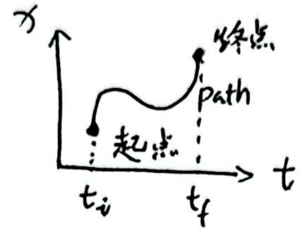
Field Theory Fundamentals

考虑一个1D粒子的运动



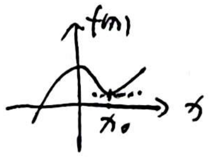
Lagrangian: $L = \underbrace{\frac{1}{2}m\left(\frac{dx}{dt}\right)^2}_{\text{动能}} - \underbrace{U(x)}_{\text{势能}}$

Action: $S = \int_{t_i}^{t_f} dt\, L.$ Action是能量的积分

理念: 变换路径后S要几乎不变 才说明S是"真实的情况"

S在"真实路径"下为最小或"局部最小"或"鞍点"

理解: eg. $f(x)$:



在x_0处 $f'(x_0)=0$

则 $df\big|_{x=x_0} = f'(x_0)\cdot \underset{\text{步长}\,dx}{\varepsilon} = 0$

说明在此处轻微(也许剧烈也行)扰动后f的值几乎不变.

实操:

原来认为的真实(或者说猜测的)路径为 $x(t)$

扰动后为 $x(t)+\varepsilon(t)$

我们期望通过"扰动后S几乎不变"这一特征找到$x(t)$的解.

将L看作 $\frac{dx}{dt}$, x 的函数. 同时 $\frac{dx}{dt}(t)$, $x(t)$, 作为第二层变量关系.

$$dL = \frac{1}{2}m\cdot 2\underbrace{\left(\frac{dx}{dt}\right)}\, d\left(\frac{dx}{dt}\right) - \frac{\partial U}{\partial x}dx$$

注意　这里不能换成 $\frac{d\varepsilon}{dt}$.

$$dL = \frac{1}{2}m\cdot 2\frac{dx}{dt}\frac{d\varepsilon}{dt} - \frac{\partial U}{\partial x}d\varepsilon$$

又∵ $\frac{d}{dt}\left(m\frac{d\varepsilon}{dt}\varepsilon\right) = m\frac{d^2x}{dt^2}\varepsilon + m\frac{dx}{dt}\frac{d\varepsilon}{dt}$

∴ $dL = \cancel{m} - m\frac{d^2x}{dt^2}\varepsilon + \frac{d}{dt}\left(m\frac{dx}{dt}\varepsilon\right) - \frac{\partial U}{\partial x}\varepsilon$

∴ $dS = \int_{t_i}^{t_f} dt \left\{ -\left(m\frac{d^2x}{dt^2} + \frac{dU}{dx}\right)\varepsilon + \frac{d}{dt}\left(m\frac{dx}{dt}\varepsilon\right) \right\}$

↑

这一项通过边界条件 $\varepsilon(t_i) = \varepsilon(t_f) = 0$ vanish.

∴ $dS = \int_{t_i}^{t_f} dt \left\{ -\left(m\frac{d^2x}{dt^2} + \frac{dU}{dx}\right)\varepsilon \right\} = 0$

则对任ε都成立只有 $m\frac{d^2x}{dt^2} + \frac{dU}{dx} = 0$.

一个稍复杂的例子：

$\phi(x,t)$ eg.温度既是空间又是时间的函数.

$$S=\int_{t_i}^{t_f} dt \int dx \, \mathcal{L}(\phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial t})$$

从上文的变成 locality, Lagrangian density

因为"dx"被积分出去了.

类比动能势能：

$$\mathcal{L} = \frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \left\{\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{2}k^2\phi^2\right\}$$

"Gradient energy"

构建的.类似弹簧的 $\frac{1}{2}kx^2$

Klein-Gordon Field Theory

if define $m=\frac{\hbar k}{c}$, it becomes "Quantize", m is Relativistic scalar particles of mass

$$\phi(x,t) \longrightarrow \phi(x,t)+\varepsilon(x,t), \quad ds=\int dt\int dx \, \delta\mathcal{L}=0$$

$$d\mathcal{L} = \frac{1}{c^2}\frac{\partial\phi}{\partial t} d\left(\frac{\partial\phi}{\partial t}\right) - \frac{\partial\phi}{\partial x} d\left(\frac{\partial\phi}{\partial x}\right) - k^2\phi \, d\phi$$

$$= \frac{1}{c^2}\frac{\partial\phi}{\partial t}\frac{\partial\varepsilon}{\partial t} - \frac{\partial\phi}{\partial x}\frac{\partial\varepsilon}{\partial x} - k^2\phi\varepsilon$$

分部积分,边界项 vanish 一下.

则 $ds=\int dt\int dx\left\{-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^2\phi}{\partial x^2}-k^2\phi\right\}\varepsilon=0$. {}里为0.即

Klein-Gordon Equation.

求解方程. 猜 $\phi(x,t) = A\cos(kx-\omega t)$

为了方便. $\phi(x,t) = Ae^{i(kx-\omega t)}$ 最后算完了取个实部即可.

代进去发现:

$\frac{\omega^2}{c^2} - k^2 = K^2$, 则满足此方程的 k 对应的 $A\cos(kx-\omega t)$ 即为解.

General solution = sum of plane waves.

在量子力学里.

$\psi(x,t) \propto e^{i(px-Et)/\hbar}$, $p=\hbar k$, $E=\hbar\omega$.

则 $\frac{E^2}{c^2} - p^2 = \hbar^2 K^2$

即: $E = \sqrt{(\frac{\hbar K}{c})^2 c^4 + p^2c^2}$, einstein's equation for the energy of a particle of mass m and momentum p where $m = \frac{\hbar K}{c}$ in the case.

若 $p=0$, $E=mc^2$

下面用一些新的notation来描述 $(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2})\phi = K^2\phi$

$$X^\mu = \begin{bmatrix} X^0 \\ X^1 \\ X^2 \\ X^3 \end{bmatrix} = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$ position 4-vector.

$$\frac{\partial}{\partial X^\mu} = \begin{bmatrix} \frac{\partial}{\partial X^0} \\ \frac{\partial}{\partial X^1} \\ \frac{\partial}{\partial X^2} \\ \frac{\partial}{\partial X^3} \end{bmatrix} = \begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \qquad \eta^{\mu\nu} = \begin{bmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{bmatrix}$$

Minkowski metric

제

$$\underbrace{\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right)}\phi = K^2\phi$$

$$\underbrace{\sum_{\mu,\nu=0}^{3} \eta^{\mu\nu}\frac{\partial}{\partial X^\mu}\frac{\partial}{\partial X^\nu}} = \eta^{00}\frac{\partial}{\partial X^0}\frac{\partial}{\partial X^0} + \eta^{11}\frac{\partial}{\partial X^1}\frac{\partial}{\partial X^1} + \cdots$$

d' Alembertian $\partial^2 = \Box$

$$\Box\phi = K^2\phi.$$