

## δ(x) 定义

数学上有若干种:

1.  $\delta(\vec{r}) = \frac{1}{-4\pi} \nabla^2 \frac{1}{r}$  ,  $\delta(\vec{r}) = \delta(x)\delta(y)\delta(z)$

2.  $\delta(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}$  , 适合数值计算

3.  $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx} dk$  , 可用于求  $\sin \omega t$ ,  $e^{i\omega t}$  的 FFT.

1.  $r = \sqrt{x^2 + y^2 + z^2}$

$\nabla^2(\frac{1}{r}) = \frac{1}{r^3} \frac{\partial}{\partial r}$   $r \neq 0$  时:

$\nabla^2(\frac{1}{r}) = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r \cdot \frac{1}{r}) = 0$

$\int \nabla^2(\frac{1}{r}) dV = \int \nabla \cdot (\nabla \frac{1}{r}) dV = \oint (\nabla \frac{1}{r}) \cdot d\vec{S} = -\oint \frac{1}{r^2} \frac{\vec{r}}{r} \cdot d\vec{S} = -4\pi.$

得:  $\nabla^2(\frac{1}{r}) = -4\pi \delta(\vec{r})$

2.  $\int_{-\infty}^{+\infty} \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2} dx = \frac{1}{\pi} \int_{-\infty}^{+\infty} \lim_{\epsilon \rightarrow 0} \frac{1}{(\frac{x}{\epsilon})^2 + 1} d(\frac{x}{\epsilon}) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \left[ \tan^{-1} \frac{x}{\epsilon} \right]_{-\infty}^{+\infty} = 1$

$= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \tan^{-1} \frac{x}{\epsilon} \Big|_{-\infty}^{+\infty} = 1$

3.  $\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx} dk = \frac{1}{2\pi} \lim_{\eta \rightarrow 0^+} \left( \int_0^{+\infty} e^{ik(x+i\eta)} dk + \int_{-\infty}^0 e^{ik(x-i\eta)} dk \right)$

$= \lim_{\eta \rightarrow 0^+} \frac{1}{\pi} \frac{\eta}{x^2 + \eta^2} = \delta(x)$

↑  
上文已证.

↑  
证明合理. 见背面.

$$\int_{0^-}^{0^+} e^{ik(x+i\eta)} dk = 0. \text{ 显然成立.}$$

$$\lim_{\eta \rightarrow 0^+} \lim_{M \rightarrow \infty} \int_0^M e^{ik(x+i\eta)} dk = \text{无穷大. } x+i\eta \rightarrow x.$$

$$\text{若 } M\eta \rightarrow 0, \text{ 即 } \eta \text{ 足够小, 则 } \int_0^M e^{ik(x+i\eta)} dk \rightarrow \int_0^M e^{ikx} dk$$

但这证明也有不严谨处, 比如,  $M\eta \rightarrow 0$  为什么成立, 也可能趋于  $\infty$ .

$$x+i- = 2i \cdot \frac{1}{2} \cdot \frac{1}{x-i} = 2i \left( \frac{1}{x} + i \right) = \frac{2i}{x} - 2 = -2 + \frac{2i}{x}$$

$$\frac{1}{x+i-} = \frac{1}{-2 + \frac{2i}{x}} = \frac{x}{-2x + 2i}$$

$$1 = \lim_{M \rightarrow \infty} \left[ \frac{1}{2\pi i} \int_{\gamma} \frac{e^{ik(x+i\eta)} dk}{k} \right] = \lim_{M \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma} \frac{e^{ik(x+i\eta)} dk}{k}$$

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$$\left( \frac{1}{2\pi i} \int_{\gamma} \frac{e^{ik(x+i\eta)} dk}{k} \right) = \frac{1}{2\pi i} \int_{\gamma} \frac{e^{ik(x+i\eta)} dk}{k} = \frac{1}{2\pi i} \int_{\gamma} \frac{e^{ik(x+i\eta)} dk}{k}$$

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