Spy Problem 4

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1 Problem

(Sweet Revenge is close at hand!!) The spy has tracked Eco Li and is ready to pay him back for killing his buddy, Sal Manella. The spy is in a Cartesian (air) plane flying at $10,000\ ft$ when he spots the cave of h is nemesis. There is no place to land, so decides to para(bola?)chute into Li's domain. In order to controol where he lands, he freefalls (is this a Six Flags ride?) for a whlie before opening his parachute. He realizes that, to land safely, he must meet two criteria:

a) he must open the chute at an altitude of about $1500\ ft$ (to pick the safest spot)

b) the total descent cannot exceed 2 minutes (otherwise Li will spot him and he could experience the same fate as poor Sal Manella)

Does our heor land safely?
Note: the weight of the spy is $W = mg = 175\ lb$, gravitational acceleration is $g = 32\ \frac{ft}{s^2}$, and the constants of air resistance are $k = 0.003$ during freefall and $k = 0.56$ during the chute portion.
At minimum, your solution should list the initial conditions for each phase, the solution functions for the velocity and the distance fallen for each phase; tell how long (nearest second) he should free fall before opening the chute and the distance fallen at that time, the total elapsed time the descent took place.

2 Solution

2.1 Setting Up the Solution

For the convenience of the problem, let us define the upwards direction as positive and the downwards direction as negative. Thus, the force of gravity, along with the gravitational acceleration and velocity of the object, is negative. On the other hand, the resistive force is positive.

2.2 Modeling the Problem

To model the problem, we examine the forces upon the spy: the downwards force of gravity and the upwards force of air resistance. Thus,

$$F = ma = m\left(\frac{dv}{dt}\right) = -W + kv^2$$

After some algebraic manipulation to turn the model into a first order homogenous seperable differential equation, we get:

$$m\left(\frac{dv}{dt}\right) = -W + kv^2$$
$$\rightarrow \frac{dv}{dt} = g - \frac{k}{m}v^2$$
$$\rightarrow \frac{dv}{dt} = g\left(\frac{k}{mg}v^2 - 1\right)$$
Let us define $a^2 = \frac{k}{mg}$
$$\rightarrow \frac{dv}{dt} = g(a^2v^2 - 1)$$

2.3 Solving the Differential Equation

$$\rightarrow \frac{dv}{dt} = g(a^2v^2 - 1)$$
$$\rightarrow \frac{dv}{a^2v^2-1} = gdt$$
$$\rightarrow \int \frac{dv}{a^2v^2-1} = \int gdt$$
We apply partial fraction decomposition.
$$\rightarrow \int \frac{-1}{2}\frac{1}{av+1} + \frac{1}{2}\frac{1}{av-1}dv = \int gdt$$
$$\rightarrow \frac{-1}{2a}ln(av+1) + \frac{1}{2a}ln(av-1) = gt + C_1$$

$$\rightarrow \frac{1}{2a} ln\left(\frac{av-1}{av+1}\right)$$

$$\rightarrow \begin{cases} \frac{-1}{a} tanh^{-1}(av) & |v| \leq \frac{1}{a} \\ \frac{-1}{a} coth^{-}1(av) & |v| < \frac{1}{a} \end{cases} = gt + C$$

$$\rightarrow v = \begin{cases} \frac{1}{a} tanh(-agt + C) & |v| \leq \frac{1}{a} \\ \frac{1}{a} coth(-agt + C) & |v| > \frac{1}{a} \end{cases}$$

2.4 Solving the First Portion of Descent

In the first portion of descent, the spy is in freefall so $k = 0.003$. Thus, $a_1 = \sqrt{\frac{k}{mg}} = \sqrt{\frac{0.003}{175}} \approx 0.00414$. In this portion, the initial velocity $v_{1_0} = 0$ and $x_{1_0} = 10000$.

2.4.1 Solving for the Velocity Function

We can easily tell that in this portion, the velocity cannot exceed $\frac{1}{a_1^2}$. Thus, the equation that models this portion of descent is,

$$v_1(t) = \frac{1}{a_1} tanh(-a_1 gt + C_1)$$

We substitute in the initial condition of $v_{1_0} = 0$ to find,

$$v_1(0) = 0 = \frac{1}{a_1} tanh(-a_1 g * 0 + C_1)$$
$$\rightarrow C_1 = 0$$

Now we have,

$$v_1(t) = \frac{1}{a_1} tanh(-a_1 gt)$$

2.4.2 Solving for the Distance Function

Since we have the velocity function, we can find the distance function easily.

$$v_1(t) = \frac{dv}{dt} = \frac{1}{a_1} tanh(-a_1 gt)$$
$$\rightarrow dv = \frac{1}{a_1} tanh(-a_1 gt) dt$$
$$\rightarrow \int dv = \int \frac{1}{a_1} tanh(-a_1 gt) dt$$
$$x_1(t) = \frac{1}{2a_1 g} ln(cosh(-a_1 gt)) + C_2$$

We substitute in the initial condition of $x_{1_0} = 10000$ to find,

$$x_1(0) = 10000 = \frac{1}{2a_1 g} ln(cosh(-a_1 gt)) + C_2$$
$$\rightarrow C_2 = 10000$$

Now we have,

$$x_1(t) = \frac{1}{2a_1 g} ln(cosh(-a_1 gt)) + 10000$$

2.5 Solving the Second Portion of Descent

In the second portion of descent, the spy is in freefall so $k = 0.56$. Thus, $a_2 = \sqrt{\frac{k}{mg}} = \sqrt{\frac{0.56}{175}} \approx 0.05657$.

2.5.1 Solving for the Velocity Function

We can easily tell that in this portion, the velocity has already exceeded $\frac{1}{a_2^2}$. Thus, the equation that models this portion of descent is,

$$v_2(t) = \frac{1}{a_2} coth(-a_2 gt + C_3)$$

2.5.2 Solving for the Distance Function

Since we have the velocity function, we can find the distance function easily.

$$v_2(t) = \frac{dv}{dt} = \frac{1}{a_2} tanh(-a_2 gt + C_3)$$
$$\rightarrow dv = \frac{1}{a_2} tanh(-a_2 gt + C_3) dt$$
$$\rightarrow \int dv = \int \frac{1}{a_2} tanh(-a_2 gt + C_3) dt$$
$$x_2(t) = \frac{1}{2a_2 g} ln(cosh(-a_2 gt + C_3)) + C_4$$

2.5.3 Solving for the Time to the Ground

Since we do not know the actual location to release the parachute, it is best to use trial and error to find the correct height. To do so, I have created an excel spreadsheet that calculates the two times of descent and I tested values until I got an answer as close to 120 s as possible but not exceeding it.
Every value is calculated using the above derived functions or their inverse function. The derivation is just subtituting in numbers into the above functions and their inverses again and again.

Data		Constants		Constants of Integration		Initial Values	
Test Height:	1420.362156	g	32	C	0	v_2o	-241.5130856
T1	40.75451361	mg	175	D	10000	v_1o	0
T2	79.24548634	k1	0.003	E	0.07306519536	x_2o	1420.362156
TT	119.9999999	k2	0.56	F	1394.820031	x_1o	10000
D1	8579.637844	a1	0.004140393356				
D2	1420.362156	a2	0.05656854249				

Test height is simply an independent variable that I changed to test for the correct height. $T1$ is the time taken during the first portion of descent and $T2$ for the second portion. $D1$ is the distance traveled in the first portion and $D2$ is the distance traveled in the second portion. The constants are the constants provided and the constants are integration are C_1, C_2, C_3, C_4 respectively from top to bottom. The initial conditions for each state is also clearly listed.
The time the spy takes to reach the ground will be approximately 120 s.

2.6 Final Remarks

As you can see, if the spy releases his parachute at 1420 ft from the ground, he will successfully make it to the ground within 2 min. In fact, the spy will reach the ground at approximately 120 s. Although 1420 ft isn't neccessarily close to 1500 ft, it is the first height he can survive at so the spy has to be on it being close enough.