Wave equation 25.01.07

学自维基词条"wave equation"

Plane-wae eignmodes 思路

"traveling" means that the shape of these individual arbitrary functions with respect to x stays constant, however, the functions are translated left and right with time at the speed of c

$$U_w(x,t) = e^{-iwt} f(x)$$

分离变数. $f(x)$是振幅. 一维情况.

$$U_{tt} = c^2 U_{xx}$$

代入U_w, 左边 $= \frac{\partial}{\partial t^2}\left(e^{-iwt} f(x)\right) = -f(x) w^2 e^{-iwt}$

右边 $= c^2 \frac{\partial}{\partial x^2}\left(e^{-iwt} f(x)\right) = c^2 e^{-iwt} \frac{\partial^2 f(x)}{\partial x^2}$

则: $-\left(\frac{w}{c}\right)^2 f(x) = \frac{\partial^2}{\partial x^2} f(x)$,

其解为: $f(x) = Ae^{\pm ikx}$, $k = \frac{w}{c}$

代回$U_w(x,t)$, $U_w(x,t) = e^{-iwt}\left(Ae^{-ikx} + Be^{ikx}\right) = Ae^{-i(kx+wt)} + Be^{i(kx-wt)}$

A, B 取决于 initial and boundary condition.

每一个 eignmodes 都有一项 phase factor e^{-iwt}, 波动方程的解可以 decomposed into an

eignmode expansion:

$$u(x,t) = \int_{-\infty}^{+\infty} s(w) U_w(x,t)\, dw$$

也可为平面波形式: $= \int_{-\infty}^{+\infty} S_+(w) e^{-i(kx+wt)} dw + \int_{-\infty}^{+\infty} S_-(w) e^{i(kx-wt)} dw$. $S_\pm(w)$是 Fourier component, 取决于 initial and boundary conditions.

$$= \int_{-\infty}^{+\infty} S_+(w) e^{-ik(x+ct)} dw + \int_{-\infty}^{+\infty} S_-(w) e^{ik(x-ct)} dw$$

$$= F(x-ct) + G(x+ct)$$

用到了时数FFT. "$x+ct$" 即为时间项 这里与x无关.

Scalar wave equation in 3D 思路

"轴对称"下的波动方程解，是从一个更复杂的球谐展开简化下来的。

振幅只和 radial distance 有关：

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi(r,t) = 0$$

$$\Downarrow$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)u(r,t) = 0$$

也即：$\dfrac{\partial^2(ru)}{\partial t^2} - c^2\dfrac{\partial^2(ru)}{\partial r^2} = 0$

类比 1D 的解：

$$ru = F(r-ct) + G(r+ct)$$

则 $u = \frac{1}{r}F(r-ct) + \frac{1}{r}G(r+ct)$，几何扩散 $\frac{1}{r}$

$ru = Ae^{i(\omega t \pm kr)}$，$u = \frac{1}{r}Ae^{i(\omega t \pm kx)}$

Amplitude：$I = |u(r,t)|^2 = \dfrac{A^2}{r^2}$

Solution of a general initial-value problem

接上文，将 F 变为 delta function $\delta(r-ct)$，将源放在 (ξ,η,ζ)，$\varphi(\xi,\eta,\zeta)$ 是任意函数（源），r 是传播距离。
δ 上的强度，也是 "weighting function"。

$$r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$$

则波场是狄拉克函数在空间上的积分，φ 不完全是源的强度，它被抽出了一个系数 $\frac{1}{4\pi c}$，for convenience，则：

$$u(t,x,y,z) = \frac{1}{4\pi c}\iiint \varphi(\xi,\eta,\zeta)\frac{\delta(r-ct)}{r}\,d\xi\,d\eta\,d\zeta$$

体积分可转成先沿径向，再在单位面积上积分：

$$\iiint dv = \oint_\Omega \left(\int_0^R r^2 \, dr \right) d\Omega$$

则：

$$u(t,x,y,z) = \frac{1}{4\pi c}\oint_\Omega \left(\underbrace{\int_0^R r^2 \varphi(\xi,\eta,\zeta)\frac{\delta(r-ct)}{r} \, dr}_{\text{运用}\,\delta\,\text{的积分性质可把}\; r=ct} \right) d\Omega$$

$$= \frac{1}{4\pi c}\oint_\Omega \left(\int_0^R r \varphi(\xi,\eta,\zeta)\, \delta(r-ct) \, dr \right) d\Omega$$

$$= \frac{1}{4\pi c}\oint_\Omega \left(ct \, \underbrace{\varphi(\xi,\eta,\zeta)\Big|_{r=ct}} \right) d\Omega$$

即：$r=ct$ 时，$\varphi = \varphi(x+ct\alpha,\ y+ct\beta,\ z+ct\gamma)$，$\alpha, \beta, \gamma$ 是单位球面上的坐标投影系数.

$$= \frac{t}{4\pi}\oint_\Omega \varphi(x+\underbrace{ct\alpha}_{x\text{方向上传播距离}},\ y+ct\beta,\ z+ct\gamma)$$

也可记作：$tM_{ct}[\phi]$

Scalar wave equation in 2D:

$u_{tt} = c^2(u_{xx}+u_{yy})$，初始条件：$u(0,x,y)=0$，$u_t(0,x,y)=\phi(x,y)$

↑ 和上两行的 $M_{ct}[\phi]$ 中 ϕ 不同

有解：

$$u(t,x,y) = \frac{t}{4\pi}\oint_S \phi(x+ct\alpha,\ y+ct\beta)\, dS$$

也可写成：

$$u(t,x,y) = \frac{1}{2\pi c}\iint_D \frac{\phi(x+\xi,\ y+\eta)}{\sqrt{(ct)^2-\xi^2-\eta^2}}\, d\xi\, d\eta, \quad (x-\xi)^2+(y-\eta)^2 = c^2t^2$$