Random Variables and their Probability Distributions

Random Variables

Often, we are interested in a numerical measurement of the outcome of a random experiment.

For example, we might want to know the number of heads observed when tossing a coin 3 times.

However, the observed number varies depending on the actual result of the experiment. This type of variable is called a **random variable**.

Definition: (Random Variable, R.V.)

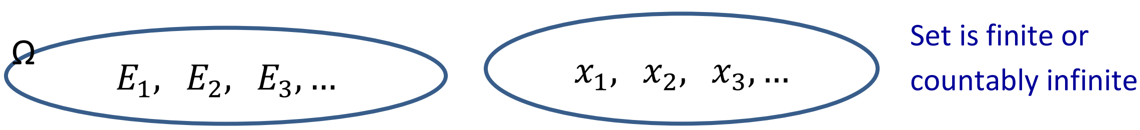
A **random variable** is a function that associates a real number with each element in the sample space.

That is, a random variable is a function from a sample space, Ω, into the real numbers, $\mathbb{R}$.

Note: Capitol letters such as $X, Y, Z,$ and W represent random variables while small letters such as $x, y, z,$ and w represent a value of the R.V.

Identifying the type(s) of random variable(s) we are interested in helps us decide which methods and procedures are most appropriate for certain problems and answering specific questions. In general, there are two types of random variables:

- **Discrete Random Variable**: If the **sample space of a random variable is countably finite or countably indefinite**, then the random variable is a discrete random variable.



Ex 1: Roll two fair dice and the random variable X is the number of times 1 observed. Ω has 36 elements and countably finite. The support of random variable X has only 3 values (0, 1, 2) and countably finite.

Ex 2 (Optical communication systems): Since the output of a photodetector is a random number of photoelectrons, the random variable W is the number of photoelectrons. The support of random variable W is $\{0, 1, 2, 3, 4, \ldots\}$ is countably infinite.

- **Continuous Random Variable**: If the **sample space of a random variable is uncountably infinite**, then the random variable is a continuous random variable. A continuous random variable can take on any value in an interval.

Ex 3: Let X be the waiting time for a flight. It can be any value from 0 to ∞. Support of the random variable X is $[0,\ \infty)$.
Ex 4 (Wireless communication systems): Noncoherent receivers measure the energy of the incoming waveform. Let Y be this energy. The support of Y is [0,∞).

Probability Mass Function (PMF)

A mathematical function called **probability mass function (pmf)** characterizes a **discrete** random variable by listing possible values of random variable and associated probabilities with those numbers.

Definition: (Probability Mass Function, pmf)

The **probability mass function** or **pmf** of a discrete random variable X, denoted by $p(x)$ or $p_X(x)$, is given by

$$\boldsymbol{p(x) = P(X = x) \quad \textbf{for all } x}$$

such that

- $0 < p(x) < 1$ for all x in X and
- $\sum_{all\ x} p(x) = 1$, where the summation is over all values of x with nonzero probability.

Ex 6: **A Betting Game**

A player pays $1.50 to play the following game: A coin is tossed three times and the number of heads is counted. Let X be the number of heads observed.

The player receives $1 if X = 2 and $8 if X = 3 but nothing otherwise. Let Y be the reward to the player. Y is a function of the random variable X and their outcomes can be related back to the sample space of the underlying random experiment as follows:

Ω								
X								
Y								

(a) Plot of pmf of each random variable.

(b) Compute the probability that at most 2 heads are observed in this game.

(c) Compute the probability that player is rewarded some money when playing this game.

Expected Value (or Mean) of Discrete Random Variable

It is important to know the **expected value or "average"** of a random variable and its variation.

The expected value can tell us things like the "average" amount we might win (or lose!) in a game, the typical weight range for 3-month-old babies, the price we might pay for a typical house in a new city.

The expected value of a random variable is the weighted (in probability) average of possible values of variable. It is a single value to represent the variable. It is also called mean or expectation of variable.

Definition: (Expected Value of a Discrete random variable)

Let X be a **discrete** random variable and $p(x)$ is its pmf. Then the expected value of X is denoted by $\boldsymbol{E}[\boldsymbol{X}]$ or $\boldsymbol{\mu}$ and given by

$$\boldsymbol{\mu} = \boldsymbol{E}[\boldsymbol{X}] = \sum_{\boldsymbol{all\ x}} [\boldsymbol{x} \times \boldsymbol{p}(\boldsymbol{x})]$$

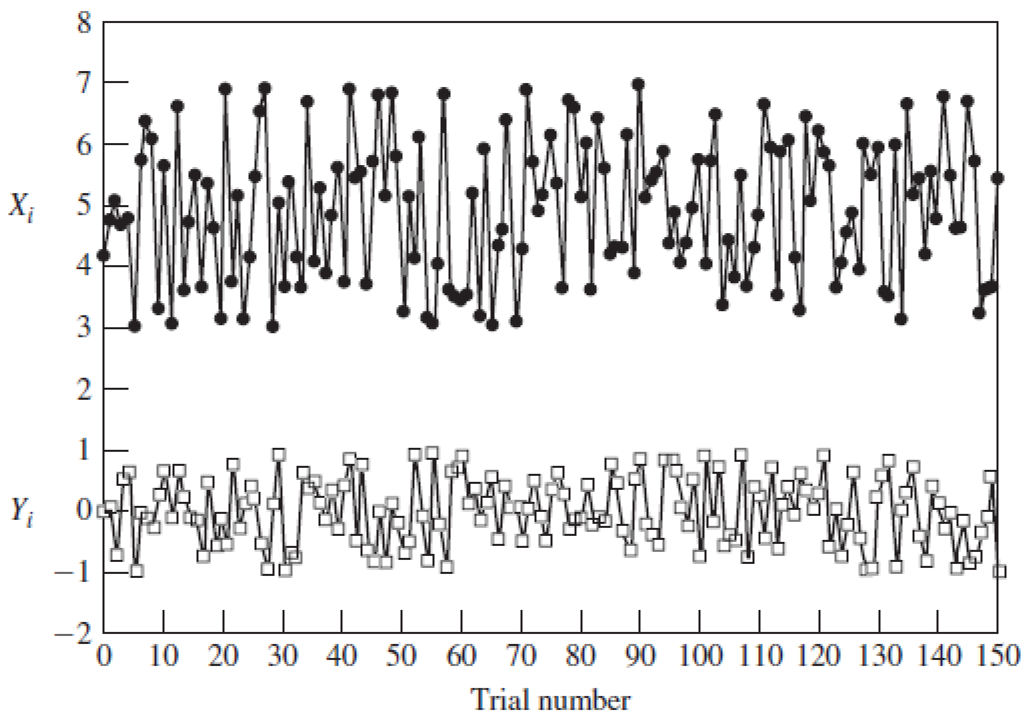


Figure 1: This graph shows 150 repetitions of the experiments yielding values of X and Y variables.
It is clear that X is centered at about 5 while Y is centered at about 0.
It is also clear that X is more spread out (or variable) than Y.

Ex 6 Contd.: **A Betting Game**
Compute the expected amount the player is rewarded when playing this game.

Variance and Standard Deviation of Discrete Random Variable

Variance explains the spread of values of random variable compared to mean of random variable.

Definition: (Variance of a Discrete random variable)

Let X be a **discrete** random variable and $p(x)$ is its pmf. Then the variance of X is denoted by $Var[X]$ or $\boldsymbol{\sigma}^2$ and given by

$$\boldsymbol{\sigma^2 = Var[X] = E\left[\left(X - E(X)\right)^2\right] \; = E[(X-\mu)^2]}$$

$$\boldsymbol{= \sum_{all\ x} [(x-\mu)^2 \times p(x)] \quad = \sum_{all\ x} [x^2 \times p(x)] - \mu^2}$$

The units of variance are squared units of random variable.

The square root of variance is the **standard deviation** of random variable and it explains average spread of random variable values from its mean. The units of standard deviation are same as units of random variable.

Definition: (Standard Deviation of a random variable)

If $Var[X]$ is the variance of random variable X, then the **standard deviation** of X is denoted by $\boldsymbol{\sigma}$ or σ_X and given by

$$\boldsymbol{\sigma = \sqrt{Var[X]}}$$

Ex 6 Contd.: **A Betting Game**
Compute the variance and standard deviation of amount the player receives.