5; % Define a range of value for Y%
[X,Y] = meshgrid(x,y); %Create a 2-D grid coordinates based on the
coordinates contained in vectors x and y%
V = [X(:) Y(:)];
pdf = mvnpdf(V,mu,sigma); %Obtain the joint Normal (Gaussian) pdf %
f = reshape(pdf,length(x),length(y)); %obtain the joint pdf values at the
values of X and Y %

% To obtain 3-D figure of joint pdf %
figure(1)
surf(x,y,f)
caxis([min(f(:))-0.5*range(f(:)),max(f(:))])
%axis([-5 5 -5 5 0 0.2])
xlabel('X')
ylabel('Y')
zlabel('Joint Probability Density Function')
grid on; % Adds grid on the plot %

% To obtain the plot of contours%
figure(2)
contour(X,Y,f)
colorbar
xlabel('X')
ylabel('Y')
grid on; % Adds grid on the plot %
```





Ex 10: Let $X_1, X_2, \dots, X_n$ be independent and identically distributed normal random variables with mean μ and variance σ^2. Write the joint pdf of these variables.