Comprehensive Guide to Calculating Derivatives

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1 Introduction

Derivatives are fundamental in calculus, allowing us to understand rates of change. This paper explores different methods of computing derivatives: the limit definition, L'Hôpital's Rule, rule-based differentiation, and forward-mode automatic differentiation.

2 Derivative Definition via Limits

The derivative of a function $f(x)$ at a point $x = a$ is defined as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{1}$$

To illustrate, consider $f(x) = x^2$. Using the limit definition:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \to 0} (2x + h) = 2x. \end{aligned}$$

Thus, $\frac{d}{dx} x^2 = 2x$.

3 L'Hôpital's Rule

If a function results in an indeterminate form such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$, L'Hôpital's Rule states:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}, \tag{2}$$

provided the limit on the right exists.

Consider:

$$\lim_{x \to 0} \frac{\sin x}{x}. \tag{3}$$

Since both numerator and denominator approach zero:

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1. \tag{4}$$

4 Rule-Based Differentiation

Several fundamental differentiation rules simplify computations:

- **Power Rule:** $\frac{d}{dx}x^n = nx^{n-1}$.
- **Product Rule:** $(fg)' = f'g + fg'$.
- **Quotient Rule:** $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.
- **Chain Rule:** $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$.

Example: Compute $\frac{d}{dx}(x^3 \sin x)$. Using the product rule:

$$\begin{aligned}(x^3 \sin x)' &= (x^3)' \sin x + x^3(\sin x)' \\ &= 3x^2 \sin x + x^3 \cos x.\end{aligned}$$

5 Forward-Mode Automatic Differentiation

Automatic differentiation (AD) uses dual numbers to compute derivatives efficiently. A dual number is defined as:

$$x_\epsilon = x + \epsilon dx, \quad \text{where } \epsilon^2 = 0. \tag{5}$$

For a function $f(x)$, we substitute x_ϵ:

$$f(x_\epsilon) = f(x) + \epsilon f'(x). \tag{6}$$

Thus, $f'(x)$ is extracted directly.

Example: Compute $f(x) = x^2$ using dual numbers:

$$\begin{aligned}f(x_\epsilon) &= (x + \epsilon)^2 = x^2 + 2x\epsilon + \epsilon^2 \\ &= x^2 + 2x\epsilon. \quad (\text{Since } \epsilon^2 = 0)\end{aligned}$$

Hence, $f'(x) = 2x$.

6 Conclusion

This paper covered multiple derivative computation techniques, from the fundamental limit definition to advanced automatic differentiation. Each method serves different purposes, from theoretical rigor to computational efficiency.