$\delta(x)$ 定义

数学上有若干种:

1. $\delta(\vec{r}) = \frac{1}{-4\pi} \nabla^2 \frac{1}{r}$, $\delta(\vec{r}) = \delta(x)\delta(y)\delta(z)$

2. $\delta(x) = \lim\limits_{\varepsilon \to 0} \frac{1}{\pi} \frac{\varepsilon}{x^2+\varepsilon^2}$, 适合数值计算

3. $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx}\, dk$, 可用于求 $\sin\omega_0 t$, $e^{i\omega_0 t}$ 的FFT.

1. $r = \sqrt{x^2+y^2+z^2}$

~~$\nabla^2(\frac{1}{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}$~~ $r \neq 0$时:

$$\nabla^2(\frac{1}{r}) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r \cdot \frac{1}{r}) = 0$$

$$\int \nabla^2(\frac{1}{r})dV = \int \nabla \cdot (\nabla \frac{1}{r})\, dV = \oint (\nabla \frac{1}{r}) d\vec{S} = -\oint \frac{1}{r^2}\frac{\vec{r}}{r} \cdot d\vec{S} = -4\pi.$$

则: $\nabla^2(\frac{1}{r}) = -4\pi\delta(\vec{r})$

2. $$\int_{-\infty}^{+\infty} \lim_{\varepsilon \to 0} \frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}dx = \frac{1}{\pi}\int_{-\infty}^{+\infty} \lim_{\varepsilon \to 0} \frac{1}{(\frac{x}{\varepsilon})^2+1} d(\frac{x}{\varepsilon}) = \cancel{\frac{1}{\pi}\lim_{\varepsilon \to 0}\tan\frac{x}{\varepsilon}\Big|_{-\infty}^{+\infty}} = 1$$

$$= \frac{1}{\pi}\lim_{\varepsilon \to 0} \tan\frac{x}{\varepsilon}\Big|_{-\infty}^{+\infty} = 1$$

3. $$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}dk = \frac{1}{2\pi}\lim_{\eta \to 0^+}\left(\int_0^{\infty} e^{ik(x+i\eta)}dk + \int_{-\infty}^{0} e^{ik(x-i\eta)}dk\right)$$

证明合理,见背面.

$$= \lim_{\eta \to 0^+} \frac{1}{\pi}\frac{\eta}{x^2+\eta^2} = \delta(x)$$

上文已证.

$\int_{0^-}^{0^+} e^{ik(x+i\eta)} dk = 0$. 显然成立.

$\lim\limits_{\eta\to 0^+} \lim\limits_{M\to+\infty} \int_0^M e^{ik(x+i\eta)} dk$. 当 $x\neq 0$. $x+i\eta \to x$.

若 $M\eta\to 0$. 即 η 足够小. 则 $\int_0^M e^{ik(x+i\eta)} dk \to \int_0^M e^{ikx} dk$

但这证明也有不严谨处, 比如. $M\eta\to 0$ 为什么成立, 也可能趋于 ∞.