Fréchet derivative 和 8X(cm)

Jacobian Matrix.

一. Fréchet derivative 定义:

If $F: \mathbb{R}^n \to \mathbb{R}^m$ and $\exists L: \mathbb{R}^n \to \mathbb{R}^m$ linear transfor matrix such that the following holds:

$$\lim_{h \to 0} \frac{\| F(\vec{a}+\vec{h}) - F(\vec{a}) - L(\vec{h})\vec{h} \|}{\| \vec{h} \|} = 0 \quad \cdots\cdots\cdots \text{①}$$

用范数因为分母不能放矢量

可类比理解标量情况:

$$0 = \lim_{x \to a} \frac{|f(x) - f(a) - f'(a)(x-a)|}{|x-a|}$$

then we say F is differentiable at $a \in \mathbb{R}^m$ ~~and write $dF_a(h) = L(h)$~~

Fréchet derivative 是唯一的. 证明 (TODO) 待学。

一般用算子 D_f, $D_{f(x)}$, $Df(x)$, 甚至 A, L, 表示 Fréchet derivative.

$Df(x)(h)$ 表示 D_f 乘 h, h 是"步长" eg dx.

根据定义, 可以对 $f(x)$ 作一阶展开:

$$f(x+h) = f(x) + Ah + o(h) \quad \cdots\cdots\cdots \text{②}$$

也可极量表达: $f(x+h) = f(x) + Ah + \|o(h)\|$

二. Fréchet derivative 计算:

用 Jacobian Matrix, which is the best linear approximation.

$$\mathbb{F}(x) \triangleq \mathbb{F}(a) + J_F(a)(x-a)$$

↑

表两项乘积、,

or:

$$\mathbb{F}(\cancel{x} + \overset{h}{\cancel{a}}) \triangleq \mathbb{F}(a) + J_F(a)h$$

对 Fréchet derivatie，更直观地有：

$$\underbrace{Df(a)(h)}_{\text{表 } Df\cdot h\text{，也表 } A\cdot h} = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(a) \quad , \quad n\text{为}x\text{的维度} \quad \cdots\cdots ③$$

Jacobian Matrix 定义：

矢量

与 $\frac{\partial f(x)}{\partial x_n}$ 对应的矢量

$$J = \frac{df(x)}{dx} = \left[\frac{\partial f(x)}{\partial x_1} \cdots \boxed{\frac{\partial f(x)}{\partial x_n}}\right] = \left[\begin{matrix} \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_2}{\partial x_1} \\ \vdots \\ \frac{\partial f_n}{\partial x_1} \end{matrix} \cdots\cdots \boxed{\begin{matrix} \frac{\partial f_1}{\partial x_n} \\ \vdots \\ \frac{\partial f_n}{\partial x_n} \end{matrix}}\right]$$

三、例和 $\delta\chi(m)$

若 $\chi(m) = \|s(m)\|^2$，其中 χ 是标量，s 是 N 维矢量，m 是 N 维矢量。

求 $\delta\chi(m)$

解：由③和②有：

$$\chi(m+dm) = \chi(m) + \sum_{i=1}^{N} dm_i \frac{\partial \chi}{\partial m_i} + \|o(dm)\|$$

[其实用"全微分"的概念也能理解]

$dy = dx_1 \frac{\partial y}{\partial x_1} + dx_2 \frac{\partial y}{\partial x_2} + \cdots$

则 $\delta\chi(m) = dm_j \frac{\partial \chi}{\partial m_j}$（用爱因斯坦求和约定）

注意!!

χ是S的函数，$S = S_i \hat{e}_i$，所以其实χ是S_i, $i=1\cdots N$的函数.

这样就可以用标量求导来理解了.

则: $\delta\chi(m) = dm_i \dfrac{\partial \chi(S)}{\partial S_j} \dfrac{\partial S_j}{\partial m_i}$ （对每一个m_i，都要遍历所有S_j）

即: $\dfrac{\partial \chi}{\partial m_1} = \dfrac{\partial \chi}{\partial S_1}\dfrac{\partial S_1}{\partial m_1} + \dfrac{\partial \chi}{\partial S_2}\dfrac{\partial S_2}{\partial m_1} + \cdots$

$\because \chi = \|S(m)\|^2 = S_1^2 + S_2^2 + \cdots S_N^2$

$\therefore \dfrac{\partial \chi}{\partial S_j} = 2S_j$

则 $\delta\chi = dm_i \cdot 2S_j \dfrac{\partial S_j}{\partial m_i}$

$= 2S_j\, dS_j$

$= 2S \cdot dS$

□.

错误理解:

1. $\dfrac{d}{dx}$，啊，对矢量求导，不会求啊

2. $\dfrac{\partial}{\partial} = \dfrac{XX(x+dx) - XX(x)}{XX(x+dx) - XX(x)}$ 试图用最浅薄的方式去求解，但定义、链式法则等都没用对.