Bayesian Method

$B_1, B_2, \ldots, B_k$ is a partition

Bayes' Rule

If $B_1, B_2, \ldots, B_k$ is a collection of **mutually exclusive** ($B_i \cap B_j = \emptyset$ for all $i \neq j$) and **exhaustive** ($B_1 \cup B_2 \cup \ldots \cup B_k = \Omega$) events, and if $P(B_i) > 0$ for all i, then for any event A,

$$P(B_j|A) = \frac{P(A \cap B_j)}{P(A)} = \frac{P(A|B_j) \times P(B_j)}{P(A)} = \frac{P(A|B_j) \times P(B_j)}{\sum_{i=1}^{\infty}[\, P(A|B_i) \times P(B_i)\,]}$$

Ex 19 Contd.: Linda sees 20% of the patients, while Minnie and Robin each see 40% of the patients. Suppose that 60% of Linda's patients receive flu shots, while 30% of Minnie's patients receive flu shots and 10% of Robin's patients receive flu shots. Find the conditional probability that the patient was seen by Robin given that patient received a flu shot.

$$P(R|F) = \frac{P(F|R) \times P(R)}{P(F)}$$

$$= \frac{0.1 \times 0.4}{0.28} = \underline{0.142857} \text{ or } \underline{14.2\,\%}$$

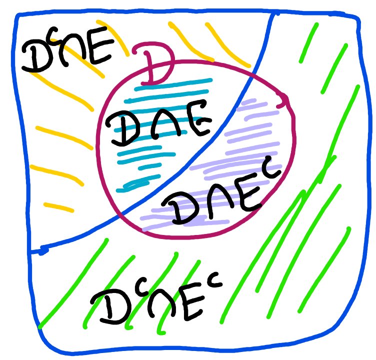
Ex 21: Due to an Internet configuration error, packets sent from New York to Los Angeles are routed through El Paso, Texas with probability 3/4. $P(E) = 3/4$

Given that a packet is routed through El Paso, suppose it has conditional probability 1/3 of being dropped. $P(D|E) = 1/3$

Given that a packet is not routed through El Paso, suppose it has conditional probability 1/4 of being dropped.

Let E = Packet is routed through El Paso. $P(D|E^c) = 1/4$

D = Packet is dropped.



(a) Find the probability that a packet is dropped.

Total Probability Law

$$P(D) = [P(D|E) \times P(E)] + [P(D|E^c) \times P(E^c)]$$

$$= \left[\frac{1}{3} \times \frac{3}{4}\right] + \left[\frac{1}{4} \times \left(1 - \frac{3}{4}\right)\right]$$

$$= \underline{0.3125}$$

(b) Find the probability that a packet is routed through El Paso given that it is not dropped.

$$P(E|D^c) = \frac{P(E \cap D^c)}{P(D^c)}$$

$$= \frac{P(D^c|E) \times P(E)}{1 - P(D)}$$

$$= \frac{[1 - P(D|E)] \times P(E)}{1 - P(D)}$$

$$\frac{\left(1 - \frac{1}{3}\right) \times \left(\frac{3}{4}\right)}{(1 - 0.3125)}$$

$$= \underline{0.72727}$$

Likely

Ex 22: A computer virus spreads around the world, all reporting to a master computer. The good people capture the master computer and find that one million out of 1000 million computers have been infected (but don't know which ones). Governments decide to take action!

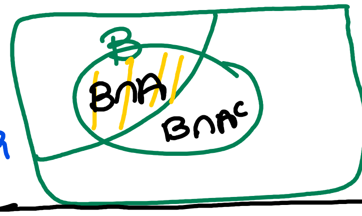
No one can use the internet until their computer passes the "virus-free" test.

The test gives a positive result 99% of the times given that the computer actually has the virus. But the test has a **false positive rate** of 2%. That is, 2% of the time it says the computer has the virus when it doesn't have the virus.

If a randomly selected computer is tested positive, then how likely that computer actually has the virus? (That is, what percent of computers actually has the virus if those were tested positive?)

Let A = Computer has been infected

B = Test gives positive result.



Given:

$P(A) = \frac{1}{1000} = 0.001 \Rightarrow P(A^c) = 1 - 0.001 = 0.999$

- $P(B|A) = 99\% = 0.99$
- $P(B|A^c) = 2\% = 0.02$

$B = (B \cap A) \cup (B \cap A^c)$

$P(B) = P[B \cap A] + P(B \cap A^c)$

$= [P(B|A) \times P(A)] + [P(B|A^c) \times P(A^c)]$

Find $P(A|B)$

From Bayes' Rule

$$P(A|B) = \frac{P(B|A) \times P(A)}{[P(B|A) \times P(A)] + P[(B|A^c) \times P(A^c)]}$$

$$= \frac{0.99 \times 0.001}{(0.99 \times 0.001) + (0.02 \times 0.999)} = \underline{0.04721} \text{ unlikely.}$$

Ex 23: A student dormitory in a college consists of the following:

30% are freshman of whom 10% own a car F = Freshman ; $P(F) = 30\%$; $P(C|F) = 10\%$

40% are sophomore of whom 20% own a car S_1 = Sophomore ; $P(S_1) = 40\%$; $P(C|S_1) = 20\%$

20% are juniors of whom 40% own a car J = Junior ; $P(J) = 20\%$; $P(C|J) = 40\%$

60% of seniors own a car S_2 = Senior. ; $P(C|S_2) = 60\%$

Assume that there are no other types of students at this dorm. A student is randomly selected from this dormitory. C = Student owns a car.

(a) What is the probability that this student is a senior?

$P(S_2) = 1 - [P(F) + P(S_1) + P(J)] = \underline{0.1}$

(b) If the student owns a car, find the probability that this student is a senior. From Bayes' rule,

$$P(S_2|C) = \frac{P(C|S_2) \times P(S_2)}{[P(C|F) \times P(F)] + [P(C|S_1) \times P(S_1)] + [P(C|J) \times P(J)] + [P(C|S_2) \times P(S_2)]}$$

$$= \frac{0.06}{0.25}$$

$$= \underline{0.24}$$

From everyone who has a car, 24% will be seniors.

Multiplicative Rule: $P(A\cap B) = P(A|B) \times P(B)$

(*) Independent events can never be disjoint.

(*) Disjoint events cannot happen at the same time. Therefore, disjoint events are always NOT independent.

Independence in Probability

Sometimes the occurrence of one event, B, will have no effect on the probability of another event, A. If A and B are unrelated, then intuitively it should be the case that

$$P(A|B) = P(A)$$

Similarly, $P(B|A) = P(B)$

Also, it follows that

$$P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B)$$

Statistical Independence

Two events, A and B, are statistically **independent** if and only if

$$\boldsymbol{P(A \cap B) = P(A) \times P(B)}$$

$$\boldsymbol{P(A|B) = P(A)}$$

$$\boldsymbol{P(B|A) = P(B)}$$

(*) To decide events are independent or not, check one of these 3 equations is true or false.

Ex 24: Three bits are transmitted across a noisy channel and the number of correct receptions is noted. Find the probability that the number of correctly received bits is two, assuming bit errors are mutually independent and that on each bit transmission the probability of correct reception is λ for some fixed 0 ≤λ ≤ 1. Let A = event that single bit is correctly received.

$P(A) = \lambda$

$$P\left(\begin{matrix}\text{2 of 3 bits are}\\ \text{correctly received}\end{matrix}\right) = P\left[(AAA^c) \cup (AA^cA) \cup (A^cAA)\right]$$

These are disjoint.

$$= P(AAA^c) + P(AA^cA) + P(A^cAA)$$

independent events

$$= \boxed{\underbrace{P(A)}_{\lambda} \times \underbrace{P(A)}_{\lambda} \times \underbrace{P(A^c)}_{(1-\lambda)}} + \left[P(A) \times P(A^c) \times P(A)\right] + \left[P(A^c) \times P(A) \times P(A)\right]$$

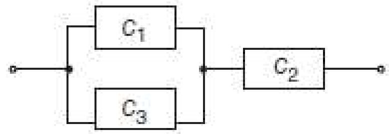
$$= \underline{\underline{3\lambda^2(1-\lambda)}}$$

$R_i = P(i\text{th component works}) = P(C_i)$

C_i = ith component works

Ex 25: Given three components with respective reliabilities $R_1 = 0.8$, $R_2 = 0.75$, and $R_3 = 0.98$, compute the reliabilities of each system shown in figure below. Assume all components fail independently.

Series



1st

$$\begin{aligned} P(\text{System works}) &= P(\text{All components work}) \\ &= P(C_1 \cap C_2 \cap C_3) \quad \text{independent} \\ &= P(C_1) \times P(C_2) \times P(C_3) \\ &= R_1 \times R_2 \times R_3 \\ &= 0.8 \times 0.75 \times 0.98 \\ &= \underline{0.588} \end{aligned}$$

2nd

$$\begin{aligned} P(\text{System works}) &= P[(C_1 \cup C_3) \cap C_2] \quad \text{independent} \\ &= P[C_1 \cup C_3] \times P(C_2) \\ &= [P(C_1) + P(C_3) - P(C_1 \cap C_3)] \times P(C_2) \\ &= [P(C_1) + P(C_3) - [P(C_1) \times P(C_3)]] \times P(C_2) \\ &= [R_1 + R_3 - R_1 R_3] \times R_2 = \underline{0.747} \end{aligned}$$

In general

Reliability of system with components in series: $\boxed{R_1 \times R_2 \times \dots \times R_k = \prod_{i=1}^{k} R_i}$

Reliability of system with components in parallel: $1 - [(1-R_1) \times (1-R_2) \times \dots \times (1-R_k)]$

$$= \boxed{1 - \prod_{i=1}^{k} (1 - R_i)}$$

If A and B, are statistically **independent**, then

$\mathbf{A}$ **and** $\mathbf{B^c}$ **are independent, that is** $\mathbf{P(A \cap B^c) = P(A) \times P(B^c)}$

$\mathbf{A^c}$ **and** $\mathbf{B}$ **are independent, that is** $\mathbf{P(A^c \cap B) = P(A^c) \times P(B)}$

$\mathbf{A^c}$ **and** $\mathbf{B^c}$ **are independent, that is** $\mathbf{P(A^c \cap B^c) = P(A^c) \times P(B^c)}$

A collection of events $A_1, A_2, \dots, A_n$ are **pairwise independent** if and only if

$$\boldsymbol{P(A_i \cap A_j) = P(A_i) \times P(A_j)} \ \textbf{for all } \boldsymbol{i \neq j}$$