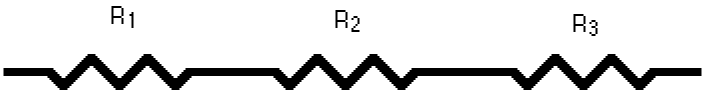
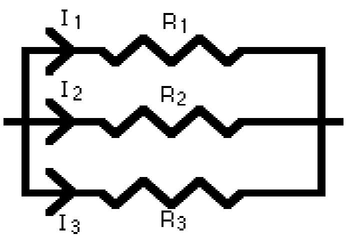
Convolution: Sum of Independent Random Variables

Suppose X and Y are independent random variables and $Z = X + Y$. Then the probability distribution of $Z = X + Y$ is called the **convolution.**

Let R_1, R_2 and R_3 be resistance of randomly selected 3 resistors each with resistance independent from others.





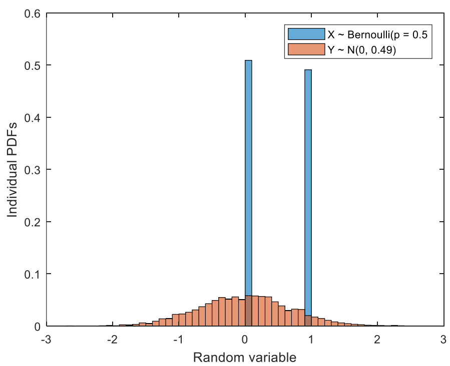
If X and Y are integer-valued (that is, discrete) random variables with pmfs $p_X(x)$ and $p_Y(y)$, respectively, then the pmf of $Z = X + Y$ is

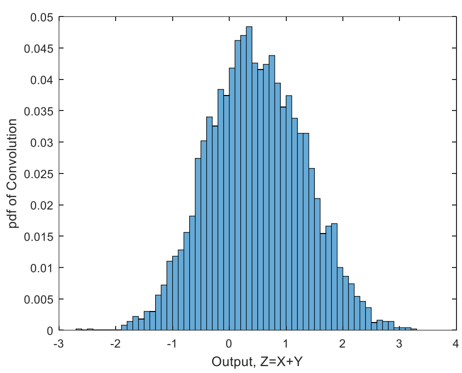
$$\boldsymbol{p_Z(z) = \sum_{all\ x} [\ p_X(x) \times\ p_Y(z-x)\ ]}$$

If X and Y are continuous random variables with pdfs $f_X(x)$ and $f_Y(y)$, respectively, then the pdf of $Z = X + Y$ is

$$\boldsymbol{f_Z(z) = \int_{-\infty}^{\infty} [\, f_X(x) \times f_Y(z-x)\,]\ dx}$$

Ex 1: A Noisy Channel. A transmitter sends a sequence of binary messages given by X which has a Bernoulli distribution with p = 0.5. A receiver on the other end of channel observes Z = X+Y where Y is a Gaussian (Normal) noise with zero mean and variance of 0.49. X, Y are independent.

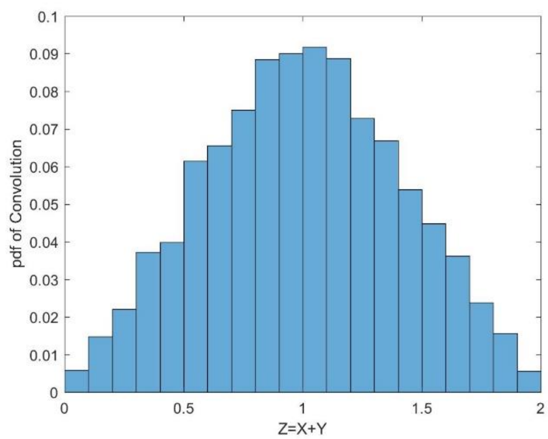


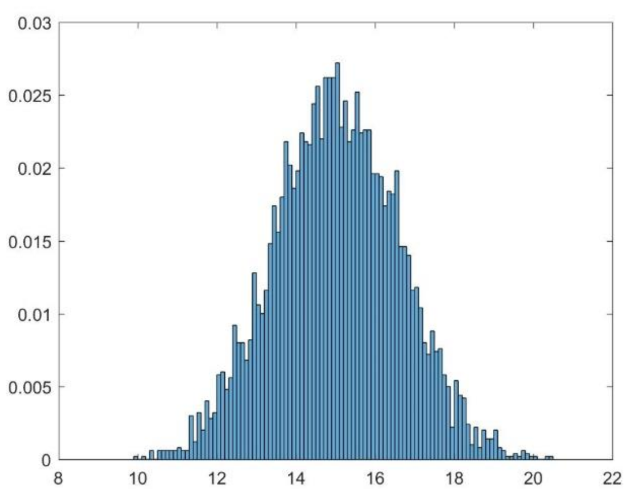



```
% Use Monte Carlo simulation for convolution of Bernoulli and Gaussian
distributions;
N = 5000;  % Number of replications = number of observations in the
sample;
% Generate one random sample of size N from the Bernoulli (p)%;
p=0.5; %Set the probability of success of Bernoulli trial
X = binornd(1,p, 1, N); % Generate a row vector of N values from
Bernoulli(p) distribution

% Generate one random sample of size N from the Normal (Mean, Var)%;
Mean_Y=0;  % Set the mean of Y
Var_Y=0.49; % Set the variance of Y
%Note, randn() function generates data from Normal (mean=0, var=1)
distribution %
Y = Mean_Y + sqrt(Var_Y)*randn(1,N); % Generate a row vector of N values
from N(Mean_Y,Var_Y) distribution;

% Compute Z = X + Y ;
Z = X + Y; %The output is sum of input binary bit and Gaussian noise;
% To plot empirical pdf of X and Y;
figure(1); histogram(X,'Normalization', 'probability','BinWidth',0.1);
hold on; histogram(Y,'Normalization', 'probability','BinWidth',0.1);
ylabel('Individual PDFs'); xlabel('Random variable'); hold off;
legend(['X ~ Bernoulli(p = ' num2str(p) ] , ['Y ~ N(0, ' num2str(Var_Y)
')'])
% To plot empirical pdf of the Convolution of X and Y;
figure(2); histogram(Z,'Normalization', 'probability','BinWidth',0.1);
ylabel('pdf of Convolution'); xlabel('Output, Z=X+Y');% ylim([0,0.5]);
mean(Z)   %Compute the mean of Z;
ans
       =0.4788

var(Z)    %Compute the variance of Z;
ans
       =0.7082
```

❖ If X_1, X_2 are **independent** continuous random variables so that $X_i \sim Uniform(0\,,1)$ for $i = 1, 2$, then the sum of those random variables

$$Y = X_1 + X_2 \text{ has the pdf of } \begin{cases} y\ ;\ 0 < y \le 1 \\ 2 - y\ ;\ 1 < y < 2 \\ 0\ ;\ \text{otherwise} \end{cases}$$ which is the function of triangle with base from 0 to 2 and height of 1





```
%%
clear %Clear the items in the Workspace Window %
clc %Clear the Command Window %
% Use Monte Carlo simulation to obtain convolution of
Continuous Uniform (0,1) RVs;
N = 5000;  % Number of replications = number of
observations in the sample;

% Generate two independent random samples each of
size N from the Continuous Uniform(0,1) distribution;
X = rand(1, N);
Y = rand(1, N);

% To plot empirical pdf of X and Y;
figure(1); histogram(X,'Normalization',
'probability','BinWidth',0.1);
figure(2); histogram(Y,'Normalization',
'probability','BinWidth',0.1);
ylabel('Empirical pdf'); xlabel('Random variable');

% Compute Z = X + Y ;
Z = X + Y; % The output is sum of two independent
Uniform RVs;
% To plot empirical pdf of the Convolution of X and
Y;
figure(3); histogram(Z,'Normalization',
'probability','BinWidth',0.1);
ylabel('pdf of Convolution'); xlabel('Z=X+Y');%
ylim([0,0.5]);
mean(Z)  %Compute the mean of Z;
ans =
    1.0003

var(Z)   %Compute the variance of Z;
ans =
    0.1669

% Use a for loop to obtain empirical pdf of
convolution of several;
% independent Continuous Uniform (0,1) RVs;
k = 30; %Set the number of RVs to be added %

Sum = zeros(1, N); %Initialize the Sum %
for i = 1:k
    x = rand(1, N); % Generate a row vector of N
values from Continuous Uniform(0,1) distribution
    Sum = Sum + x; % Compute the sum of independent
RVs %
    end

figure(4); histogram(Sum,'Normalization',
'probability','BinWidth',0.1);

mean(Sum)
ans =
   15.0339

var(Sum)
ans =
    2.4736
```

❖ If $X_1, X_2, \dots, , X_k$ are **independent** random variables so that $X_i \sim B(n_i\ ,\ p)$ for $i = 1, 2, \dots, k$, then the sum of those random variables

$$Y = X_1 + X_2 + \cdots + X_k \sim B\left(\sum_{i=1}^{k} n_i \quad , \quad p\right)$$

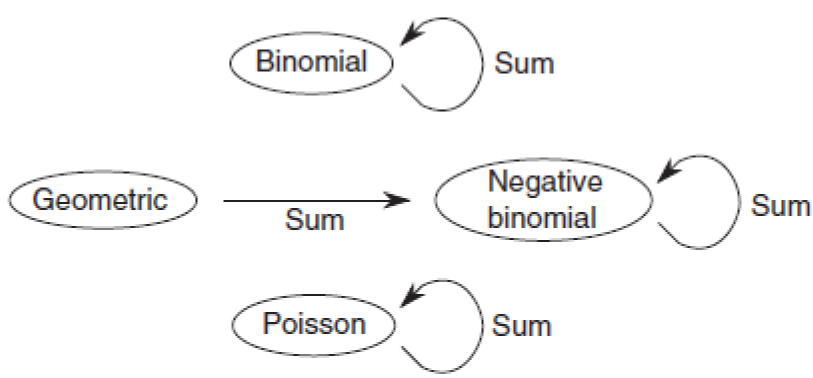
❖ If $X_1, X_2, \dots, , X_k$ are **independent** random variables so that $X_i \sim P(\lambda_i)$ for $i = 1, 2, \dots, k$, then the sum of those random variables

$$Y = X_1 + X_2 + \cdots + X_k \sim P\left(\sum_{i=1}^{k} \lambda_i\right)$$

Ex 2: (Poisson channel)

The Poisson(λ) random variable is a good model for the number of photoelectrons generated in a photodetector when the incident light intensity is λ . Now suppose that an additional light source of intensity μ is also directed at the photodetector. Then we expect that the number of photoelectrons generated should be related to the total light intensity λ +μ.

If X and Y are independent Poisson random variables with respective parameters λ and μ, then Z = X +Y has a Poisson(λ + μ).



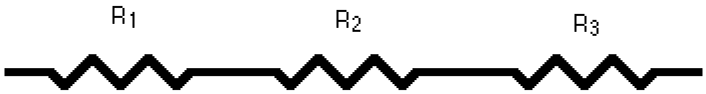
❖ If $X_1, X_2, \dots, , X_k$ are **independent** random variables so that $X_i \sim N(\mu_i\ ,\ \sigma_i^2)$ for $i = 1, 2, \dots, k$, then the sum of those random variables

$$Y_1 = X_1 + X_2 + \cdots + X_k \sim N\left(\sum_{i=1}^{k} \mu_i \quad , \quad \sum_{i=1}^{k} \sigma_i^2\right)$$

and the linear function of those random variables

$$Y_2 = a_1 X_1 + a_2 X_2 + \cdots + a_k X_k \sim N\left(\sum_{i=1}^{k} a_i \mu_i \quad , \quad \sum_{i=1}^{k} a_i^2 \sigma_i^2\right)$$

Ex 3: Let R_1, R_2 and R_3 be resistance of randomly selected 3 resistors each with resistance independent from others.



The total resistance is $R = R_1 + R_2 + R_3$.

Each resistance has a normal distribution with mean of 10 ohms and standard deviation of 0.4 ohms.

(a) Compute the probability that total resistance is more than 31 ohms.

(b) Compute the probability that total resistance is within 1.5 standard deviations of the mean total resistance.

Ex 4: If X, Y are independent random variables such that $E(X) = E(Y) = \mu,\ Var(X) = Var(Y) = \sigma^2$, find

$$E\left(\frac{X+Y}{2}\right)$$

$$Var\left(\frac{X+Y}{2}\right)$$

The above results can be extended to n independent random variables:
In general, if $X_1, X_2, \dots, X_n$ are independent random variables with $E(X_i) = \mu$, and $Var(X_i) = \sigma^2$, then the **mean (or average)** of these random variables is a random variable and given by

$$\overline{\boldsymbol{X}} = \frac{\boldsymbol{X_1} + \boldsymbol{X_2} + \cdots + \boldsymbol{X_n}}{\boldsymbol{n}} = \frac{\sum_{i=1}^{n} \boldsymbol{X_i}}{\boldsymbol{n}}$$

and

$$\boldsymbol{E}(\,\overline{\boldsymbol{X}}\,) =$$

$$\boldsymbol{Var}(\,\overline{\boldsymbol{X}}\,) =$$