1D wave equation with damping 25.01.02.

1D damping 波动方程

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$u(x,t) = \int_{-\infty}^{+\infty} \frac{dw}{2\pi} \hat{u}(x,w) e^{iwt}$$

代入： $$-w^2 \hat{u} + iw\alpha \hat{u} - c^2 \frac{\partial^2 \hat{u}}{\partial x^2} = 0$$

$$\hat{u} = \underbrace{A_w e^{iq_w x}}_{\text{left going}} + \underbrace{B_w e^{-iq_w x}}_{\text{right going}},$$

✓

so: $A = 0$. $c^2 q_w^2 = w^2 - iw\alpha$

Boundary condition: $u(x=0,t) = f(t)$.

而 $$u(x=0,t) = \int_{-\infty}^{+\infty} \frac{dw}{2\pi} B_w e^{i(wt - q_w x)} \overset{x=0}{=\!=} \int_{-\infty}^{+\infty} \frac{dw}{2\pi} B_w e^{iwt} \overset{BC}{=} f(t)$$

So: $B_w = \tilde{f}(w)$

So: $$u(x,t) = \int_{-\infty}^{+\infty} \frac{dw}{2\pi} \tilde{f}(w) e^{i(wt - q_w x)}, \quad q_w = \left[\frac{w^2}{c^2}\left(1 - \frac{i\alpha}{w}\right)\right]^{\frac{1}{2}}$$

$$\simeq \frac{w}{c} - \frac{i\alpha}{2c}, \text{ where } \frac{\alpha}{w} \ll 1$$

$$u(x,t) = \int_{-\infty}^{+\infty} \frac{dw}{2\pi} \tilde{f}(w) e^{iw\left(t - \frac{x}{c} + \frac{i\alpha x}{2cw}\right)}$$

$$= e^{-\frac{\alpha x}{2c}} \int_{-\infty}^{+\infty} \tilde{f}(w) e^{iw\left(t - \frac{x}{c}\right)}$$

$$= e^{-\frac{\alpha x}{2c}} f\left(t - \frac{x}{c}\right)$$