Continuous Random Variables and Probability Density Functions

Recall: If the sample space of experiment or support of a random variable is uncountably infinite, then the random variable is a **continuous random variable.**

Probability density functions are associated with (absolutely) continuous random variables. For continuous random variables, the probability of any point value is zero (i.e., $P(X = a) = 0$). As a result, we define the probability density function (pdf) for a continuous random variable differently.

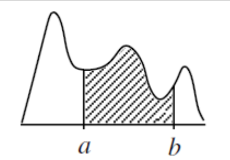
Definition: (Probability Density Function, pdf)

The **probability density function** or **pdf** of a continuous random variable X, denoted by $f(x)$ or $f_X(x)$, is such that

- $f(x) \geq 0$ for all x in X
- $\int_{-\infty}^{\infty} f(x)dx = 1$

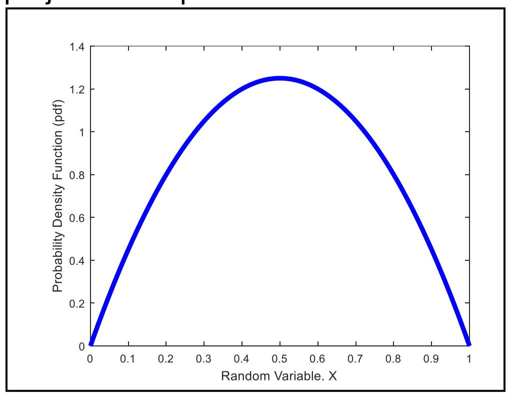
The **pdf** is a curve that describes the probability of observing X in some range of values, such as between a and b where $a < b$. The probability is defined as:

$$\boldsymbol{P(a < X < b) = \int_a^b f(x)\, dx}$$



Ex 1: The time (measured in years), X, required to complete a software project has a pdf of the form:

$$f(x) = \begin{cases} kx(1-x)\,; & 0 \leq x \leq 1 \\ 0\,; & \text{Otherwise} \end{cases}$$



(a) Find the value of k so that $f(x)$ is a valid pdf.

(b) Compute the probability that the project will be completed in less than 4 months.

(c) Compute the probability that the project will be completed in between 3 and 6 months.

Expected Value (Mean) of a Continuous Random Variable

Similar to the **expected value or "average"** of a discrete random variable, we can define it for a continuous random variable as shown below:

Definition: (Expected Value of a Continuous random variable)

Let X be a **continuous** random variable and $f(x)$ is its pdf. Then the expected value of X is denoted by $\boldsymbol{E}[\boldsymbol{X}]$ or $\boldsymbol{\mu}$ and given by

$$\boldsymbol{\mu} = \boldsymbol{E}[\boldsymbol{X}] = \int_{-\infty}^{\infty} [\, \boldsymbol{x} \times \boldsymbol{f}(\boldsymbol{x}) \,] \boldsymbol{dx}$$

Ex 1 Contd.: Compute the expected (or mean) project completion time.

Variance of a Continuous Random Variable

Variance explains the spread of values of random variable compared to mean of random variable.

Definition: (Variance of a Continuous random variable)

Let X be a **continuous** random variable and $f(x)$ is its pdf. Then the variance of X is denoted by $Var[X]$ or $\boldsymbol{\sigma}^2$ and given by

$$\boldsymbol{\sigma}^2 = \boldsymbol{Var}[\boldsymbol{X}] = \boldsymbol{E}\left[\left(\boldsymbol{X} - \boldsymbol{E}(\boldsymbol{X})\right)^2\right] = \boldsymbol{E}[(\boldsymbol{X} - \boldsymbol{\mu})^2]$$

$$= \int_{-\infty}^{\infty} [\, (\boldsymbol{x} - \boldsymbol{\mu})^2 \times \boldsymbol{f}(\boldsymbol{x}) \,] \, \boldsymbol{dx} = \int_{-\infty}^{\infty} [\, \boldsymbol{x}^2 \times \boldsymbol{f}(\boldsymbol{x}) \,] \boldsymbol{dx} - \boldsymbol{\mu}^2$$

Ex 1 Contd.: Compute the variance and standard deviation of the project completion time.

Ex 2: The lifetime of a device is a random variable denoted by T (in hours). The pdf of T is given by

$$f(t) = \begin{cases} e^{-t}\,; & t > 0 \\ 0\,; & \text{Otherwise} \end{cases}$$

The expected lifetime of a device is called the **Mean Time To Failure (MTTF).** Compute the MTTF for this device.

Ex 3: In coherent radio communication, the phase difference between the transmitter and the receiver, denoted by Y, is modeled as having the following pdf:

$$f(y) = \begin{cases} \dfrac{1}{2\pi} \; ; \; -\pi < y < \pi \\ 0 \; ; \; \text{Elsewhere} \end{cases}$$

(a) Compute the probability that there is no phase difference.

(b) Compute the probability that the phase difference is positive.

(c) Compute the probability that the phase difference is less than $\pi/2$.

(d) What is the expected phase difference?