Complexity

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Topic Overview

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- Terminology

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  - TM Recognizable/Decidable

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  - Based on Pseudocode Notation
  - Based on TM Descriptions

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- A decider recognizing a language:
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- A language is said to be “Decidable”
  if some TM decides it.

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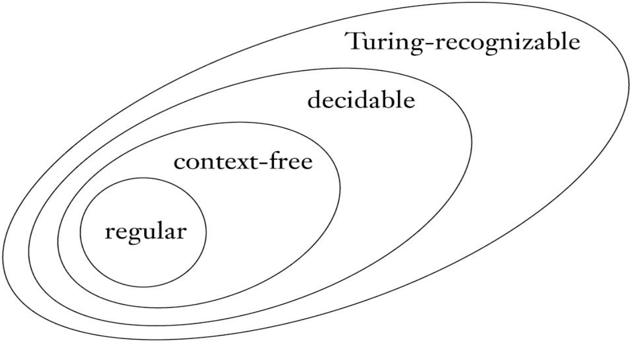

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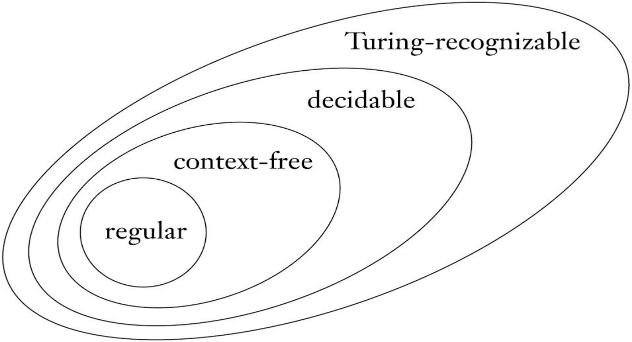

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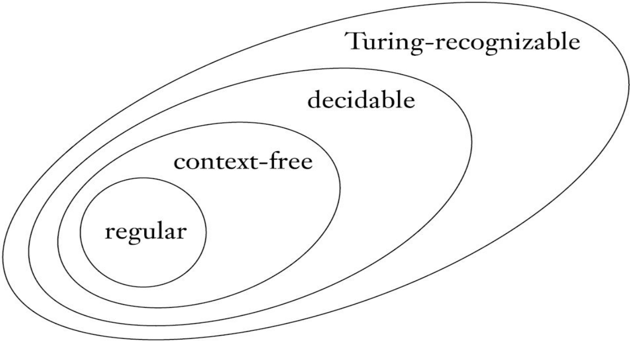

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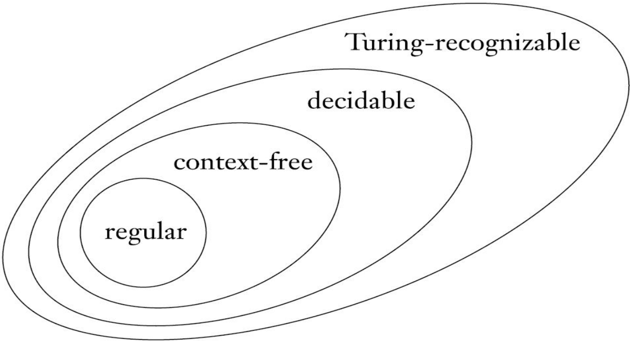

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- All examples of the languages we have seen are decidable.

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- Customarily we use n to represent the length of the input.

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- We suppress constant factors.

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- Therefore $f(n) = O(n^3)$.
- Note that $f(n) = O(n^k)$ for all $k \geq 3$.
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$$f(n) = O(n^2) + O(n) \Rightarrow f(n) = O(n^2)$$

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- $\log_a n = O(\log n)$ as they differ by a constant factor.
- Suppose $f(n) = 3n\log_2 n + \log_2 \log_2 n + 2$.
- Then, $f(n) = O(n \log n)$.
- Other forms of growth functions:

$$f(n) = O(n^2) + O(n) \Rightarrow f(n) = O(n^2)$$

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- For every integer $n \geq n_0$, it is true that $f(n) < c\, g(n)$.

Analysis of Algorithms

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Algorithm Analysis: Examples

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Hello is printed k times, with $2^{k-1} \leq n < 2^k$.
This amounts to $\lfloor \log_2 \mathtt{n} \rfloor + 1$ times.

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Hello is printed k times, with $3^{k-1} \leq n < 3^k$.

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Hello is printed $(n^2 + n)/2$ times.

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Trace Table: Integer Multiplication

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m = 13; n = 77
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n>0?	odd(n)	c	m	n

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n>0?	odd(n)	c	m	n
–	–	0	13	77
Yes	Yes	13	26	38

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Yes	No	13	52	

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```
m = 13; n = 77
p = 0
while (n > 0)
  if (odd(n))
    {p = p + m}
  m = m * 2
  n = n / 2
```

n>0?	odd(n)	c	m	n
–	–	0	13	77
Yes	Yes	13	26	38
Yes	No	13	52	19
Yes	Yes	65	104	9
Yes	Yes	169		

Trace Table: Integer Multiplication

<u>Algorithm A</u>:

```
m = 13; n = 77
p = 0
while (n > 0)
  p = p + m
  n = n - 1
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```

<u>Algorithm B</u>:

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Yes	No	13	52	19
Yes	Yes	65	104	9
Yes	Yes	169	208	

Trace Table: Integer Multiplication

Algorithm A:

```
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---------------
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Algorithm B:

```
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p = 0
while (n > 0)
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Yes	Yes	13	26	38
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Trace Table: Integer Multiplication

Algorithm A:

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m = 13; n = 77
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Algorithm B:

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m = 13; n = 77
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while (n > 0)
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Trace Table: Integer Multiplication

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Trace Table: Integer Multiplication

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Trace Table: Integer Multiplication

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Yes	No	169	416	

Trace Table: Integer Multiplication

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Yes	No	169	416	2

Trace Table: Integer Multiplication

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Trace Table: Integer Multiplication

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Trace Table: Integer Multiplication

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Yes	Yes	13	26	38
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Yes	Yes	65	104	9
Yes	Yes	169	208	4
Yes	No	169	416	2
Yes	No	169	832	

Trace Table: Integer Multiplication

Algorithm A:

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Trace Table: Integer Multiplication

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Yes	No	169	832	1
Yes				

Trace Table: Integer Multiplication

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Yes	Yes			

Trace Table: Integer Multiplication

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while (n > 0)
   if (odd(n))
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Yes	Yes	169	208	4
Yes	No	169	416	2
Yes	No	169	832	1
Yes	Yes	1001		

Trace Table: Integer Multiplication

Algorithm A:

```
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----------------
```

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```
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Yes	Yes	169	208	4
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Yes	No	169	832	1
Yes	Yes	1001	1664	

Trace Table: Integer Multiplication

Algorithm A:

```
m = 13; n = 77
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Algorithm B:

```
m = 13; n = 77
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Trace Table: Integer Multiplication

Algorithm A:

```
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```
m = 13; n = 77
p = 0
while (n > 0)
  if (odd(n))
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Yes	Yes	65	104	9
Yes	Yes	169	208	4
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Yes	No	169	832	1
Yes	Yes	1001	1664	0
No				

Trace Table: Integer Multiplication

Algorithm A:

```
m = 13; n = 77
p = 0
while (n > 0)
  p = p + m
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---------------
```

Algorithm B:

```
m = 13; n = 77
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while (n > 0)
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n>0?	odd(n)	c	m	n
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Yes	Yes	65	104	9
Yes	Yes	169	208	4
Yes	No	169	416	2
Yes	No	169	832	1
Yes	Yes	1001	1664	0
No	–	–	–	–

Trace Table: Integer Multiplication

Algorithm A:

```
m = 13; n = 77
p = 0
while (n > 0)
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Algorithm B:

```
m = 13; n = 77
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while (n > 0)
  if (odd(n))
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n>0?	odd(n)	c	m	n
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Yes	Yes	65	104	9
Yes	Yes	169	208	4
Yes	No	169	416	2
Yes	No	169	832	1
Yes	Yes	1001	1664	0
No	–	–	–	–

`p=p+m` executed 77 times by A.

Trace Table: Integer Multiplication

Algorithm A:

```
m = 13; n = 77
p = 0
while (n > 0)
   p = p + m
   n = n - 1
----------------
```

Algorithm B:

```
m = 13; n = 77
p = 0
while (n > 0)
   if (odd(n))
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n>0?	odd(n)	c	m	n
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Yes	Yes	169	208	4
Yes	No	169	416	2
Yes	No	169	832	1
Yes	Yes	1001	1664	0
No	–	–	–	–

p=p+m executed 77 times by A.

p=p+m executed 4 times by B.

Trace Table: Integer Multiplication

Algorithm A:

```
m = 13; n = 77
p = 0
while (n > 0)
  p = p + m
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---------------
```

Algorithm B:

```
m = 13; n = 77
p = 0
while (n > 0)
  if (odd(n))
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Yes	Yes	1001	1664	0
No	–	–	–	–

p=p+m executed 77 times by A.
p=p+m executed 4 times by B.
A is of time complexity $O(n)$.

Trace Table: Integer Multiplication

Algorithm A:

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m = 13; n = 77
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m = 13; n = 77
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Yes	Yes	1001	1664	0
No	–	–	–	–

p=p+m executed 77 times by A.
p=p+m executed 4 times by B.
A is of time complexity $O(n)$.
B is of time complexity of $O(\log n)$.

Trace Table: Integer Exponentiation

Trace Table: Integer Exponentiation

Algorithm A:

Trace Table: Integer Exponentiation

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```
m = 13; n = 77
p = 1
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Trace Table: Integer Exponentiation

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Algorithm B:

Trace Table: Integer Exponentiation

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Algorithm B:

```
m = 13; n = 77
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while (n > 0)
  if (odd(n))
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Trace Table: Integer Exponentiation

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m = 13; n = 77
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m = 13; n = 77
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```

n>0?	odd(n)	c	m	n

Trace Table: Integer Exponentiation

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```
m = 13; n = 77
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while (n > 0)
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m = 13; n = 77
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Trace Table: Integer Exponentiation

Algorithm A:

```
m = 13; n = 77
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Trace Table: Integer Exponentiation

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n>0?	odd(n)	c	m	n
–	–	1	13	77
Yes	Yes	13^1		

Trace Table: Integer Exponentiation

Algorithm A:

```
m = 13; n = 77
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Trace Table: Integer Exponentiation

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Yes	No	13^1	13^4	

Trace Table: Integer Exponentiation

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Trace Table: Integer Exponentiation

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Trace Table: Integer Exponentiation

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while (n > 0)
  p = p * m
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```

Algorithm B:

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m = 13; n = 77
p = 1
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n>0?	odd(n)	c	m	n
–	–	1	13	77
Yes	Yes	13^1	13^2	38
Yes	No	13^1	13^4	19
Yes	Yes	13^5		

Trace Table: Integer Exponentiation

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Yes	Yes	13^{13}		

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Yes	Yes	13^{13}	13^{16}	

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Yes	Yes	13^{13}	13^{16}	4

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Yes	No			

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Yes	Yes	13^{13}	13^{16}	4
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Yes	No	13^{13}	13^{64}	

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No	–	–	–	–

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No	–	–	–	–

`p=p*m` executed 77 times by A.

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No	–	–	–	–

p=p*m executed 77 times by A.

p=p*m executed 4 times by B.

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No	–	–	–	–

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A is of time complexity $O(n)$.

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Yes	Yes	13^{77}	13^{128}	0
No	–	–	–	–

`p=p*m` executed 77 times by A.
`p=p*m` executed 4 times by B.
A is of time complexity $O(n)$.
B is of time complexity of $O(\log n)$.

The GCD Algorithm

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- Computing the *greatest common divisor*

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- The divisors of 18 are 1, 2, 3, 6, 9, and 18.

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- For example, $gcd(18, 12) = 6$.
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- The divisors of 12 are 1, 2, 3, 6, and 12.

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- There may be several common divisors.
- Which is the greatest?
- For example, $gcd(18, 12) = 6$.
- The divisors of 18 are 1, 2, 3, 6, 9, and 18.
- The divisors of 12 are 1, 2, 3, 6, and 12.
- Of the common divisors 1, 2, 3, and 6, 6 is the greatest.

The GCD Algorithm (cont.)

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- If d divides n and m, d also divides $n \mod m$.

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$$gcd(n,m) = \begin{cases} n, & \text{if } m = 0; \\ gcd(m, n \mod m), & \text{if } m \neq 0. \end{cases}$$

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- $gcd(18, 12) =$

The GCD Algorithm (cont.)

- If d divides n and m, d also divides $n \mod m$.

$$gcd(n, m) = \begin{cases} n, & \text{if } m = 0; \\ gcd(m, n \mod m), & \text{if } m \neq 0. \end{cases}$$

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- Remainder sequence: 18, 12, 6, 0.
- For $n = 7366$ and $m = 242$, the remainder sequence is:
- 7366, 242, 106,

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- Remainder sequence: 18, 12, 6, 0.
- For $n = 7366$ and $m = 242$, the remainder sequence is:
- 7366, 242, 106, 30,

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- Remainder sequence: 18, 12, 6, 0.
- For $n = 7366$ and $m = 242$, the remainder sequence is:
- 7366, 242, 106, 30, 16,

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- 7366, 242, 106, 30, 16, 14,

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- Remainder sequence: 18, 12, 6, 0.
- For $n = 7366$ and $m = 242$, the remainder sequence is:
- 7366, 242, 106, 30, 16, 14, 2, 0,
- so that $gcd(7366, 242) = 2$.

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- The remainder sequence is:
- 987, 610, 377,

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- Consider $n = 987$, $m = 610$.
- The remainder sequence is:
- 987, 610, 377, 233,

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- Consider $n = 987$, $m = 610$.
- The remainder sequence is:
- 987, 610, 377, 233, 144,

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- Consider $n = 987$, $m = 610$.
- The remainder sequence is:
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- Consider $n = 987$, $m = 610$.
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- 987, 610, 377, 233, 144, 89, 55,

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  34,

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  34, 21,

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  34, 21, 13,

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  34, 21, 13, 8,

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  34, 21, 13, 8, 5,

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- Thus, $gcd(987, 610) = 1$

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- Since $F_k \approx (\phi)^k$, where $\phi \approx 1.618$, $k \approx \log_\phi n = O(\log n)$.

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Total time: $O(n) + O(n^2) + O(n) = O(n^2)$.

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$$\mathbf{NTIME}(\boldsymbol{t(n)}) = \{L \mid L \text{ is a language decided by an } O(t(n)) \text{ time nondeterministic Turing machine}\}.$$

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1. P is invariant for all models of computation that are polynomially equivalent to the deterministic single-tape Turing machine, and
2. P roughly corresponds to the class of problems that are realistically solvable on a computer.

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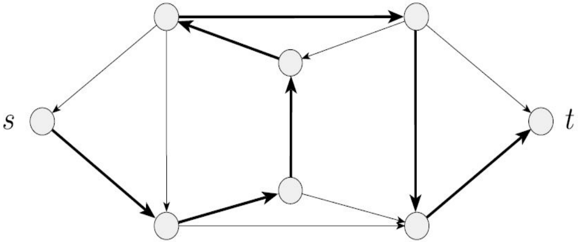

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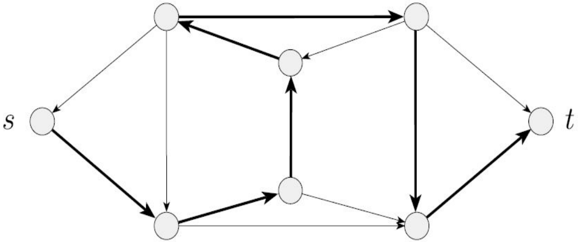

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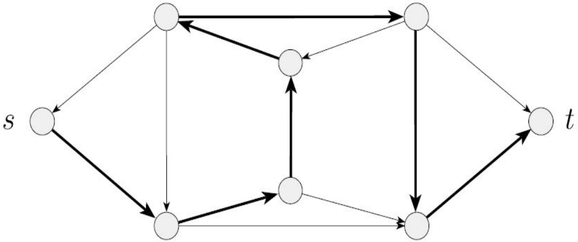

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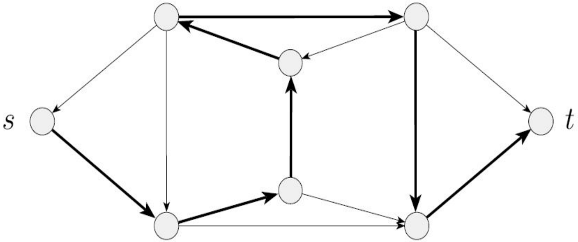

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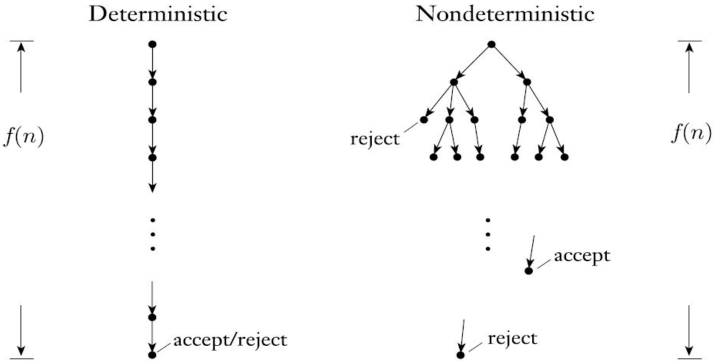

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1. Write a list of m numbers, $p_1, \ldots, p_m$, where m is the number of nodes in G. Each number in the list is nondeterministically selected to be between 1 and m.
2. Check for repetitions in the list. If any are found, *reject*.
3. Check whether $s = p_1$ and $t = p_m$. If either fail, *reject*.
4. For each i between 1 and $m - 1$, check whether (p_i, p_{i+1}) is an edge of G. If any are not, *reject*. Otherwise, all tests have been passed, so *accept*.

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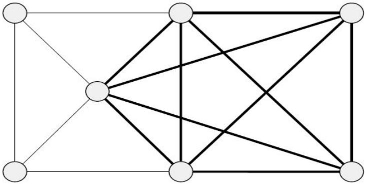

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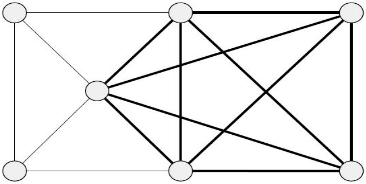

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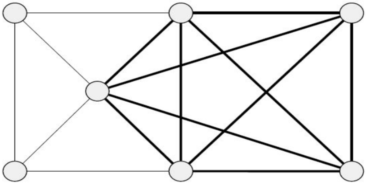

CLIQUE: Given an undirected graph G and an integer k, Determine whether G contains a k-clique.

$CLIQUE \in$ NP (cont.)

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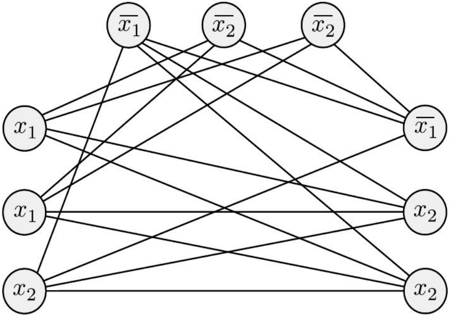

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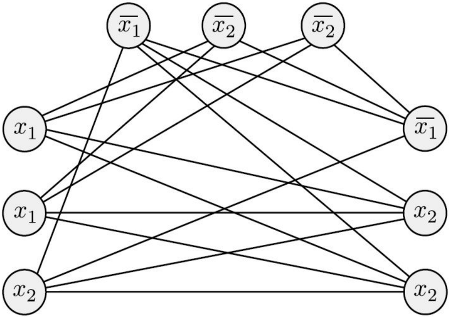

HAMPATH and *SUBSET-SUM* are NP-complete.