Functions of Random Variables and Their Expected Values

Similar to the univariate case, we can also find the **expected value (average or mean)** of functions of multiple random variables.

Expected value of a function of two random variables (Bivariate Expectations)

Suppose $g(x, y)$ is a real-valued function. If X and Y are random variables with joint pmf of $p_{X,Y}(x, y)$ or pdf of $f_{X,Y}(x, y)$, then $\boldsymbol{E}[\ \boldsymbol{g}(\boldsymbol{X},\ \ \boldsymbol{Y})\ ]$ denotes the expected value of $\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y})$ and is computed as

$$\boldsymbol{E}[\ \boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y})\ ] = \sum_{all\ x} \sum_{all\ y} [\ \boldsymbol{g}(\boldsymbol{x},\ \ \boldsymbol{y}) \times \boldsymbol{p}_{X,Y}(\boldsymbol{x}, \boldsymbol{y})\ ] \qquad ;\ \textbf{if } X \text{ and } Y \textbf{ are DISCRETE}$$

$$\boldsymbol{E}[\ \boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y})\ ] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}) \times \boldsymbol{f}_{X,Y}(\boldsymbol{x},\boldsymbol{y})\ \boldsymbol{dx}\,\boldsymbol{dy} \qquad ;\ \textbf{if } X \text{ and } Y \textbf{ are CONTINUOUS}$$

Ex 1: The input (X) and output (Y) of a system subject to random perturbations are described probabilistically by the joint pmf given in the table below:

		Output (Y)				
		1	**2**	**3**	**4**	**5**
Input (X)	**1**	**7/71**	**2/71**	**8/71**	**5/71**	**4/71**
	2	**4/71**	**2/71**	**5/71**	**5/71**	**9/71**
	3	**2/71**	**4/71**	**8/71**	**5/71**	**1/71**

(a) Compute the expected value of absolute difference between input and output of the system.

(b) Compute the E(XY).

(c) Compute the expected value of ratio between input and output.

(d) Compute the E[(X-1)/Y].

Ex 2: Consider the following function.

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{5}{3}(1+x^2y) & ; \quad 0 < y \leq x \leq 1 \\ 0 & ; \text{ Elsewhere} \end{cases}$$

(a) Compute E(XY).

(b) Compute the expected value of ratio between Y and X.

Independence in probability can make our lives easier in lots of ways!

Let $\boldsymbol{g(X)}$ is a function of X, and $\boldsymbol{h(Y)}$ is a function of Y. **If** two random variables $\boldsymbol{X}$ **and** $\boldsymbol{Y}$ **are independent**, then

$$\boldsymbol{E[\ g(X) \times h(Y)\ ] = E[\ g(X)\ ] \times E[\ h(Y)\ ]}$$

Ex 3: The amplitudes of two signals X and Y have joint pdf:

$$f_{X,Y}(x,y) = \begin{cases} e^{-x/2}\ y\ e^{-y^2} & ; \quad x > 0\ ,\ \ y > 0 \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

(a) Are X, Y independent?

(b) If $E(X) = 2$ and $E(Y) = \sqrt{\pi}/2$, then compute the $E(XY)$.

(c) Compute $E(X^k)$ where k is any integer.

(d) Use your answer in part (c) to compute $Var(X)$.

(e) Compute the expected value of ratio between signal X and signal Y.

(f) Compute $E(e^{sX})$ where s is a constant.

The **transform** associated with a random variable X (also referred to as the associated **moment generating function, mgf**) is a function $M_X(s)$ of a scalar parameter s, defined by

$$\boldsymbol{M_X(s)} \; = \; \boldsymbol{E[\,e^{sX}\,]} = \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right.$$

It is important to realize that the transform is not a number but rather a function of a parameter s .

From Transforms to Moments

If $E[\,X^n\,]$ of the random variable X exists, then it can be found by differentiating mgf n times with respect to s and then setting $s = 0$ as shown below:

$$\boldsymbol{E[\,X^n\,]} \; = \; \frac{\boldsymbol{d^n}}{\boldsymbol{ds^n}} \boldsymbol{M_X(s)} \Big|_{\boldsymbol{s=0}}$$

Ex 4: The joint pmf of two variables is given in the table below:

$X \setminus Y$	0	2
0	0.04	0.16
1	0.16	0.64

(a) Are X, Y are independent? Explain.

(b) Compute E[X/(Y+1)].

(b) Compute the transform $M_Y(s) = E(e^{sY})$.

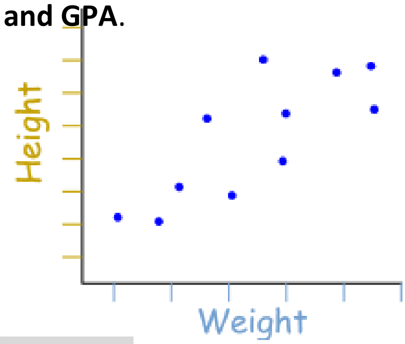
(c) Then use $M_Y(s) = E(e^{sY})$ to compute E(Y).

<u>Covariance and Correlation</u>

If two random variables are **not independent**, we are often interested in **quantifying the strength of the relationship** between them.

> For example, we could quantify the strength of the relationship between a randomly chosen person's height and weight or the strength of the relationship between a randomly chosen person's height and GPA.

We would expect the relationship between a randomly chosen person's **height and weight** would be much stronger than that between a randomly chosen person's **height and GPA**.



<u>Covariance and correlation</u> are two commonly used **numerical measures of the nature of a <u>linear</u> relationship between two random variables**.

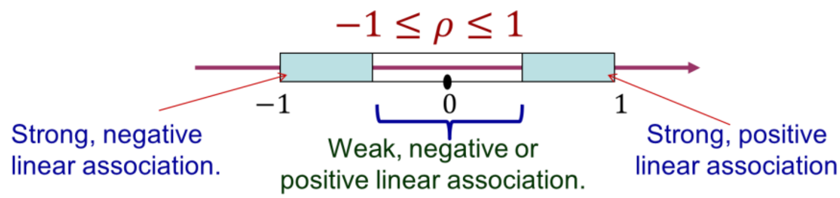
Correlation describes **both strength and nature (direction)** of a **<u>linear</u>** relationship between two random variables.

Let X and Y be random variables such that $(X) = \mu_X$, $Var(X) = \sigma_X^2$, $E(Y) = \mu_Y$, and $Var(Y) = \sigma_Y^2$

Covariance : $\boldsymbol{Cov\,(X\,,\;Y) \;=\; E[\,(X-\mu_X)(Y-\mu_Y)\,] \;=\; E(XY) - [E(X)E(Y)]}$

Correlation: $\boldsymbol{Corr\,(X\,,\;Y) \;=\; \rho_{XY} \;=\; \dfrac{Cov\,(X\,,\;Y)}{\sigma_X\,\sigma_Y}}$

Note: $\boldsymbol{-1 \le \rho_{XY} \le 1}$



Note: $\boldsymbol{Cov}\ (\boldsymbol{X}\ ,\ \boldsymbol{Y}) = \boldsymbol{Cov}\ (\boldsymbol{Y}\ ,\ \boldsymbol{X})$**.**

What is $\boldsymbol{Cov}\ (\boldsymbol{X}\ ,\ \boldsymbol{X})$**?**

If $\boldsymbol{X}$ **and** $\boldsymbol{Y}$ are **independent** random variables, then $\boldsymbol{Cov}\ (\boldsymbol{X}\ ,\ \boldsymbol{Y}) = \mathbf{0}$

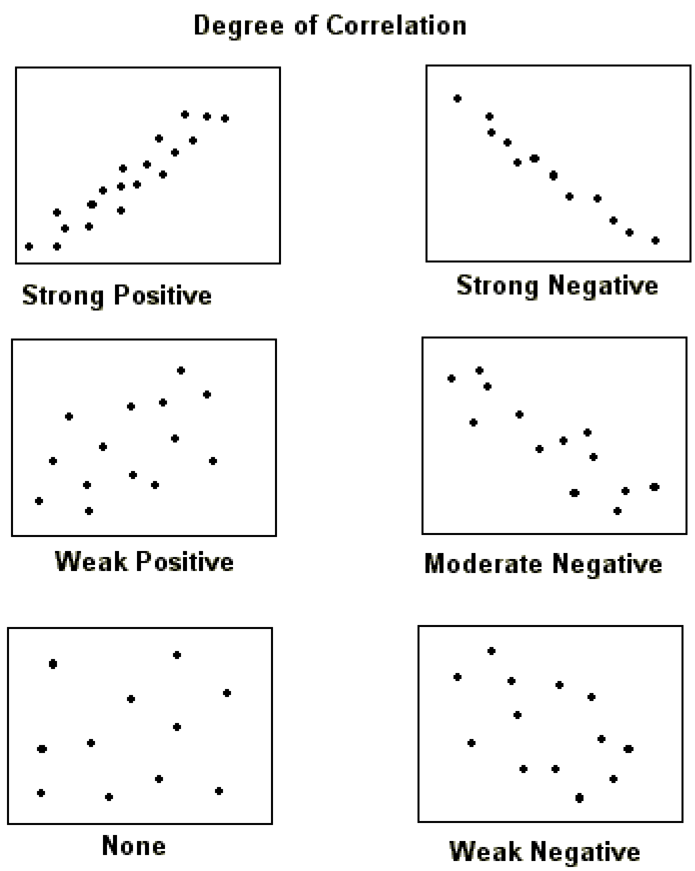
$$\text{because} \quad E(XY) - [E(X)E(Y)] = E(X)E(Y) - [E(X)E(Y)]$$

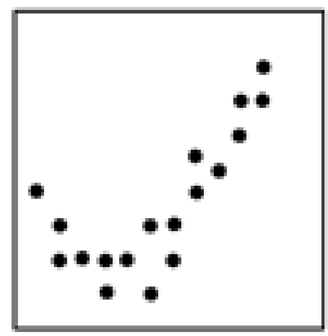
Thus, **independent random variables must be uncorrelated**.

However, the **converse is not true**. That is, **if the covariance** between two random variables **is zero**, **this does NOT necessarily imply they are independent**.

Ex 5: Compute $E(X)$, $E(Y)$, $E(XY)$ and then show that $E(XY) = E(X)E(Y)$ though $Y = |X|$ if the joint pmf of X and Y is as given below:

$$p_{X,Y}(x,y) = \begin{cases} 1/3\ ; & (x,y) = (-1,1), (0,0), (1,1) \\ 0\ ; & \text{Otherwise} \end{cases}$$





Ex 3 Contd.: The amplitudes of two signals X and Y have joint pdf:

$$f_{X,Y}(x,y) = \begin{cases} e^{-x/2}\, y\, e^{-y^2} \;; & x > 0,\; y > 0 \\ 0 \;; & \text{Otherwise} \end{cases}$$

Compute covariance between amplitudes of two signals.

Ex 2 Contd.: Consider the following function.

$$f_{X,Y}(x,y) = \begin{cases} \frac{5}{3}(1+x^2y) \ ; & 0 < y \le x \le 1 \\ 0 \ ; & \text{Elsewhere} \end{cases}$$

(a) Compute covariance between two variables and then explain the relationship between X and Y.

(b) Compute variance of X.

(c) If the variance of Y is 625/10206, compute the correlation between two variables and then explain the relationship between X and Y.

Covariance is also useful when calculating the variance of linear functions of random variables.

Expected Values of Linear Function of Random Variables

If X and Y are random variables, and $a, b,$ and c are constants, then $\boldsymbol{aX + bY + c}$ is a linear function of X, Y, and

$$\boldsymbol{E(aX + bY + c) = \ a\, E(X) + b\, E(Y) + c}$$

$$\boldsymbol{Var(aX + bY + c) = \ a^2\, Var(X) \ + \ b^2\, Var(Y) + 2ab\, Cov\,(X,\ \ Y)}$$

If $\boldsymbol{X}$ **and** $\boldsymbol{Y}$ are **independent** random variables, then $\boldsymbol{Cov\,(X,\ \ Y) = 0}$

Ex 2 Contd.: The input (X) and output (Y) of a system subject to random perturbations are described probabilistically by the joint pmf given in the table below:

$\boldsymbol{X \backslash Y}$	**1**	**2**	**3**	**4**	**5**
1	**7/71**	**2/71**	**8/71**	**5/71**	**4/71**
2	**4/71**	**2/71**	**5/71**	**5/71**	**9/71**
3	**2/71**	**4/71**	**8/71**	**5/71**	**1/71**

(a) Compute the covariance between input and output of the system. That is, compute $Cov(X, Y)$.

(b) Compute the expected value of difference between input and output of the system. That is, compute $E(X - Y)$.

(c) Compute the standard deviation of difference between input and output of the system. That is, compute $\sqrt{Var(X - Y)}$.

(d) Compute the correlation between input and output of the system and interpret it.

Ex 6: Let X be a fluctuating current in an electric circuit shown below and the average current is 1 Amp. Let the sum of another current of 5 Amp and the fluctuating current flows through a 1-ohm resistor and the average power dissipated over the resistor is 40 watts. Compute the variance and standard deviation of the fluctuating current X. (Note: Power = I^2R if I is current and R is resistance)