The PoA of 2 is tight for efficient welfare for proportional allocation

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1 PROBLEM

The problem consists of n buyers, where each buyer i has a budget B_i. Additionally, the buyers have an evaluation function $V_i : [0,1]- > \mathbb{R}^+$, which is non-decreasing and concave. The auctioneer lists one item only, for which the buyers can submit a private bid b_i, such that $0 \le b_i \le B_i$. From the bid b_i, buyer i receives a fraction $d_i = {}^{b_i}/_B$ of the item, where $B = \sum_i b_i$. The utility of buyer i is $U_i(\vec{b}) = V_i(d_i) - b_i$. The goal is to maximize the effective welfare of the buyers: $EW_i(\vec{b}) = \min\{B_i,\ V_i(d_i)\}$

2 FORMULATION

The following formulation transforms the initial problem to finding the solution that maximizes the effective welfare among all possible solutions. For simplicity, the fractions of assignments are discretized ($d_i \in k_\epsilon : 0 \le k \le {}^1/_\epsilon$). A solution $S = \{(i,d_i) : 0 \le d_i \le 1\}$ assigns a fraction to each buyer, such that $\sum_{(i,d_i)\in S} d_i \le 1$. The variable z_S indicates which solution is chosen.

$$\max \sum_{S\subseteq \mathcal{S}} c_S\, z_S$$

$$(\beta) \qquad \sum_{i=1}^{n}\sum_{d_i} d_i \sum_{S:(i,d_i)\in S} z_S = 1$$

$$(\gamma) \qquad \sum_{S} z_S = 1$$

$$z_S \ge 0 \qquad \forall i, \forall S \subseteq \mathcal{S}$$

$$\min \beta + \gamma$$

$$\sum_{(i,d_i)\in S} d_i\beta + \gamma \ge c_S \quad \forall S \subseteq \mathcal{S}$$

3 SETTING THE DUAL VARIABLES

Consider a Nash equilibrium $\mathbf{b}$. Let d_i^* be the fraction received by i in the equilibrium ($d_i^* = {}^{b_i}/_B$). Using the KKT conditions (similar to Chapter 21 of AGT book), for any bidder i with the equilibrium bid $0 < b_i < B_i$, we have

$$\hat{V}_i'\left(\frac{b_i}{B}\right) = V_i'\left(\frac{b_i}{B}\right)\cdot\left(1 - \frac{b_i}{B}\right) = B$$

where $B = \sum_{j=1}^{n} b_j$ is the sum of the bids. Note that for all bidders i and i' such that $0 < b_i, b_{i'} < B_i$ then $\hat{V}_i'(d_i^*) = \hat{V}_{i'}'(d_{i'}^*)$. Let us define the dual variables β and γ as the following:

$$\beta = \hat{V}_i'(d_i^*) = B$$

$$\gamma = \sum_{i=1}^{n} \gamma_i$$

$$\gamma_i = \begin{cases} B_i \text{ if } V_i(d_i^*) \ge B_i, \\ 2V_i(d_i^*) - d_i^*\hat{V}_i'(d_i^*) \end{cases}$$

Feasibility. The dual constraints reads

$$\sum_{(i,d_i)\in S} d_i\beta + \gamma \ge c_S$$

$$\Leftrightarrow \sum_{(i,d_i)\in S} d_i\hat{V}_i'(d_i^*) + \sum_{i=1}^{n}\gamma_i \ge \sum_{(i,d_i)\in S} \min\{V_i(d_i), B_i\}$$

We prove the above inequality for each term i. If $\gamma_i = B_i$ then it is trivial. Now assume that $\gamma_i = 2V_i(d_i^*) - d_i^* \hat{V}_i'(d_i^*)$ (meaning that $V_i(d_i^*) < B_i$). We have

$$\begin{aligned}
\hat{V}_i'(d_i^*) + \gamma_i &= d_i \hat{V}_i'(d_i^*) + 2V_i(d_i^*) - d_i^* \hat{V}_i'(d_i^*) = 2V_i(d_i^*) + \hat{V}_i'(d_i^*)(d_i - d_i^*) \\
&= 2V_i(d_i^*) + V_i'(d_i^*)(1 - d_i^*)(d_i - d_i^*) \\
&= V_i(d_i^*) + V_i'(d_i^*)(d_i - d_i^*) + V_i(d_i^*) - V_i'(d_i^*) d_i^* (d_i - d_i^*) \\
&\geq V_i(d_i) + V_i(d_i^*) - V_i'(d_i^*) d_i^* (d_i - d_i^*) \\
&\geq V_i(d_i) + V_i(d_i^*) - V_i'(d_i^*) \frac{d_i^*}{d_i} (d_i - d_i^*)
\end{aligned}$$

The first inequality is due to the concavity of V_i and the second holds since $d_i \leq 1$. It remains to prove that $V_i(d_i^*) - V_i'(d_i^*) \frac{d_i^*}{d_i}(d_i - d_i^*) \geq 0$. If $d_i \leq d_i^*$ then the inequality follows immediately. Assume that $d_i = \rho \cdot d_i^*$ for $\rho > 1$. Therefore,

$$\begin{aligned}
V_i(d_i^*) - V_i'(d_i^*) \frac{d_i^*}{d_i}(d_i - d_i^*) &= V_i(d_i^*) - \frac{\rho - 1}{\rho} V_i'(d_i^*) d_i^* \\
&\geq V_i(d_i^*) - V_i'(d_i^*) d_i^* \geq V_i(0) \geq 0
\end{aligned}$$

The second inequality follows the concavity of V_i. We deduce that the feasibility holds.

Primal and Dual. The ratio between primal and dual is at most 2.

$$2 \sum_{(i,d_i) \in S} \min\{V_i(d_i^*), B_i\} \geq \beta + \gamma$$

$$\begin{aligned}
\forall\, i: &\qquad 2 \min\{V_i(d_i^*), B_i\} \geq b_i + \gamma_i \\
\text{if } V_i(d_i^*) \geq B_i: &\qquad 2B_i \geq b_i + B_i \\
\text{otherwise}: &\qquad 2V_i(d_i^*) \geq b_i + 2V_i(d_i^*) - d_i^* \hat{V}_i'(d_i^*) \\
\text{since}: &\qquad b_i - d_i^* \hat{V}_i'(d_i^*) = b_i - \frac{b_i}{B} B = 0 \\
&\qquad 2V_i(d_i^*) \geq 2V_i(d_i^*)
\end{aligned}$$

Remark. To complete the analysis, one must consider cases where all equilibrium bids are either B_i or 0. The PoA, in this case, is 1.