# COT3210–Computability and Automata
# Answers to HW4 Supplementary Exercises

1. For each of the fifteen unordered pairs of functions $f$ and $g$ chosen from the functions given below, determine whether $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$, or $f(n) = \Theta(g(n))$.

| | | |
|---|---|---|
| a. $n^3$ | b. $2^{n\log_2 n}$ | c. $n^6$ |
| d. $n2^n$ | e. $n^3 \log_2 n$ | f. $2^{2^n}$ |

For any two functions $f(n)$ and $g(n)$, we wish to determine constants $K$ and $n_0$ such that a relationship of the form

$$f(n) \le Kg(n) \text{ for } n \ge n_0$$

is satisfied in order to conclude $f(n) = O(g(n))$. We may also conclude in this case that $g(n) = \Omega(f(n))$. Also, if $f(n) = O(g(n))$ and $g(n) = O(f(n))$, we conclude that $f(n) = \Theta(g(n))$ and $g(n) = \Theta(f(n))$.

We consider the fifteen pairs below, one pair at a time.

- Let $f(n) = n^3$ and $g(n) = 2^{n\log_2 n}$. Since $g(n) = n^n$, we can conclude that $n^3 \le n^n$ for $n \ge 3$. Therefore, $n^3 = O(2^{n\log_2 n})$ and $2^{n\log_2 n} = \Omega(n^3)$.

- Let $f(n) = n^3$ and $g(n) = n^6$. We have $n^3 \le n^6$ for all $n \ge 1$. Hence $n^3 = O(n^6)$ and $n^6 = \Omega(n^3)$.

- Let $f(n) = n^3$ and $g(n) = n2^n$. We would like to determine $n_0$ such that $n^3 \le n2^n$ for all $n \ge n_0$. Since $n^3 \le n2^n \;\Rightarrow\; n^2 \le 2^n$ for $n \ge 4$, it follows that $n^3 = O(n2^n)$ and $n2^n = \Omega(n^3)$.

- Let $f(n) = n^3$ and $g(n) = n^3 \log_2 n$. Clearly, $n^3 \le n^3 \log_2 n$ for $n \ge 2$. Therefore, $n^3 = O(n^3 \log_2 n)$ and $n^3 \log_2 n = \Omega(n^3)$.

- Let $f(n) = n^3$ and $g(n) = 2^{2^n}$. Clearly, $n^3 \le 2^{2^n}$ for $n \ge 1$. Therefore, $n^3 = O(2^{2^n})$ and $2^{2^n} = \Omega(n^3)$.

- Let $f(n) = 2^{n\log_2 n}$ and $g(n) = n^6$. Since $2^{n\log_2 n} = n^n$, and since $n^n \ge n^6$ for $n \ge 6$, it follows that $2^{n\log_2 n} \ge n^6$, and we conclude $2^{n\log_2 n} = \Omega(n^6)$ and $n^6 = O(2^{n\log_2 n})$.

- Let $f(n) = 2^{n\log_2 n}$ and $g(n) = n2^n$. Since $2^{n\log_2 n} = n^n$, and since $n^n \ge n2^n$ for $n \ge 3$, it follows that $2^{n\log_2 n} \ge n2^n$, and we conclude $2^{n\log_2 n} = \Omega(n2^n)$ and $n2^n = O(2^{n\log_2 n})$.

- Let $f(n) = 2^{n\log_2 n}$ and $g(n) = n^3 \log_2 n$. Since $2^{n\log_2 n} = n^n$, and since $n^n \ge n^3 \log_2 n$ for $n \ge 3$, it follows that $2^{n\log_2 n} \ge n^3 \log_2 n$ for $n \ge 3$, and we conclude $2^{n\log_2 n} = \Omega(n^3 \log_2 n)$ and $n^3 \log_2 n = O(2^{n\log_2 n})$.

- Let $f(n) = 2^{n\log_2 n}$ and $g(n) = 2^{2^n}$. Since $2^{n\log_2 n} = n^n$, and since $n^n \le 2^{2^n}$ for $n \ge 1$, it follows that $2^{n\log_2 n} \le 2^{2^n}$ for $n \ge 1$, and we conclude $2^{n\log_2 n} = O(2^{2^n})$ and $2^{2^n} = \Omega(2^{n\log_2 n})$.

- Let $f(n) = n^6$ and $g(n) = n2^n$. In order to find $n_0$ such that $n^6 \leq n2^n$ (or, equivalently, $n^5 \leq 2^n$) for $n \geq n_0$. We can either plot $n^5$ and $2^n$ and determine this value, or plot $5\log_2 n$ and $n$ to determine this value. A quick solution to this problem can be obtained by looking at values of $n$ which are powers of 2. We see that for $n \geq 32$ it is true that $5\log_2 n \leq n$. A tighter bound can be obtained by actually plotting these functions and inding out where they intersect. This yields $n_0 = 23$. We conclude $n^6 = O(n2^n)$ and $n2^n = \Omega(n^6)$.

- Let $f(n) = n^6$ and $g(n) = n^3\log_2 n$. Clearly, $n^6 \geq n^3\log_2 n$ for $n \geq 1$. Therefore $n^6 = \Omega(n^3\log_2 n)$.

- Let $f(n) = n^6$ and $g(n) = 2^{2^n}$. Clearly, $n^6 \leq 2^{2^n}$ for $n \geq 1$. Thereore, $n^6 = O(2^{2^n})$ and $2^{2^n} = \Omega(n^6)$.

- Let $f(n) = n2^n$ and $g(n) = n^3\log_2 n$. By plotting these functions we see that $n2^n \geq n^3\log_2 n$ for $n \geq 8$. Therefore, $n2^n = \Omega(n^3\log_2 n)$ and $n^3\log_2 n = O(n2^n)$.

- Let $f(n) = n2^n$ and $g(n) = 2^{2^n}$. Clearly, $n2^n \leq 2^{2^n}$ for $n \geq 1$. Therefore, $n2^n = O(2^{2^n})$ ans $2^{2^n} = \Omega(n2^n)$.

- Let $f(n) = n^3\log_2 n$ and $g(n) = 2^{2^n}$. Clearly, $n^3\log_2 n \leq 2^{2^n}$ for $n \geq 1$. Therefore, $n^3\log_2 n = O(2^{2^n})$ and $2^{2^n} = \Omega(n^3\log_2 n)$.

The following table summarizes the results for the fifteen pairs of functions in this exercise.

| $f(n)$ | $g(n)$ | Inequality | $K$ | $n_0$ | Relationship |
|---|---|---|---|---|---|
| $n^3$ | $2^{n\log_2 n}$ | $n^3 \leq 2^{n\log_2 n}$ | 1 | 3 | $f = O(g)$ |
| $n^3$ | $n^6$ | $n^3 \leq n^6$ | 1 | 1 | $f = O(g)$ |
| $n^3$ | $n2^n$ | $n^3 \leq n2^n$ | 1 | 4 | $f = O(g)$ |
| $n^3$ | $n^3\log_2 n$ | $n^3 \leq n^3\log_2 n$ | 1 | 2 | $f = O(g)$ |
| $n^3$ | $2^{2^n}$ | $n^3 \leq 2^{2^n}$ | 1 | 1 | $f = O(g)$ |
| $2^{n\log_2 n}$ | $n^6$ | $2^{n\log_2 n} \geq n^6$ | 1 | 6 | $f = \Omega(g)$ |
| $2^{n\log_2 n}$ | $n2^n$ | $2^{n\log_2 n} \geq n2^n$ | 1 | 3 | $f = \Omega(g)$ |
| $2^{n\log_2 n}$ | $n^3\log_2 n$ | $2^{n\log_2 n} \geq n^3\log_2 nn^6$ | 1 | 3 | $f = \Omega(g)$ |
| $2^{n\log_2 n}$ | $2^{2^n}$ | $2^{n\log_2 n} \leq 2^{2^n}$ | 1 | 1 | $f = O(g)$ |
| $n^6$ | $n2^n$ | $n^6 \leq n2^n$ | 1 | 23 | $f = O(g)$ |
| $n^6$ | $n^3\log_2 n$ | $n^6 \geq n^3\log_2 n$ | 1 | 1 | $f = \Omega(g)$ |
| $n^6$ | $2^{2^n}$ | $n^6 \leq 2^{2^n}$ | 1 | 1 | $f = O(g)$ |
| $n2^n$ | $n^3\log_2 n$ | $n2^n \geq n^3\log 2n$ | 1 | 8 | $f = \Omega(g)$ |
| $n2^n$ | $2^{2^n}$ | $n2^n \leq 2^{2^n}$ | 1 | 1 | $f = O(g)$ |
| $n^3\log_2 n$ | $2^{2^n}$ | $n^3\log_2 n \leq 2^{2^n}$ | 1 | 1 | $f = O(g)$ |

2. Assume $f(n) = O(g(n))$. Then there exist constants $K$ and $n_0$ satisfying that $f(n) \leq Kg(n)$ for $n \geq n_0$. This means that $g(n) \geq \frac{1}{K} f(n)$ for $n \geq n_0$, or $g(n) = \Omega(f(n))$. In a similar manner, we can show that if $g(n) = \Omega(f(n))$, then $f(n) = O(g(n))$.

3. Assume that $f(n) = \Theta(g(n))$. Then there exist constants $c_1$, $c_2$, and $n_0$ satisfying $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for $n \geq n_0$. This means that $\frac{1}{c_2} f(n) \leq g(n) \leq \frac{1}{c_1} f(n)$, or $g(n) = \Theta(f(n))$. In a similar manner, we can show that if $g(n) = \Theta(f(n))$, then $f(n) = \Theta(g(n))$.

4. We start with the basic fact that $n^j \leq n^m$ for all $j \leq m$. Now, if we let $g(n) = \sum_{j=0}^{k} a_j n^j$, we can conclude that $g(n) \leq \sum_{j=0}^{k} |a_j| n^m$ for all $k \leq m$. Thus, $g(n) \leq K n^m$ with $K = \sum_{j=0}^{k} |a_j|$, or $g(n) = O(n^m)$.

5. By replacing $n$ with $2^n$, the comparison of $\log n$ and $n^k$ is the same as comparing $n$ and $2^{kn}$. Since $n = O(2^{kn})$ it follows that $\log n = O(n^k)$.

6. $n^k \leq n^{\log n}$ for $n \geq b^k$, where $b$ is the base of the logarithm used. Therefore we may conclude that $n^k = O(n^{\log n})$. However, since $k$ is a constant and does not vary with $n$, it is not possible to find an $n_0$ such that $\log n \leq k$ for all $n \geq n_0$. Therefore, $n^{\log n}$ can never be less than $n^k$, or $n^{\log n}$ can never be $O(n^k)$.

7. Instead of comparing $c^n$ and $n^{\log n}$ as $n$ grows, we will compare the corresponding functions that we obtain by replacing $\log n$ with $k$. We may assume that the base of the logarithm is 2 without loss of generality, and the functions we need to compare are $c^{2^k}$ and $(2^k)^k = 2^{k^2}$ as $k$ grows. If $f(k) = c^{2^k}$ and $g(k) = 2^{k^2}$, we have $\log(f) = 2^k \log c$ and $\log(g) = k^2$. It is clear that $\log(f) = \Omega(\log(g))$, implying $c^n = \Omega(n^{\log n})$. Of course, since $k^2$ can never be $\Omega(2^k)$, it follows that $n^{\log n}$ can never be $\Omega(c^n)$.

8. The answers are shown below:

   a. $\dfrac{n}{100000000} + 999999999 = O(n)$
   b. $\log(n^2 + 1) = O(\log n)$
   c. $\sqrt{n^2 + 1} = O(n)$
   d. $(n^2 + 1)(n \log n + 1) = O(n^3 \log n)$
   e. $10^{1000} = O(1)$
   f. $\dfrac{n+3}{n+1} = O(1)$
   g. $\dfrac{n^3 + 1}{n + 1} = O(n^2)$
   h. $2^{3 \log n} + n^3 + 4 = O(n^3)$
   i. $\dfrac{n!}{999999999} + 999999999999 \cdot 2^n = O(n!)$
   j. $\log_{10} 2^n + 10^{10} n^2 = O(n^2)$

9. The answers are shown below:

   a. $n^3 - 47n^{5/2} + 17n - 7 \leq (1 + 47 + 17 + 7)n^3 = 72n^3 = O(n^3)$; Answer: 72

   b. $n \log n + n^2 \leq (1 + 1)n^2 = 2n^2 = O(n^2)$; Answer: 2

   c. $7n \log n + n^{3/2} \leq (7 + 1)n^2 = 8n^2 = O(n^2)$; Answer: 8

   d. $n \log n^7 + n^{3/2} \leq (7 + 1)n^{3/2} = 8n^{3/2} = O(n^{3/2})$; Answer: 8

10. Since $i$ gets squared ater each execution of the loop, the value of $i$ after $k$ executions of the loop will be $2^{2^k}$. Therefore, we must estimate $k$ that satisfies $2^{2^k} \approx n$. We get $k \approx \log_2(\log_2 n)$.

11. The trace table is shown below:

| $m$ | $n$ | $p$ |
|---|---|---|
| 2 | 67 | 1 |
| $2^2$ | 33 | 2 |
| $2^4$ | 16 | $2^3$ |
| $2^8$ | 8 | $2^3$ |
| $2^{16}$ | 4 | $2^3$ |
| $2^{32}$ | 2 | $2^3$ |
| $2^{64}$ | 1 | $2^3$ |
| $2^{128}$ | 0 | $2^{67}$ |

12. The answers are provided below (the maximum number of division steps are also provided):

    a. 227, 143, 84, 59, 25, 9, 7, 2, 1, 0; 8 division steps $\leq \log 227/\log 1.618 \approx 11.27$

    b. 131, 71, 60, 11, 5, 1, 0; 5 division steps $\leq \log 131/\log 1.618 \approx 10.13$

    c. 259, 93, 73, 20, 13, 7, 6, 1, 0; 7 division steps $\leq \log 259/\log 1.618 \approx 11.55$

13. The graph is shown below: