Solutions Manual

to accompany

Probability, Random Variables and Stochastic Processes

Fourth Edition

Athanasios Papoulis
Polytechnic University

S. Unnikrishna Pillai
Polytechnic University

Boston Burr Ridge, IL Dubuque, IA Madison, WI New York San Francisco St. Louis
Bangkok Bogotá Caracas Kuala Lumpur Lisbon London Madrid Mexico City
Milan Montreal New Delhi Santiago Seoul Singapore Sydney Taipei Toronto

McGraw-Hill Higher Education
A Division of The McGraw-Hill Companies

Solutions Manual to accompany
PROBABILITY, RANDOM VARIABLES AND STOCHASTIC PROCESSES, FOURTH EDITION
ATHANASIOS PAPOULIS

Published by McGraw-Hill Higher Education, an imprint of The McGraw-Hill Companies, Inc., 1221 Avenue of the Americas, New York, NY 10020. Copyright © 2002 by The McGraw-Hill Companies, Inc. All rights reserved.

The contents, or parts thereof, may be reproduced in print form solely for classroom use with PROBABILITY, RANDOM VARIABLES AND STOCHASTIC PROCESSES, FOURTH EDITION, provided such reproductions bear copyright notice, but may not be reproduced in any other form or for any other purpose without the prior written consent of The McGraw-Hill Companies, Inc., including, but not limited to, in any network or other electronic storage or transmission, or broadcast for distance learning.

www.mhhe.com

CHAPTER 2

2-1 We use De Morgan's law:

(a) $\overline{\bar{A}+\bar{B}} + \overline{\bar{A}+B} = AB + A\bar{B} = A(B+\bar{B}) = A$

(b) $(A+B)(\overline{AB}) = (A+B)(\bar{A}+\bar{B}) = A\bar{B} + B\bar{A}$

because $A\bar{A} = \{\emptyset\}$ $B\bar{B} = \{\emptyset\}$

2-2 If $A = \{2 \le x \le 5\}$ $B = \{3 \le x \le 6\}$ $S = \{-\infty < x < \infty\}$ then

$A+B = \{2 \le x \le 6\}$ $AB = \{3 \le x \le 5\}$

$$(A+B)(\overline{AB}) = \{2 \le x \le 6\}\ [\{x < 3\} + \{x > 5\}]$$
$$= \{2 \le x < 3\} + \{5 < x \le 6\}$$

2-3 If $AB = \{\emptyset\}$ then $A \subset \bar{B}$ hence

$$P(A) \le P(\bar{B})$$

2-4 (a) $P(A) = P(AB) + P(A\bar{B})$ $P(B) = P(AB) + P(\bar{A}B)$

If, therefore, $P(A) = P(B) = P(AB)$ then

$P(A\bar{B}) = 0$ $P(\bar{A}B) = 0$ hence

$$P(\bar{A}B + A\bar{B}) = P(\bar{A}B) + P(A\bar{B}) = 0$$

(b) If $P(A) = P(B) = 1$ then $1 = P(A) \le P(A+B)$ hence

$$1 = P(A+B) = P(A) + P(B) - P(AB) = 2 - P(AB)$$

This yields $P(AB) = 1$

2-5 From (2-13) it follows that

$$P(A+B+C) = P(A) + P(B+C) - P[A(B+C)]$$
$$P(B+C) = P(B) + P(C) - P(BC)$$
$$P[A(B+C)] = P(AB) + P(AC) - P(ABC)$$

because $ABAC = ABC$. Combining, we obtain the desired result.

Using induction, we can show similarly that

$$P(A_1 + A_2 + \cdots + A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$
$$- P(A_1A_2) - \cdots - P(A_{n-1}A_n)$$
$$+ P(A_1A_2A_3) + \cdots + P(A_{n-2}A_n)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\pm P(A_1A_2 \cdots A_n)$$

2-6 Any subset of S contains a countable number of elements, hence, it can be written as a countable union of elementary events. It is therefore an event.

2-7 Forming all unions, intersections, and complements of the sets $\{1\}$ and $\{2,3\}$, we obtain the following sets:

$$\{\emptyset\},\ \{1\},\ \{4\},\ \{2,3\},\ \{1,4\},\ \{1,2,3\},\ \{2,3,4\},\ \{1,2,3,4\}$$

2-8 If $A \subset B$, $P(A) = 1/4$, and $P(B) = 1/3$, then

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/4}{1/3} = \frac{3}{4}$$

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

2-9

$$P(A|BC)P(B|C) = \frac{P(ABC)}{P(BC)}\frac{P(BC)}{P(C)}$$

$$= \frac{P(ABC)}{P(C)} = P(AB|C)$$

$$P(A|BC)P(B|C)P(C) = \frac{P(ABC)}{P(BC)}\frac{P(BC)}{P(C)}P(C)$$

$$= P(ABC)$$

2-10 We use induction. The formula is true for $n = 2$ because $P(A_1A_2) = P(A_2|A_1)P(A_1)$. Suppose that it is true for n. Since

$$P(A_{n+1}A_n \cdots A_1) = P(A_{n+1}|A_n \cdots A_2A_1)P(A_1 \cdots A_n)$$

we conclude that it must be true for $n+1$.

2-11 <u>First solution</u>. The total number of m element subsets equals $\binom{n}{m}$ (see Probl. 2-26). The total number of m element subsets containing ζ_o equals $\binom{n-1}{m-1}$. Hence

$$p = \binom{n}{m}\Big/\binom{n-1}{m-1} = \frac{m}{n}$$

<u>Second solution</u>. Clearly, $P\{\zeta_o|A_m\} = m/n$ is the probability that ζ_o is in a specific A_m. Hence (total probability)

$$p = \sum P\{\zeta_o|A_m\}p(A_m) = \frac{m}{n}\sum P(A_m) = \frac{m}{n}$$

where the summation is over all sets A_m.

2-12 (a) $P\{6 \le t \le 8\} = \frac{2}{10}$

(b) $P\{6 \le t \le 8 | t > 5\} = \frac{P\{6 \le t \le 8\}}{P\{t > 5\}} = \frac{2}{5}$

2-13 From (2-27) it follows that

$$P\{t_o \le t \le t_o + t_1 | t \ge t_o\} = \int_{t_o}^{t_o + t_1} \alpha(t)dt \Big/ \int_{t_o}^{\infty} \alpha(t)dt$$

$$P\{t \le t_1\} = \int_0^{t_1} \alpha(t)dt$$

Equating the two sides and setting $t_1 = t_0 + \Delta t$ we obtain

$$\alpha(t_o) \Big/ \int_{t_o}^{\infty} \alpha(t)dt = \alpha(0)$$

for every t_o. Hence,

$$-\ln \int_{t_o}^{\infty} \alpha(t)dt = \alpha(0)t_o \qquad \int_{t_o}^{\infty} \alpha(t)dt = e^{-\alpha(0)t_o}$$

Differentiating the setting $c = \alpha(0)$, we conclude that

$$\alpha(t_o) = c\,e^{ct} \qquad P\{t \le t_1\} = 1 - e^{-ct_1}$$

2-14 If A and B are independent, then $P(AB) = P(A)P(B)$. If they are mutually exclusive, then $P(AB) = 0$. Hence, A and B are mutually exclusive and independent iff $P(A)P(B) = 0$.

2-15 Clearly, $A_1 = A_1A_2 + A_1\bar{A}_2$ hence

$$P(A_1) = P(A_1A_2) + P(A_1\bar{A}_2)$$

If the events A_1 and $\bar{A}_2$ are independent, then

$$P(A_1\bar{A}_2) = P(A_1) - P(A_1A_2) = P(A_1) - P(A_1)P(A_2)$$
$$= P(A_1)[1 - P(A_2)] = P(A_1)P(\bar{A}_2)$$

hence, the events A_1 and $\bar{A}_2$ are independent. Furthermore, S is independent with any A because SA = A. This yields

$$P(SA) = P(A) = P(S)P(A)$$

Hence, the theorem is true for $n = 2$. To prove it in general we use induction: Suppose that A_{n+1} is independent of $A_1,\ldots,A_n$. Clearly, A_{n+1} and $\bar{A}_{n+1}$ are independent of $B_1,\ldots,B_n$. Therefore

$$P(B_1 \cdots B_nA_{n+1}) = P(B_1 \cdots B_n)P(A_{n+1})$$
$$P(B_1 \cdots B_n\bar{A}_{n+1}) = P(B_1 \cdots B_n)P(\bar{A}_{n+1})$$

2.16 The desired probabilities are given by (a)

$$\frac{\binom{m-1}{k-1}}{\binom{n}{k}}$$

(b)

$$\frac{\binom{m}{k}}{\binom{n}{k}}$$

2.17 Let A_1, A_2 and A_3 represent the events

$$\begin{aligned} A_1 &= \text{"ball numbered less than or equal to } m \text{ is drawn"} \\ A_2 &= \text{"ball numbered } m \text{ is drawn"} \\ A_3 &= \text{"ball numbered greater than } m \text{ is drawn"} \end{aligned}$$

$$\begin{aligned} P(A_1 \text{ occurs } n_1 = k-1, \quad & A_2 \text{ occurs } n_2 = 1 \text{ and } A_3 \text{ occurs } n_3 = 0) \\ &= \frac{(n_1+n_2+n_3)!}{n_1!\,n_2!\,n_3!} p_1^{n_1} p_2^{n_2} p_3^{n_3} \\ &= \frac{k!}{(k-1)!} \left(\frac{m}{n}\right)^{k-1} \left(\frac{1}{n}\right) \\ &= \frac{k}{n} \left(\frac{m}{n}\right)^{k-1} \end{aligned}$$

2.18 All cars are equally likely so that the first car is selected with probability $p = 1/3$. This gives the desired probability to be

$$\binom{10}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^7 = 0.26$$

2.19 $P\{\text{"drawing a white ball "}\} = \frac{m}{m+n}$

$$\begin{aligned} P(\text{"atleat one white ball in } k \text{ trials "}) \\ &= 1 - P(\text{"all black balls in } k \text{ trials"}) \\ &= 1 - \frac{\binom{n}{k}}{\binom{m+n}{k}} \end{aligned}$$

2.20 Let $D = 2r$ represent the penny diameter. So long as the center of the penny is at a distance of r away from any side of the square, the penny will be entirely inside the square. This gives the desired probability to be

$$\frac{(1-2r)^2}{1} = \left(1 - \frac{3}{4}\right)^2 = \frac{1}{16}.$$

2.21 Refer to Example 3.14.
(a) Using (3.39), we get

$$P(\text{“}all\ one-digit\ numbers\text{”}) = \frac{\binom{9}{6}\binom{42}{0}}{\binom{51}{6}} = 5 \times 10^{-6}.$$

(b)

$$P(\text{“}two\ one\text{–}digit\ and\ four\ two\text{–}digit\ numbers\text{”}) = \frac{\binom{9}{2}\binom{42}{4}}{\binom{51}{6}} = 0.224.$$

2-22 The number of equations of the form $P(A_iA_k) = P(A_i)P(A_k)$ equals $\binom{n}{2}$. The number of equations involving r sets equals $\binom{n}{r}$. Hence the total number N of such equations equals

$$N = \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n}$$

And since

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = (1+1)^n = 2^n$$

we conclude that

$$N = 2^n - \binom{n}{0} - \binom{n}{1} = 2^n - 1 - n$$

2-23 We denote by B_1 and B_2 respectively the balls in boxes 1 and 2 and by R the set of red balls. We have (assumption)

$$P(B_1) = P(B_2) = 0.5 \qquad P(R|B_1) = 0.999 \qquad P(R|B_2) = 0.001$$

Hence (Bayes' theorem)

$$P(B_1|R) = \frac{P(R|B_1)P(B_1)}{P(R|B_1)P(B_1) + P(R|B_2)P(B_2)} = \frac{0.999}{0.999 + 0.001} = 0.999$$

2-24 We denote by B_1 and B_2 respectively the ball in boxes 1 and 2 and by D all pairs of defective parts. We have (assumption)

$$P(B_1) = P(B_2) = 0.5$$

To find $P(D|B_1)$ we proceed as in Example 2-10:

First solution. In box B_1 there are 1000×999 pairs. The number of pairs with both elements defective equals 100×99. Hence,

$$P(D|B_1) = \frac{100 \times 99}{1000 \times 999}$$

Second solution. The probability that the first bulb selected from B_1 is defective equals 100/1000. The probability that the second is defective assuming the first was effective equals 99/999. Hence,

$$P(D|B_1) = \frac{100}{1000} \times \frac{99}{999}$$

We similarly find

$$P(D|B_2) = \frac{100}{2000} \times \frac{99}{1999}$$

(a) $P(D) = P(D|B_1)P(B_1) + P(D|B_2)P(B_2) = 0.0062$

(b) $P(B_1|D) = \dfrac{P(D|B_1)P(B_1)}{P(D)} = 0.80$

2-25 Reasoning as in Example 2-13, we conclude that the probability that the bus and the train meet equals

$$(10+x)60 - \frac{10^2}{2} - \frac{x^2}{2}$$

Equating with 0.5, we find $x = 60 - 10\sqrt{11}$.

2-26 We wish to show that the number $N_n(k)$ of the element subsets of S equals

$$N_n(k) = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdots k}$$

This is true for $k = 1$ because the number of 1-element subsets equals n. Using induction in k, we shall show that

$$N_n(k+1) = N_n(k)\,\frac{n-k}{k+1} \qquad 1 < k < n \qquad \text{(i)}$$

We attach to each k-element subset of S one of the remaining $n-k$ elements of S. We, then, form $N_n(k)(n-k)$ $k+1$-element subsets. However, these subsets are not all different. They form groups each of which has $k+1$ identical elements. We must, therefore, divide by $k+1$.

2-27 In this experiment we have 8 outcomes. Each outcome is a selection of a particular coin and a specific sequence of heads or tails; for example fhh is the outcome "we selected the fair coin and we observed hh". The event F = {the selected coin is fair} consists of the four outcomes fhh, fht, fth and fhh. Its complement $\overline{F}$ is the selection of the two-headead coin. The event HH = {heads at both tosses} consists of two outcomes. Clearly,

$$P(F) = P(\overline{F}) = \frac{1}{2} \qquad P(HH|F) = \frac{1}{4} \qquad P(HH|\overline{F}) = 1$$

Our problem is to find P(F|HH). From (2-41) and (2-43) it follows that

$$P(HH) = P(HH|F)P(F) + P(HH|\overline{F})P(\overline{F}) = \frac{5}{8}$$

$$P(F|HH) = \frac{P(HH|F)P(F)}{P(HH)} = \frac{1/4 \times 1/2}{5/8} = \frac{1}{5}$$

3.1 (a) $P(\text{A } \textit{occurs atleast twice in } n \textit{ trials})$

$= 1 - P(\text{A } \textit{never occurs in } n \textit{ trials}) - P(\text{A } \textit{occurs once in } n \textit{ trials})$

$= 1 - (1-p)^n - np(1-p)^{n-1}$

(b) $P(\text{A } \textit{occurs atleast thrice in } n \textit{ trials})$

$= 1 - P(\text{A never occurs in } n \text{ trials}) - P(\text{A occurs once in } n \text{ trials})$

$\quad -P(\text{A occurs twice in } n \text{ trials})$

$= 1 - (1-p)^n - np(1-p)^{n-1} - \frac{n(n-1)}{2} p^2(1-p)^{n-2}$

3.2

$$P(doublesix) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

$P(\text{"double six atleast three times in n trials"})$

$= 1 - \binom{50}{0}\left(\frac{1}{36}\right)^0\left(\frac{35}{36}\right)^{50} - \binom{50}{1}\left(\frac{1}{36}\right)\left(\frac{35}{36}\right)^{49} - \binom{50}{2}\left(\frac{1}{36}\right)^2\left(\frac{35}{36}\right)^{48}$

$= 0.162$

3–3 If A = {seven}, then

$$P(A) = \frac{6}{36} \qquad P(\overline{A}) = \frac{5}{6}$$

If the dice are tossed 10 times, then the probability that $\overline{A}$ will occur 10 times equals $(5/6)^{10}$. Hence, the probability p that {seven} will show at least once equals

$$1 - (5/6)^{10}$$

3–4 If k is the number of heads, then

$$P\{\text{even}\} = P\{k = 0\} + P\{k = 2\} + \cdots$$
$$= q^n + \binom{n}{2}p^2q^{n-2} + \binom{n}{4}p^4q^{n-4} + \cdots$$

But

$$1 = (q + q)^n = q^n + \binom{n}{1}p\, q^{n-1} + \binom{n}{2}p^2q^{n-2} + \cdots$$
$$(p - q)^n = q^n - \binom{n}{1}p\, q^{n-1} + \binom{n}{2}p^2\, q^{n-2} - \cdots$$

Adding, we obtain

$$1 + (p - q)^n = 2\, P\{\text{even}\}$$

3–5 In this experiment, the total number of outcomes is the number $\binom{N}{n}$ of ways of picking n out of N objects. The number of ways of picking k out of the K good components equals $\binom{K}{k}$ and the number of ways of picking n-k out of the N-K defective components equals $\binom{N-K}{n-k}$. Hence, the number of ways of picking k good components and n-k deafective components equals $\binom{K}{k}\binom{N-K}{n-k}$. From this and (2-25) it follows that

$$p = \binom{K}{k}\binom{N-K}{n-k} \Big/ \binom{N}{n}$$

3.6 (a)

$$p_1 = 1 - \left(\frac{5}{6}\right)^6 = 0.665$$

(b)

$$1 - \left(\frac{5}{6}\right)^{12} - \binom{12}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{11} = 0.619$$

(c)

$$1 - \left(\frac{5}{6}\right)^{18} - \binom{18}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{17} - \binom{18}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{16} = 0.597$$

3.7 (a) Let n represent the number of wins required in 50 games so that the net gain or loss *does not* exceed \$1. This gives the net gain to be

$$-1 < n - \frac{50-n}{4} < 1$$

$$16 < n < 17.3$$

$$n = 17$$

$P(\text{net gain } \textit{does not} \text{ exceed \$1}) = \binom{50}{17}\left(\frac{1}{4}\right)^{17}\left(\frac{3}{4}\right)^{33} = 0.432$

$P(\text{net gain or loss exceeds \$1}) = 1 - 0.432 = 0.568$

(b) Let n represent the number of wins required so that the net gain or loss *does not* exceed \$5. This gives

$$-5 < n - \frac{(50-n)}{2} < 5$$

$$13.3 < n < 20$$

$P(\text{net gain } \textit{does not} \text{ exceed \$5}) = \sum_{n=14}^{19}\binom{50}{n}\left(\frac{1}{4}\right)^{n}\left(\frac{3}{4}\right)^{50-n} = 0.349$

$P(\text{net gain or loss exceeds \$5}) = 1 - 0.349 = 0.651$

3.8 Define the events

A=" r successes in n Bernoulli trials"

B="success at the ith Bernoulli trial"

C="$r-1$ successes in the remaining $n-1$ Bernoulli trials excluding the ith trial"

$$P(A) = \binom{n}{r} p^r \, q^{n-r}$$

$$P(B) = p$$

$$P(C) = \binom{n-1}{r-1} p^{r-1} \, q^{n-r}$$

We need

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(BC)}{P(A)} = \frac{P(B)\,P(C)}{P(A)} = \frac{r}{n}.$$

3.9 There are $\binom{52}{13}$ ways of selecting 13 cards out of 52 cards. The number of ways to select 13 cards of any suit (out of 13 cards) equals $\binom{13}{13} = 1$. Four such (mutually exclusive) suits give the total number of favorable outcomes to be 4. Thus the desired probability is given by

$$\frac{4}{\binom{52}{13}} = 6.3 \times 10^{-12}$$

3.10 Using the hint, we obtain

$$p\,(N_{k+1} - N_k) = q\,(N_k - N_{k-1}) - 1$$

Let

$$M_{k+1} = N_{k+1} - N_k$$

so that the above iteration gives

$$\begin{aligned} M_{k+1} &= \frac{q}{p}\,M_k - \frac{1}{p} \\ &= \begin{cases} \left(\frac{q}{p}\right) M_1 - \frac{1}{p-q}\left\{1 - (\frac{q}{p})^i\right\}, & p \neq q \\ M_1 - \frac{k}{p}, & p = q \end{cases} \end{aligned}$$

This gives

$$\begin{aligned} N_i &= \sum_{k=0}^{i-1} M_{k+1} \\ &= \begin{cases} \left(M_1 + \frac{1}{p-q}\right) \sum_{k=0}^{i-1} \left(\frac{q}{p}\right)^k - \frac{i}{p-q}, & p \neq q \\ iM_1 - \frac{i(i-1)}{2p}, & p = q \end{cases} \end{aligned}$$

where we have used $N_o = 0$. Similarly $N_{a+b} = 0$ gives

$$M_1 + \frac{1}{p-q} = \frac{a+b}{p-q} \cdot \frac{1 - q/p}{1 - (q/p)^{a+b}}.$$

Thus

$$N_i = \begin{cases} \frac{a+b}{p-q} \cdot \frac{1 - (q/p)^i}{1 - (q/p)^{a+b}} - \frac{i}{p-q}, & p \neq q \\ i(a+b-i), & p = q \end{cases}$$

which gives for $i = a$

$$
\begin{aligned}
N_a &= \begin{cases} \dfrac{a+b}{p-q} \cdot \dfrac{1-(q/p)^a}{1-(q/p)^{a+b}} - \dfrac{a}{p-q}, & p \neq q \\ ab, & p = q \end{cases} \\
&= \begin{cases} \dfrac{b}{2p-1} - \dfrac{a+b}{2p-1} \cdot \dfrac{1-(q/p)^b}{1-(q/p)^{a+b}} - \dfrac{a}{p-q}, & p \neq q \\ ab, & p = q \end{cases}
\end{aligned}
$$

3.11

$$P_n = pP_{n+\alpha} + qP_{n-\beta}$$

Arguing as in (3.43), we get the corresponding iteration equation

$$P_n = P_{n+\alpha} + qP_{n-\beta}$$

and proceed as in Example 3.15.

3.12 Suppose one bet on $k = 1, 2, \cdots, 6$.
Then

$$
\begin{aligned}
p_1 &= P(k \textit{ appears on one dice}) = \binom{3}{1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^2 \\
p_2 &= P(k \textit{ appear on two dice}) = \binom{3}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right) \\
p_3 &= P(k \textit{ appear on all the tree dice}) = \left(\frac{1}{6}\right)^3 \\
p_0 &= P(k \textit{ appear none}) = \left(\frac{5}{6}\right)^3
\end{aligned}
$$

Thus, we get

$$\text{Net gain} = 2p_1 + 3p_2 + 4p_3 - p_0 = 0.343.$$

CHAPTER 4

4-1 From the evenness of f(x): $1 - F(x) = F(-x)$.

From the definition of x_u: $u = F(x_u)$, $1 - u = F(x_{1-u})$. Hence

$$1 - u = 1 - F(x_u) = F(-x_u) = F(x_{1-u}) \qquad -x_u = x_{1-u}$$

4-2 From the symmetry of f(x): $1 - F(\eta+a) = F(\eta-a)$. Hence [see (4-8)]

$$P\{\eta-a < \underset{\sim}{x} < \eta+a\} = F(\eta+a) - F(\eta-a) = 2F(\eta+a) - 1$$

This yields

$$1-\alpha = 2F(\eta+a) - 1 \qquad F(\eta+a) = 1 - \alpha/2 \qquad \eta+a = x_{1-\alpha/2}$$

$$F(a-\eta) = \alpha/2 \qquad a-\eta = x_{\alpha/2}$$

4-3 (a) In a linear interpolation:

$$x_u \simeq x_a + \frac{x_b-x_a}{u_b-u_a}(u-u_a) \qquad \text{for } x_a < x_u < x_b$$

From Table 4-1 page 106

$$z_{0.9} \simeq 1.25 + \frac{0.00565}{0.00885} \times 0.05 = 1.2819$$

Proceeding simiplarly, we obtain

u =	0.9	0.925	0.95	0.975	0.99
z_u =	1.282	1.440	1.645	1.960	2.327

(b) If $\underset{\sim}{z}$ is such that $\underset{\sim}{x} = \eta + \sigma\underset{\sim}{z}$ then $\underset{\sim}{z}$ is N(0,1) and $G(z) = F_x(\eta+\sigma z)$. Hence,

$$u = G(z_u) = F_x(\eta+\sigma z_u) = F_x(x_u) \qquad x_u = \eta+\sigma z_u$$

4-4 $p_k - 2G(k) = 1 = 2\ \mathrm{erf}\,k$

(a) From Table 4-1

k =	1	2	3
p_k=	0.6827	0.9545	0.9973

(b) From Table 3-1 with linear interpolation:

p_k =	0.9	0.99	0.999
k =	1.282	2.32	3.090

(c) $P\{\eta - z_u\sigma < \underset{\sim}{x} < \eta + z_u\sigma\} = 2G(z_u) - 1 = \gamma$

Hence, $G(z_u) = (1+\gamma)/2$ $\qquad u = (1+\gamma)/2$

4-5 (a) $F(x) = x$ for $0 \le x \le 1$; hence, $u = F(x_u) = x_u$

(b) $F(x) = 1-e^{-2x}$ for $x \ge 0$; hence, $u = 1-e^{-2x_u}$

$$x_u = -\frac{1}{2}\,\ell n(1-u)$$

u =	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
x_u =	0.0527	0.1116	0.1783	0.2554	0.3466	0.4581	0.6020	0.847	1.1513

4-6 Percentage of units between 96 and 104 ohms equals 100p where $p = P\{96 < \underset{\sim}{R} < 104\} = F(104) - F(96)$

(a) $F(R) = 0.1(R-95)$ for $95 \le R \le 105$. Hence,

$p = 0.1(104-95) - 0.1(96-95) = 0.8$

(b) $p = G(2.5) - G(-2.5) = 0.9876$

4-7 From (4-34), with $\alpha = 2$ and $\beta = 1/\lambda$ we get $f(x) = c^2\, xe^{-cx}U(x)$

$$F(x) = c^2 \int_0^x ye^{-cy}dy = 1 - e^{-cx} - cxe^{-cx}$$

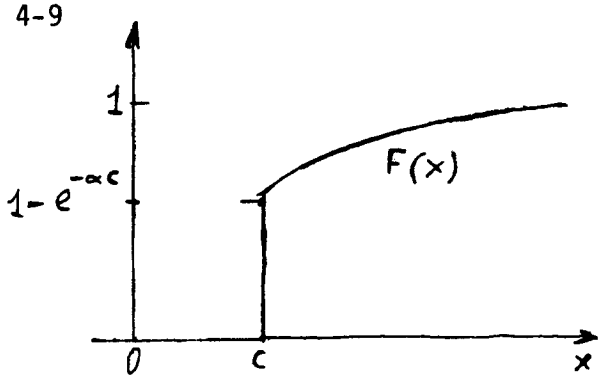
4-8 $\{(\underset{\sim}{x} - 10)^2 < 4\} = \{8 < \underset{\sim}{x} < 12\}$

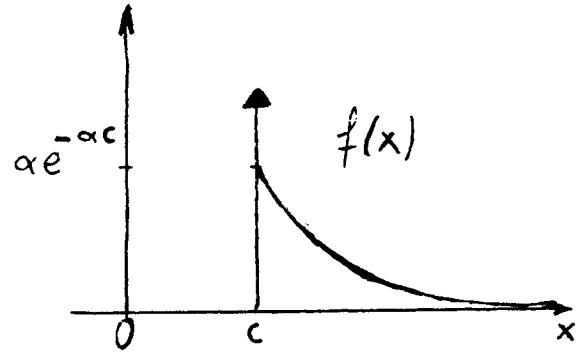
$P\{(\underset{\sim}{x} - 10)^2 < 4\} = G(12 - 10) - G(8 - 10) = 0.954$

$$f(x|(\underset{\sim}{x} - 10)^2 < 4\} = \frac{f(x)}{P\{8 < \underset{\sim}{x} < 12\}} = \frac{1}{0.954\sqrt{2\pi}} e^{-\frac{(x-10)^2}{2}}$$

for $8 < x < 12$ and zero otherwise

4-9





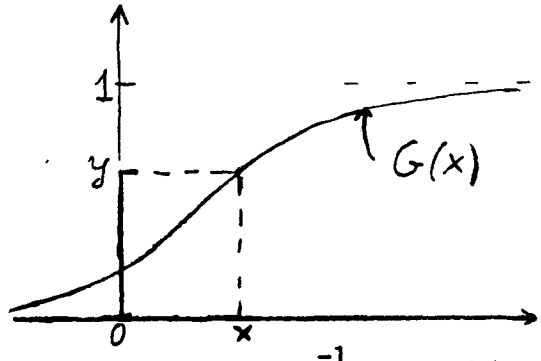
$F(x) = (1 - e^{-\alpha x})U(x-c)$ $\qquad$ $f(x) = (1 - e^{-\alpha c})\delta(x-c) + e^{-\alpha x}U(x-c)$

4-10 (a) $P\{1 \le x \le 2\} = G(\frac{2}{2}) - G(\frac{1}{2}) = 0.1499$

(b) $P\{1 \le \underset{\sim}{x} \le 2 | \underset{\sim}{x} \ge 1\} = \frac{G(1) - G(0.5)}{1 - G(0.5)} = \frac{0.1499}{0.3085} = 0.4857$

because $\{1 \le \underset{\sim}{x} \le 2, \underset{\sim}{x} \ge 1\} = \{1 \le \underset{\sim}{x} \le 2\}$

4-11



$x = G^{-1}(y) = H(y)$

If $\underset{\sim}{x}(t_1) \le x$

then

$t_i \le y = G(x)$

Hence,

$P\{\underset{\sim}{x} \le x\} = P\{t_i \le y\} = y = G(x)$

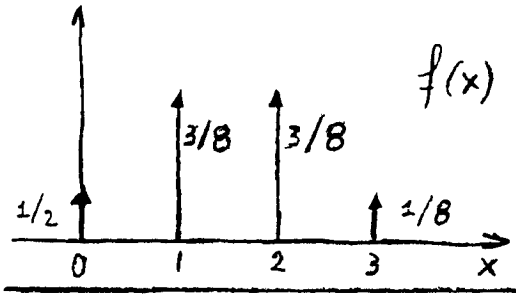
4-12 (a) $P\{\underset{\sim}{x} < 1024\} = G(\frac{1024 - 1000}{20}) = G(1.2) = 0.8849$

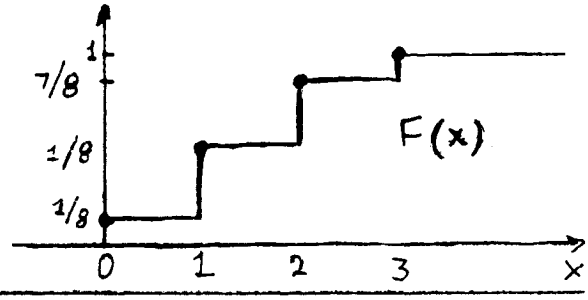
(b) $P\{\underset{\sim}{x} < 1024 | \underset{\sim}{x} > 961\} = \frac{P\{961 < \underset{\sim}{x} < 1024\}}{P\{\underset{\sim}{x} > 961\}}$

$$= \frac{G(1.2) - G(-1.95)}{1 - G(-1.95)} = 0.8819$$

(c) $P\{31 < \sqrt{\underset{\sim}{x}} \leq 32\} = P\{961 < \underset{\sim}{x} \leq 1024\} = 0.8593$

4-13 $P\{\underset{\sim}{x} = 0\} = \frac{1}{8}$ $P\{\underset{\sim}{x} = 1\} = \frac{3}{8}$ $P\{\underset{\sim}{x} = 2\} = \frac{3}{8}$ $P\{\underset{\sim}{x} = 3\} = \frac{1}{8}$





4-14 (a) 1. $f_x(x) = \frac{1}{2^{900}} \sum_{k=0}^{900} \binom{900}{k} \delta(x-k)$

2. $f_x(x) = \frac{1}{15\sqrt{2\pi}} \sum_{k=0}^{900} e^{-(k-450)^2/450} \delta(x-k)$

(b) $P\{435 \leq x \leq 460\} = G(\frac{10}{15}) - G(-\frac{15}{15}) = 0.5888$

4-15 If $x > b$ then $\{\underset{\sim}{x} \leq x\} = S$ $F(x) = 1$

If $x < a$ then $\{\underset{\sim}{x} \leq x\} = \{\emptyset\}$ $F(x) = 0$

4-16 If $\underset{\sim}{y}(\zeta_i) \leq w$, then $\underset{\sim}{x}(\zeta_i) \leq w$ because $\underset{\sim}{x}(\zeta_i) \leq \underset{\sim}{y}(\zeta_i)$.

Hence,

$\{\underset{\sim}{y} \leq w\} \subset \{\underset{\sim}{x} \leq w\}$ $P\{\underset{\sim}{y} \leq w\} \leq P\{\underset{\sim}{x} \leq w\}$

Therefore $F_y(w) \leq F_x(w)$

4-17 From (4-80)

$$f(x) = kx\, e^{-\int_0^x kt\,dt} = kx\, e^{-kx^2/2}$$

4-18 It follows from (2-41) with

$$A_1 = \{\underset{\sim}{x} \le x\} \qquad A_2 = \{\underset{\sim}{x} > x\}$$

4-19 It follows from

$$F_x(x|A) = \frac{P\{\underset{\sim}{x} \le x, A\}}{P(A)} \qquad P\{A|\underset{\sim}{x} \le x\} = \frac{P\{\underset{\sim}{x} \le x, A\}}{P\{\underset{\sim}{x} \le x\}}$$

4-20 We replace in (4-80) all probabilities with conditional probabilities assuming $\{\underset{\sim}{x} \le x_o\}$. This yields

$$\int_{-\infty}^{\infty} P(A|\underset{\sim}{x} = x,\ \underset{\sim}{x} \le x_o) f(x|\underset{\sim}{x} \le x_o)dx = P(A|\underset{\sim}{x} \le x_o)$$

But $f(x|\underset{\sim}{x} \le x_o) = 0$ for $x > x_o$ and

$\{\underset{\sim}{x} = x,\ \underset{\sim}{x} \le x_o\} = \{\underset{\sim}{x} = x\}$ for $x \le x_o$. Hence,

$$\int_{-\infty}^{x_o} P(A|\underset{\sim}{x} = x) f(x|\underset{\sim}{x} \le x_o)dx = P(A|\underset{\sim}{x} \le x_o)$$

Writing a similar equation for $P(B|\underset{\sim}{x} \le x_o)$ we conclude that, if $P(A|\underset{\sim}{x} = x) = P(B|\underset{\sim}{x} = x)$ for $x \le x_o$, then $P(A|\underset{\sim}{x} \le x_o) = P(B|\underset{\sim}{x} \le x_o)$

4-21 (a) Clearly, $f(p) = 1$ for $0 \le p \le 1$ and 0 otherwise; hence

$$P\{0.3 \le \underset{\sim}{p} \le 0.7\} = \int_{0.3}^{0.7} dp = 0.4$$

(b) We wish to find the conditional probability $P\{0.3 \le \underset{\sim}{p} \le 0.7|A\}$ where A = {6 heads in 10 tosses}. Clearly $P\{A|\underset{\sim}{p}=p\} = p^6(1-p)^4$. Hence, [see (4-81)]

$$f(p|A) = \frac{p^6(1-p)^4}{\int_0^1 p^6(1-p)^4 dp} = \frac{p^6(1-p)^4}{4329\times10^{-7}}$$

This yields

$$P\{0.3 \le \underset{\sim}{p} \le 0.7|A\} = \int_{0.3}^{0.7} f(p|A)dp = \frac{10^7}{4329}\int_{0.3}^{0.7} p^6(1-p)^4 dp = 0.768$$

4-22 (a) In this problem, $f(p) = 5$ for $0.4 \le \underset{\sim}{p} \le 0.6$ and zero otherwise; hence [see(4-82)]

$$P(H) = 5\int_{0.4}^{0.6} p\,dp = 0.5$$

(b) With A = {60 heads in 100 tosses} it follows from (4-82) that

$$f(p|A) = p^{60}(1-p)^{40} \Big/ \int_{0.4}^{0.6} p^{60}(1-p)^{40} dp$$

for $0.4 \le p \le 0.6$ and 0 otherwise. Replacing $f(p)$ by $f(p|A)$ in (4-82), we obtain

$$P(H|A) = \int_{0.4}^{0.6} p f(p|A) dp = 0.56$$

4-23 $n = 900$ $p = q = 0.5$ $np = 450$ $\sqrt{npq} = 15$

$k_1 = 420$ $k_2 = 465$ $\dfrac{k_2 - np}{\sqrt{npq}} = 1$ $\dfrac{k_1 - np}{\sqrt{npq}} = -2$

$$P\{420 \le k \le 465\} = G(1) - [1 - G(-2)] = G(1) + G(2) - 1$$
$$= 0.819$$

4-24 For a fair coin $\sqrt{npq} = \sqrt{n}/2$. If

$k_1 = 0.49n$ and $k_2 = 0.52n$ then

$$\frac{k_2 - np}{\sqrt{npq}} = \frac{0.52n - n/2}{\sqrt{n}/2} = 0.04\sqrt{n} \qquad \frac{k_1 - np}{\sqrt{npq}} = -0.02\sqrt{n}$$

$$P\{k_1 \le k \le k_2\} = G(0.04\sqrt{n}) + G(0.02\sqrt{n}) - 1 \ge 0.9$$

From Table 4-1 (page 106) it follows that

$$0.02\sqrt{n} > 1.3 \qquad n > 65^2$$

4-25

(a) Assume $n = 1{,}000$ (Note correction to the problem)

$P(A) = 0.6 \qquad np = 600 \qquad npq = 240 \qquad k_2 = 650 \qquad k_1 = 550$

$$\frac{k_2 - np}{\sqrt{npq}} = \frac{50}{\sqrt{240}} = 3.23 \qquad \frac{k_1 - np}{\sqrt{npq}} = -3.23$$

$P\{550 \le k \le 650\} = 2G(3.23) - 1 = 0.999$

(b) $P\{0.59n \le k \le 0.61n\} = 2G\left(\frac{0.01n}{\sqrt{0.24n}}\right) - 1$

$= 2G\left(\sqrt{\frac{n}{2400}}\right) - 1 = 0.476$

Hence, (Table 3-1) $\quad n \simeq 9220$

4-26 With $a = 0$, $b = T/4$ it follows that

$p = 1-e^{-1/4} = 0.22 \qquad np = 220 \qquad npq = 171.6 \qquad k_2 = 100$

$$\frac{k_2 - np}{\sqrt{npq}} = -9.16 \text{ and (4-100) yields}$$

$P\{0 \le k \le 100\} \simeq G(-9.16) \simeq 0.$

4-27 The event

$A = \{k \text{ heads show at the first } n \text{ tossings but not earlier}\}$

occurs iff the following two events occur

$B = \{k-1 \text{ heads show at the first } n-1 \text{ tossing}\}$

$C = \{\text{heads show at the nth tossing}\}$

And since these two events are independent and

$$P(B) = \binom{n-1}{k-1} p^{k-1} q^{n-1-(k-1)} \qquad P(C) = p$$

we conclude that

$$P(A) = P(B)P(C) = \binom{n-1}{k-1} p^k q^{n-k}$$

4-28 $$-\frac{d}{dx}\left(\frac{1}{x}e^{-x^2/2}\right) = \left(1+\frac{1}{x^2}\right)e^{-x^2/2} > e^{-x^2/2}$$

Multiplying by $1/\sqrt{2\pi}$ and integrating from x to ∞, we obtain

$$\frac{1}{x\sqrt{2\pi}}e^{-x^2/2} > \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\zeta^2/2}\,d\zeta = 1 - G(x)$$

because

$$\frac{1}{x}e^{-x^2/2} \xrightarrow[x\to\infty]{} 0$$

The first inequality follows similarly because

$$-\frac{d}{dx}\left[\left(\frac{1}{x}-\frac{1}{x^3}\right)e^{-x^2/2}\right] = \left(1-\frac{3}{x^4}\right)e^{-x^2/2} < e^{-x^2/2}$$

4-29 If $P(A) = p$ then $P(\bar{A}) = 1-p$. Clearly $P_1 = 1-Q_1$ where Q_1 equals the probability that A does not occur at all. If $pn \ll 1$, then $Q_1 = (1-p)^n \simeq 1 - np$ $P_1 \simeq p$

4-30 With $p = 0.02$, $n = 100$, $k = 3$, it follows from (4-107) that the unknown probability equals

$$\binom{100}{3}(0.02)^3(0.98)^{97} \simeq \frac{2^3}{3!}e^{-2} = \frac{4}{3}e^{-2}$$

4-31 With $n = 3$, $r = 3$, $k_1 = 2$, $k_2 = 2$, $k_3 = 1$, $p_1 = p_2 = p_3 = 1/6$, it follows from (4-102) that the unknown probability equals

$$\frac{5!}{1!2!2!}\frac{1}{6^6} = 0.00386$$

4-32 With $r = 2$, $k_1 = k$, $k_2 = n-k$, $p_1 = p$, $p_2 = 1-p = q$, we obtain

$$k_1 - np_1 = k - np \qquad k_2 - np_2 = n-k-nq = np - k$$

Hence, the bracket in (4-103) equals

$$\frac{(k_1-np_1)^2}{np_1} + \frac{(k_2-np_2)^2}{np_2} = \frac{(k-np)^2}{n}\left(\frac{1}{p}+\frac{1}{q}\right) = \frac{(k-np)^2}{npq}$$

as in (4-90).

4-33 $P(M) = 2/36$ $P(\overline{M}) = 34/36$. The events M and $\overline{M}$ form a partition, hence, [see (2-41)]

$$P(A) = P(A|M)P(M) + P(A|\overline{M})P(\overline{M}) \qquad \text{(i)}$$

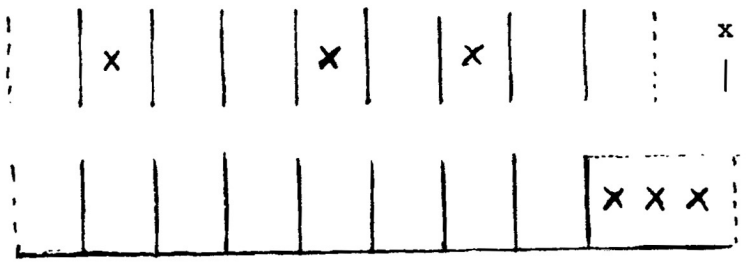
Clearly, $P(A|M) = 1$ because, if M occurs at first try, X wins. The probability that X wins after the first try equals $P(A|\overline{M})$. But in the experiment that starts at the second rolling, the first player is Y and the probability that he wins equals $P(\overline{A}) = 1-p$. Hence, $P(A|\overline{M}) = P(\overline{A}) = 1-p$. And since $P(M) = 1/18$ $P(\overline{M}) = 17/18$ (i) yields

$$p = \frac{1}{18} + (1-p)\frac{17}{18} \qquad p = \frac{18}{35}$$

4-34

(a) Each of the n particles can be placed in any one of the m boxes. There are n particles, hence, the number of possibilities equals $N - m^n$. In the m preselected boxes, the particles can be placed in $N_A = n!$ ways (all permutations of n objects). Hence $p = n!/m^n$.

(b)



x n particles

| m-1 interior walls

All possibilities are obtained by permuting the $n+m-1$ objects consisting of the m-1 interior walls with and n particles. The $(m-1)!$ permutations of the walls and the $n!$ permutations of the particles must count as one. Hence

$$N = \frac{(n+m-1)!}{n!\,(m-1)!} \qquad N_A = 1$$

(c) Suppose that S is a set consisting of the m boxes. Each placing of the particles specifies a subset of S consisting of n elements (box). The number of such subsets equals $\binom{m}{n}$ (see Prob. 2-26). Hence,

$$N = \binom{m}{n} \qquad N_A = 1$$

4-35 If $k_1 + k_2 << n$, then $k_3, \simeq n$ and

$$k_3(p_1 + p_2) = [n - (k_1 + k_2)](p_1 + p_2) \simeq n(p_1 + p_2)$$

$$p_3 = 1 - (p_1 + p_2) \simeq e^{-(p_1+p_2)} \qquad p_3^{k_3} \simeq e^{-n(p_1+p_2)}$$

$$\frac{n!}{k_1!k_2!k_3!} = \frac{n(n-1)\cdots(n-k_3+1)}{k_1!k_2!} \simeq \frac{n^{k_1+k_2}}{k_1!k_2!}$$

Hence,

$$\frac{n!}{k_1!k_2!k_3!} p_1^{k_1} p_2^{k_2} p_3^{k_3} \simeq e^{-np_1} \frac{(np_1)^{k_1}}{k_1!} e^{-np_2} \frac{(np_2)^{k_2}}{k_2!}$$

4-36 The probability p that a particular point is in the interval (0,2) equals 2/100. (a) From (3-13) it follows that the probability p_1 that only one out of the 200 points is in the interval (0,2) equals

$$p_1 = \binom{200}{1} \times 0.02 \times 0.09^{199}$$

(b) With $np = 200 \times 0.02 = 4$ and $k = 1$, (3-41) yields $p_1 \simeq e^{-4} \times 4 = 0.073$

CHAPTER 5

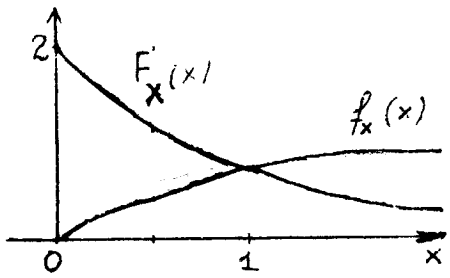
5-1 $\eta = 2\eta_x + 4 = 14 \qquad \sigma_y^2 = 4\sigma_x^2 = 16$

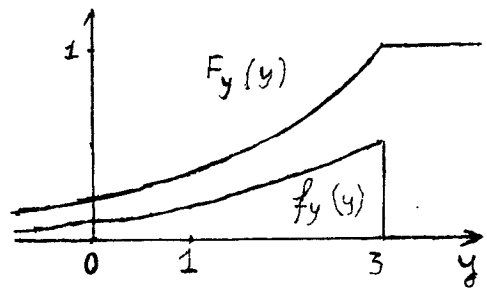
5-2 $\{\underset{\sim}{y} \le y\} = \{-4\underset{\sim}{x} + 3 \le \underset{\sim}{y}\}\ \{\underset{\sim}{x} \le (y-3)/4\}$. Hence

$$F_y(y) = P\left\{\underset{\sim}{x} \ge \frac{3-y}{4}\right\} = 1 - F_x\left(\frac{3-y}{4}\right) \qquad f_y(y) = \frac{1}{4} f_x\left(\frac{3-y}{4}\right)$$

Since $F_x(x) = (1-e^{-2x})U(x)$, this yields

$$F_y(y) = e^{(y-3)/2}U\left(\frac{y-3}{2}\right) \qquad f_y(y) = \frac{1}{2} e^{(y-3)/2}U\left(\frac{y-3}{2}\right)$$

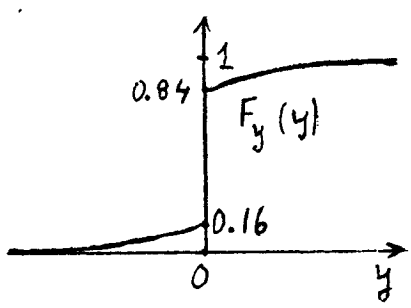


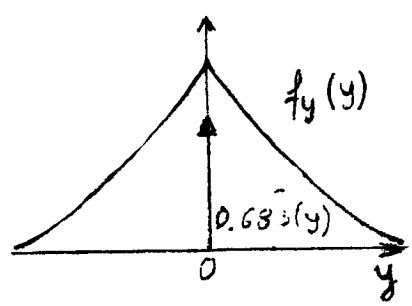


5-3 From Example 5-3 with $F_x = G(x/c)$:

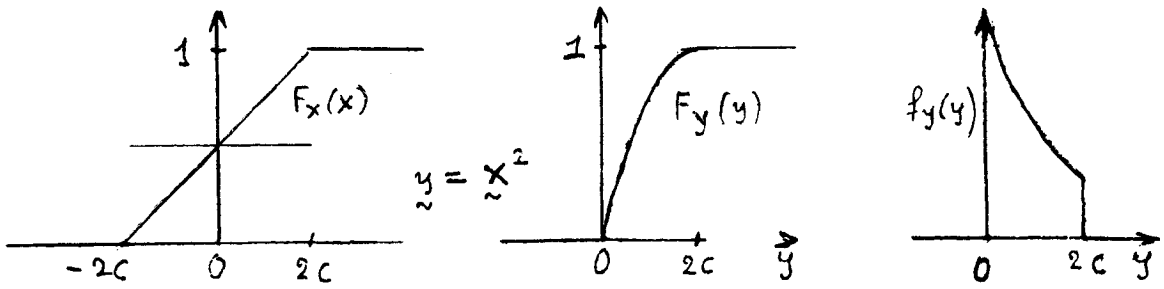
$$f_y(y) = \begin{cases} G(y/c+1) & y \ge 0 \\ G(y/c-1) & y < 0 \end{cases}$$

$$f_y(y) = 0\ 0.68\ \delta(y) + \frac{1}{c\sqrt{2\pi}}\left[e^{-(y+c)^2/2c^2}U(y) + e^{-(y-c)^2/2c^2}U(-y)\right]$$



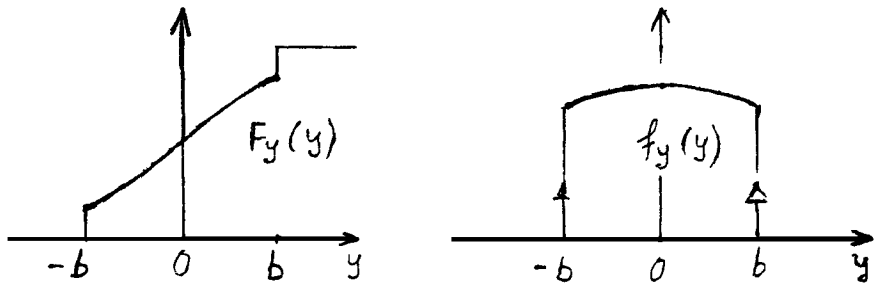


5-4 If $\underset{\sim}{y} = \underset{\sim}{x}^2$ and $F_x(x) = (x+2c)/4c$ for $|x| \le 2c$, then (see Example 5-2) $F_y(y) = \sqrt{y}/2c$ and $f_y(y) = 1/4\sqrt{y}$ for $0 < y < 2c$.



5-5 From Example 5-4 with $F_x(x) = G(x/b)$: For $|x| \le b$ $F_u(y) = G(y/b)$ and

$$f_y(y) = 0.16\delta(y+b) + \frac{1}{b\sqrt{2\pi}} e^{-y^2/2b^2} + 0.16\delta(y-b)$$



5-6 The equation $y = -\ln x$ has a single solution $x = e^{-y}$ for $y > 0$ and no solutions for $y < 0$. Furthermore, $g'(x) = -1/x = -e^y$. Hence

$$f_y(y) = \frac{f_x(e^{-y})}{e^y} U(y) = e^{-y}U(y)$$

5-7 Clearly, $\underset{\sim}{z} \le z$ iff the number $\underset{\sim}{n}(0,z)$ of the points in the interval (0,z) is at least one. Hence,

$$F_z(z) = P\{\underset{\sim}{z} \le z\} = P\{\underset{\sim}{n}(0,z) > 0\} = 1 - P\{\underset{\sim}{n}(0,z) = 0\}$$

The probability p that a particular point is in the integral (0,z) equals z/100. With n = 200, k = 0, and p = z/100, (3-21) yields $P\{\underset{\sim}{n}(0,z) = 0\} = (1-p)^{200}$. Hence,

(a) $F_z(z) = 1 - \left(1 - \frac{z}{100}\right)^{100}$

(b) From (4-107) it follows thata $F_z(z) \simeq 1 - e^{-2z}$ for $z << 100$.

5.8

$$Y = \sqrt{X} \quad \Rightarrow \quad x_1 = y^2$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2y}$$

Thus

$$f_Y(y) = \frac{1}{\left|\frac{dy}{dx}\right|} f_X(x_1) = 2y f_X(y^2)$$

$$\frac{2y}{\lambda} e^{-y^2/\lambda} = \begin{cases} \frac{y}{\sigma^2} e^{-y^2/2\sigma^2}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

which represents Rayleigh density function (with $\lambda = 2\sigma^2$).

5-9 For both cases, $f_y(y) = 0$ for $y < 0$.

(a) If $y > 0$ and $|x| = y$, then $x_1 = y$, $x_2 = -y$. Hence

$$f_y(y) = [f_x(y) + f_x(-y)]U(y)$$

(b) If $y > 0$ and $e^{-x}U(x) = y$, then $x = -\ln y$.

Furthermore, $P\{\underset{\sim}{y} = 0\} = P\{\underset{\sim}{x} \le 0\} = F_x(0)$. Hence

$$f_y(y) = F_x(0)\delta(y) + \frac{1}{y} f_x(-\ln y)U(y)$$

5-10 (a) If $y \geq 0$ and $(x-1)U(x-1) = y$, then $\{\underset{\sim}{y} \leq y\} = \{\underset{\sim}{x} \leq y+1\}$.

If $y < 0$, then $\{\underset{\sim}{y} < y\} = \{\emptyset\}$

$$F_y(y) = F_x(1+y)U(y) = [1 - e^{-2(y+1)}]U(y)$$

$$f_y(y) = (1 - e^{-2})\delta(y) + 2e^{-2(y+1)}U(y)$$

(b) If $y > 0$ and $y = x^2$, then $\{\underset{\sim}{y} \leq y\} = \{-\sqrt{y} \leq \underset{\sim}{x} \leq \sqrt{y}\}$

$$F_y(y) = F_x(\sqrt{y}) - F_x(-\sqrt{y}) = (1 - e^{-2\sqrt{y}})U(y)$$

$$f_y(y) = \frac{1}{\sqrt{y}}\, e^{-2\sqrt{y}}U(y)$$

5-11 If $y = \arctan x$, then $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$

$$f_y(y) = (1+x^2)f_x(\tan y) = \frac{1+x^2}{\pi(1+x^2)} = \frac{1}{\pi} \qquad \frac{\pi}{2} < y < \frac{\pi}{2}$$

5-12 (a) If $y = x^3$ then $x = \sqrt[3]{y}$ for any y

$$f_y(y) = \frac{1}{3\sqrt[3]{y^2}}\, f_x\left(\sqrt[3]{y}\right) = \frac{1}{12\pi\sqrt[3]{y^2}}$$

for $|y| < 8\pi^3$ and zero otherwise

(b) If $y = x^4$ and $y > 0$, then $x_1 = \sqrt[4]{y}$ $x_1 = -\sqrt[4]{y}$

$$f_y(y) = \frac{1}{4\sqrt[4]{y^3}}\left[f_x(\sqrt[4]{y}) + f_x(-\sqrt[4]{y})\right] = \frac{1}{8\pi\sqrt[4]{y^3}}$$

for $0 < y < 16\pi^4$ and zero otherwise

(c) If $y = 2\sin(3x + 40°)$ and $|y| < 2$ then $x = x_i$ as shown.

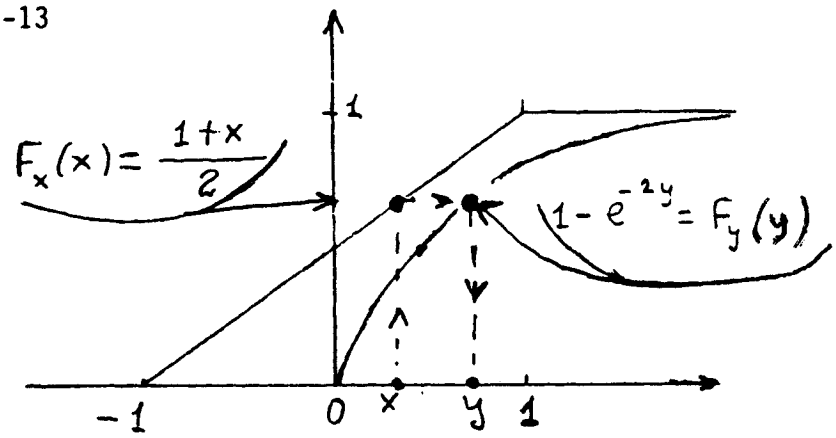
$$\frac{dy}{dx} = \frac{1}{6\sqrt{1 - y^2/4}}$$

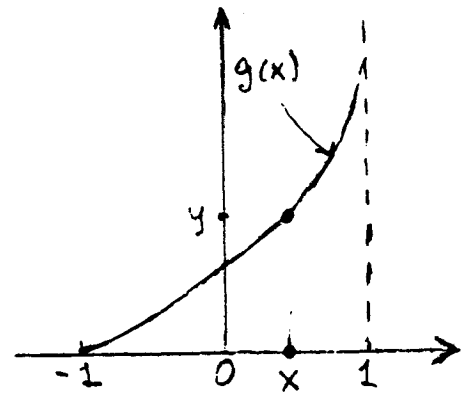
In the interval $(-2\pi, 2\pi)$ there are 12 x_i's. Hence

$$f_y(y) = \frac{1}{3\sqrt{4 - y^2}} \sum_i f_x(x_i) = \frac{12}{12\pi\sqrt{4 - y^2}} = \frac{1}{\pi\sqrt{4 - y^2}}$$

for $|y| < 2$ and zero otherwise.

5-13





As in (5-43) $\qquad F_y[g(x)] = F_x(x)$

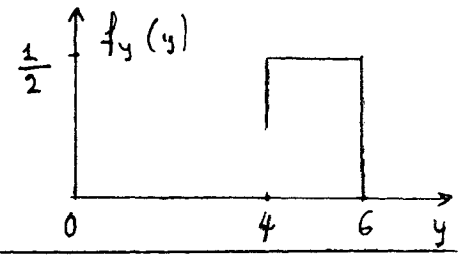
$$\frac{1+x}{2} = 1 - e^{-2y} \qquad\qquad y = g(x) = -\frac{1}{2}\ln\frac{1-x}{2}$$

for $|x| < 1$. For $x \le -1$, $g(x) = 0$; for $x \ge 1$, $g(x) = \infty$.

5-14 (a) $g(x) = 2F_x(x) + 4 \qquad g'(x) = 2f_x(x)$

If $4 < y < 6$ then $y = 2F_x(x) + 4$ has a unique solution x_1 and

$$f_y(y) = \frac{f_x(x_1)}{2f_x(x_1)} = \frac{1}{2}$$



(b) Similarly $g(x) = 2F_x(x) + 8$

5-15 (a) The RV $\underset{\sim}{x}$ takes the values $k = 0,1,\ldots,10$ and

$$P\ x = k = p_k = \binom{10}{k}\frac{1}{2^{10}} \qquad 0 \le k \le 10$$

$F_x(x)$ is a staircase function with discontinuities at the points $x = k$ and jumps equal to p_k.

(b) The RY $\underset{\sim}{y} = (\underset{\sim}{x} - 3)^2$ takes the values $y = k^2$ for $k = 0,1,\ldots,7$ and probabilities $P\{y = k^2\} = q_k$.

$k =$	0	1	2	3	4	5	6	7
$q_k =$	p_3	p_2+p_4	p_1+p_5	p_0+p_6	p_7	p_8	p_9	p_{10}

5.16

$$X \sim Beta(\alpha, \beta) \quad \text{gives}$$

$$f_X(x) = \frac{1}{B(\alpha,\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1}, \quad 0 < x < 1.$$

$$Y = 1 - X \quad \Rightarrow \quad x_1 = 1 - y, \quad |\,\frac{dy}{dx}\,| = 1$$

$$\Rightarrow F_Y(y) = \frac{1}{|\frac{dy}{dx}|}\, f_X(1-y) = \begin{cases} \frac{1}{B(\beta,\alpha)}\, y^{\beta-1}\,(1-y)^{\alpha-1}, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

This gives

$$Y \sim Beta(\beta, \alpha).$$

5.17

$$X \sim \chi^2(n) \Rightarrow$$

$$f_X(x) = \frac{1}{2^{n/2}\Gamma(n/2)}\, x^{n/2-1}\, e^{-x/2}\, U(x)$$

$$y = \sqrt{x} \quad \Rightarrow \quad x_1 = y^2$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

Thus

$$f_Y(y) = 2y f_X(y^2) = \frac{y^{n-1}}{2^{n/2-1}\,\Gamma(n/2)}\, e^{-y^2/2}\, U(y)$$

and it represents the chi-distribution.

5.18

$$X \sim U(0,1)$$

$$Y = -2logX \quad \Rightarrow \quad x_1 = e^{-y/2}$$

$$\frac{dy}{dx} = -\frac{2}{x} = -2e^{y/2}$$

$$f_Y(y) \;= \frac{1}{|\frac{dy}{dx}|}\, f_X(x_1) = \tfrac{1}{2} e^{-y/2}\, U(y)$$

$$\sim \;\text{Exponential}(2) \equiv \chi^2(2)$$

5.19

$$f_X(x) = \lambda\, e^{-\lambda x}\, u(x)$$

$$Y = X^{1/\beta} \quad \Rightarrow \quad x_1 = y^\beta$$

$$|\frac{dy}{dx}| = \frac{1}{\beta}\, x^{1/\beta-1} = \frac{1}{\beta}\, y^{1-\beta}$$

$$f_Y(y) = \frac{1}{|\,\frac{dy}{dx}\,|}\, f_X(x_1) = \lambda\,\beta\, y^{\beta-1}\, e^{-\lambda y^\beta}\, U(y)$$

and it represents Weibull distribution

5-20 For $|y| < a$ the equation $y = a \sin\omega t$ has infinitely many solutions τ_i; in each interval of length $2\pi/\omega$ there are two such solutions. Furthermore, $y'(t) = \omega\sqrt{a^2-y^2}$

$$\tau_i = \frac{1}{\omega}\sin^{-1}\frac{y}{a} \qquad \tau_{i+2} - \tau_i \quad \frac{2\pi}{\omega} \xrightarrow[\omega\to\infty]{} 0$$

Hence,

$$\frac{1}{\omega\sqrt{a^2-y^2}} \sum_{i=-\infty}^{\infty} f_t(\tau_i) \xrightarrow[\omega\to\infty]{} \frac{1}{\sqrt{a^2-y^2}} \frac{2}{2\pi} \int_{-\infty}^{\infty} f_t(\tau)d\tau = \frac{1}{\pi\sqrt{a^2-y^2}}$$

5-21 If $y > 0$ then

$$F_y(y|\underset{\sim}{x} \geq 0) = F_x(\sqrt{y}|\underset{\sim}{x} \geq 0) + F_x(-\sqrt{y}|\underset{\sim}{x} \geq 0) = F_x(\sqrt{y}|\underset{\sim}{x} \geq 0)$$

$$F_x(\sqrt{y}|x \geq 0) = \frac{P\{0 < x < \sqrt{y}\}}{P\{x \geq 0\}} = \frac{F_x(\sqrt{y}) - F_x(0)}{1 - F_x(0)}$$

$$f_y(y|x \geq 0) = \frac{d}{dy} F_y(\sqrt{y}|x \geq 0) = \frac{f_x(\sqrt{y})}{2\sqrt{y}[1-F_x(0)]}$$

5-22 (a) $\eta_y = a\,\eta_x + b \qquad \sigma_y^2 = E\{[a\,\underset{\sim}{x} + b - (a\,\eta_x + b)]^2\}$

$$\sigma_y^2 = E\{a^2(\underset{\sim}{x} - \eta_x)^2\} = a^2\sigma_x^2$$

(b) $\underset{\sim}{y} = \dfrac{\underset{\sim}{x}-\eta_x}{\sigma_x} \qquad E\{\underset{\sim}{y}\} = 0 \qquad \sigma_y^2 = \dfrac{\sigma_x^2}{\sigma_x^2} = 1$

5-23 If $\underset{\sim}{x}$ has a Rayleigh density, then [see (5-76)]

$$E\{\underset{\sim}{x}^2\} = 2\alpha^2 \qquad E\{\underset{\sim}{x}^4\} = 8\alpha^4$$

If $\underset{\sim}{y} = b + c\underset{\sim}{x}^2$, then

$$E\{\underset{\sim}{y}\} = b + 2\alpha^2 c \qquad E\{\underset{\sim}{y}^2\} = b^2 + 4\alpha^2 bc + 8\alpha^4 c^2$$

$$\sigma_y^2 = E\{\underset{\sim}{y}^2\} - E^2\{\underset{\sim}{y}\} = 4\alpha^4 c^2$$

5-24 $\underset{\sim}{y} = 3\underset{\sim}{x}^2$ $E\{\underset{\sim}{x}^2\} = \sigma_x^2 = 4$ $E\{\underset{\sim}{x}^4\} = 3\sigma_x^4 = 48$

$E\{\underset{\sim}{y}\} = 12$ $E\{y^2\} = 9 \times 48 = 432$ $\sigma_y^2 = 432 - 144 = 288$

If $y > 0$ then $3x^2 = y$ for $x = \pm\sqrt{y/3}$ $y' = 6x$

$$f_y(y) = \frac{24}{\sqrt{12y}} f_x\left(\sqrt{\frac{y}{3}}\right) = \frac{1}{\sqrt{24\pi y}} e^{-y/24} U(y)$$

5.25

$$X \sim B(n,p) \Rightarrow P(X=k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0,1,2,\cdots n.$$

a)

$$\begin{aligned} E(X) &= \sum_{k=0}^{n} k\, P(X=k) = \sum_{k=1}^{n} k \frac{n!}{k!\,(n-k)!} p^k q^{n-k} \\ &= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} q^{n-k} \\ &= np\,(p+q)^{n-1} = np. \end{aligned}$$

b)

$$\begin{aligned} E[X(X-1)] &= \sum_{k=2}^{n} k\,(k-1) \frac{n!}{k!\,(n-k)!} p^k q^{n-k} \\ &= n(n-1)p^2 \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!\,(n-k)!} p^{k-2} q^{n-k} \\ &= n(n-1)p^2\,(p+q)^{n-2} \\ &= n(n-1)p^2 \end{aligned}$$

c)

$$\begin{aligned} E[X(X-1)(X-2)] &= \sum_{k=3}^{n} k(k-1)(k-2) \frac{n!}{k!\,(n-k)!} p^k q^{n-k} \\ &= n(n-1)(n-2)\,p^3 \sum_{k=3}^{n} \frac{(n-3)!}{(k-3)!\,(n-k)!} p^{k-3} q^{n-k} \\ &= n(n-1)(n-2)\,p^3\,(p+q)^{n-3} \\ &= n(n-1)(n-2)\,p^3 \end{aligned}$$

$$E(X^2) = E\,(X(X-1)) + E(X) = n^2p^2 + npq$$

$$\begin{aligned} E(X^3) &= E\,(X(X-1)(X-2)) + 3E(X^2) - 2E(X) \\ &= n(n-1)(n-2)\,p^3 + 3(n^2p^2 + npq) - 2np \\ &= n^3p^3 + 3n^2p^2q + npq(q-p). \end{aligned}$$

5.26

$$X \sim P(\lambda) \quad \Rightarrow \quad P(X=k) = e^{\lambda}\,\frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \cdots$$

a)

$$E(X) = \lambda, \quad \mathrm{Var}(X) = \sigma_X^2 = \lambda$$

From Chebyshev's inequality (5-88)

$$P\left(|X-\mu| < \lambda\right) > 1 - \frac{\sigma^2}{\lambda^2} = 1 - \frac{1}{\lambda}$$

But

$$|X-\mu| < \lambda = |X-\lambda| < \lambda \quad \Rightarrow \quad 0 < X < 2\lambda$$

which gives

$$P(0 < X < 2\lambda) > 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda}.$$

b)

$$\begin{aligned} E\left[X(X-1)\right] &= \textstyle\sum_{k=2}^{\infty} k(k-1)\, e^{-\lambda}\, \frac{\lambda^k}{k!} \\ &= e^{-\lambda}\,\lambda^2 \textstyle\sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} = e^{-\lambda}\,\lambda^2\, e^{\lambda} = \lambda^2. \end{aligned}$$

$$\begin{aligned} E\left[X(X-1)(X-2)\right] &= \textstyle\sum_{k=3}^{\infty} k(k-1)(k-2)\, e^{-\lambda}\, \frac{\lambda^k}{k!} \\ &= e^{-\lambda}\,\lambda^3 \textstyle\sum_{k=3}^{\infty} \frac{\lambda^{k-3}}{(k-3)!} = \lambda^3. \end{aligned}$$

5-27 Follows from (4-74)

$$E\{\underset{\sim}{x}\} = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x \sum_i f(x|A_i)P(A_i)\,dx$$

because $$E\{\underset{\sim}{x}|A_i\} = \int_{-\infty}^{\infty} x f(x|A_i)\,dx$$

5-28 From (5-89) with $\alpha = \sqrt{\eta}$:

$$P\{\underset{\sim}{x} \geq \sqrt{\eta}\} \leq \eta/\sqrt{\eta} = \sqrt{\eta}$$

5-29 From (5-86) with $g(x) = x^3$ $g''(x) = 6x$:

$$E\{\underset{\sim}{x}^3\} = \eta^3 + 6\eta \frac{\sigma^2}{2} = 1120$$

5-30 (a) If $y = x^3$, then $x = \sqrt[3]{y}$ $g'(x) = 3x^2 = 3\sqrt[3]{y^2}$

But $f_x(x) = 0.5$ for $10 < x < 12$, i.e., for $10^3 < y < 12^3$

and (5-16) yields

$$f_y(y) = \frac{0.5}{3\sqrt[3]{y^2}} \qquad 10^3 < y < 12^3$$

and zero otherwise.

(b) 1.

$$E\{\underset{\sim}{x}^3\} = 0.5 \int_{10}^{12} x^3 dx = 1342$$

2. With $g(x) = x^3$ $E\{\underset{\sim}{x}\} = 11$ $\sigma_x^2 = 1/3$, (5-86) yields

$$E\{x^3\} \simeq 11^3 + 6 \times 11 \times \frac{1}{6} \simeq 1342$$

5-31 With $g(x)=1/x$, $g''(x)=2/x^3$, $\eta=100$, and $\sigma=3$, (5-55) yields

$$E\left\{\frac{1}{\underset{\sim}{x}}\right\} \simeq \frac{1}{100} + \frac{9}{2} \times \frac{2}{100^3} = 0.010009$$

5-32

$$\frac{\partial|x-a|}{\partial a} = \begin{cases} 1 & x < a \\ -1 & x > a \end{cases} \qquad \text{If } I(a) = E\{|\underset{\sim}{x}-a|\} \text{ then}$$

$$\frac{dI(a)}{da} = E\,\frac{\partial|\underset{\sim}{x}-a|}{\partial a} = 1\,P\{\underset{\sim}{x} < a\} - 1\,P\{\underset{\sim}{x} > a\}$$
$$= 2\,F(a) - 1$$

(a)
$$I(a) = I(m) + \int_m^a I'(\alpha)d\alpha = I(m) + \int_m^a [2\,F(\alpha) - 1]d\alpha$$

$$= E\{|\underset{\sim}{x} - m|\} - 2\int_m^a x\,f(x)dx$$

because

$$\int_m^a F(\alpha)d\alpha = a\,F(a) - m\,F(m) - \int_m^a x f(x)dx$$

$$F(m) = \frac{1}{2} \qquad \int_m^a f(x)dx = F(a) - F(m)$$

(b) $I(a) = E\{|\underset{\sim}{x} - a|\}$ is minimum if

$I'(a) = 2F(a) - 1 = 0$ i.e. if $F(a) = \frac{1}{2}$ $\quad a = m$

5-33
$$E\{|\underset{\sim}{x}|\} = \int_0^\infty xf(x)dx - \int_{-\infty}^0 xf(x)dx$$

$$\eta = E\{\underset{\sim}{x}\} = \int_0^\infty xf(x)dx + \int_{-\infty}^0 xf(x)dx$$

$$\frac{E\{|\underset{\sim}{x}|+\eta\}}{2} = \int_0^\infty xf(x)dx = \frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty x\,e^{-(x-\eta)^2/2\sigma^2}dx$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty (x+\eta) e^{-(x-\eta)^2/2\sigma^2} dx = \frac{1}{\sigma\sqrt{2\pi}} \int_\eta^\infty y\, e^{-y^2/2\sigma^2} dy = \frac{\sigma}{\sqrt{2\pi}} e^{-\eta^2/2\sigma^2}$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty e^{-(x-\eta)^2/2\sigma^2} dx = G(\frac{\eta}{\sigma})$$

Multiplying the last line by η and subtracting from the fourth line, we obtain

$$\frac{E\{|x|+\eta\}}{2} = \frac{\sigma}{\sqrt{2\pi}} e^{-\eta^2/2\sigma^2} + G(\frac{\eta}{\sigma})$$

5-34 The proof is given in sec 14-3: [see (14-100)].

5-35 (a) Follows from (5-89) (b) $e^{sx} \geq e^{sA}$ iff $x \geq A$ for $s > 0$ and $x \leq A$ for $s < 0$.

5.36 See proof for Lyapunov inequality (Ch.5, Eq.(5-92).)

5-37 (a) If $\Phi(\omega) = e^{-\alpha|\omega|}$ then [see (5-102)]

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha|\omega|}\, e^{j\omega x} d\omega = \frac{1}{\pi}\int_{0}^{\infty} \cos\omega x e^{-\alpha\omega} d\omega = \frac{\alpha}{\pi(\alpha^2+x^2}$$

(b) If $f(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, then [see (5-94)]

$$\Phi(\omega) = \frac{\alpha}{2}\int_{-\infty}^{\infty} e^{-\alpha|x|} e^{-j\omega x} dx = \alpha\int_{0}^{\infty} e^{-\alpha x}\cos\omega x dx = \frac{\alpha^2}{\alpha^2+\omega^2}$$

5.38 a) On comparing Eq.(4-34) with Eq.(5-106), Example 5-29, we get

$$X \sim G(\alpha,\beta) \quad \Rightarrow \quad \phi_X(\omega) = (1-j\beta\omega)^{-\alpha}$$

$$\phi_X'(\omega) = -\alpha(1-j\beta\omega)^{-(\alpha+1)}(-j\beta)$$

so that

$$E(X) = \frac{1}{j}\phi_X'(0) = \alpha\beta.$$

Similarly

$$\phi_X''(\omega) = j\alpha\beta(\alpha+1)(1-j\beta\omega)^{-(\alpha+2)}(j\beta)$$

and hence

$$E(X^2) = \frac{1}{j^2}\phi_X''(0) = \alpha\beta^2(\alpha+1).$$

Thus

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \alpha\beta^2.$$

b)

$$X \sim \chi^2(n) \quad \Rightarrow \quad \alpha = \frac{n}{2}, \quad \beta = 2$$

in Gamma(α,β). This gives

$$\begin{aligned}\phi_X(\omega) &= (1-j2\omega)^{-n/2}\\ E(X) &= n\\ \mathrm{Var}(X) &= 2n.\end{aligned}$$

c)

$$X \sim B(n,p).$$

From Prob 5-25 (a)-(b)

$$\begin{aligned}E(X) &= np\\ \mathrm{Var}(X) &= E(X(X-1)) + E(X) = npq.\end{aligned}$$

and

$$\begin{aligned}\phi_X(\omega) &= \textstyle\sum_{k=0}^{n} e^{jk\omega} P(X=k)\\ &= \textstyle\sum_{k=0}^{n}\binom{n}{k}(p\,e^{j\omega})^k q^{n-k} = (pe^{j\omega}+q)^n.\end{aligned}$$

d)

$$X \sim N\,Binomial\,(r,p).$$

From (4-64)

$$\begin{aligned}\phi_X(\omega) &= \sum_{k=0}^{\infty} e^{jk\omega}\,P(X=k)\\ &= \sum_{k=0}^{\infty}\binom{r+k-1}{k}p^r\,(qe^{j\omega})^k\\ &= p^r\sum_{k=0}^{\infty}\binom{-r}{k}(-qe^{j\omega})^k\\ &= p^r\,(1-qe^{j\omega})^{-r}.\end{aligned}$$

5-39

$$\Gamma(z) = \sum_{k=0}^{\infty} p\,q^k z^k = \frac{p}{1-qz} \qquad q = 1-p$$

$$\Gamma'(z) = \frac{pq}{(1-qz)^2} \qquad \Gamma'(1) = \frac{pq}{(1-q)^2} = \frac{p}{q} = \eta_x$$

$$\Gamma''(z) = \frac{2pq^2}{(1-qz)^3} \qquad \Gamma''(1) = \frac{2q^2}{p^2} = m_2 - m_1$$

$$\sigma^2 = m_2 - m_1^2 = 2\frac{q^2}{p^2} + m_1 - m_1^2 = \frac{q}{p^2}$$

5-40

$$\Gamma(z) = p^n \sum_{k=0}^{\infty}\binom{-n}{k}(-q)^k z^k = p^n(1-qz)^{-n}$$

(binomial expansion with negative exponent)

$$\Gamma'(z) = \frac{n\,p^n q}{(1-qz)^{n+1}} \qquad \Gamma'(1) = \frac{nq}{p} = \eta_x$$

$$\Gamma''(z) = \frac{n(n+1)p^n q^2}{(1-qz)^{n+2}} \qquad \Gamma''(1) = \frac{n(n+1)q^2}{p^2} = m_2 - m_1$$

$$\sigma_x^2 = \Gamma''(1) + m_1 - m_1^2 = \frac{nq}{p^2}$$

5.41 We have

$$P(X=k)=\binom{k-1}{r-1}p^r\,q^{k-r},\quad k=r,r+1,\cdots$$

Let $k=n+r$ so that

$$\begin{aligned}
P(X=n+r) &= \binom{n+r-1}{r-1}p^r\,q^n,\quad n=0,1,2,\cdots\\
&= \frac{(n+r-1)!}{n!\,(r-1)!}\,p^r\,(1-p)^n\\
&= \frac{1}{n!}\,\frac{(n+r-1)(n+r-2)\cdots(r)}{r^n}\,[r(1-p)]^n\,p^r\\
&= \frac{\lambda^n}{n!}\left\{\left(1+\frac{n-1}{r}\right)\left(1+\frac{n-2}{r}\right)\cdots\right\}\left(1-\frac{r(1-p)}{r}\right)^r\\
&= \frac{\lambda^n}{n!}\left\{\prod_{k=1}^{n}\left(1+\frac{n-k}{r}\right)\right\}\left(1-\frac{\lambda}{r}\right)^r,
\end{aligned}$$

where $\lambda=r(1-p)$. Thus

$$\begin{aligned}
\lim_{r\to\infty}P(X=n+r) &= \frac{\lambda^n}{n!}\left\{\lim_{r\to\infty}\prod_{k=1}^{n}\left(1+\frac{n-k}{r}\right)\right\}\lim_{r\to\infty}\left(1-\frac{\lambda}{r}\right)^r\\
&\to \frac{\lambda^n}{n!}e^{-\lambda}\sim P(\lambda).
\end{aligned}$$

5-42 $E\{e^{s\underset{\sim}{x}}\} = e^{s\eta}\, E\{e^{s(\underset{\sim}{x}-\eta)}\} = e^{s\eta}\, E\left\{\sum_{n=0}^{\infty} \frac{s^n}{n!} (\underset{\sim}{x}-\eta)^n\right\}$

$$= e^{s\eta} \sum_{n=0}^{\infty} \frac{s^n}{n!} \mu_n$$

5-43 If $\Phi(\omega_1) = 0$, then [see also (9-176]

$$\int_{-\infty}^{\infty} (1-e^{j\omega_1 x}) f(x)dx = 0, \text{ hence, } f(x) = \sum_{n=\infty}^{\infty} p_n \delta(x - \frac{2\pi n}{\omega_1})$$

5-44 (a) If $\eta = 0$, then $m_n = \mu_n \qquad \lambda_1 = \eta = 0$

$$\Phi(s) = \sum_{n=0}^{\infty} \frac{\mu_n}{n!} s^n \qquad\qquad \Psi(s) = \sum_{n=2}^{\infty} \frac{\lambda_n}{n!} s^n$$

$$1 + \frac{\mu_2}{2!}s^2 + \frac{\mu_3}{3!} s^3 + \frac{\mu_4}{4!} s^4 + \cdots = \exp\left\{\frac{\lambda_2}{2!} s^2 + \frac{\lambda_3}{3!} s^3 + \frac{\lambda_4}{4!} s^4 + \cdots\right\}$$

Expanding the exponential and equating powers of s, we obtain

$$\mu_2 = \lambda_2 \qquad \mu_3 = \lambda_3 \qquad\qquad \frac{\mu_4}{4!} = \frac{\lambda_4}{4!} + \frac{1}{2!} \left(\frac{\lambda_2}{2!}\right)^2$$

(b) If $\underset{\sim}{y}$ is $N(0;\sigma_y)$ then

$$\Psi_y(s) = \frac{\lambda_2}{2} s^2 \text{ , hence, } \lambda_n = 0 \text{ for } n \geq 3$$

5-45 $$P\{\underset{\sim}{y} = 0\} = P\{\underset{\sim}{x} \le 1\} = p_o + p_1$$

$$P\{\underset{\sim}{y} = k\} = P\{\underset{\sim}{x} = k + 1\} = p_{k+1} \qquad k \ge 1$$

$$\Gamma_y(z) = p_o + p_1 + \sum_{k=1}^{\infty} p_{k+1} z^k = p_o + z^{-1}[\Gamma_x(z) - p_o]$$

$$\eta_y = \sum_{k=1}^{\infty} k\, p_{k+1} = \sum_{r=1}^{\infty} r\, p_r - \sum_{r=1}^{\infty} p_r = \eta_x - 1 + p_o$$

$$E\{\underset{\sim}{y}^2\} = \sum_{k=1}^{\infty} k^2 p_{k+1} = \sum_{r=1}^{\infty} (r-1)^2 p_r = E\{\underset{\sim}{x}^2\} - 2\eta_x + 1 - p_o$$

5-46 $$0 \le E\left\{\left|\sum_{i=1}^{n} a_i e^{j\omega_i \underset{\sim}{x}}\right|^2\right\} = E\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* e^{j(\omega_i-\omega_j)\underset{\sim}{x}}\right\}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* \Phi(\omega_i - \omega_j)$$

5-47 From the assumptions it follows that

$$g'(-x) = -g'(x) \qquad g''(x) \ge 0 \qquad f(x-\eta) = f(\eta-x)$$

Hence, if $I(a) = E\{g(\underset{\sim}{x}-a)\}$, then

$$I'(a) = -\int_{-\infty}^{\infty} g'(x-a) f(x)\,dx \qquad I'(\eta) = 0$$

$$I''(a) = \int_{-\infty}^{\infty} g''(x-a) f(x)\,dx \ge 0 \qquad \text{all } a$$

Hence, $I(a)$ is minimum for $a = \eta$.

5-48 $$f(x,v) = \frac{1}{\sqrt{2\pi v}} e^{-x^2/2v}$$

$$\sqrt{2\pi}\,\frac{\partial f}{\partial v} = \frac{-1 + x^2/v}{2v\sqrt{v}} e^{-x^2/2v}$$

$$\sqrt{2\pi}\,\frac{\partial^2 f}{\partial x^2} = \frac{-1 + x/.}{v\sqrt{v}} e^{-x^2/2v}$$

Hence

(see also (6-198) - (6-199)) $$\boxed{\frac{\partial f}{\partial v} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}} \qquad (1)$$

(a) Integrating by parts, using (1) and assuming that $g^{(k)}(x)f(x) \to 0$ as $|x| \to \infty$, $k = 0, 1, 2$, we obtain

$$E\{g''(x)\} = \int_{-\infty}^{\infty} \frac{d^2 g}{dx^2} f dx = \int_{-\infty}^{\infty} g \frac{\partial^2 f}{\partial x^2} dx = 2\int_{-\infty}^{\infty} g \frac{\partial f}{\partial v} dx$$

$$= 2\frac{d}{dv}\int_{-\infty}^{\infty} g\, f dx = 2\frac{d}{dv} E\{g(\underset{\sim}{x})\}$$

(b) The moments $\mu_n(u) = E\{\underset{\sim}{x}^n\}$ of $\underset{\sim}{x}$ depend on the variance v of $\underset{\sim}{x}$ and (i) yields

$$\mu_n'(v) = \frac{d}{dv} E\{\underset{\sim}{x}^n\} = \frac{1}{2} E\{n(n-1)\underset{\sim}{x}^{n-2}\} = \frac{n(n-1)}{2}\mu_{n-2}(v)$$

Furthermore, $\mu_n(0) = 0$ because, if $v = 0$, then $\underset{\sim}{x} = 0$.

Hence

$$\mu_n(v) = \frac{n(n-1)}{2}\int_0^v \mu_{n-2}(\beta)d\beta$$

5-49 The function

$$\Gamma(e^{j\omega}) = E\{e^{jx\omega}\} = \sum_{k=0}^{\infty} p_k \, e^{jk\omega}$$

is periodic with period 2π and Fourier series coefficients $p_k = E\{\underset{\sim}{x} = k\}$.

5.50 The event $\{X = 1\}$ is given by the disjoint union "$TH \cup HT$". Similarly, the event "$X = k$" is given by the union of the disjoint events (k "T"s followed by "H" or k "H"s followed by "T")

$$\text{“}TT\cdots TTH'' \cup \text{“}HH\cdots HHT'', \qquad k = 1, 2, \cdots$$

Thus

$$\begin{aligned} P(X = k) &= P(\text{“}TT\cdots TH'' \cup \text{“}HH\cdots HT'') \\ &= P(TT\cdots TH) + P(HH\cdots HT) = q^k p + p^k q, \quad k = 1, 2, \cdots \end{aligned}$$

Also

$$\begin{aligned} E(X) &= \textstyle\sum_{k=1}^{\infty} kP(X = k) \\ &= \textstyle\sum_{k=1}^{\infty} kq^k p + \sum_{k=1}^{\infty} kp^k q = pq\left\{\sum_{k=1}^{\infty} kq^{k-1} + \sum_{k=1}^{\infty} kp^{k-1}\right\} \\ &= pq\left\{\frac{\partial}{\partial q}\textstyle\sum_{k=1}^{\infty} q^k + \frac{\partial}{\partial p}\sum_{k=1}^{\infty} p^k\right\} = pq\left\{\frac{\partial}{\partial q}\left(\frac{q}{1-q}\right) + \frac{\partial}{\partial p}\left(\frac{p}{1-p}\right)\right\} \\ &= pq\left\{\frac{1}{p^2} + \frac{1}{q^2}\right\} = \frac{p}{q} + \frac{q}{p}. \end{aligned}$$

5.51 (a) When samples are drawn with replacement, probability of each item being defective is given by

$$p = \frac{M}{N} < 1 \quad \text{(constant)}$$

and

$$q = 1 - p = \frac{N - M}{M} < 1$$

represents the constant probability that the chosen item is not defective. In that case (with replacement), there are $\binom{n}{k}$ possible ways of arranging k defective items among n chosen items, and each such arrangement has probability $p^k q^{n-k}$. This gives

$$P(X = k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \cdots n$$

which represents the Binomial distribution.
(b) If the samples are drawn without replacement, there are $\binom{M}{k}$ possible ways of choosing k defective item from a total of M defective items, and $\binom{N-M}{n-k}$ possible ways of choosing $n-k$ "good" items from $(N-M)$ "good" items independently. This gives

$$\binom{M}{k}\binom{N-M}{n-k}$$

to be the total number of ways of selecting k defective items and $n-k$ "good" items from a subsample of M and $N-M$ items respectively (favorable ways). But there are a total of $\binom{N}{n}$ ways of selecting n items among N items. This gives

$$P(X = k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}},$$

since $0 \le k \le M$ and $n - k \le N - M, n - k \ge 0$, i.e. $0 \le k \le M, k \le n, k \ge n + M - N$.
(c) From (b)

$$P(X = k) = \frac{M!}{k!(M-k)!} \frac{(N-M)!}{(n-k)!(N-M-n+k)!} \frac{n!(N-n)!}{N!}$$

$$= \binom{n}{k} \frac{M(M-1)\cdots(M-k+1)}{N(N-1)\cdots(N-k+1)} \frac{(N-M)(N-M-1)\cdots(N-M-n+k+1)}{(N-k)(N-k-1)\cdots(N-n+1)} \frac{(I}{(I}$$

$$\simeq \binom{n}{k} \left(\frac{M}{N}\right)^k \left(\frac{N-M}{N}\right)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \cdots n$$

since $N \to \infty, M \to \infty$ such that $M/N \to p$, and $n \ll N$. Thus

$$P(X = k) \to \text{Binomial}(n, p = M/N)$$

under the above conditions.

5.52 (a) Refer to discussions in problem 5.51 (a) if sampleing is done with replacement, then

$$p = \frac{n}{n+m}$$

represents the probability of selecting a white marble on any trial. The event "$X = k''$ is given by "$r-1$ white mables among the first $k-1$ trials" followed by "a white marble at the kth trial". But from problem 5.51 (a), the event $r-1$ white mables among the first $k-1$ trials has a binomial distribution whose probability is given by $\binom{k-1}{r-1}p^{r-1}q^{k-r}$. Thus

$$P(X=k) = \binom{k-1}{r-1}p^{r-1}q^{k-r}p = \binom{k-1}{r-1}p^r q^{k-r}, \quad k = r, r+1, \cdots$$

which represents the Negative-binomial distribution

(b) If sampling is done with replacement, then the favarable ways of choosing the white balls are given by:

(i) $\binom{k-1}{r-1}$ ways of selecting $r-1$ white balls among the first $k-1$ trials/balls.

(ii) One ways of selecting (the rth) white ball at the kth trial

(iii) $\binom{m+n-k}{n-r}$ ways of selecting the remaining $n-r$ white balls among the remaining $m+n-k$ balls.

This gives $\binom{k-1}{r-1} \cdot 1 \cdot \binom{m-n-k}{n-r}$ to be the total number of favorable ways of selecting the white balls. Since there are $n+m$ balls there are a total of $\binom{n+m}{n}$ ways of selecting n white balls. This gives

$$P(X=k) = \binom{k-1}{r-1}\frac{\binom{m+n-k}{n-r}}{\binom{n+m}{n}}, \qquad k = r, r+1, \cdots$$

(c) From (b)

$$P(X=k) = \binom{k-1}{r-1}\frac{(m+n-k)!}{(n-r)!\,(m-k+r)!}\frac{n!m!}{(m+n)!}$$

$$= \binom{k-1}{r-1}\left(\frac{n}{m+n}\right)\left(\frac{n-1}{m+n-1}\right)\cdots\left(\frac{n-r+1}{m+n-r+1}\right)\left(\frac{m!(m+n-k)!}{(m+n-r)!\,(m-k+r)!}\right)$$

$$\simeq \binom{k-1}{r-1}\left(\frac{n}{m+n}\right)^r\left(\frac{m}{m+n-r}\right)\left(\frac{m-1}{m+n-r-1}\right)\cdots\left(\frac{m-k+r+1}{m+n-k+1}\right)$$

$$\simeq \binom{k-1}{r-1}\left(\frac{n}{m+n}\right)^r\left(\frac{m}{m+n}\right)^{k-r} \text{ as } m+n \to \infty$$

$$= \binom{k-1}{r-1}p^r q^{k-r}, \qquad k = r, r+1, \cdots, \quad q = 1-p$$

$$\sim \mathrm{N}B\left(r, p = n/(n+m)\right).$$

CHAPTER 6

6.1 (a) Define

$$Z = X + Y$$

Note that both X and Y positive random variables hence (use Eq. (6-45))

$$\begin{aligned} f_Z(z) &= \int_0^z f_{XY}(z-y,y)dy = \int_0^z e^{-(z-y+y)}dy \\ &= z\,e^{-z}U(z). \end{aligned}$$

(b)

$$Z = X - Y$$

Z ranges over the entire real axis for the random variables X and Y (see Eq. (6-55))

$$F_Z(z) = \begin{cases} \int_0^\infty \int_0^{z+y} f_{XY}(x,y)dx\,dy, & z > 0 \\ \int_{-z}^\infty \int_0^{z+y} f_{XY}(x,y)dx\,dy, & z < 0 \end{cases}$$

Differrentiation gives

$$f_Z(z) = \begin{cases} \int_0^\infty f_{XY}(z+y,y)\,dy, & z > 0 \\ \int_{-z}^\infty f_{XY}(z+y,y)\,dy, & z < 0 \end{cases}$$

$$f_Z(z) = \begin{cases} \int_0^\infty e^{-(z+y+y)}\,dy = e^{-z}\int_0^\infty e^{-2y}\,dy = \frac{1}{2}\,e^{-z}, & z > 0 \\ \int_{-z}^\infty e^{-(z+y+y)}\,dy = e^{-z}\int_{-z}^\infty e^{-2y}\,dy = \frac{1}{2}\,e^{z}, & z < 0 \end{cases}$$

or

$$f_Z(z) = \frac{1}{2}\,e^{-|z|}, \qquad -\infty \le z \le \infty.$$

(c)

$$Z = XY.$$

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{XY \le z\} \\ &= \int_0^\infty \int_0^{z/y} f_{XY}(x,y)dx\,dy \end{aligned}$$

or (see Eq. (6-148))

$$f_Z(z) = \int_0^\infty \frac{1}{y} f_{XY}(\frac{z}{y},y)dy = \int_0^\infty \frac{1}{y} e^{-((z/y)+y)}dy$$

(d)

$$Z = X/Y$$

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{\tfrac{X}{Y} \le z\} \\ &= \int_0^\infty \int_0^{yz} f_{XY}(x,y)dx\,dy \end{aligned}$$

(use Eq. (6-60))

$$\begin{aligned} f_Z(z) &= \int_0^\infty y f_{XY}(yz,y)dy = \int_0^\infty y e^{y(z+1)}dy = \int_0^\infty y e^{(1+z)y} \\ &= \left[y\frac{e^{-(1+z)y}}{-(1+z)}\right]_0^\infty + \left(\frac{1}{1+z}\right)\int_0^\infty e^{(1+z)y}dy \\ &= \left(\frac{1}{1+z}\right)\left[\frac{e^{-(1+z)y}}{-(1+z)}\right]_0^\infty = \frac{1}{(1+z)^2}\,U(z) \end{aligned}$$

(e)

$$Z = \min\,(X,Y)$$

$$\begin{aligned} F_Z(z) &= P\{\min\,(X,Y) \le z\} \\ &= 1 - P\{Z > z, Y > z\} \\ &= 1 - [1 - F_X(z)]\,[1 - F_Y(z)] \\ &= F_X(z) + F_Y(z) - F_X(z)\,F_Y(z) \end{aligned}$$

(see Eq. (6-81))

$$f_Z(z) = f_X(z) + f_Y(z) - F_X(z)f_Y(z) - f_X(z)F_Y(z).$$

We have

$$f_X(z) = f_Y(z) = e^{-z}\,U(z)$$

so that

$$F_X(z) = \int_0^z e^{-x}dx = (1 - e^{-z})\,U(z) = F_Y(z)$$

$$\begin{aligned} f_Z(z) &= [e^{-z} + e^{-z} - 2(1 - e^{-z})\,e^{-z}]U(z) \\ &= 2e^{-z}\,[1 - 1 + e^{-z}]\,U(z) \\ &= 2e^{-2z}\,U(z) \sim \text{Exponential (2)}. \end{aligned}$$

(f)

$$Z = \max(X,Y)$$

$$\begin{aligned} F_Z(z) &= P\{\max(X,Y) \le z\} = P\{X \le z, Y \le z\} \\ &= P\{X \le z\}\,P\{Y \le z\} = F_X(z)\,F_Y(z) \end{aligned}$$

$$\begin{aligned} f_Z(z) &= F_X(z)\,f_Y(z) + f_X(z)\,F_Y(z) \\ &= e^{-z}\,(1 - e^{-z}) + e^{-z}\,(1 - e^{-z}) \\ &= 2e^{-z}\,(1 - e^{-z})\,U(z) \end{aligned}$$

(g)

$$Z = \frac{\min(X,Y)}{\max(X,Y)}, \quad 0 < z < 1$$

$$\begin{aligned}
F_Z(z) &= P\left\{\left(\frac{\min(X,Y)}{\max(X,Y)} \le z\right) \cap ((X \le Y) \cup (X > Y))\right\} \\
&= P\left\{\left(\frac{\min(X,Y)}{\max(X,Y)} \le z\right) \cap (X \le Y)\right\} + P\left\{\left(\frac{\min(X,Y)}{\max(X,Y)} \le z\right) \cap (X > Y)\right\} \\
&= P\{\tfrac{X}{Y} \le z, X \le Y\} + P\{\tfrac{Y}{X} \le z, X > Y\} \\
&= P\{X \le Yz, X \le Y\} + P\{Y \le Xz, X > Y\} \\
&= \int_0^\infty \int_0^{yz} f_{XY}(x,y)\,dx\,dy + \int_0^\infty \int_0^{xz} f_{XY}(x,y)\,dy\,dx
\end{aligned}$$

$$\begin{aligned}
f_Z(z) &= \int_0^\infty y\, f_{XY}(yz,y)dy + \int_0^\infty x f_{XY}(x,xz)dx \\
&= \int_0^\infty y\, f_{XY}(yz,y)dy + \int_0^\infty y\, f_{XY}(y,yz)dy \\
&= \int_0^\infty y\left(e^{-(yz+y)} + e^{-(y+yz)}\right)dy \\
&= 2\int_0^\infty y e^{-y(1+z)}dz = \begin{cases} \dfrac{2}{(1+z)^2}, & 0 \le z \le 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

6.2

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) = \frac{1}{a^2}, \qquad 0 < x \le a, \quad 0 < y \le a$$

(a)

$$F_Z(z) = P\left\{\frac{X}{Y} \le z\right\} = P\{X \le zY\}$$

(i) $z < 1$

$$\begin{aligned}
F_Z(z) &= P\{X \le zY\} \\
&= \int_0^a \int_0^{zy} \frac{1}{a} \cdot \frac{1}{a}\,dx\,dy = \frac{z}{2}, \quad z \le 1
\end{aligned}$$

(ii) $z \ge 1$

$$\begin{aligned}
F_Z(z) &= P\{X \le zY\} \\
&= 1 - \int_0^a \int_0^{x/z} \frac{1}{a} \cdot \frac{1}{a}\,dy\,dx \\
&= 1 - \int_0^1 \frac{x}{z}\,dx = 1 - \frac{1}{2z} \quad z > 1
\end{aligned}$$

$$f_Z(z) = \begin{cases} \frac{1}{2}, & z \le 1 \\ \frac{1}{2z^2}, & z > 1 \end{cases}$$

(b)

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P\left\{\frac{Y}{X+Y} \le z\right\} \\ &= P\left\{\frac{X}{Y} \ge \frac{1}{z} - 1\right\} = 1 - P\left(\frac{X}{Y} \le \frac{1-z}{z}\right) \\ &= \begin{cases} \frac{1}{2}\left(\frac{z}{1-z}\right), & 0 < z \le 1/2 \\ 1 - \frac{1}{2}\left(\frac{1-z}{z}\right), & 1/2 < z < 1 \end{cases} \end{aligned}$$

$$f_Z(z) = \begin{cases} \dfrac{1}{2(1-z)^2}, & 0 < z \le 1/2 \\ \dfrac{1}{2z^2}, & 1/2 < z < 1 \end{cases}$$

(c)

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{|X-Y| \le z\} \\ &= P\{\{|X-Y| \le z\} \bigcap (X \ge Y)\} + P\{\{|X-Y| \le z\} \bigcap (X < Y)\} \\ &= P\{X - Y \le z, X \ge Y\} + P\{Y - X \le z, X < Y\} \\ &= \int_0^\infty \int_y^{y+z} f_{XY}(x,y)\,dx\,dy + \int_0^\infty \int_x^{x+z} f_{XY}(x,y)\,dy\,dx \\ &= \int_0^\infty \int_y^{y+z} f_{XY}(x,y)\,dx\,dy + \int_0^\infty \int_y^{y+z} f_{XY}(y,x)\,dx\,dy \\ &= \int_0^\infty \int_y^{y+z} \{f_{XY}(x,y) + f_{XY}(y,x)\}\,dx\,dy. \end{aligned}$$

In general

$$\begin{aligned} f_Z(z) &= \int_0^\infty \frac{d}{dz} \int_y^{y+z} f_{XY}(x,y) + f_{XY}(y,x)\,dx\,dy \\ &= \int_0^\infty \{f_{XY}(y+z,y) + f_{XY}(y,y+z)\}\,dy. \end{aligned}$$

Here

$$X \sim U(0,a), \qquad Y \sim U(0,a)$$

$$F_Z(z) = 1 - \frac{1}{a^2} \cdot 2 \cdot \frac{(a-z)^2}{2} = 1 - \left(1 - \frac{z}{a}\right)^2$$

and

$$f_Z(z) = \frac{2}{a}\left(1 - \frac{z}{a}\right) \qquad 0 \le z \le a.$$

6.3

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{X+Y \le z\} \\ &= \frac{1}{2} - \frac{z^2}{2}, \quad -1 < z < 0, \end{aligned}$$

(which represents the area below the line $X+Y=z$.)

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{X+Y \le z\} \\ &= \frac{1}{2} + \frac{z^2}{2}, \quad 0 \le z < 1 \end{aligned}$$

$$f_Z(z) = \begin{cases} -z, & -1 \le z < 0 \\ z, & 0 \le z < 1 \end{cases}$$

6.4

$$Z = X - Y$$

For $z < 0$

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} \\ &= \int_0^{(1+z)/2} \int_{x-z}^{1-x} f_{XY}(x,y)\,dy\,dx = \int_0^{(1+z)/2} \int_{x-z}^{1-x} 6x\,dy\,dx \\ &= \int_0^{(1+z)/2} 6x\,[y]_{x-z}^{1-x}\,dx = \int_0^{(1+z)/2} 6x(1-x-x+z)\,dx \\ &= 6\left[(1+z)\frac{x^2}{2} - \frac{2x^3}{3}\right]_0^{(1+z)/2} = 6\left[\frac{(1+z)^3}{8} - \frac{(1+z)^3}{12}\right] \\ &= \frac{(1+z)^3}{4}, \quad z \le 0. \end{aligned}$$

For $z > 0$

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = 1 - P\{Z > z\} \\ &= 1 - \int_0^{(1-z)/2} \int_{z+y}^{1-y} f_{XY}(x,y)\,dx\,dy = 1 - \int_0^{(1-z)/2} \int_{z+y}^{1-y} 6x\,dy \\ &= 1 - \int_0^{(1-z)/2} \left[\frac{6x^2}{2}\right]_{z+y}^{1-y} dy = 1 - 3\int_0^{(1-z)/2} \left[(1-y)^2 - (z-y)^2\right] dy \\ &= 1 - 3\,(1+z)\left[\frac{(1-z)^2}{2} - \frac{(1-z)^2}{4}\right] = 1 - \frac{3}{4}\,(1+z)(1-z)^2 \quad z \le 0. \end{aligned}$$

$$f_Z(z) = \begin{cases} \frac{3}{4}\,(1-z)(1+3z), & 0 \le z \le 1 \\ \frac{3}{4}\,(1+z)^2, & -1 < z < 0 \end{cases}$$

6.5 (a) See Example 6-15 for solutions
(b) See Example 6-14 for solutions
(c)

$$U = X - Y \sim N(0, 2\sigma^2)$$

since linear combinations of jointly Gaussian random variables are Gaussian random variables (see Eq. (6-120) Text.). Here $Var(U) = Var(X) + Var(Y) = 2\sigma^2$.

6.6

$$Z = XY$$

$$\begin{aligned} F_Z(z) &= P(XY \le z) = 1 - P(XY > z) \\ &= 1 - \int_z^1 \int_{z/y}^1 f_{XY}(x,y)\,dx\,dy \\ f_Z(z) &= 1 + \int_z^1 \frac{1}{y} f_{XY}(z/y, y)\,dy = 1 + \int_z^1 \left\{ \frac{2}{y} - \frac{2z}{y^2} \right\} dy \\ &= 1 - 2\ln z + 2z, \quad 0 \le z \le 1 \end{aligned}$$

6.7 (a)

$$Z_1 = X + Y$$

$$F_{Z_1}(z) = P(X+Y \le z) = \begin{cases} \displaystyle\int_0^z \int_0^{z-y} f_{XY}(x,y)\,dx\,dy, & 0 < z < 1 \\ \displaystyle 1 - \int_{z-1}^1 \int_{z-y}^1 f_{XY}(x,y)\,dx\,dy, & 1 < z < 2 \end{cases}$$

$$f_{Z_1}(z) = \begin{cases} \displaystyle\int_0^z f_{XY}(z-y, y)\,dy, & 0 < z < 1 \\ \displaystyle\int_{z-1}^1 f_{XY}(z-y, y)\,dy, & 1 < z < 2 \end{cases}$$

$$= \begin{cases} z^2, & 0 < z < 1 \\ z(2-z), & 1 < z < 2 \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$Z_2 = XY$$

$$F_{Z_2}(z) = P(XY \le z) = 1 - \int_z^1 \int_{z/y}^1 f_{XY}(x,y)\,dx\,dy$$

$$\begin{aligned} f_{Z_2}(z) &= \int_z^1 \frac{1}{y} f_{XY}(z/y, y)\,dy = \int_z^1 \frac{1}{y} \left(\frac{z}{y} + y \right) dy \\ &= 2(1-z), \quad 0 < z < 1 \end{aligned}$$

(c)

$$Z_3 = \frac{Y}{X}$$

$$F_{Z_3}(z) = P(Y/X \le z) = \begin{cases} \displaystyle\int_0^1 \int_0^{zx} f_{XY}(x,y)\,dy\,dx, & 0 < z < 1 \\ \displaystyle 1 - \int_0^1 \int_0^{y/z} f_{XY}(x,y)\,dx\,dy, & z > 1 \end{cases}$$

$$f_{Z_3}(z) = \begin{cases} \int_0^1 x\, f_{XY}(x, zx)\, dx, & 0 < z < 1 \\ \int_0^1 \frac{y}{z^2}\, f_{XY}(y/z, y)\, dy, & z > 1 \end{cases}$$

$$= \begin{cases} \frac{1+z}{3}, & 0 < z < 1 \\ \frac{1+z}{3z^3}, & z > 1 \end{cases}$$

(d)

$$Z_4 = Y - X$$

$$F_{Z_4}(z) = P(Y - X \le z) = \begin{cases} 1 - \int_z^1 \int_0^{y-z} f_{XY}(x, y)\, dx\, dy & 0 < z < 1 \\ \int_0^{z+1} \int_{y-z}^1 f_{XY}(x, y)\, dx\, dy, & -1 < z < 0 \end{cases}$$

$$f_{Z_4}(z) = \begin{cases} \int_z^1 f_{XY}(y - z, y)\, dy, & 0 < z < 1 \\ \int_0^{z+1} f_{XY}(y - z, y)\, dy, & -1 < z < 0 \end{cases}$$

$$= \begin{cases} 1 - z, & 0 < z < 1 \\ 1 + z, & -1 < z < 0 \end{cases} = 1 - |z|, \quad |z| < 1$$

6.8

$$F_Z(z) = P(X + Y \le z)$$

$$= \begin{cases} \int_0^{z/3} \int_{2y}^{z-y} f_{XY}(x, y)\, dx\, dy = \frac{z^2}{6}, & 0 < z < 2 \\ 1 - \int_{2z/3}^2 \int_{z-x}^{x/2} f_{XY}(x, y)\, dy\, dx = 2z - \frac{z^2}{3} - 2, & 2 < z < 3 \end{cases}$$

Thus

$$f_Z(z) = \begin{cases} \int_0^{z/3} f_{XY}(z - y, y)\, dy & 0 < z < 2 \\ \int_{2z/3}^2 f_{XY}(x, z - x)\, dx & 2 < z < 3 \end{cases}$$

$$f_Z(z) = \begin{cases} \frac{1}{3} z, & 0 < z < 2 \\ 2 - \frac{2z}{3}, & 2 < z < 3 \\ 0, & \text{otherwise} \end{cases}$$

6.9 (a)

$$Z = \frac{X}{Y}, \qquad z \geq 1$$

$$F_Z(z) = P(X \leq Yz) = \int_0^1 \int_{x/z}^x f_{XY}(x,y)\,dy\,dx$$

$$f_Z(z) = \int_0^1 \frac{x}{z^2} f_{XY}(x, x/z)\,dx = \frac{1}{z^2}, \quad z \geq 1$$

(b)

$$W = XY$$

$$\begin{aligned} F_W(w) &= P(W \leq w) = P(XY \leq w) = 1 - P(XY > w) \\ &= 1 - \int_{\sqrt{w}}^1 \int_{w/x}^x f_{XY}(x,y)\,dy\,dx \end{aligned}$$

Hence

$$\begin{aligned} f_W(w) &= \int_{\sqrt{w}}^1 \frac{1}{x} f_{XY}(x, w/x)\,dx = \int_{\sqrt{w}}^1 \frac{2}{x}\,dx \\ &= \ln(1/w), \quad 0 < w \leq 1 \end{aligned}$$

6.10 (a)

$$Z = X + Y$$

$$F_Z(z) = \int_0^{z/2} \int_x^{2-x} f_{XY}(x,y)\,dx = \frac{z^2}{4}, \quad 0 < z < 2$$

$$f_Z(z) = \frac{z}{2}, \qquad 0 < z < 2$$

(b)

$$W = X - Y$$

$$F_W(w) = \frac{1}{2}(2+w)(1+\frac{w}{2}) = \left(1 + \frac{w}{2}\right)^2$$

$$f_W(w) = \begin{cases} 1 + \frac{w}{2}, & -2 < w < 0 \\ 0, & \text{otherwise} \end{cases}$$

6.11 (a) The charaterristic function of $X+Y$ is given by

$$\begin{aligned}\phi_{X+Y}(\omega) &= \phi_X(\omega)\,\phi_Y(\omega) = \tfrac{1}{(1-j\omega\beta)^\alpha}\cdot\tfrac{1}{(1-j\omega\beta)^\alpha}\\ &= \tfrac{1}{(1-j\omega\beta)^{2\alpha}} \sim \text{Gamma}\,(2\alpha,\beta)\end{aligned}$$

(b)

$$f_{XY}(x,y) = f_X(x)\,f_Y(y) = \frac{(xy)^{(\alpha-1)}}{(\Gamma(\alpha)\beta^\alpha)^2}\,e^{(x+y)/\beta}, \quad x>0,\, y>0$$

Let

$$Z = \frac{X}{Y}$$

Using (Eq. 6-60) we get

$$\begin{aligned}f_Z(z) &= \int_0^\infty y\,\frac{(y^2z)^{(\alpha-1)}}{(\Gamma(\alpha)\beta^\alpha)^2}\,e^{-(1+z)y/\beta}\,dy\\ &= \frac{z^{(\alpha-1)}}{(\Gamma(\alpha)\beta^\alpha)^2}\int_0^\infty y^{(2\alpha-1)}\,e^{-(1+z)y/\beta}\,dy\\ &= \frac{z^{(\alpha-1)}}{(\Gamma(\alpha))^2\,\beta^{2\alpha}}\,\frac{\beta^{(2\alpha-1)}}{(1+z)^{2\alpha-1}}\,\frac{\beta}{(1+z)}\int_0^\infty u^{2\alpha-1}\,e^{-u}\,du\\ &= \frac{(\Gamma(2\alpha))\;z^{\alpha-1}}{(\Gamma(\alpha))^2\,(1+z)^{2\alpha}}, \quad z>0\end{aligned}$$

(see also Example 6-27 for the answer).
(c)

$$W = \frac{X}{X+Y} = \frac{X/Y}{X/Y+1} = \frac{Z}{Z+1}$$

$$F_W(w) = P\left(\frac{Z}{Z+1}\le w\right) = P\left(Z\le\frac{w}{1-w}\right) = F_Z\left(\frac{w}{1-w}\right)$$

This gives

$$\begin{aligned}f_W(w) &= \frac{1}{(1-w)^2}\,f_Z\left(\frac{w}{1-w}\right)\\ &= \frac{\Gamma(2\alpha)}{(\Gamma(\alpha))^2}\,w^{\alpha-1}\,(1-w)^{\alpha-1}\\ &\sim\; Beta\,(\alpha,\alpha)\end{aligned}$$

where we have used results from (b) above.

6.12

$$X\sim U(0,1), \quad Y\sim U(0,1), \quad X,Y \text{ are independent, and}$$

$$U = X+Y, \qquad V = X-Y \;\Rightarrow\; |v|<u<2.$$

U and V have one pair of solutions given by

$$x_1 = \frac{u+v}{2}, y_1 = \frac{u-v}{2}.$$

Also the Jacobian is given by

$$J = \begin{vmatrix} 1 & 1\\ 1 & -1\end{vmatrix} = -2$$

so that

$$f_{UV}(u,v) = \frac{1}{|J|}\,f_{XY}(x_1,y_1) = \frac{1}{2}, \quad 0<|v|<u<2$$

6.13

$$f_{XY}(x,y) = \frac{xy}{\sigma^4}\, e^{-(x^2+y^2)/2\sigma^2}, \quad x, y \geq 0$$

$$Z = \frac{X}{Y}$$

$$F_Z(z) = P(Z \leq z) = P(X/Y \leq z) = \int_0^\infty \int_0^{zy} f_{XY}(x,y)\, dx\, dy.$$

This gives the density function of z to be

$$\begin{aligned} f_Z(z) &= \int_0^\infty y\, f_{XY}(zy, y)\, dy = \int_0^\infty \frac{zy^3}{\sigma^4}\, e^{-(z^2y^2+y^2)/2\sigma^2}\, dy \\ &= \frac{z}{\sigma^4}\int_0^\infty y^3\, e^{-y^2(z^2+1)/2\sigma^2}\, dy \quad \text{Let, } t = y^2(z^2+1)/2\sigma^2 \\ &= \frac{2z}{(z^2+1)^2}\int_0^\infty t\, e^{-t}\, dt = \frac{2z}{(z^2+1)^2}, \quad 0 \leq z \leq \infty. \end{aligned}$$

6-14 $\quad \underset{\sim}{z} = \underset{\sim}{x} + \underset{\sim}{y} \qquad f_z(z) = f_x(z) * f_y(z)$

For $z > 0$

$$c^2 z\, e^{-cz} = \int_0^z c\, e^{-c(z-y)}\, f_y(y)\, dy$$

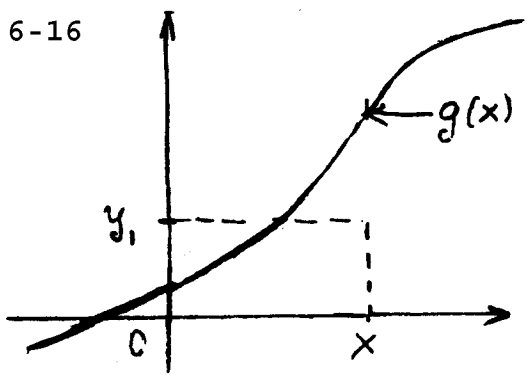
$$c\, z = \int_0^z e^{cy}\, f_y(y)\, dy \qquad c = e^{c\,z}\, f_y(z)$$

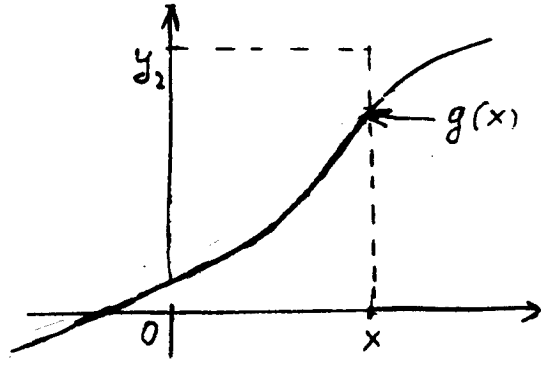
(differentiation). Hence, $f_y(z) = c\, e^{-cz}$; and zero for $z < 0$.

6-15 $\quad f_z(z) = \int_{-\infty}^{\infty} f_x(x) f_y(z-x)\, dx = \int_{z-1}^{z} f_x(x)\, dx = F_x(z) - F_x(z-1)$

because $f_y(z-x) = 1$ for $z - 1 < x < z$ and zero otherwise.

6-16





All probability masses are on the line $y = g(x)$.

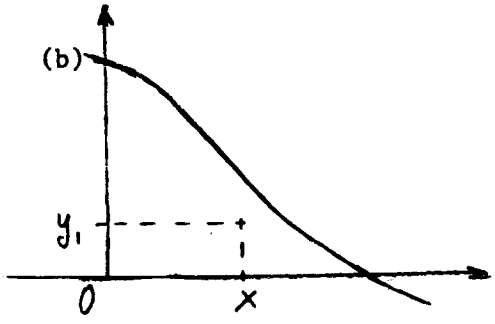
(a) If $y = y_1 < g(x)$ then

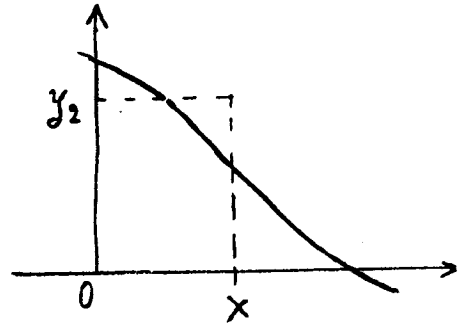
$$F(x,y) = P\{\underset{\sim}{x} \le x,\ \underset{\sim}{y} \le y_1\} = P\{\underset{\sim}{y} \le y_1\} = F_y(y_1).$$

If $y = y_2 > g(x)$ then

$$F(x,y) = P\{\underset{\sim}{x} \le x,\ \underset{\sim}{y} \le y_2\} = P\{\underset{\sim}{x} \le x\} = F_x(x)$$

(b)





If $y = y_1 < g(x)$ then

$$F(x,y) = P\{\underset{\sim}{x} \le x,\ \underset{\sim}{y} \le y_1\} = 0$$

If $y = y_2 > g(x)$ then

$$F(x,y) = P\{\underset{\sim}{x} \le x,\ \underset{\sim}{y} \le y_2\} = P\{\underset{\sim}{x} \le x\} - P\{\underset{\sim}{y} > y_2\}$$

$$= F_x(x) - [1 - F_y(y_2)]$$

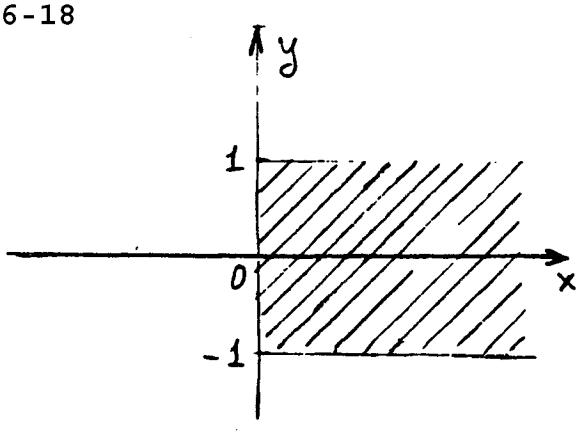
6-17 (a) If $\underset{\sim}{z} = 2\underset{\sim}{x} + 3\underset{\sim}{y}$ then $E\{\underset{\sim}{z}\} = 0$ $\quad \sigma_z^2 = 4\sigma_x^2 + 9\sigma_y^2 = 5^2$

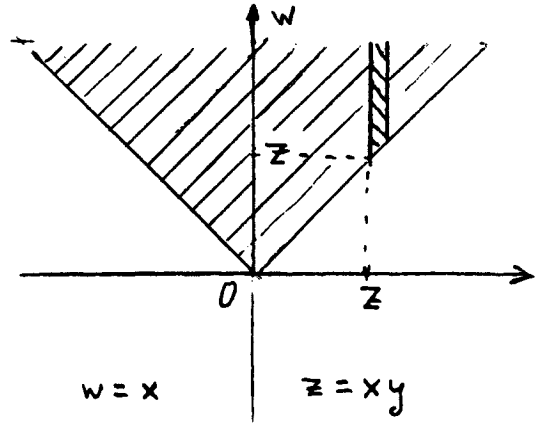
Hence, $\underset{\sim}{z}$ is $N(0;\sqrt{52})$

(b) If $\underset{\sim}{z} = \underset{\sim}{x}/\underset{\sim}{y}$, then from (6-63) with $\sigma_1 = \sigma_2 = 2$, $r = 0$

$$F_z(z) = \frac{1}{2} + \frac{1}{\pi}\arctan z \qquad f_z(z) = \frac{1}{\pi(1+z^2)}$$

6-18





$$f_{zw}(z,w) = \frac{1}{|x|} f_{xy}(x,y) \qquad x = w \qquad y = z/w$$

The function $f_{zw}(z,w)$ is different from zero in the shaded areas shown. Hence, with $w^2 - z^2 = s^2$

$$f_z(z) = \frac{1}{\pi\alpha^2} \int_{|z|}^{\infty} e^{-w^2/2\alpha^2} \frac{dw}{\sqrt{1 - z^2/w^2}}$$

$$= \frac{1}{\pi\alpha^2} \int_0^{\infty} e^{-(z^2+s^2)/2\alpha^2}\, ds = \frac{1}{\alpha\sqrt{2\pi}} e^{-z^2/2\alpha^2}$$

6-19 (a) $\underset{\sim}{z} = \underset{\sim}{x}/\underset{\sim}{y}$ $\qquad \underset{\sim}{w} = \underset{\sim}{y}$ $\qquad J = 1/y$

$$f_z(z) = \int_{-\infty}^{\infty} |w| f_x(zw) f_y(w)\,dw \qquad z > 0$$

$$= \frac{z}{\alpha^2\beta^2} \int_0^{\infty} w^3 e^{-cw^2} dw = \frac{z}{2\alpha^2\beta^2 c^2} \qquad c = \frac{z^2}{2\alpha^2} + \frac{1}{2\beta^2}$$

$$= \frac{2\alpha^2}{\beta^2} \frac{z}{(z^2 + \alpha^2/\beta^2)^2} \qquad \text{for } z > 0 \text{ and zero otherwise}$$

(b) $$F_z(z) = \int_0^z \frac{2\alpha^2 z\,dz}{\beta^2(z^2 + \alpha^2/\beta^2)^2} = \frac{\alpha^2}{\beta^2} \int_{\alpha^2/\beta^2}^{z^2 + \alpha^2/\beta^2} \frac{dt}{t^2}$$

$$= \frac{z^2}{z^2 + \alpha^2/\beta^2} = P\{\underset{\sim}{z} \le z\} = P\{\underset{\sim}{x} \le zy\}$$

6-20 1. The density of $2\underset{\sim}{x}$ equals $\frac{1}{2} f_x(\frac{x}{2})$. Hence, if $\underset{\sim}{z} = 2\underset{\sim}{x} + \underset{\sim}{y}$, then

$$f_z(z) = \int_0^z \frac{\alpha}{2} e^{-\alpha x/2} \beta e^{-\beta(z-x)} dx = \frac{\alpha\beta}{\alpha - 2\beta}(e^{-\beta z} - e^{-\alpha z/2})U(z)$$

2. The density of $\underset{\sim}{y}$ equals $f_y(-y)$. Hence, if $\underset{\sim}{z} = \underset{\sim}{x} - \underset{\sim}{y}$, then

$$f_z(z) = f_x(z) * f_y(-z)$$

$$= \alpha\beta \begin{cases} \int_z^\infty e^{-\alpha x} e^{-\beta(x-z)} dx = \frac{\alpha\beta}{\alpha+\beta} e^{-\alpha z} & z > 0 \\ \int_0^\infty e^{-\alpha x} e^{-\beta(x-z)} dx = \frac{\alpha\beta}{\alpha+\beta} e^{\beta z} & z < 0 \end{cases}$$

3. $\underset{\sim}{z} = \underset{\sim}{x}/\underset{\sim}{y}$ $\qquad \underset{\sim}{w} = \underset{\sim}{y}$ $\qquad J = 1/y$

$$f_z(z) = \alpha\beta \int_0^\infty w e^{-\alpha z w} e^{-\beta w} dw = \frac{\alpha\beta}{(\alpha z + \beta)^2} U(z)$$

4. $\underset{\sim}{z} = \max(\underset{\sim}{x}, \underset{\sim}{y})$ $\qquad F_z(z) = F_{xy}(z,z) = F_x(z)F_y(z)$

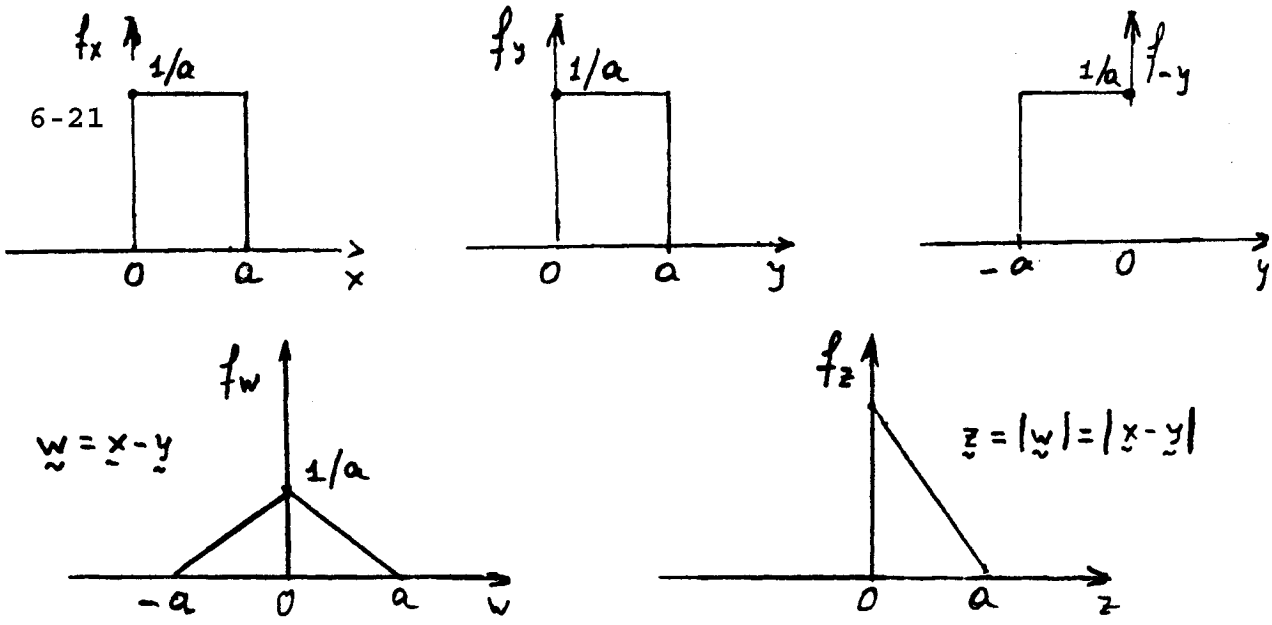
$$f_z(z) = f_x(z)F_y(z) + f_y(z)F_x(z)$$

$$= \left[\alpha e^{-\alpha z}(1 - e^{-\beta z}) + \beta e^{-\beta z}(1 - e^{-\alpha z})\right] U(z)$$

5. $\underset{\sim}{z} = \min(\underset{\sim}{x}, \underset{\sim}{y})$ $\qquad F_z(z) = F_x(z) + F_y(z) - F_x(z)F_y(z)$

$$f_z(z) = f_x(z)[1 - F_y(z)] + f_y(z)[1 - F_x(z)] = (\alpha + \beta)e^{-(\alpha+\beta)z}U(z)$$

6-21



6-22 (a)

$$\alpha y^2 + \beta(x-y)^2 = (\alpha+\beta)\left(y - \frac{\beta x}{\alpha+\beta}\right)^2 + \frac{\alpha\beta}{\alpha+\beta}x^2$$

$$e^{-\alpha x^2} * e^{-\beta x^2} = \int_{-\infty}^{\infty} e^{-\alpha y^2 - \beta(x-y)^2} dy$$

$$= e^{-\alpha\beta x^2/(\alpha+\beta)} \int_{-\infty}^{\infty} e^{-(\alpha+\beta)\left(y - \frac{\beta x}{\alpha+\beta}\right)^2} dy = \sqrt{\frac{\pi}{\alpha+\beta}}\, e^{\frac{-\alpha\beta x^2}{\alpha+\beta}}$$

(b)

$$\frac{\alpha/\pi}{x^2+\alpha^2} * \frac{\beta/\pi}{x^2+\beta^2} = \frac{\alpha\beta}{\pi^2}\int_{-\infty}^{\infty} \frac{dy}{(y^2+\alpha^2)[(x-y)^2+\beta^2]} = \frac{(\alpha+\beta)/-}{x^2+(\alpha+\beta)^2}$$

Characteristic functions lead to a simpler derivation of the above [see (6-192)]

6-23 We introduce the auxiliary variable w=y. The Jacobian of the transformation z=nx/my, w=y equals n/my. Since x=mzw/n, y=w and the RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are independent, (6-113) yields

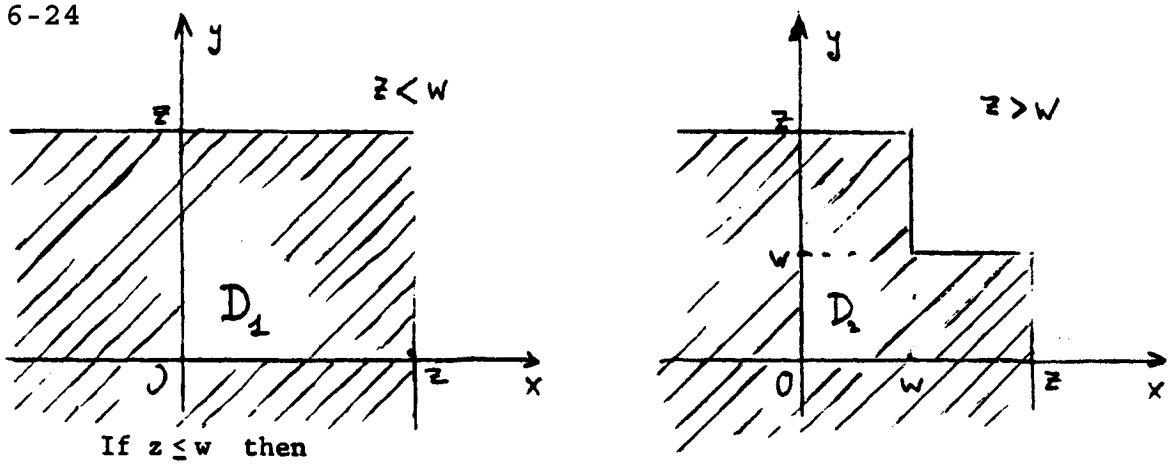
$$f_{zw}(z,w) = \frac{mw}{n} f_x\left(\frac{m}{n}zw\right) f_y(w) \sim w(zw)^{m/2-1}e^{-mzw/2}w^{n/2-1}e^{-w/2}$$

for z>0, w>0 and 0 othrwise. Integrating with respect to w, we obtain

$$f_z(z) \sim z^{m/2-1}\int_0^\infty w^{(m+n)/2-1}\exp\left\{-\frac{w}{2}\left(1+\frac{m}{n}z\right)\right\}dw$$

$$\sim \frac{z^{m/2-1}}{(1+mz/n)^{(m+n/2)}}\int_0^\infty q^{(m+n)/2}e^{-q}dq$$

6-24



If $z \le w$ then

$$P\{\underset{\sim}{z} \le z,\ \underset{\sim}{w} \le w\} = P\{\underset{\sim}{z} \le z\} = P\{(\underset{\sim}{x},\underset{\sim}{y}) \in D_1\} = F_{xy}(z,z)$$

If $z > w$ then

$$P\{\underset{\sim}{z} \le z,\ \underset{\sim}{w} \le w\} = P\{(\underset{\sim}{x},\underset{\sim}{y}) \in D_2\}$$

$$= F_{xy}(z,w) + F_{xy}(w,z) - F_{xy}(w,w)$$

6.25

$$X \sim \text{Exponential}(\lambda), \quad Y \sim \text{Exponential}(\lambda)$$

X and Y are independent so that

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) = \frac{1}{\lambda^2}\, e^{-(x+y)/\lambda}\, U(x)\, U(y)$$

$$Z = X + Y$$

$$\phi_Z(\omega) = \phi_X(\omega)\, \phi_Y(\omega) \frac{1}{(1 - j\omega\lambda)^2}$$

$$Z \sim \text{ Gamma } (2, \lambda)$$

This gives

$$f_Z(z) = \frac{z}{\lambda^2}\, e^{-z/\lambda} U(z)$$

$$P(Z > 2\lambda) = \int_{2\lambda}^{\infty} \frac{z}{\lambda^2}\, e^{-z/\lambda}\, dz = \int_2^{\infty} x e^{-x}\, dx = 3e^{-2} = 0.406$$

Let,

$$W = Y - X$$

Then

$$P(Y - X > \lambda) = P(W > \lambda) = \int_{\lambda}^{\infty} f_W(w) dw$$

Notice that $F_W(w)$ is given by (6-55).
For $w > 0$, this gives

$$\begin{aligned} f_W(w) &= \int_0^{\infty} \frac{1}{\lambda^2}\, e^{-(w+2y)/\lambda}\, dy = \frac{1}{\lambda^2}\, e^{-w/\lambda} \int_0^{\infty} e^{-2y/\lambda}\, dy \\ &= \frac{1}{2\lambda}\, e^{-w/\lambda}, \quad w > 0 \end{aligned}$$

Hence

$$P(Y - X > \lambda) = P(W > \lambda) = \int_{\lambda}^{\infty} \frac{1}{2\lambda}\, e^{-w/\lambda}\, dw = \frac{1}{2e}$$

6.26 (a)

$$\begin{aligned} R &= W - Z \\ &= \max(\mathrm{X}, \mathrm{Y}) \ - \ \min(\mathrm{X}, \mathrm{Y}) \\ &= \begin{cases} X - Y, & X \geq Y \\ Y - X, & X < Y \end{cases} \end{aligned}$$

$$\begin{aligned} F_R(r) &= P\{R \leq r\} \\ &= P\{R \leq r, X \geq Y\} + P\{R \leq r, X < Y\} \\ &= P\{X - Y \leq r, X \geq Y\} + P\{Y - X \leq r, X < Y\} \\ &= 1 - 2\frac{(1-r)^2}{2} = 1 - (1-r)^2, \quad 0 \leq r \leq 1 \end{aligned}$$

$$f_R(r) = \begin{cases} 2(1-r), & 0 \leq r \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$\begin{aligned} S &= W + Z \\ &= \max(\mathrm{X}, \mathrm{Y}) \ + \ \min(\mathrm{X}, \mathrm{Y}) = X + Y \end{aligned}$$

Case 1: $0 < s < 1$

$$F_S(s) = P\{S \leq s\} = P\{X + Y \leq s\} = \frac{s^2}{2}, \quad 0 < s < 1$$

Case 2: $1 \leq s \leq 2$

$$F_S(s) = P\{S \leq s\} = P\{X + Y \leq s\} = 1 - \frac{(2-s)^2}{2}, \quad 1 \leq s \leq 2$$

$$F_S(s) = \begin{cases} s, & 0 \leq s \leq 1 \\ (2-s), & 1 \leq s \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

6.27 (a) X, Y are independent, identically distributed exponential random variables.

$$Z = \frac{Y}{\max(X,Y)} = \begin{cases} \frac{Y}{X}, & X \geq Y \\ 1, & X < Y \end{cases} \quad \Rightarrow \; 0 < z \leq 1.$$

$0 < z < 1$

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P\left\{\tfrac{Y}{X} \leq z,\, X > Y\right\} \\ &= P\{Y \leq Xz,\, X > Y\} = \int_0^\infty \int_0^{xz} f_{XY}(x,y)\, dy\, dx \end{aligned}$$

$$f_Z(z) = \int_0^\infty x\, f_{XY}(x, xz)\, dx = \int_0^\infty \frac{x}{\lambda^2}\, e^{-(1+z)x/\lambda}\, dx = \frac{1}{(1+z)^2}, \quad 0 < z < 1.$$

Also

$$P(Z = 1) = P(X < Y) = \int_0^\infty \int_0^y \frac{1}{\lambda^2}\, e^{-(x+y)/\lambda}\, dx\, dy = \frac{1}{2}$$

(b)

$$W = \frac{X}{\min(X, 2Y)} = \begin{cases} \frac{X}{2Y}, & X \geq 2Y \\ 1, & X < 2Y \end{cases} \quad \Rightarrow \; 1 \leq w < \infty$$

$$F_W(w) = P(X \leq 2Yw,\, X > 2Y) = \int_0^\infty \int_{2y}^{2wy} f_{XY}(x,y)\, dx\, dy$$

This gives

$$\begin{aligned} f_W(w) &= \int_0^\infty 2y\, f_{XY}(2wy, y) dy = \int_0^\infty \frac{2y}{\lambda^2}\, e^{-(1+2w)y/\lambda} dy \\ &= \frac{2}{(1+2w)^2}, \quad w > 1 \end{aligned}$$

Also

$$P(W = 1) = P(X < 2Y) = \int_0^\infty \int_0^{2y} \frac{1}{\lambda^2}\, e^{-(x+y)/\lambda}\, dx\, dy = \frac{2}{3}$$

Note that the p.d.f. of Z as well as W has an impulse at $z = 1$ and $w = 1$ respectively.

6.28 X, Y are independent identically distributed exponential random variables.

$$Z = \frac{X}{X+Y}$$

$$\begin{aligned} F_Z(z) &= P\left(\frac{X}{X+Y} \le z\right) = P\left(\frac{X}{Y} \le \frac{z}{1-z}\right) \\ &= P\left\{X \le \frac{zY}{1-z}\right\} = \int_0^\infty \int_0^{(zy)/(1-z)} f_{XY}(x,y)\,dx\,dy \end{aligned}$$

$$\begin{aligned} f_Z(z) &= \int_0^\infty \frac{y}{(1-z)^2} f_{XY}(zy/(1-z), y)\,dy \\ &= \frac{1}{(1-z)^2} \int_0^\infty y \frac{1}{\lambda^2} e^{-(z/(1-z)+1)(y/\lambda)}\,dy \\ &= \frac{1}{(1-z)^2} \int_0^\infty \frac{y}{\lambda^2} e^{-[y/(1-z)\lambda]}\,dy \\ &= \int_0^\infty u\,e^{-u}\,du = 1, \quad 0 < z < 1 \\ &\Rightarrow \frac{X}{X+Y} \sim U(0,1) \end{aligned}$$

6.29 Let

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda} U(x), \quad f_Y(y) = \frac{1}{\lambda} e^{-y/\lambda} U(y).$$

$$Z = \min(X, Y)$$

$$W = \max(X, Y) - \min(X, Y)$$

$$Z = \begin{cases} Y, & X \ge Y \\ X, & X < Y \end{cases}$$

$$W = \begin{cases} X - Y, & X \ge Y \\ Y - X, & X < Y \end{cases}$$

$Z = \min(X, Y)$. See Example 6-18, Eq.(6-82) for solution. From there (replace λ by $1/\lambda$ in (6-82))

$$f_Z(z) = \frac{2}{\lambda} e^{-2z/\lambda} U(z).$$

$$\begin{aligned} F_W(w) &= P(X - Y \le w, X \ge Y) + P(Y - X \le w, X < Y) \\ &= \int_0^\infty \int_y^{y+w} f_{XY}(x,y)\,dx\,dy \\ &+ \int_0^\infty \int_x^{x+w} f_{XY}(x,y)\,dy\,dx, \quad w > 0 \end{aligned}$$

This gives

$$\begin{aligned} F_W(w) &= \int_0^\infty f_{XY}(y+w, y)\,dy + \int_0^\infty f_{XY}(x, x+w)\,dx \\ &= 2\int_0^\infty \frac{1}{\lambda^2} e^{(2y+w)/\lambda}\,dy \\ &= \frac{2}{\lambda^2} e^{-w/\lambda} \left.\frac{e^{-2y/\lambda}}{-2/\lambda}\right|_0^\infty = \frac{1}{\lambda} e^{-w/\lambda}, \quad w > 0 \end{aligned}$$

Also

$$\begin{aligned} F_{ZW}(z,w) &= P\{Z \le z, W \le w\} \\ &= P\{Y \le z, X - Y \le w, X \ge Y\} \\ &+ P\{X \le z, Y - X \le w, X < Y\} \\ &= \int_0^z \int_y^{y+w} f_{XY}(x,y)\,dx\,dy + \int_0^z \int_x^{x+w} f_{XY}(x,y)\,dy\,dx \end{aligned}$$

Repeated use of (6-39)-(6-40) gives

$$\begin{aligned} f_{ZW}(z,w) &= f_{XY}(z+w,z) + f_{XY}(z,z+w) \\ &= \frac{2}{\lambda^2}\, e^{-(2z+w)/\lambda} = \frac{2}{\lambda}\, e^{-2z/\lambda} \frac{1}{\lambda}\, e^{-w/\lambda} \\ &= f_Z(z)\, f_W(w) \end{aligned}$$

Thus Z and W are independent exponential random variables.

6.30 (a) Let

$$U = X + Y, \qquad 0 < u < 2\beta.$$

The probability density function of U can be computed as in (6-48)-(6-50). Using Fig. 6-11, for $0 < u \le \beta$, we have

$$F_U(u) = \int_0^u \int_0^{u-x} f_{XY}(x,y)\,dy\,dx$$

which gives

$$\begin{aligned} f_U(u) &= \int_0^u f_{XY}(x, u-x)dx = \alpha^2 \beta^{-2\alpha} \int_0^u x^{\alpha-1}\,(u-x)^{\alpha-1}\,dx \\ &= \alpha^2 \beta^{-2\alpha} u^{2\alpha-1} \int_0^1 y^{\alpha-1}\,(1-y)^{\alpha-1}\,dy \\ &= B(\alpha,\alpha)\,\alpha^2 \beta^{-2\alpha} u^{2\alpha-1} \quad 0 < u \le \beta \end{aligned}$$

where we have substituted $y = ux$ and made use of the beta function defied in (4-49)-(4-51). Similarly for $\beta < u \le 2\beta$, we get (see (6-49))

$$F_U(u) = 1 - \int_{u-\beta}^{\beta} \int_{u-x}^{\beta} f_{XY}(x,y)\,dydx$$

and hence

$$\begin{aligned} f_U(u) &= \int_{u-\beta}^{\beta} f_{XY}(x, u-x)dx = \alpha^2 \beta^{-2\alpha} \int_{u-\beta}^{\beta} x^{\alpha-1}(u-x)^{\alpha-1}\,dx \\ &= \alpha^2 \beta^{-2\alpha} u^{2\alpha-1} \int_{1-\beta/u}^{\beta/u} y^{\alpha-1}(1-y)^{\alpha-1}\,dy, \quad \beta < u \le 2\beta \end{aligned}$$

(b)

$$Z = \min(X,Y), \qquad W = \max(X,Y)$$

We can proceed as in Example 6-21 to complete this problem. From (6-92) and (6-93), we get

$$F_{ZW}(z,w) = \begin{cases} F_{XY}(z,w) + F_{XY}(w,z) - F_{XY}(z,z), & w \ge z \\ F_{XY}(w,w), & w < z \end{cases}$$

which gives

$$f_{ZW}(z,w) = f_X(z)f_Y(w) + f_X(w)f_Y(z), \quad 0 < z \le w < \beta$$

$$f_{ZW}(z,w) = \begin{cases} 2\alpha^2\beta^{-2\alpha}z^{\alpha-1}w^{\alpha-1}, & 0 < z \le w < \beta \\ 0, & \text{otherwise} \end{cases}$$

check:

$$\begin{aligned} \int_0^\beta \int_0^w f_{ZW}(z,w)dzdw &= 2\alpha^2\beta^{-2\alpha} \int_0^\beta w^{\alpha-1} \left(\frac{z^\alpha}{\alpha}\bigg|_0^w \right) dw \\ &= 2\alpha\beta^{-2\alpha} \int_0^\beta w^{2\alpha-1}dw = 1 \end{aligned}$$

Note: Z and W are not independent random variables, since

$$f_Z(z) = 2\alpha\beta^{-2\alpha}\, z^{\alpha-1}\,(\beta^\alpha - z^\alpha), \quad 0 < z < \beta$$

and

$$f_W(w) = 2\alpha\beta^{-2\alpha}\, w^{2\alpha-1}, \quad 0 < w < \beta$$

(c) Let

$$V = \frac{Z}{W} = \frac{\min(X,Y)}{\max(X,Y)} = \begin{cases} \frac{Y}{X}, & X \ge Y \\ \frac{X}{Y}, & X < Y \end{cases}$$

and

$$W = \max(X,Y) = \begin{cases} X, & X \ge Y \\ Y, & X < Y \end{cases}$$

For $0 < v < 1,\ 0 < w < \beta$

$$\begin{aligned} F_{VW}(v,w) &= P(V \le v,\, W \le w) \\ &= P\left\{V \le v,\, W \le w,\ (X \ge Y) \bigcup (X < Y)\right\} \\ &= P\left\{Y \le Xv,\, X \le w,\, X \ge Y\right\} \\ &+ P\left\{X < Yv,\, Y \le w,\, X < Y\right\} \\ &= \int_0^w \int_0^{xv} f_{XY}(x,y)\,dy\,dx + \int_0^w \int_0^{yv} f_{XY}(x,y)\,dx\,dy \end{aligned}$$

Hence

$$\begin{aligned} f_{VW}(v,w) &= \frac{\partial^2 F_{VW}(v,w)}{\partial v\,\partial w} \\ &= \frac{\partial}{\partial v}\left\{\int_0^{vw} f_{XY}(w,y)\,dy + \int_0^{vw} f_{XY}(x,w)\,dx\right\} \\ &= w\left\{f_{XY}(w,vw) + f_{XY}(vw,w)\right\} \\ &= 2\alpha^2\,\beta^{-2\alpha}\,w^{2\alpha-1}\,v^{\alpha-1}, \qquad 0<v<1,\ 0<w<\beta \end{aligned}$$

Hence

$$f_V(v) = \int_0^{\beta} f_{VW}(v,w)dw = \alpha\,v^{\alpha-1}, \quad 0<v<1$$

$$f_W(w) = \int_0^{1} f_{VW}(v,w)dv = 2\alpha\,\beta^{-2\alpha}\,w^{2\alpha-1}, \quad 0<w<\beta$$

and

$$f_{VW}(v,w) = f_V(v)\,f_W(w).$$

Thus V and W are independent random variables.

6.31 (a) Solved in Examples 6-27 and 6-12.
(b) Solved in Example 6-27.
(c)

$$Z = X+Y, \qquad W = \frac{X}{X+Y}$$

$$x_1 = zw, \qquad y_1 = z - x_1 = z(1-w)$$

$$J = \begin{vmatrix} 1 & 1 \\ \dfrac{y}{(x+y)^2} & -\dfrac{x}{(x+y)^2} \end{vmatrix} = \frac{1}{x+y} = \frac{1}{z}$$

$$\begin{aligned} f_{ZW}(z,w) &= \frac{z}{\alpha^{m+n}\Gamma(m)\Gamma(n)}\,(zw)^{m-1}\left\{z(1-w)\right\}^{n-1} \\ &= \left(\frac{z^{m+n-1}}{\alpha^{m+n}\Gamma(\alpha+\beta)}\,e^{-z/\alpha}\right)\left(\frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}\,w^{m-1}\,(1-w)^{n-1}\right) \\ &= f_Z(z)\,f_W(w) \end{aligned}$$

Thus Z and W are independent random variables.

6.32 (a)

$$Z = \frac{X}{|Y|}, \qquad W = \frac{|X|}{|Y|} = |Z|$$

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(X \le |Y|z) = \int_{-\infty}^{\infty} \int_0^{|y|z} f_{XY}(x,y)\,dx\,dy \\ &= 2\int_0^{\infty} |y| f_{XY}(|y|z, y)dy = \frac{2}{2\pi\sigma^2}\int_0^{\infty} y e^{-(z^2+1)y^2/2\sigma^2}\,dy \\ &= \frac{1/\pi}{1+z^2}, \qquad -\infty < z < \infty \end{aligned}$$

Thus Z is a Cauchy random variable. Interestingly, the random variable X/Y is also a Cauchy random variable (see Example 6-11).

$$W = |Z|$$

so that

$$\begin{aligned} F_W(w) &= P(W \le w) = P(|Z| \le w) \\ &= P(-w < Z < w) = F_Z(w) - F_Z(-w) \end{aligned}$$

and hence

$$f_W(w) = f_Z(w) + f_Z(-w) = \frac{2/\pi}{1+w^2}, \quad w > 0.$$

(b)

$$U = X + Y \sim N(0,2)$$
$$V = X^2 + Y^2 \sim \text{Exponential } (2)$$

(see Example 6-14). Here U, V are *not* independent, since

$$J(x,y) = \begin{vmatrix} 1 & 1 \\ 2x & 2y \end{vmatrix} = -2(x-y) = 2\sqrt{2v-u^2}$$

and

$$\begin{aligned} f_{UV}(u,v) &= \frac{1}{2\sqrt{2v-u^2}}\,\frac{1}{2\pi\sigma^2}\,e^{-v/2\sigma^2} \\ &\ne f_U(u)\,f_V(v), \qquad -\infty < u < \infty, \;\; v > 0. \end{aligned}$$

6.33

$$Z = X + Y, \qquad W = X - Y$$

are jointly normal random variables. Hence if they are uncorrelated, then they are also independent.

$$\begin{aligned} Cov(Z,W) &= E[(Z-\mu_Z)(W-\mu_W)] \\ &= E\left[\{(X-\mu_X)+(Y-\mu_Y)\}\{(X-\mu_X)-(Y-\mu_Y)\}\right] \\ &= \text{Var}(X) - \text{Var}(Y) = \sigma_X^2 - \sigma_Y^2. \end{aligned}$$

The random variables Z and W are uncorrelated implies that $Cov(Z,W) = 0$. Hence $\sigma_X^2 = \sigma_Y^2$ is the necessary and sufficient condition for the independence of $X+Y$ and $X-Y$.

6.34 (a)-(b) Let

$$R = \sqrt{X^2 + Y^2}, \qquad \theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

Form Example 6-22, R and θ are independent random variables with joint p.d.f. as in (6-128). (see (6-131)). In term of R and θ, we have $X = R\cos\theta, Y = R\sin\theta$ and hence we obtain

$$U = \frac{X^2 - Y^2}{\sqrt{X^2 + Y^2}} = R\cos 2\theta$$

$$V = \frac{2XY}{\sqrt{X^2 + Y^2}} = R\sin 2\theta$$

This gives

$$J = \begin{vmatrix} \cos 2\theta & -2r\sin 2\theta \\ \sin 2\theta & 2r\cos 2\theta \end{vmatrix} = 2r = 2\sqrt{u^2 + v^2}$$

$$r = \sqrt{u^2 + v^2}, \quad \theta_1 = \frac{1}{2}\tan^{-1}\left(\frac{v}{u}\right), \quad 2\theta_2 = \pi + 2\theta_1.$$

There are two sets of solutions (r, θ_1) and (r, θ_2). Substituting into (6-128) we get

$$\begin{aligned} f_{UV}(u, v) &= \frac{1}{|J|}\left\{f_{r,\theta}(r, \theta_1) + f_{r,\theta}(r, \theta_2)\right\} = \frac{2}{|J|} f_{r,\theta}(r, \theta_1) \\ &= \frac{2}{2\sqrt{u^2 + v^2}} \frac{\sqrt{u^2 + v^2}}{2\pi\sigma^2} e^{-(u^2+v^2)/2\sigma^2} \\ &= \frac{1}{2\pi\sigma^2} e^{-(u^2+v^2)/2\sigma^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-u^2/2\sigma^2} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-v^2/2\sigma^2} \\ &= f_U(u) f_V(v) \end{aligned}$$

Thus U and V are independent normal random variables. Hence it follows that $U = \dfrac{X^2 - Y^2}{\sqrt{X^2 + Y^2}}$ and $V/2 = \dfrac{XY}{\sqrt{X^2 + Y^2}}$ are independent random variables.

(c)

$$\begin{aligned} Z &= \frac{(X - Y)^2 - 2Y^2}{\sqrt{X^2 + Y^2}} = \frac{(X^2 - Y^2) - 2XY}{\sqrt{X^2 + Y^2}} \\ &= \frac{X^2 - Y^2}{\sqrt{X^2 + Y^2}} - \frac{2XY}{\sqrt{X^2 + Y^2}} \\ &= U - V \sim N(0, 2\sigma^2). \end{aligned}$$

6.35 (a) $Z \sim F(m,n)$ is given by (6-157) Let

$$Y = \frac{1}{Z}$$

Then

$$\begin{aligned}
F_Y(y) &= \frac{1}{|dy/dz|} f_Z(1/y) \\
&= \frac{1}{y^2} \frac{(m/n)^{m/2}}{\beta(m/2,n/2)} \frac{1}{y^{m/2-1}} \frac{1}{(1+m/ny)^{m+n/2}} \\
&= \frac{(n/m)^{n/2}}{\beta(n/2,m/2)} y^{n/2-1} \left(1+\frac{n}{my}\right)^{-(m+n)/2} \\
&\sim F(n,m).
\end{aligned}$$

(b)

$$W = \frac{Zm}{Zm+n}$$

$$\begin{aligned}
F_W(w) &= P(W \le w) = P\left(\frac{Zm}{Zm+n} \le w\right) \\
&= P\left(Z \le \frac{nw}{m(1-w)}\right) = F_Z\left(\frac{nw}{m(1-w)}\right)
\end{aligned}$$

which gives

$$\begin{aligned}
f_W(w) &= \frac{n}{m(1-w)^2} f_Z\left(\frac{nw}{m(1-w)}\right) \\
&= \frac{n}{m(1-w)^2} \frac{(m/n)^{m/2}}{\beta(m/2,n/2)} \left(\frac{nw}{m(1-w)}\right)^{m/2-1} \left(1+\frac{w}{(1-w)}\right)^{-(m+n)/2} \\
&= \frac{1}{\beta(m/2,n/2)} w^{m/2-1} (1-w)^{n/2-1}, \quad 0 < w < 1.
\end{aligned}$$

Thus W has Beta distribution.

6.36

$$\begin{aligned}
Z = X+Y > 0, \qquad & W = X-Y > 0 \\
x_1 = \frac{z+w}{2}, \qquad & y_1 = \frac{z-w}{2}
\end{aligned}$$

is the only solution. Moreover

$$J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

so that

$$f_{ZW}(z,w) = \frac{1}{|J|} f_{XY}(x_1,y_1) = \frac{1}{2} e^{-(z+w)/2}, \quad 0 < w < z < \infty$$

$$\begin{aligned}
F_Z(z) &= \int_0^z f_{ZW}(z,w)\,dw = \frac{1}{2} e^{-z/2} \left.\frac{e^{-w/2}}{(-1/2)}\right|_0^z \\
&= \frac{1}{2} e^{-z/2} \left.\frac{e^{-w/2}}{(-1/2)}\right|_0^z = e^{-z/2}(1-e^{-z/2}), \quad z > 0
\end{aligned}$$

6.37

$$Z = X + Y > 0, \qquad W = \frac{Y}{X} > 1$$

$$y = xw, \quad x(1+w) = z, \quad x_1 = \frac{z}{1+w}, \quad y_1 = \frac{zw}{1+w}$$

is the only solution. Also

$$J = \begin{vmatrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{x+y}{x^2} = \frac{(1+w)^2}{z}$$

This gives

$$\begin{aligned} f_{ZW}(z,w) &= \frac{1}{|J|} f_{XY}(x_1, y_1) \\ &= \frac{z}{(1+w)^2}\, 2\, e^{-z}, \quad z > 0,\ w > 1 \\ &= z\, e^{-z} \frac{2}{(1+w)^2} = f_Z(z)\, f_W(w) \end{aligned}$$

since

$$\begin{aligned} f_Z(z) &= \int_1^\infty f_{ZW}(z,w)\, dw \\ &= 2ze^{-z} \int_1^\infty \frac{1}{(1+w)^2}\, dw = z\, e^{-z}, \quad z > 0 \end{aligned}$$

and

$$\begin{aligned} f_w(w) &= \int_0^\infty f_{ZW}(z,w)\, dz \\ &= \frac{2}{(1+w)^2} \int_0^\infty ze^{-z}\, dz = \frac{2}{(1+w)^2}, \quad w > 1. \end{aligned}$$

Thus Z and W are independent random variables.

6-38 $\quad \underset{\sim}{z} = \underset{\sim}{x}\,\underset{\sim}{y} \qquad \underset{\sim}{y} = \cos(\omega t + \underset{\sim}{\theta})$

$$\underset{\sim}{w} = \underset{\sim}{y} \qquad J = |y| \qquad f_y(y) = \begin{cases} \dfrac{1}{\pi\sqrt{1-y^2}} & |y| < 1 \\ 0 & |y| > 1 \end{cases}$$

The RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are independent. Hence,

$$f_{zw}(z,w) = \frac{1}{|w|} f_x\left(\frac{z}{w}\right) f_y(w)$$

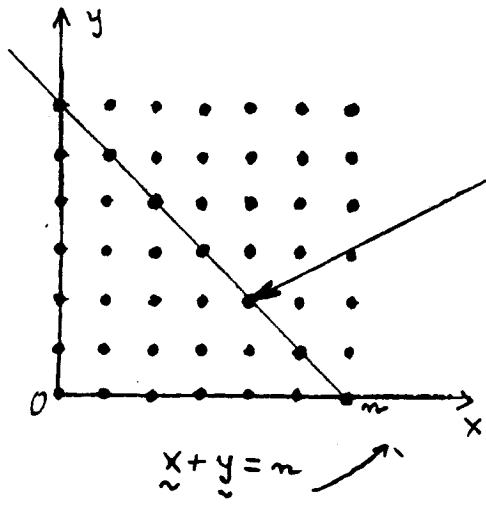
$$f_z(z) = \frac{1}{\pi}\int_{-1}^{1} \frac{f_x(z/w)}{|w|\sqrt{1-w^2}}\,dw = \frac{1}{\pi}\int_{|x|>z} \frac{f_x(x)}{\sqrt{x^2-z^2}}\,dx$$

6-39 $\quad \underset{\sim}{z} = \underset{\sim}{x} + \underset{\sim}{s} \qquad \underset{\sim}{s} = a\cos \underset{\sim}{y}$

$$f_z(z) = f_x(z) * f_s(z) \qquad f_s(s) = \begin{cases} \dfrac{1}{\pi\sqrt{a^2-s^2}} & |s| < a \\ 0 & |s| > a \end{cases}$$

$$f_z(z) = \frac{1}{\pi\sigma\sqrt{2\pi}}\int_{-a}^{a} \frac{e^{-(z-s)^2/2\sigma^2}}{\sqrt{a^2-s^2}}\,ds = \frac{1}{\pi\sigma\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-(z-a\cos y)^2/2\sigma^2}\,dy$$

6-40



Point masses

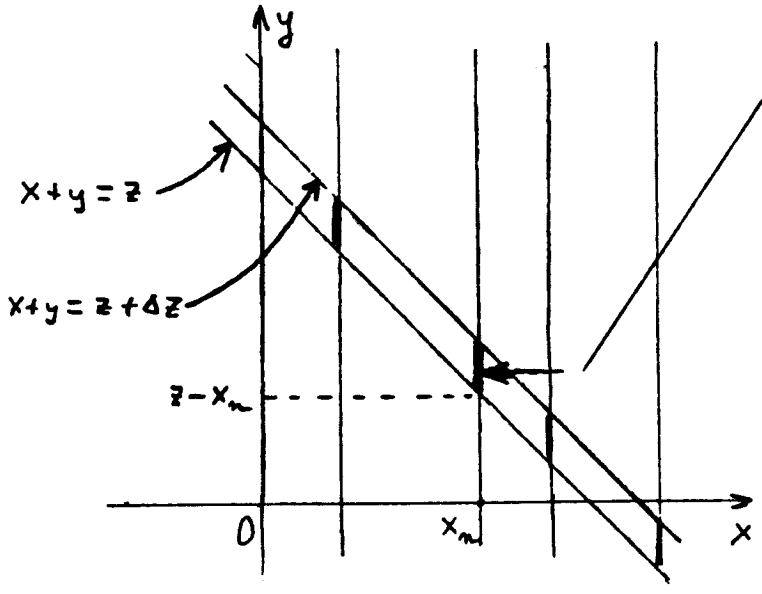
$$P\{\underset{\sim}{x} = k,\ \underset{\sim}{y} = n-k\} = a_k b_{n-k}$$

$$\{\underset{\sim}{z} = n\} = \sum_{k=0}^{n} \{\underset{\sim}{x} = k,\ \underset{\sim}{y} = n-k\}$$

$$P\{\underset{\sim}{z} = n\} = \sum_{k=0}^{n} P\{\underset{\sim}{x} = k,\ \underset{\sim}{y} = n-k\}$$

6-41 (a)

Line masses



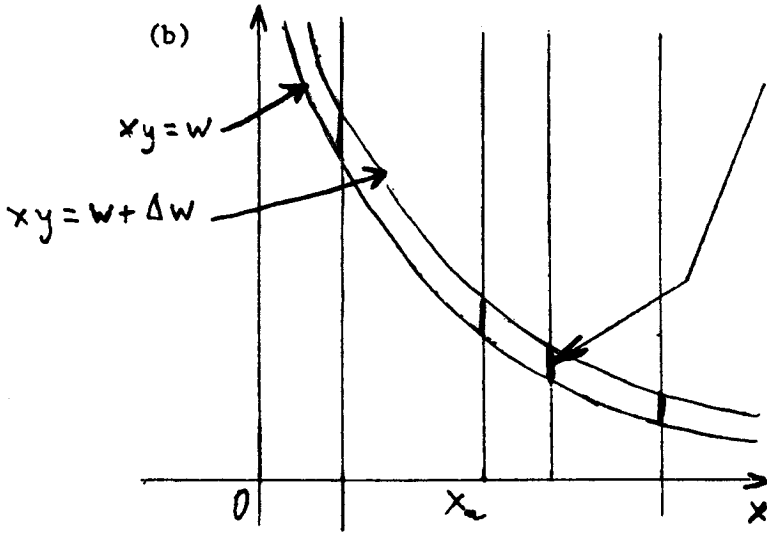
$$P\{\underset{\sim}{x} = x_n,\ z - x_n < \underset{\sim}{y} \le z - x_n + \Delta z\}$$

$$= p_n f_y(z - x_n)\Delta z$$

$$\{z < \underset{\sim}{z} \le z + \Delta z\} = \sum_n \{\underset{\sim}{x} = x_n,\ z - x_n < \underset{\sim}{y} \le z - x_n + \Delta z\}$$

$$f_z(z)\Delta z = \sum_n p_n f_y(z - x_n)\Delta z$$

(b)



$$P\{\underset{\sim}{x} = x_n,\ \frac{w}{x_n} < \underset{\sim}{y} \le \frac{w + \Delta w}{x_n}\}$$

$$= p_n f_y\left(\frac{w}{x_n}\right)\frac{\Delta w}{x_n}$$

$$\{w < \underset{\sim}{w} \le w + \Delta w\} = \sum_n \{\underset{\sim}{x} = x_n,\ \frac{w}{x_n} < \underset{\sim}{y} \le \frac{w + \Delta w}{x_n}\}$$

$$f_w(w)\Delta w = \sum_n p_n f_y\left(\frac{w}{x_n}\right)\Delta w$$

6.42 X, Y are independent geometric random variables. Thus

$$\begin{aligned} P\{X=k, Y=m\} &= P\{X=k\}\, P\{Y=m\} \\ &= (pq^k)\,(pq^m) = p^2 q^{k+m}, \qquad k, m = 0, 1, 2, \cdots \end{aligned}$$

(a) Let

$$Z = X + Y$$

$$\begin{aligned} P\{Z=n\} &= P\{X+Y=n\} = \textstyle\sum_k P\{X=k, Y=n-k\} \\ &= \textstyle\sum_{k=0}^{n} P\{X=k, Y=n-k\} \\ &= \textstyle\sum_{k=0}^{n} P\{X=k\}\, P\{Y=n-k\} \\ &= \textstyle\sum_{k=0}^{n} pq^k\, pq^{n-k} = \sum_{k=0}^{n} p^2\, q^n \\ &= (n+1)\, p^2\, q^n, \quad n = 0, 1, 2, \cdots \end{aligned}$$

(b) Let

$$W = X - Y$$

Case 1: $W \geq 0 \Rightarrow X \geq Y$. Thus for $m \geq 0$

$$\begin{aligned} P\{W=m\} &= P\{X-Y=m\} = \textstyle\sum_{k=0}^{\infty} P\{X=m+k, Y=k\} \\ &= \textstyle\sum_{k=0}^{\infty} P\{X=m+k, Y=k\} \\ &= \textstyle\sum_{k=0}^{\infty} P\{X=m+k\}\, P\{Y=k\} \\ &= \textstyle\sum_{k=0}^{\infty} (pq^{m+k})\,(pq^k) = p^2 q^m \sum_{k=0}^{\infty} q^{2k} \\ &= p^2 q^m\,(1 + q^2 + q^4 + \cdots) = \frac{p^2 q^m}{(1-q^2)} \\ &= \frac{pq^m}{1+q}, \qquad m = 0, 1, 2, \cdots \qquad (1) \end{aligned}$$

Case 2: $W < 0 \Rightarrow X < Y$. Thus for $m < 0$

$$\begin{aligned} P\{W=m\} &= P\{X-Y=m\} = \textstyle\sum_k P\{X=k, Y=k-m\} \\ &= \textstyle\sum_{k=0}^{\infty} P\{X=k, Y=k-m\} \\ &= \textstyle\sum_{k=0}^{\infty} P\{X=k\}\, P\{Y=k-m\} \\ &= \textstyle\sum_{k=0}^{\infty} (pq^k)\,(pq^{k-m}) = p^2 q^{-m} \sum_{k=0}^{\infty} q^{2k} \\ &= \frac{p^2 q^{-m}}{(1-q^2)} = \frac{pq^{-m}}{1+q}, \quad m = -1, -2, \cdots \qquad (2) \end{aligned}$$

Thus combining (1) and (2) we can write

$$P\{W=m\} = \frac{pq^{|m|}}{1+q}, \qquad m = 0, \pm 1, \pm 2, \cdots$$

6.43 We have X and Y are independent and $P(X=k)=P(Y=k)=p_k$. Also

$$\begin{aligned} P(X=k|X+Y=k) &= \frac{P(X=k,Y=0)}{P(X+Y=k)} \\ &= \frac{p_k p_0}{\sum_{i=0}^{k} p_i p_{k-i}} = \frac{1}{k+1}. \end{aligned} \tag{1}$$

Also

$$\begin{aligned} &P(X=k-1|X+Y=k) \\ &= \frac{P(X=k-1,Y=1)}{P(X+Y=k)} \frac{p_{k-1}p_1}{\sum_{i=0}^{k} p_i p_{k-i}} = \frac{1}{k+1}. \end{aligned} \tag{2}$$

From (1) and (2),

$$\frac{p_k}{p_{k-1}} = \frac{p_1}{p_0} \Rightarrow p_k = \lambda p_{k-1} = \lambda^k p_0$$

where $\lambda \triangleq p_1/p_0$. Since $\sum_{k=0}^{\infty} p_k = 1$, we must have $\lambda < 1$, and this gives

$$\sum_{k=0}^{\infty} p_k = \frac{p_0}{1-\lambda} = 1 \rightarrow p_0 = 1-\lambda.$$

Thus

$$p_k = p_0\,\lambda^k = (1-\lambda)\lambda^k, \quad k=0,1,2,\cdots,\ 0<\lambda<1$$

represents a geometric distribution. Thus X and Y are geometric random variables.

6.44 The moment generating functions of X and Y are given by (see (5-117))

$$\Gamma_X(z) = (pz+q)^n, \qquad \Gamma_Y(z) = (pz+q)^n$$

Also

$$\Gamma_{X+Y}(z) = E[z^{X+Y}] = \Gamma_X(z)\Gamma_Y(z) = (pz+q)^{2n} \sim \text{Binomial}\,(2n,p)$$

6.45 (a) Let

$$Z = \min(X, Y), \qquad W = X - Y$$

$$\begin{aligned}
&P\{Z = k, W = m\} \\
&= P\{\min(X, Y) = k, X - Y = m\} \\
&= P\{(\min(X, Y) = k, X - Y = m) \bigcap (X \geq Y \bigcup X < Y)\} \\
&= P\{Y = k, X - Y = m, X \geq Y\} + P\{X = k, X - Y = m, X < Y\} \\
&= P\{X = m + k, Y = k, X \geq Y\} + P\{X = k, Y = k - m, X < Y\}
\end{aligned}$$

Note that $k \geq 0$, and m takes both positive, zero and negative values. Hence

$$\begin{aligned}
P\{Z = k, W = m\} &= \begin{cases} P\{X = k + m, Y = k, X \geq Y\}, & k \geq 0,\ m \geq 0 \\ P\{X = k, Y = k - m, X < Y\}, & k \geq 0,\ m < 0 \end{cases} \\
&= \begin{cases} pq^{k+m}\, pq^{k}, & k \geq 0,\ m \geq 0 \\ pq^{k}\, pq^{k-m}, & k \geq 0,\ m < 0 \end{cases}
\end{aligned}$$

$$P\{Z = k, W = m\} = p^2 q^{2k+|m|}, \qquad k = 0, 1, 2, \cdots, \quad m = 0, \pm 1, \pm 2, \cdots$$

Also

$$\begin{aligned}
P\{Z = k\} &= \sum_{m=-\infty}^{\infty} P\{Z = k, W = m\} \\
&= p^2 q^{2k} \sum_{m=-\infty}^{\infty} q^{|m|} = p^2 q^{2k} \left(1 + 2\sum_{m=1}^{\infty} q^m\right) \\
&= p^2 q^{2k} \left(1 + \frac{2q}{p}\right) = p(1 + q) q^{2k}, \qquad k = 0, 1, 2, \cdots
\end{aligned}$$

and

$$\begin{aligned}
P\{W = m\} &= \sum_{k=0}^{\infty} P\{Z = k, W = m\} \\
&= p^2 q^{|m|} \sum_{k=0}^{\infty} q^{2k} \\
&= \frac{p}{1 + q} q^{|m|}, \qquad m = 0, \pm 1, \pm 2, \cdots
\end{aligned}$$

Note that

$$P\{Z = k, W = m\} = P\{Z = k\}\, P\{W = m\}$$

and hence Z and W are independent random variables.
(b) Let

$$Z = \min(X, Y), \quad W = \max(X, Y) - \min(X, Y)$$

Proceeding as in (a), we obtain

$$\begin{aligned}
&P\{Z = k, W = m\} \\
&= P(Y = k, X - Y = m, X \geq Y) + P(X = k, Y - X = m, X < Y) \\
&= P(X = k + m, Y = k, X \geq Y) + P(X = k, Y = k + m, X < Y) \\
&= \begin{cases} pq^{k+m}\, pq^k + pq^k\, pq^{k+m}, & k = 0, 1, 2, \cdots,\ m = 1, 2, \cdots \\ pq^{k+m}\, pq^k, & k = 0, 1, 2, \cdots,\ m = 0 \end{cases} \\
&= \begin{cases} 2p^2 q^{2k+m}, & k = 0, 1, 2, \cdots,\ m = 1, 2, \cdots \\ p^2 q^{2k}, & k = 0, 1, 2, \cdots,\ m = 0 \end{cases}
\end{aligned}$$

This gives

$$\begin{aligned}
P\{Z = k\} &= \textstyle\sum_{m=0}^{\infty} P\{Z = k, W = m\} \\
&= p^2 q^{2k} \left(1 + 2\textstyle\sum_{m=1}^{\infty} q^m\right) = p^2 q^{2k} \left(1 + \dfrac{2q}{p}\right) \\
&= p(1 + q) q^{2k}, \quad k = 0, 1, 2, \cdots
\end{aligned}$$

Also

$$\begin{aligned}
P\{W = m\} &= \textstyle\sum_{k=0}^{\infty} P\{Z = k, W = m\} \\
&= \begin{cases} \dfrac{p}{1+q}, & m = 0 \\ \dfrac{2p}{1+q}\, q^m, & m = 1, 2, \cdots \end{cases}
\end{aligned}$$

Notice that

$$P\{Z = k, W = m\} = P\{Z = k\}\, P\{W = m\}$$

and hence Z and W are also indepedndent random variables in this case also.

6.46 The moment generating function of X and Y are given by (see (5-119))

$$\Gamma_X(z) = e^{\lambda_1(z-1)}, \qquad \Gamma_Y(z) = e^{\lambda_2(z-1)}$$

Also

$$\Gamma_{X+Y}(z) = \Gamma_X(z)\Gamma_Y(z) = e^{(\lambda_1+\lambda_2)(z-1)}$$

so that

$$Z \sim P(\lambda_1 + \lambda_2)$$

Thus

$$P(X+Y=k) = e^{-(\lambda_1+\lambda_2)}\,\frac{(\lambda_1+\lambda_2)^k}{k!}$$

and

$$\begin{aligned}
&P(X=k|X+Y=n)\\
&\quad= \frac{P(X=k, X+Y=n)}{P(X+Y=n)} = \frac{P(X=k)P(Y=n-k)}{P(X+Y=n)}\\
&\quad= \frac{e^{-\lambda_1}(\lambda_1^k/k!)\,e^{-\lambda_2}(\lambda_2^{n-k}/(n-k)!)}{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n/n!}\\
&\quad= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}, \quad k = 0, 1, 2, \cdots n\\
&\quad\sim \text{Binomial}(n,p), \text{ where } p = \frac{\lambda_1}{\lambda_1+\lambda_2}.
\end{aligned}$$

See also (6-222). From there the converse is also true (proceed as in Example 6-43).

6-47

$$C = \begin{bmatrix} \sigma_1^2 & r\sigma_1\sigma_2 \\ r\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \qquad \Delta = \sigma_1^2\sigma_2^2(1-r^2)$$

$$C^{-1} = \begin{bmatrix} \dfrac{1}{(1-r^2)\sigma_1^2} & \dfrac{r}{(1-r^2)\sigma_1\sigma_2} \\ \dfrac{r}{(1-r^2)\sigma_1\sigma_2} & \dfrac{1}{(1-r^2)\sigma^2} \end{bmatrix}$$

$$XC^{-1}X^t = \frac{1}{(1-r^2)}\left(\frac{x_1^2}{\sigma_1^2} - 2r\,\frac{x_1x_2}{\sigma_1\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right)$$

6-48

$$\{\underset{\sim}{x}\,\underset{\sim}{y} < 0\} = \{\underset{\sim}{x} < 0,\ \underset{\sim}{y} > 0\} + \{\underset{\sim}{x} > 0,\ \underset{\sim}{y} < 0\}$$

$$P\{\underset{\sim}{x}\,\underset{\sim}{y} < 0\} = F_x(0)[1 - F_y(0)] + [1 - F_x(0)]F_y(0)$$

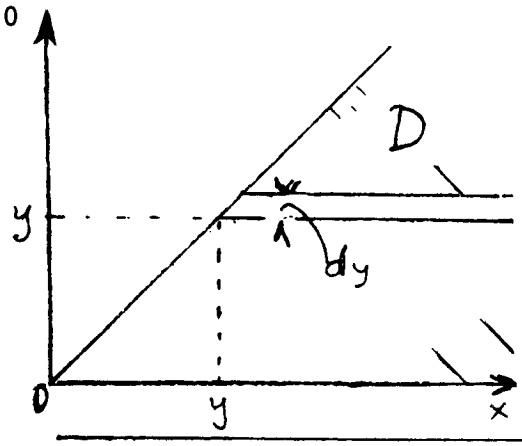
$$F_x(0) = 1 - G\left(\frac{\eta_x}{\sigma_x}\right) \qquad F_y(0) = 1 - G\left(\frac{\eta_y}{\sigma_y}\right)$$

6-49 If $\underset{\sim}{w} = \underset{\sim}{x} - \underset{\sim}{y}$, then $E\{\underset{\sim}{w}\} = 0$ $\quad \sigma_w^2 = \sigma_x^2 + \sigma_y^2 = 2\sigma^2$

Thus, $\underset{\sim}{w} = 1, N(0; \sigma\sqrt{2})$ and [see (5-74)]

$$E\{\underset{\sim}{z}\} = E\{|\underset{\sim}{w}|\} = \sqrt{2}\,\sigma\sqrt{\frac{2}{\pi}} \qquad E\{\underset{\sim}{z}^2\} = E\{\underset{\sim}{w}^2\} = 2\sigma^2$$

6-50



$$E\{\underset{\sim}{z}\} = \iint_D (x-y)f(x,y)\,dxdy$$

$$= \int_0^\infty \int_y^\infty (x-y)e^{-x}\,e^{-y}\,dxdy = \frac{1}{2}$$

6-51 Since $|E\{\underset{\sim}{x}\,\underset{\sim}{y}\}| \le E\{|\underset{\sim}{x}||\underset{\sim}{y}|\}$, we can assume that the RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are real

(a) $D \le E\{[z\,\underset{\sim}{x} - \underset{\sim}{y}]^2\} = z^2\,E\{\underset{\sim}{x}^2\} - 2z\,E\{\underset{\sim}{x}\,\underset{\sim}{y}\} + E\{\underset{\sim}{y}^2\}$

The above is a non-negative quadratic in z for any z. Hence, its discriminant is non-positive.

(b) Using (a), we obtain

$$E\{\underset{\sim}{x}^2\} + E\{\underset{\sim}{y}^2\} + 2\sqrt{E\{\underset{\sim}{x}^2\}E\{\underset{\sim}{y}^2\}}$$

$$\ge E\{\underset{\sim}{x}^2\} + E\{\underset{\sim}{y}^2\} + 2\,E\{\underset{\sim}{x}\,\underset{\sim}{y}\} = E\{(\underset{\sim}{x} + \underset{\sim}{y})^2\}$$

6-52 If $r_{xy} = 1$ then

$$E^2\{(\underset{\sim}{x} - \eta_x)(\underset{\sim}{y} - \eta_y)\} = E\{(\underset{\sim}{x} - \eta_x)^2\}E\{(\underset{\sim}{y} - \eta_y)^2\}$$

i.e., the discriminant of the quadratic

$$E\{[z(\underset{\sim}{x} - \eta_x) - (\underset{\sim}{y} - \eta_y)]^2\}$$

is zero. This is possible only if the quadratic is zero for some $z = z_o$. This shows that $z(\underset{\sim}{x} - \eta_x) - (\underset{\sim}{y} - \eta_y) = 0$ in the MS sense.

6-53 **If $E\{\underset{\sim}{x}\} = E\{\underset{\sim}{y}^2\} = E\{\underset{\sim}{x}\,\underset{\sim}{y}\}$, then**

$$E\{(\underset{\sim}{x} - \underset{\sim}{y})^2\} = E\{\underset{\sim}{x}^2\} + E\{\underset{\sim}{y}^2\} - 2\,E\{\underset{\sim}{x}\,\underset{\sim}{y}\} = 0.$$

Hence, $\underset{\sim}{x} = \underset{\sim}{y}$ in the MS sense.

6-54 **If $\underset{\sim}{x}$ has a Cauchy density, then (Prob. 5-31)**

$$E\{e^{j\omega\underset{\sim}{x}}\} = e^{-\alpha|\omega|} \qquad E\{e^{j\omega k\underset{\sim}{x}}\} = e^{-\alpha k|\omega|}$$

Hence, [see (6-240)]

$$\Phi_z(\omega) = E\{e^{j\omega\underset{\sim}{n}\underset{\sim}{x}}\} = E\{E\{e^{j\omega\underset{\sim}{n}\underset{\sim}{x}}|\underset{\sim}{n}\}\} =$$

$$\sum_{k=0}^{\infty} E\{e^{j\omega k\underset{\sim}{x}}\}e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty} e^{-\alpha k|\omega|}\frac{k}{k!} = e^{-\lambda}e^{-\lambda e^{-\alpha|\omega|}}$$

6.55 If $X = k$, then

$$Y = n - k$$

and

$$Z = X - Y = 2X - n,$$

where Z takes the values $-n, -(n-2), \cdots n-2, n$.

$$\begin{aligned} P\{Z = z\} &= P\{2X - n = z\}P\{X = \tfrac{n+z}{2}\} \\ &= \binom{n}{n + z/2} p^{(n+z)/2}\, q^{(n-z)/2}. \end{aligned}$$

Also

$$E(Z) = E[2X - n] = 2np - n = n(2p - 1).$$

$$\mathrm{Var}(Z) = E[(z - \mu_z)^2] = 4E[(X - np)^2] = 4\mathrm{Var}(X) = 4npq$$

6.56 (a)

$$\begin{aligned}\phi_Z(\omega) &= E[e^{j\omega Z}] = E[e^{j\omega(aX+bY+c)}] \\ &= \phi_X(a\omega)\,\phi_Y(b\omega)e^{j\omega c} = e^{j\omega c-(a^2\sigma_1^2+b^2\sigma_2^2)\omega^2/2}\end{aligned}$$

(see (5-100)).
(b) On comparing with (5-100) we obtain

$$Z \sim \mathrm{N}(c, a^2\,\sigma_1^2 + b^2\,\sigma_2^2)$$

(c)

$$E[Z] = c, \qquad \mathrm{Var}(Z) = a^2\,\sigma_1^2 + b^2\,\sigma_2^2$$

6.57

$$P(X = k|Y = n) = \binom{n}{k} p_1^k\, q_1^{n-k}, \quad k = 0, 1, 2, \cdots n$$

$$E[e^{j\omega X}|Y = n] = \sum_{k=0}^{n} e^{j\omega k}\, P(X = k|Y = n) = \left(p_1 e^{j\omega} + q_1\right)^n$$

use (5-117). Also

$$\begin{aligned}\phi_X(\omega) &= E[e^{j\omega X}] = E\left\{E[e^{j\omega X}|Y = n]\right\} \\ &= \textstyle\sum_{n=0}^{M} E[e^{j\omega X}|Y = n]\, P(Y = n) \\ &= \textstyle\sum_{n=0}^{\infty} (p_1 e^{j\omega} + q_1)^n \binom{M}{n} p_2^n\, q_2^{M-n} \\ &= \textstyle\sum_{n=0}^{M} \binom{M}{n} [p_2(p_1\, e^{j\omega} + q_1)]^n\, q_2^{M-n} \\ &= (p_2 p_1 e^{j\omega} + q_1 p_2 + q_2)^M\end{aligned}$$

But

$$1 - p_1 p_2 = 1 - (1 - q_1)(1 - q_2) = q_1 p_2 + q_2$$

Hence

$$\phi_X(\omega) = \left(pe^{j\omega} + q\right)^M$$

where $p = p_1 p_2$. Thus

$$X \sim \mathrm{Binomial}(M, p_1 p_2).$$

6.58

$$\int\int f_{XY}(x,y)\,dx\,dy = \int_0^1 \int_x^1 kx\,dy\,dx = k\int_0^1 x(1-x)\,dx$$

$$\frac{k}{6} = 1 \quad \Rightarrow \quad k = 6.$$

$$f_X(x) = \int_x^1 6x\,dy = 6x(1-x), \quad 0 < x < 1.$$

$$f_Y(y) = \int_0^y 6x\,dy = 3y^2, \quad 0 < y < 1.$$

$$E[X] = \int_0^1 x\,f_X(x)\,dx = 6\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^1 = \frac{1}{2}.$$

$$E[X^2] = \int_0^1 x^2\,f_X(x)\,dx = 6\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{3}{10}.$$

$$\mathrm{Var}(X) = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}.$$

$$E[Y] = \int_0^1 y\,f_Y(y)\,dy = 3\,\frac{y^4}{4}\Bigg|_0^1 = \frac{3}{4}.$$

$$E[Y^2] = \int_0^1 y^2\,f_Y(y)\,dy = 3\,\frac{y^5}{5}\Bigg|_0^1 = \frac{3}{5}.$$

$$\mathrm{Var}(Y) = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}.$$

$$\begin{aligned} E[XY] &= \int\int xy\,f_{XY}(x,y)\,dy\,dx \\ &= \int_0^1\int_x^1 xy\,6x\,dy\,dx = \int_0^1 3x^2\,(1-x^2)\,dx \\ &= 3\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1 = 3\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2}{5} \end{aligned}$$

$$\begin{aligned} Cov(X,Y) &= E(XY) - E(X)E(Y) \\ &= \frac{2}{5} - \frac{1}{2}\,\frac{3}{4} = \frac{1}{40} \end{aligned}$$

6.59 (a)

$$\begin{aligned} \phi_{X,Y}(\omega_1,\omega_2) &= E[e^{j(\omega_1 X + \omega_2 Y)}] \\ &= E[e^{j\omega_1 X}]\;E[e^{j\omega_2 Y}] = \phi_X(\omega_1)\,\phi_Y(\omega_2) \\ &= e^{\lambda(e^{j\omega_1}-1)}\;e^{(j\mu\omega_2 - \sigma^2\omega_2^2/2)} \end{aligned}$$

(b)

$$\begin{aligned} \phi_Z(\omega) &= E[e^{j\omega Z}] \\ &= E[e^{j\omega(X+Y)}] = \phi_{X,Y}(\omega,\omega) \\ &= e^{\{\lambda(e^{j\omega}-1)+(j\mu\omega - \sigma^2\omega^2/2)\}} \end{aligned}$$

6.60 (a)

$$Z = \min(X, Y)$$

From Example 6-18, we have

$$f_Z(z) = 2\lambda\, e^{-2\lambda z}, \qquad z \geq 0$$

and hence

$$E[Z] = E[\min(X, Y)] = \frac{1}{2\lambda}$$

(b)

$$\begin{aligned}
E[\max(2X, Y)] &= \int\int \max(2x, y)\, f_{XY}(x, y)\, dx\, dy \\
&= \int\int_{2x \geq y} 2x\, f_{XY}(x, y)\, dx\, dy + \int\int_{2x < y} y\, f_{XY}(x, y)\, dx\, dy \\
&= \int_0^\infty \int_0^{2x} 2x\, \lambda^2\, e^{-\lambda x}\, e^{-\lambda y}\, dy\, dx + \int_0^\infty \int_0^{y/2} y\, \lambda^2\, e^{-\lambda x}\, e^{-\lambda y}\, dx\, dy \\
&= \lambda \int_0^\infty 2x\, e^{-\lambda x}(1 - e^{-2\lambda x})\, dx + \lambda \int_0^\infty y\, e^{-\lambda y}(1 - e^{-\lambda y/2})\, dy \\
&= 2\lambda \int_0^\infty \left(x e^{-\lambda x} + 2x e^{-2\lambda x} - 3x e^{-3\lambda x} \right) dx \\
&= \frac{2}{\lambda} \int_0^\infty \left(u e^{-u} + 2u e^{-2u} - 3u e^{-3u} \right) du \\
&= \frac{2}{\lambda} \left(1 + \frac{2}{4} - \frac{3}{9} \right) = \frac{7}{3\lambda}.
\end{aligned}$$

6.61 (a)

$$Z = X - Y \quad \rightarrow \quad -1 < z < 1.$$

$z > 0$

$$\begin{aligned}
F_Z(z) &= P(X - Y \leq z) = 1 - P(X - Y > z) \\
&= 1 - \int_0^{(1-z)/2} \int_{y+z}^{1-y} f_{XY}(x, y)\, dx\, dy \\
&= 1 - \int_0^{(1-z)/2} \left(\int_{y+z}^{1-y} 6x dx \right) dy \\
&= 1 - 3 \int_0^{(1-z)/2} \{(1 - z^2) - 2(1 + z)y\}\, dy \\
&= 1 - \frac{3}{4}\,(1 + z)(1 - z)^2, \quad z \geq 0.
\end{aligned}$$

$z < 0$

$$\begin{aligned}
F_Z(z) &= P(X - Y \leq z) \\
&= \int_0^{(1+z)/2} \int_{x-z}^{1-x} 6x\, dy\, dx = \int_0^{(1+z)/2} 6x\,(1 + z - 2x)\, dx \\
&= \frac{(1 + z)^3}{4}, \quad z < 0.
\end{aligned}$$

This gives

$$f_Z(z) = \begin{cases} \frac{3}{4}(1-z)(1+3z), & 0 < z < 1 \\ \frac{3(1+z)^2}{4}, & -1 < z < 0 \end{cases}$$

(b)

$$f_X(x) = \int_0^{1-x} 6x\,dy = 6x(1-x), \quad 0 < x < 1$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{F_X(x)} = \frac{1}{1-x}, \quad 0 < y \le 1-x$$

(c)

$$W = X + Y$$

we have

$$F_W(w) = P(X+Y \le w) = \int_0^w \left(\int_0^{w-x} 6x\,dy\right) dx = w^3,$$

and

$$f_W(w) = \int_0^w 6x dx = 3w^2, \quad 0 < w < 1$$

$$E[W] = \frac{3}{4}$$

$$E[W^2] = \frac{3}{5}$$

$$\mathrm{Var}(X+Y) = \mathrm{Var}(W) = E(W^2) - (E(W))^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}.$$

6.62

$$X = \frac{1}{Z}.$$

where Z represents a Chi-square random variable. Thus (see (4-39))

$$f_Z(z) = \frac{z^{-1/2}}{\sqrt{2}\Gamma(1/2)}\, e^{-z/2} = \frac{z^{-1/2}}{\sqrt{2\pi}}\, e^{-z/2}$$

or

$$f_X(x) = \frac{1}{\left|\frac{dx}{dz}\right|} f_Z(1/x) = \frac{1}{x^2}\frac{x^{1/2}}{\sqrt{2\pi}}\, e^{-1/2x} = \frac{1}{\sqrt{2\pi}x^{3/2}}\, e^{-1/2x}, \quad x > 0$$

Also it is given that

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi x}}\, e^{-y^2/2x}$$

so that

$$f_{XY}(x,y) = f_{Y|X}(y|x)\, f_X(x) = \frac{1}{2\pi x^2}\, e^{-(1+y^2)/2x}$$

and hence

$$\begin{aligned} f_Y(y) &= \int_0^\infty f_{XY}(x,y)\,dx \\ &= \frac{1}{2\pi}\int_0^\infty \frac{1}{x^2}\, e^{-(1+y^2)/2x}\, dx \\ &= \frac{1}{2\pi}\frac{2}{1+y^2}\int_0^\infty e^{-u}\,du = \frac{1/\pi}{1+y^2}, \quad -\infty < y < \infty. \end{aligned}$$

Thus Y represents a Cauchy random variable.

6.63 (a) For any two random variables X and Y we have

$$\begin{aligned}\sigma_{X+Y}^2 &= \mathrm{Var}(X+Y) = E[\{(X-\mu_X)+(Y-\mu_Y)\}^2] \\ &= \mathrm{Var}(X)+\mathrm{Var}(Y)+2\mathrm{Cov}(X,Y) = \sigma_X^2+\sigma_Y^2+2\sigma_X\sigma_Y\rho_{XY} \\ &\le (\sigma_X+\sigma_Y)^2\end{aligned}$$

since $|\rho_{XY}| \le 1$. Thus

$$\sigma_{X+Y} \le \sigma_X + \sigma_Y,$$

and hence it easily follows that

$$\frac{\sigma_{X+Y}}{\sigma_X+\sigma_Y} \le 1.$$

(However, (b) is not so easy!)
(b) We shall prove this result in three parts by making use of Holder's inequality.
(i) **Holder's inequality:** The function $\log x$ is concave, for $0 < \alpha < 1$, and hence we have

$$\log[\alpha x_1 + (1-\alpha)x_2] \ge \alpha \log x_1 + (1-\alpha)\log x_2$$

or

$$x_1^{\alpha}\, x_2^{1-\alpha} \le \alpha x_1 + (1-\alpha)x_2, \quad 0 < \alpha < 1. \qquad (6.63-1)$$

Let

$$x_1 = |x|^p, \quad \alpha = \frac{1}{p}, \text{ so that } 1-\alpha = 1-\frac{1}{p} \triangleq \frac{1}{q}, \quad x_2 = |y|^q \quad (6.63-2)$$

so that (6.63-1) becomes

$$|xy| \le \frac{|x|^p}{p} + \frac{|y|^q}{q}, \quad p > 1, \qquad (6.63-3)$$

the Holder's inequality. From (6.63-2), note that

$$\frac{1}{p} + \frac{1}{q} = 1, \quad p > 1, \quad q > 1 \qquad (6.63-4)$$

(ii) Define

$$x = X\left(E\{|X|^p\}\right)^{-1/p}, \quad y = Y\left(E\{|Y|^q\}\right)^{-1/q}$$

where p and q are as in (6.63-4). Substituting these into the Holder's inequality in (6.63-3), we get

$$\begin{aligned} |XY| &\leq p^{-1}\,|X|^p\,\left(E\{|X|^p\}\right)^{1/p-1}\,\left(E\{|Y|\}\right)^{1/q} \\ &+q^{-1}\,|Y|^q\,\left(E\{|Y|^q\}\right)^{1/q-1}\,\left(E\{|X|^p\}\right)^{1/p}. \end{aligned} \qquad (6.63-5)$$

Taking expected values on both sides of (6.63-5), we get

$$E\{|XY|\} \leq \left(E\{|X|^p\}\right)^{1/p}\,\left(E\{|Y|^q\}\right)^{1/q} \qquad (6.63-6)$$

which represents the generalization of the Cauchy-Schwarz inequality. (Note $p = q = 2$ corresponds to Cauchy-Schwarz inequality)
(iii) To prove the desired inequality, notice that

$$\begin{aligned} |X+Y|^p &= |X+Y||X+Y|^{p-1} \\ &\leq |X||X+Y|^{p-1} + |Y||X+Y|^{p-1}, \quad p > 1 \end{aligned}$$

and taking expected values on both sides we get

$$E\{|X+Y|^p\} \leq E\{|X||X+Y|^{p-1}\} + E\{|Y||X+Y|^{p-1}\}. \quad (6.63-7)$$

Applying (6.63-6) to each term on the right side of (6.63-7) we get

$$E\{|X||X+Y|^{p-1}\} \leq \left(E\{|X|^p\}\right)^{1/p}\,\left(E\{|X+Y|^{(p-1)q}\}\right)^{1/q} \quad (6.63-8)$$

and

$$E\{|Y||X+Y|^{p-1}\} \leq \left(E\{|Y|^p\}\right)^{1/p}\,\left(E\{|X+Y|^{(p-1)q}\}\right)^{1/q} \quad (6.63-9)$$

Using (6.63-8) and (6.63-9) together with $(p-1)q = p$ in (6.63-7) we get

$$E\{|X+Y|^p\} \leq \left[\left(E\{|X|^p\}\right)^{1/p} + \left(E\{|Y|^p\}\right)^{1/p}\right] \cdot \left(E\{|X+Y|^p\}\right)^{1/q}$$

or for $p > 1$

$$\left(E\{|X+Y|^p\}\right)^{1/p} \leq \left(E\{|X|^p\}\right)^{1/p} + \left(E\{|Y|^p\}\right)^{1/p}.$$

the desired inequality. Since $p = 1$ follows trivially, we get

$$\frac{\left(E\{|X+Y|^p\}\right)^{1/p}}{\left(E\{|X|^p\}\right)^{1/p} + \left(E\{|Y|^p\}\right)^{1/p}} \leq 1, \quad p \geq 1.$$

6.64 (a) See Example 6-41. From there

$$E(Y|X=x) = \mu_Y + \frac{\rho_{XY}\sigma_Y(x-\mu_X)}{\sigma_X}$$

(b) Similarly

$$f_{X|Y}(X|Y=y) \sim N(\mu, \sigma^2)$$

where

$$\mu = \mu_X + \frac{\rho_{XY}\sigma_X(y-\mu_Y)}{\sigma_Y}$$

and

$$\sigma^2 = \sigma_X^2(1-\rho_{XY}^2).$$

Since

$$E(X^2|Y=y) = \mathrm{Var}(X|Y=y) + (E[X|Y=y])^2$$

we obtain

$$E(X^2|Y=y) = \sigma^2 + \mu^2$$

6.65 (a) See footnote 4, Chapter 8, Page 337. From there (or directly) we have

$$\mathrm{Var}(X|Y) \triangleq E(X^2|Y) - (E\{X|Y\})^2$$

$$\mathrm{Var}\left(E\{X|Y\}\right) \triangleq E\left[E\{X|Y\}\right]^2 - (E[E\{X|Y\}])^2$$

so that

$$\begin{aligned} E[\mathrm{Var}(X|Y)] + \mathrm{Var}\left(E\{X|Y\}\right) &= E\left[E\{X^2|Y\}\right] - (E[E\{X|Y\}])^2 \\ &= E(X^2) - [E(X)]^2 = \mathrm{Var}(X) \qquad (1) \end{aligned}$$

or

$$\mathrm{Var}(X) \geq E[\mathrm{Var}\{X|Y\}]$$

Also

$$\mathrm{Var}(X) \geq \mathrm{Var}[E\{X|Y\}]$$

(b) See (1).

6.66

$$Z = aX + (1-a)Y, \qquad 0 < a < 1$$

$$\sigma_Z^2 = \mathrm{Var}(Z) = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$$

$$\frac{\partial \sigma_Z^2}{\partial a} = 2a\sigma_1^2 + 2(1-a)(-1)\sigma_2^2 = 0$$

or

$$a(\sigma_1^2 + \sigma_2^2) = \sigma_2^2$$

$$a = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} < 1$$

minimizes $\mathrm{Var}(Z)$.

6-67 **From** (6-240)

$$E\{g(\underset{\sim}{x},\underset{\sim}{y})\} = E\{E\{g(\underset{\sim}{x},\underset{\sim}{y})\,|\,\underset{\sim}{y}\}\} = E\{g(x_n,\underset{\sim}{y})P\{\underset{\sim}{x} = x_n\}\}\ .$$

From (4-74) **with** $A_n = \{\underset{\sim}{x} = x_n\}$

$$f_z(z) = \sum_n f_z(z\,|\,\underset{\sim}{x} = x_n)P\{\underset{\sim}{x} = x_n\}$$

6-68 (a) The conditional density $f(y|x)$ is $N(rx;\sigma\sqrt{1-r^2})$ [see (7-42)]. Hence

$$E\{f_y(\underset{\sim}{y}|x)\} = \int_{-\infty}^{\infty} f_y(y|x)f_y(y)dy$$

$$= \frac{1}{2\pi\sigma^2\sqrt{1-r^2}}\int_{-\infty}^{\infty} \exp\left\{\frac{-(y-rx)^2}{2\sigma^2(1-r^2)}\right\}\exp\left\{\frac{-y^2}{2\sigma^2}\right\}dy = \frac{1}{\sigma\sqrt{2\pi(2-r^2)}}\exp\left\{\frac{-r^2x^2}{2\sigma^2(2-r^2)}\right\}$$

(b) From (6-241) it follows that

$$E\{f_x(\underset{\sim}{x})f_y(\underset{\sim}{y})\} = E\{f_x(\underset{\sim}{x})E\{f_y(\underset{\sim}{y}|\underset{\sim}{x})\}\} = \int_{-\infty}^{\infty} f_x(x)\ E\{f_y(y|x)\}f_x(x)dx$$

$$= \frac{1}{2\pi\sigma^3\sqrt{2\pi(2-r^2)}}\int_{-\infty}^{\infty}\exp\left\{-\frac{x^2}{\sigma^2}\right\}\exp\left\{\frac{-r^2x^2}{2\sigma^2(2-r^2)}\right\}dx = \frac{1}{2\pi\sigma^2\sqrt{4-r^2}}$$

6-69 We shall use (6-64) and Price's theorem (10-94):

$$\frac{\partial E\{|\underset{\sim}{x}\underset{\sim}{y}|\}}{\partial\mu} = E\left\{\frac{d|\underset{\sim}{x}|}{d\underset{\sim}{x}}\ \frac{d|\underset{\sim}{y}|}{\partial\underset{\sim}{y}}\right\} = E\{\operatorname{sgn}\underset{\sim}{x}\operatorname{sgn}\underset{\sim}{y}\}$$

$$= P\{\underset{\sim}{x}\underset{\sim}{y} > 0\} - P\{\underset{\sim}{x}\underset{\sim}{y} < 0\} = \frac{2\alpha}{\pi} = \frac{2}{\pi}\arcsin\frac{\mu}{\sigma_1\sigma_2}$$

If $\mu = 0$, then the RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are independent, hence,

$$E\{|\underset{\sim}{x}\underset{\sim}{y}|\}\Big|_{\mu=0} = E\{|\underset{\sim}{x}|\}E\{|\underset{\sim}{y}|\} = \frac{2}{\pi}\sigma_1\sigma_2$$

[see (5-74)]. Integrating (i) and using the above, we obtain

$$E\{|\underset{\sim}{x}\underset{\sim}{y}|\} = \frac{2}{\pi}\int_0^{\mu}\arcsin\frac{C}{\sigma_1\sigma_2}\,dC + \frac{2}{\pi}\sigma_1\sigma_2 = \frac{2\sigma_1\sigma_2}{\pi}(\cos\alpha + \alpha\sin\alpha)$$

6-70 From Example 6-41

$$f(y|x) : N\left(\eta_2 + \frac{r\sigma_2}{\sigma_1}x;\ \sigma_2\sqrt{1-r^2}\right) = N(4+x;\ \sqrt{3})$$

$$f(x|y) : N\left(\eta_1 + \frac{r\sigma_1}{\sigma_2}y;\ \sigma_1\sqrt{1-r^2}\right) = N\left(3+\frac{y}{4};\ \sqrt{3}/2\right)$$

6-71 The mass density in the square $|\underset{\sim}{x}|\le 1$, $|\underset{\sim}{y}|\le 1$ of the xy plane equals 1/4; hence, $P\{\underset{\sim}{r}\le 1\}=\pi/4$ and $P\{\underset{\sim}{r}\le r\} = \pi r^2/4$ for $r<1$. This yields

$$P\{\underset{\sim}{r}\le r, \underset{\sim}{r}\le 1\} = \begin{cases} P\{\underset{\sim}{r}\le r\} = \pi r^2/4 & r\le 1 \\ P\{\underset{\sim}{r}\le 1\} = \pi/4 & r>1 \end{cases}$$

$$F_r(r|M) = \frac{P\{\underset{\sim}{r}\le r, M\}}{P(M)} = \begin{cases} r^2 & r\le 1 \\ 1 & r>1 \end{cases} \qquad f_r(r|m) = \begin{cases} 2r, & r<+1 \\ 0 & \text{otherwise} \end{cases}$$

6-72 $\underset{\sim}{z} = \underset{\sim}{x} + \underset{\sim}{y}$ $\underset{\sim}{w} = \underset{\sim}{x}$ $f_{xz}(x,z) = f_{xy}(x, z-x)$

If $f_{xy}(x,y) = f_x(x)f_y(y)$, then

$$f_z(z|x) = \frac{f_{xz}(x,z)}{f_x(x)} = f_y(z-x)$$

6-73 The system $z = F_x(x)$ $w = F_y(y|x)$ has a solution only if $z \le z \le 1$ and $0 \le w \le 1$. Furthermore,

$$\frac{\partial z}{\partial x} = f_x(x) \qquad \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial w}{\partial x} \qquad \frac{\partial w}{\partial y} = f_y(y|x) \qquad J = f_x(x)f_y(y|x)$$

$$f_{zw}(z,w) = \frac{f_{xy}(x,y)}{f_x(x)f_y(y|x)} = 1 \quad \text{for} \quad 0 \le z,w \le 1$$

6-74 We introduce the events C_r = {we selected the rth coin} and A_k = {heads in a specific order}. From the assumptions it follows that

$$P(C_r) = \frac{1}{m} \qquad P(A_k|C_r) = p_r^k(1-p_r)^{n-k}$$

We wish to find the probability $P(C_r|A_k)$. The events C_r form a pertition; hence,

$$P(C_r|A_k) = \frac{\frac{1}{m}P(a_k|C_r)}{\frac{1}{m}\sum_{i=1}^{m} P(A_k|C_i)}$$

6-75 We wish to show that

$$E\{\underset{\sim}{x}^2\} = \frac{n}{n-1}$$

From page 207: $\underset{\sim}{x}^2 = n\underset{\sim}{y}^2/\underset{\sim}{z}$ where $\underset{\sim}{y}$ is N(0,1) and $\underset{\sim}{z}$ is $\chi^2(n)$. Hence, $E\{\underset{\sim}{y}^2\} = 1$ and (also (4-35) and (4-39))

$$E\left\{\frac{1}{\underset{\sim}{z}}\right\} = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty z^{n/2-2}e^{-z/2}dz = \frac{2^{m/2-1}\Gamma(n/2-1)}{2^{n/2}\Gamma(n/2)}$$

From this and the independence of $\underset{\sim}{y}$ and $\underset{\sim}{z}$ it follows that

$$E\{\underset{\sim}{x}^2\} = n\, E\{\underset{\sim}{y}^2\}\, E\left\{\frac{1}{\underset{\sim}{z}}\right\} = \frac{n}{n-2}$$

6-76 From (6-222):

$$R_x(x) = \exp\left\{-\int_0^x \beta_x(t)dt\right\} = \exp\left\{-k\int_0^x \beta_y(t)dt\right\} = R_y^k(t)$$

6-77 From (5-89) it follows with $\underset{\sim}{x} = |\underset{\sim}{z}|^2$ and $\alpha = \varepsilon^2$ that

$$E\{|z|^2 > \varepsilon^2\} \le \frac{E\{|\underset{\sim}{z}|^2\}}{\varepsilon^2}$$

for any $\underset{\sim}{z}$. And the result follows with $\underset{\sim}{z} = \underset{\sim}{x} - \underset{\sim}{y}$.

6-78 $E\{U(a-\underset{\sim}{x})\} = \int_{-\infty}^{\infty} U(a-x)f(x)dx = \int_{-\infty}^{a} f(x)dx = F_x(a)$

$$E\{U(b-\underset{\sim}{y})\} = F_y(b)$$

$$E\{U(a-\underset{\sim}{x})U(b-\underset{\sim}{y}) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)dxdy = F_{xy}(a,b)$$

Hence

$$F_{xy}(a,b) = F_x(a)F_y(b)$$

6-79 From Example 6-38

$$E\{\underset{\sim}{y}|\underset{\sim}{x} \leq 0\} = \int_{-\infty}^{\infty} y\ f_y(y|\underset{\sim}{x} \leq 0\}dy = \frac{1}{F_x(0)} \int_{-\infty}^{\infty} y\ \frac{\partial F(0,y)}{\partial y}\ dy$$

From (7-41) and (7-57)

$$\int_{-\infty}^{\infty} E\{\underset{\sim}{y}|x\}f_x(x)dx = \int_{-\infty}^{\infty} y \int_{-\infty}^{0} f(x,y)dxdy = \int_{-\infty}^{\infty} y\ \frac{\partial F(0,y)}{\partial y}\ dy$$

CHAPTER 7

7-1 $0 \le P\{x_1 < \underset{\sim}{x} \le x_2,\ y_1 < \underset{\sim}{y} \le y_2,\ z_1 < \underset{\sim}{z} \le z_2\} =$

$= P\{\underset{\sim}{x} \le x_2,\ y_1 < \underset{\sim}{y} \le y_2,\ z_1 < \underset{\sim}{z} \le z_2\} - P\{\underset{\sim}{x} \le x_1,\ y_1 < \underset{\sim}{y} \le y_2,\ z_1 < \underset{\sim}{z} \le z_2\} =$

$= P\{\underset{\sim}{x} \le x_2,\ \underset{\sim}{y} \le y_2,\ z_1 < \underset{\sim}{z} \le z_2\} - P\{\underset{\sim}{x} \le x_2,\ \underset{\sim}{y} \le y_1,\ z_1 < \underset{\sim}{z} \le z_2\}$

$- P\{\underset{\sim}{x} \le x_1,\ \underset{\sim}{y} \le y_2,\ z_1 < \underset{\sim}{z} \le z_2\} + P\{\underset{\sim}{x} \le x_1,\ \underset{\sim}{y} \le y_1,\ z_1 < \underset{\sim}{z} \le z_2\} =$

$= P\{\underset{\sim}{x} \le x_2,\ \underset{\sim}{y} \le y_2,\ \underset{\sim}{z} \le z_2\} - P\{\underset{\sim}{x} \le x_2,\ \underset{\sim}{y} \le y_2,\ \underset{\sim}{z} \le z_1\}$

$- P\{\underset{\sim}{x} \le x_2,\ \underset{\sim}{y} \le y_1,\ \underset{\sim}{z} \le z_2\} + P\{\underset{\sim}{x} \le x_2,\ \underset{\sim}{y} \le y_1,\ \underset{\sim}{z} \le z_1\}$

$- P\{\underset{\sim}{x} \le x_1,\ \underset{\sim}{y} \le y_2,\ \underset{\sim}{z} \le z_2\} + P\{\underset{\sim}{x} \le x_1,\ \underset{\sim}{y} \le y_2,\ \underset{\sim}{z} \le z_1\}$

$+ P\{\underset{\sim}{x} \le x_1,\ \underset{\sim}{y} \le y_1,\ z \le z_2\} - P\{\underset{\sim}{x} \le x_1,\ \underset{\sim}{y} \le y_1,\ \underset{\sim}{z} \le z_1\}$

7-2 $P\{\underset{\sim}{x}_A = 1,\ \underset{\sim}{x}_B = 1,\ \underset{\sim}{x}_C = 1\} = P(ABC) = 1/4$

$P\{\underset{\sim}{x}_A = 1\} = P(A) = 1/2 \qquad P\{\underset{\sim}{x}_B = 1\} = P(B) = 1/2$

$P\{\underset{\sim}{x}_C = 1\} = P(C) = 1/2$ hence

$$P\{x_A = 1,\ \underset{\sim}{x}_B = 1,\ \underset{\sim}{x}_C = 1\} \ne P\{\underset{\sim}{x}_A = 1\}P\{\underset{\sim}{x}_B = 1\}P\{\underset{\sim}{x}_C = 1\}$$

hence $\underset{\sim}{x}_A$, $\underset{\sim}{x}_B$, $\underset{\sim}{x}_C$ are not independent. But

$P\{\underset{\sim}{x}_A = 1,\ \underset{\sim}{x}_B = 1\} = P(AB) = 1/4 = P\{\underset{\sim}{x}_A = 1\}P\{\underset{\sim}{x}_B = 1\}$

Similarly for any other combination, e.g.,
since $P(A) = P(AB) + P(A\bar{B})$, we conclude that
$P(A\bar{B}) = 1/2 - 1/4 = 1/4 \qquad P(\bar{B}) = 1 - P(B) = 1/2$
$P\{\underset{\sim}{x}_A = 1,\ \underset{\sim}{x}_B = 0\} = P(A\bar{B}) = 1/4$
$P\{\underset{\sim}{x}_B = 0\} = P(\bar{B}) = 1/2$ hence
$P\{\underset{\sim}{x}_A = 1,\ \underset{\sim}{x}_B = 0\} = P\{\underset{\sim}{x}_A = 1\}P\{\underset{\sim}{x}_B = 0\}$

7-3 If $\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}$ are independent in pairs, then

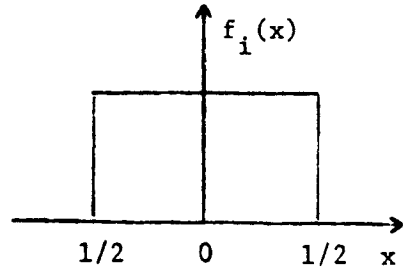
$$r_{xy} = r_{xz} = r_{yz} = 0$$

and (7-60) yields (we assume $\eta_x = \eta_y = \eta_z = 0$)

$$\Phi(\omega_1, \omega_2, \omega_3) = \exp\left\{-\frac{1}{2}\left(\sigma_1^2\omega_1^2 + \sigma_2^2\omega_2^2 + \sigma_3^2\omega_3^2\right)\right\}$$

$$f(x_1, x_2, x_3) = f(x_1)f(x_2)f(x_3)$$

7-4 $\underset{\sim}{x} = \underset{\sim}{x}_1 + \underset{\sim}{x}_2 + \underset{\sim}{x}_3$. To determine $E\{\underset{\sim}{x}^4\}$ we shall use char. functions



$$\Phi_i(\omega) = \int_{-1/2}^{1/2} e^{j\omega x} dx = \frac{2\sin(\omega/2)}{\omega}$$

$$\Phi(\omega) = \left[\frac{2\sin(\omega/2)}{\omega}\right]^3 = \left(1 - \frac{\omega^2}{24} + \frac{\omega^4}{1920} - \cdots\right)^3$$

The coefficient of ω^4 in this expansion equals

$$\frac{13}{1920} \text{ hence } \frac{1}{4!}\frac{d^4\Phi(0)}{d\omega^4} = \frac{13}{1920}$$

and [see (5-103)]

$$E\{\underset{\sim}{x}^4\} = m_4 = \frac{13 \times 4!}{1920} = \frac{13}{80}$$

7-5 (a) The joint density $f(x,y)$ has circular symmetry because

$$f(x,y) = \int_{-\infty}^{\infty} f(\sqrt{x^2 + y^2 + z^2})\, dz$$

depends only on $x^2 + y^2$. The same holds for $f(x,z)$ and $f(y,z)$. And since the RVs $\underset{\sim}{x}, \underset{\sim}{y}$, and $\underset{\sim}{z}$ are independent, they must be normal [see (6-29)].

(b) From (a) it follows that the RVs $\underset{\sim}{v}_x, \underset{\sim}{v}_y, \underset{\sim}{v}_z$ are $N(0; \sqrt{kT/m})$. With $\sigma^2 = kT/m$ and $n = 3$ it follows from (7-62)-(7-63) and (5-25) that

$$f_v(v) = \sqrt{\frac{2m^3}{\pi k^3 T^3}}\; v^2\, e^{-mv^2/2kT}\, U(v)$$

$$E\{\underset{\sim}{v}\} = 2\sqrt{\frac{2kT}{\pi m}} \qquad E\{\underset{\sim}{v}^{2n}\} = 1 \times 3 \cdots (2n+1)\left(\frac{kT}{m}\right)^n$$

7-6 From Prob. 6-52: $\underset{\sim}{y} = a\underset{\sim}{x} + b$, $\underset{\sim}{z} = c\underset{\sim}{y} + d$, hence,

$$\underset{\sim}{z} = A\underset{\sim}{x} + B \qquad \eta_z = A\eta_x + B \qquad \sigma_z = A\sigma_x$$

$$E\{(\underset{\sim}{z} - \eta_z)(\underset{\sim}{x} - \eta_x\} = E\{A(\underset{\sim}{x} - \eta_x)(\underset{\sim}{x} - \eta_x)\} = A\sigma_x^2 = \sigma_x\sigma_z$$

7-7 It follows from (6-241) with $g_1(x) = x$, $g_2(y) = y$ if we replace all densities with conditional densities assuming $\underset{\sim}{x}_3$.

7-8 Reasoning as in (7-82), we conclude that

$E\{[\underset{\sim}{y} - (a_1\underset{\sim}{x}_1 + a_2\underset{\sim}{x}_1)]^2\}$ is minimum if

$$E\{[\underset{\sim}{y} - (a_1\underset{\sim}{x}_1 + a_2\underset{\sim}{x}_2)]\underset{\sim}{x}_i\} = 0 \qquad i = 1,2$$

With $R_{oi} = E\{\underset{\sim}{y}\,\underset{\sim}{x}_i\}$, $R_{ij} = E\{\underset{\sim}{x}_i\underset{\sim}{x}_j\}$, the above yields

$$R_{01} = a_1R_{11} + a_2R_{12} \qquad\qquad R_{02} = a_1R_{12} + a_2R_{22}$$

But $\hat{E}\{\underset{\sim}{y}|\underset{\sim}{x}_1\} = A\,\underset{\sim}{x}_1 \qquad\qquad A = R_{01}/R_{11} = a_1 + a_2R_{12}/R_{11}$

$$\hat{E}\{\hat{E}\{\underset{\sim}{y}|\underset{\sim}{x}_1,x_2\}|\underset{\sim}{x}_1\} = \hat{E}\{a_1\underset{\sim}{x}_1 + a_2\underset{\sim}{x}_2|\underset{\sim}{x}_1\}$$

$$= a_1\underset{\sim}{x}_1 + a_2\hat{E}\{\underset{\sim}{x}_2|\underset{\sim}{x}_1\} = \left(a_1 + a_2\frac{R_{12}}{R_{11}}\right)\underset{\sim}{x}_1 = A\,\underset{\sim}{x}_1$$

7-9 As in Probl. 6-51

$$E^2\{\underset{\sim}{x}_i\underset{\sim}{x}_j\} \le E^2\{\underset{\sim}{x}_i\}E^2\{\underset{\sim}{x}_j\} = M^2 \qquad\qquad |E\{\underset{\sim}{x}_i\underset{\sim}{x}_j\}| \le M$$

$$E\{\underset{\sim}{s}^2|\underset{\sim}{n} = n\} = E\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\underset{\sim}{x}_i\underset{\sim}{x}_j\right\} \le Mn^2$$

Hence [see (6-240)]

$$E\{\underset{\sim}{s}^2\} = E\{E\{\underset{\sim}{s}^2|\underset{\sim}{n}\}\} < E\{M\underset{\sim}{n}^2\}$$

7-10 As we know,

$$1 + x + \cdots + x^n + \cdots = \frac{1}{1-x} \qquad |x| < 1$$

Differentiating, we obtain

$$1 + 2x + \cdots + n\,x^{n-1} + \cdots = \sum_{k=1}^{\infty} k\,x^{k-1} = \frac{1}{(1-x)^2} \qquad \text{(i)}$$

The RV $\underset{\sim}{x}_1$ equals the number of tosses until heads shows for the first time, Hence, $\underset{\sim}{x}_1$ takes the values 1,2,... with $P\{\underset{\sim}{x}_1 = k\} = pq^{k-1}$. Hence, [see (3-12) and (i)]

$$E\{\underset{\sim}{x}_1\} = \sum_{k=1}^{\infty} k\,P\{\underset{\sim}{x}_1 = k\} = \sum_{k=1}^{\infty} k\,p\,q^{k-1} = \frac{p}{(1-q)^2} = \frac{1}{p}$$

Starting the count after the first head shows, we conclude that the RV $\underset{\sim}{x}_2 - \underset{\sim}{x}_1$ has the same statistics as the RV $\underset{\sim}{x}_1$. Hence,

$$E\{\underset{\sim}{x}_2 - \underset{\sim}{x}_1\} = E\{\underset{\sim}{x}_1\} \qquad E\{\underset{\sim}{x}_2\} = 2E\{\underset{\sim}{x}_1\} = \frac{2}{p}$$

Reasoning similarly, we conclude that

$E\{\underset{\sim}{x}_n - \underset{\sim}{x}_{n-1}\} = E\{\underset{\sim}{x}_1\}$, Hence (induction)

$$E\{\underset{\sim}{x}_n\} = E\{\underset{\sim}{x}_{n-1}\} + E\{\underset{\sim}{x}_1\} = \frac{n-1}{p} + \frac{1}{p} = \frac{n}{p}$$

7-11 If n accidents occur in a day, the probability that m of them will be fatal equals $\binom{n}{m} p^m q^{n-m}$ for $m \le n$ and zero for $m > n$. Hence,

$$P\{\underset{\sim}{m} = m \mid \underset{\sim}{n} = n\} = \begin{cases} 0 & m > n \\ \binom{n}{m} p^m q^{n-m} & m \le n \end{cases}$$

This yields

$$E\{e^{j\omega \underset{\sim}{m}} \mid \underset{\sim}{n} = n\} = \sum_{m=0}^{n} e^{j\omega m} \binom{n}{m} p^m q^{n-m} = (p\,e^{j\omega} + q)^n$$

But

$$P\{\underset{\sim}{n} = n\} = e^{-a}\frac{a^n}{n!} \qquad n = 0,1,\ldots$$

Hence,

$$E\{e^{j\omega \underset{\sim}{m}}\} = E\{E\{e^{j\omega \underset{\sim}{m}} \mid \underset{\sim}{n}\}\} = E\{(p\, e^{j\omega} + q)^{\underset{\sim}{n}}\}$$

$$\sum_{n=o}^{\infty} (p\, e^{j\omega} + q)^n\, e^{-a}\, \frac{a^n}{n!} = e^{a(p\, e^{j\omega} + q)} e^{-a} = e^{a\, p\, (e^{j\omega} - 1)}$$

This shows that the RV $\underset{\sim}{m}$ is Poisson distributed with parameter $a\,p$ [see (5-119)].

7-12 We shall determine first the conditional distribution

$$F_s(s \mid \underset{\sim}{n} = n) = \frac{P\{\underset{\sim}{s} \le s,\ \underset{\sim}{n} = n\}}{P\{\underset{\sim}{n} = n\}}$$

The event $\{\underset{\sim}{s} < s,\ \underset{\sim}{n} = n\}$ consists of all outcomes such that $\underset{\sim}{n} = n$ and $\sum_{k=1}^{n} \underset{\sim}{x}_k \le s$. Since the RV $\underset{\sim}{n}$ is independent of the RVs $\underset{\sim}{x}_k$, this yields

$$F_s(s \mid \underset{\sim}{n} = n) = P\{\sum_{k=1}^{n} \underset{\sim}{x}_k \le s\} P\{\underset{\sim}{n} = n\} / P\{\underset{\sim}{n} = n\}$$

From the above and the independence of the RVs $\underset{\sim}{x}_k$ it follows that [see (7-51)]

$$f_s(s \mid \underset{\sim}{n} = n) = f_1(s) * f_2(s) * \cdots * f_n(s)$$

Setting $A_k = \{\underset{\sim}{n} = k\}$ in (4-74), we obtain

$$f_s(s) = \sum_k p_k\, [f_1(s) * \cdots * f_k(s)]$$

7-13 From the independence of the RVs n and $\underset{\sim}{x}_i$ it follows that

$$E\{e^{s\underset{\sim}{y}}|\underset{\sim}{n} = k\} = E\{e^{s(\underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_k)}\}$$

$$= E\{e^{s\underset{\sim}{x}_1}\} \cdots E\{e^{s\underset{\sim}{x}_k}\} = \Phi_x^k(s)$$

Hence,

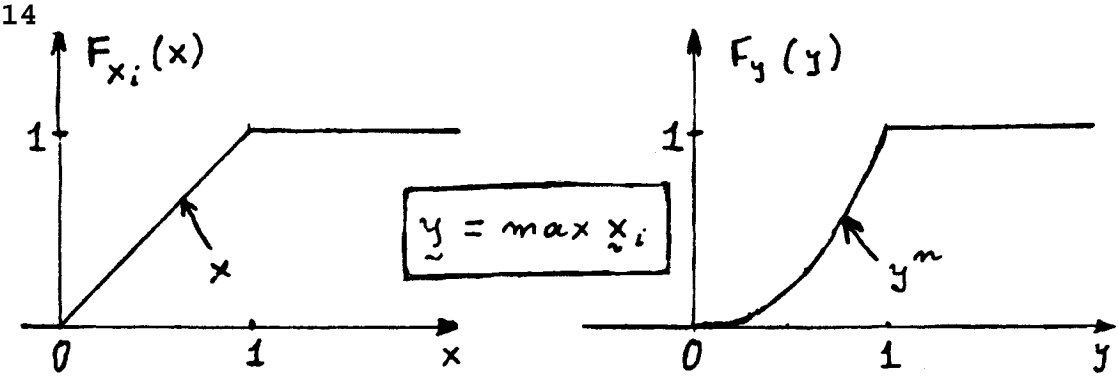
$$\Phi_y(s) = E\{e^{s\underset{\sim}{y}}\} = E\{E\{e^{s\underset{\sim}{y}}|\underset{\sim}{n}\}\} = E\{\Phi_x^{\underset{\sim}{n}}(s)\}$$

$$= \Gamma_n[\phi_x(s)] \text{ because } E\{z^{\underset{\sim}{n}}\} = \Gamma_n(z)$$

<u>Special case</u>. If $\underset{\sim}{n}$ is Poisson with parameter a, then [see (5-119)]

$$\Gamma_n(z) = e^{az - a} \qquad \Phi_y(s) = e^{a\Phi_x(s) - a}$$

7-14



$$\{\underset{\sim}{y} \le y\} = \{\underset{\sim}{x}_1 \le y, \underset{\sim}{x}_2 \le y, \ldots, \underset{\sim}{x}_n \le y\}$$

From the independence of $\underset{\sim}{x}_i$ and the above it follows that

$$F_y(y) = P\{\underset{\sim}{y} \le y\} = P\{\underset{\sim}{x}_1 \le y\} \cdots P\{\underset{\sim}{x}_n \le y\}$$

$$= F_1(y) \cdots F_n(y)$$

where $F_i(y) = y$ for $0 \le y \le 1$.

7-15 The RV $\underset{\sim}{x}$ is defined in the space S. The set

$$C = \{z < \underset{\sim}{z} \le z + dz,\ w < \underset{\sim}{w} \le w + dz\} \qquad z > w$$

is an event in the space S_n of repeated trials and its probability equals

$$P(C) = f_{zw}(z,w)dzdw$$

We introduce the events

$$D_1 = \{\underset{\sim}{x} \le w\} \qquad D_2 = \{w < \underset{\sim}{x} \le w + dw\} \qquad D_3 = \{w + dw < \underset{\sim}{x} \le z\}$$

$$D_4 = \{z < \underset{\sim}{x} \le z + dz\} \qquad D_5 = \{z + dz < \underset{\sim}{x}\}$$

These events form a partition of S and their probabilities $p_i = P(D_i)$ equal

$$F_x(w) \qquad f_x(w)dw \qquad F_x(z)-F_x(w+dw) \qquad f_z(z)dz \qquad 1-F_x(z+dz)$$

respectively. The event C occurs iff the smallest of the RVs $\underset{\sim}{x}_i$ is in the interval (w, w+dw), the largest is in the interval (z, z+dz), and, consequently, all others are between w+dw and z. This is the case iff D_1 does not occur at all, D_2 occurs once, D_3 occurs n-2 times, D_4 occurs once, and D_5 does not occur at all. With

$$k_1=0 \qquad k_2=1 \qquad k_3=n-2 \qquad k_4=1 \qquad k_5=0$$

it follows from (4-102) that

$$P(C) = \frac{n!}{(n-2)!} p_2\, p_3^{n-2}\, p_4 = n(n-1)f_x(w)dw\,[f_x(z)-F_x(w+dw)]^{n-1} f_x(z)dz$$

for z > w, and 0 otherwise.

7-16 If $\underset{\sim}{z}$ is $N(\eta,1)$ then

$$E\{e^{s\underset{\sim}{z}^2}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{sz} e^{-(z-\eta)^2/2} dz$$

$$sz^2 - \frac{(z-\eta)^2}{2} = \left(s - \frac{1}{2}\right)\left(z - \frac{\eta}{1-2s}\right)^2 + \frac{\eta^2 s}{1-1s}$$

Since

$$\frac{1}{\sqrt{2\pi}}\int_{-\eta}^{\infty} e^{-a(z-b)^2}dz = \frac{1}{\sqrt{2a}}$$

the above yields

$$E\{e^{s\underset{\sim}{z}^2}\} = \frac{1}{\sqrt{2(1/2-S)}}\exp\left\{\frac{\eta^2 S}{1-2S}\right\}$$

$$\Phi_w(s) = \frac{1}{\sqrt{1-2s}}\exp\left\{\frac{\eta_1 s}{1-2s}\right\}\cdots\frac{1}{\sqrt{1-2s}}\exp\left\{\frac{\eta_n s}{1-2s}\right\}$$

7-17 We wish to show that the RVs

$$\underset{\sim}{\bar{x}} = \frac{1}{n}\sum_{i=1}^{n}\underset{\sim}{x}_i \qquad \underset{\sim}{s}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\underset{\sim}{x}_i-\bar{x})^2$$

are independent. Since $\underset{\sim}{s}^2$ is a function of the n RVs $\underset{\sim}{x}_i-\underset{\sim}{\bar{x}}$, it suffices to show that each of these RVs is independent of $\underset{\sim}{\bar{x}}$. We assume for simplicity that $E\{\underset{\sim}{x}_i\}=0$. Clearly,

$$E\{\underset{\sim}{x}_i\bar{x}\} = \frac{1}{n}E\{\underset{\sim}{x}_i^2\} = \frac{\sigma^2}{n} \qquad E\{\underset{\sim}{\bar{x}}\underset{\sim}{\bar{x}}\} = \frac{1}{n^2}\sum_{i=1}^{n}\underset{\sim}{x}_i^2 = \frac{\sigma^2}{n}$$

because $E\{\underset{\sim}{x}_i\underset{\sim}{x}_j\}=0$ for $i\neq i$. Hence,

$$E\{(\underset{\sim}{x}_i-\underset{\sim}{\bar{x}})\underset{\sim}{\bar{x}}\} = 0$$

Thus, the RVs $\underset{\sim}{x}_i-\underset{\sim}{\bar{x}}$ and $\underset{\sim}{\bar{x}}$ are orthogonal; and since they are jointly normal, they are independent.

7-18 Since $\eta_s = \alpha_0 + \alpha_1\eta_1 + \alpha_2\eta_2$ [see (7-87)], the mean of the error

$$\underset{\sim}{\varepsilon} = \underset{\sim}{s} - (\alpha_0 + \alpha_1\underset{\sim}{x}_1 + \alpha_2\underset{\sim}{x}_2) = (\underset{\sim}{s} - \eta_s) - [\alpha_1(x_1 - \eta_1) + \alpha_2(\underset{\sim}{x}_2 - \eta_2)]$$

is zero. Furthermore, $\underset{\sim}{\varepsilon}$ is orthogonal to $\underset{\sim}{x}_i$, hence, it is also orthogonal to $\underset{\sim}{x}_i - \eta_i$.

7-19 From the orthogonality principle:

$$\hat{E}\{\underset{\sim}{y}|\underset{\sim}{x}_1,\underset{\sim}{x}_2\} = a_1\underset{\sim}{x}_1 + a_2\underset{\sim}{x}_2 \qquad \underset{\sim}{y} - \{a_1x_1 + a_2\underset{\sim}{x}_2\} \perp \underset{\sim}{x}_1, x_2$$

$$\hat{E}\{y|x_1\} = A\,\underset{\sim}{x}_1 \qquad y - A\,\underset{\sim}{x}_1 \perp \underset{\sim}{x}_1$$

Hence

$$\underset{\sim}{y} - (a_1\underset{\sim}{x}_1 + a_2\underset{\sim}{x}_2) - (\underset{\sim}{y} - A\,x_1) = a_1\underset{\sim}{x}_1 + a_2\underset{\sim}{x}_2 - A\,\underset{\sim}{x}_1 \perp \underset{\sim}{x}_1$$

From this it follows that

$$\hat{E}\{a_1x_1 + a_2\underset{\sim}{x}_2|\underset{\sim}{x}_1\} = A\,\underset{\sim}{x}_1$$

$$\hat{E}\{\hat{E}\{\underset{\sim}{y}|\underset{\sim}{x}_1,\underset{\sim}{x}_2\}|\underset{\sim}{x}_1\} = \hat{E}\{\underset{\sim}{y}|\underset{\sim}{x}_1\}$$

7-20 The event $\{\underset{\sim}{x} \le x\}$ occurs if there is at least one point in the interval $(0,x)$; the event $\{\underset{\sim}{y} \le y\}$ occurs if all the points are in the interval $(0,y)$:

$$A_x = \{\text{at least one point in } (0,x)\} = \{\underset{\sim}{x} \le x\}$$

$$B_y = \{\text{no points in } (y,1)\}$$

$$= \{\text{all points in } (0,y)\} = \{\underset{\sim}{y} \le y\}$$

Hence, for $0 \le x \le 1$, $0 \le y \le 1$

$$F_x(x) = P(A_x) = 1 - P(\bar{A}_x) = 1 - (1 - x)^n$$

$$F_y(y) = P(B_y) = y^n$$

Furthermore,

$$\{\underset{\sim}{x} \le x,\ \underset{\sim}{y} \le y\} = A_x B_y \qquad\qquad A_x B_y + \bar{A}_x B_y = B_y$$

If $x \le y$ then

$$\bar{A}_x B_y = \{\text{all points in } (x,y)\}$$

$$P(\bar{A}_x B_y) = (y - x)^n$$

If $x > y$, then $\bar{A}_x B_y = \{\emptyset\}$. Hence

$$F_{xy}(x,y) = P(A_x B_y) = \begin{cases} y^n - (y-x)^n & x \le y \\ y^n & x > y \end{cases}$$

7-21 Suppose that $E\{\underset{\sim}{x}_i\} = 0$, $E\{\underset{\sim}{x}_i^2\} = \sigma^2$, $E\{\underset{\sim}{x}_i^4\} = \mu_4$

If $\underset{\sim}{A} = \sum_{i=1}^{n} \underset{\sim}{x}_i^2$, then $E\{\underset{\sim}{A}\} = n\sigma^2$

$$E\{\underset{\sim}{A}^2\} = \sum_{i,j=1}^{n} E\{\underset{\sim}{x}_i^2 \underset{\sim}{x}_j^2\} = n\mu_4 + (n^2 - n)\sigma^4$$

because

$$E\{\underset{\sim}{x}_i^2 \underset{\sim}{x}_j^2\} = \begin{cases} \mu_4 & i = j \\ \sigma^4 & i \neq j \end{cases}$$

Furthermore

$$E\{\bar{\underset{\sim}{x}}^2 \underset{\sim}{x}_j^2\} = \frac{1}{n^2} E\left\{\left(\sum_{i=1}^{n} \underset{\sim}{x}_i\right)^2 \underset{\sim}{x}_j^2\right\} = \frac{1}{n^2}[\mu_4 + (n-1)\sigma^4]$$

$$E\{\bar{\underset{\sim}{x}}^2 \underset{\sim}{A}\} = \frac{1}{n}[\mu_4 + (n-1)\sigma^4]$$

$$E\{\bar{\underset{\sim}{x}}^4\} = \frac{1}{n^4} E\left\{\left(\sum_{i=1}^{n} \underset{\sim}{x}_i\right)^4\right\} = \frac{1}{n^4}[n\mu_4 + 3n(n-1)\sigma^4]$$

because

$$E\{\underset{\sim}{x}_i \underset{\sim}{x}_j \underset{\sim}{x}_k \underset{\sim}{x}_r\} = \begin{cases} \mu_4 & i = j = k = r & \text{[n such terms]} \\ \sigma^4 & i = j \neq k = r & \text{[3n(n-1) such terms]} \\ 0 & \text{otherwise} & \end{cases}$$

Clearly, $(n-1)\bar{\underset{\sim}{V}} = \sum_{i=1}^{n} (\underset{\sim}{x}_i - \bar{\underset{\sim}{x}})^2 = \underset{\sim}{A} - n\bar{\underset{\sim}{x}}^2$, $E\{\bar{\underset{\sim}{V}}\} = \sigma^2$. Hence

$$(n-1)^2 E\{\underset{\sim}{V}^2\} = E\{\underset{\sim}{A}^2\} - 2nE\{\bar{\underset{\sim}{x}}^2 \underset{\sim}{A}\} + n^2 E\{\bar{\underset{\sim}{x}}^4\}$$

$$= n\mu_4 + (n^2 - n)\sigma^4 - 2[\mu_4 + (n-1)\sigma^4] + \frac{1}{n}[\mu_4 + 3(n-1)\sigma^4]$$

This yields

$$E\{\bar{\underset{\sim}{V}}^2\} = \frac{\mu_4}{n} + \frac{n^2 - 2n + 3}{n(n-1)}\sigma^4 = \sigma^4 + \sigma_{\bar{V}}^2$$

<u>Note</u> If the RVs $\underset{\sim}{x}_i$ are $N(0,\sigma^2)$, then $\mu_4 = 3\sigma^4$

$$\sigma_{\bar{V}}^2 = \frac{1}{n}\left(3\sigma^4 - \frac{n-3}{n-1}\sigma^4\right) = \frac{2}{n-1}\sigma^4$$

7-22 From Prob. 6-49:

$$E\{|\underset{\sim}{x}_{2i} - \underset{\sim}{x}_{2i-1}|\} = \frac{2\sigma}{\sqrt{\pi}} \qquad E\{|\underset{\sim}{x}_{2i} - \underset{\sim}{x}_{2i-1}|^2\} = 2\sigma^2$$

Hence,

$$E\{|\underset{\sim}{x}_{2i} - \underset{\sim}{x}_{2i-1}|\,|\,\underset{\sim}{x}_{2j} - \underset{\sim}{x}_{2j-1}|\} = \begin{cases} 2\sigma^2 & i = j \\ 4\sigma^2/\pi & i \neq j \end{cases}$$

$$E\{\underset{\sim}{z}\} = \frac{\sqrt{\pi}}{2n}\,\frac{2\sigma n}{\sqrt{\pi}} = \sigma$$

$$E\{\underset{\sim}{z}^2\} = \frac{\pi}{4n^2}\,[2n\sigma^2 + \frac{4\sigma^2}{\pi}(n^2 - n)]$$

$$\sigma_z^2 = \frac{\pi}{2n}\sigma^2 + (1 - \frac{1}{n})\sigma^2 - \sigma^2 = \frac{\pi - 2}{2n}\sigma^2$$

7-23 If $R^{-1} = \begin{bmatrix} a_{11} \cdots a_{1n} \\ \\ a_{n1} \cdots a_{nn} \end{bmatrix}$ then $\sum_j a_{ij}R_{ji} = 1$

Hence,

$$E\{\underset{\sim}{X}R^{-1}\underset{\sim}{X}^t\} = E\{\sum_{i=1}^{n}\sum_{j=1}^{n} \underset{\sim}{x}_i a_{ij}\underset{\sim}{x}_j\}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}R_{ji} = \sum_{i=1}^{n} 1 = n$$

7-24 The density $f_z(z)$ of the sum $\underset{\sim}{z} = \underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_n$ tends to a normal curve with variance $\sigma_1^2 + \cdots + \sigma_n^2 \to \infty$ as $n \to \infty$ (we assume $\sigma_i > c > 0$). Hence, $f_z(z)$ tends to a constant in any interval of length 2π. The result follows as in (5-37) and Prob. 5-20.

7-25 Since $a_n - a \to 0$, we conclude that

$$E\{(\underset{\sim}{x}_n - a)^2\} = E\{[(\underset{\sim}{x}_n - a_n) + (a_n - a)]^2\}$$

$$= E\{(\underset{\sim}{x}_n - a_n)^2\} + 2(a_n - a)E\{\underset{\sim}{x}_n - a_n\} + (a_n - a)^2 \to 0$$

as $n \to \infty$.

7-26 If $E\{\underset{\sim}{x}_n \underset{\sim}{x}_m\} \to a$ as $n,m \to \infty$, then, given $\varepsilon > 0$, we can find a number n_o such that

$$E\{\underset{\sim}{x}_n \underset{\sim}{x}_n\} = a + \theta(n,m) \qquad |\theta| < \varepsilon \qquad \text{if } n,m > 0$$

Hence,

$$E\{(\underset{\sim}{x}_n - \underset{\sim}{x}_m)^2\} = E\{\underset{\sim}{x}_n^2\} + E\{\underset{\sim}{x}_m^2\} - 2E\{\underset{\sim}{x}_n \underset{\sim}{x}_m\}$$

$$= a + \theta_1 + a + \theta_2 - 2(a + \theta_3) = \theta_1 + \theta - 2\theta_3$$

and since $|\theta_1 + \theta_2 - 2\theta_3| < 4\varepsilon$ for any ε, it follows that

$E\{(\underset{\sim}{x}_n - \underset{\sim}{x}_m)^2\} \to 0$, hence (Cauchy) $\underset{\sim}{x}_n$ tends to a limit.

<u>Conversely</u> If $\underset{\sim}{x}_n \to \underset{\sim}{x}$ in the MS sense, then

$E\{(\underset{\sim}{x}_n - \underset{\sim}{x})^2\} \to 0$. Furthermore,

$E\{\underset{\sim}{x}_n^2\} \to E\{\underset{\sim}{x}^2\}$ $\qquad$ $E\{\underset{\sim}{x}\, \underset{\sim}{x}_n\} \to E\{\underset{\sim}{x}^2\}$

because (see Prob. 6-51)

$$E^2\{\underset{\sim}{x}_n^2 - \underset{\sim}{x}^2\} = E^2\{(\underset{\sim}{x}_n - \underset{\sim}{x})(\underset{\sim}{x}_n + \underset{\sim}{x})\}$$

$$\le E\{(\underset{\sim}{x}_n - \underset{\sim}{x})^2\}E\{(\underset{\sim}{x}_n + \underset{\sim}{x})^2\} \to 0$$

$$E^2\{\underset{\sim}{x}(\underset{\sim}{x}_n - \underset{\sim}{x})\} \le E\{\underset{\sim}{x}^2\}E\{(\underset{\sim}{x}_n - \underset{\sim}{x})^2\} \to 0$$

Similarly, $E\{(\underset{\sim}{x}_n - \underset{\sim}{x})(\underset{\sim}{x}_m - \underset{\sim}{x})\} \to 0$. Hence,

$$E\{\underset{\sim}{x}_n \underset{\sim}{x}_m\} + E\{\underset{\sim}{x}^2\} - E\{\underset{\sim}{x}\,\underset{\sim}{x}_n\} - E\{\underset{\sim}{x}\,\underset{\sim}{x}_m\} \to 0$$

Combining, we conclude that $E\{\underset{\sim}{x}_n \underset{\sim}{x}_m\} \to E\{\underset{\sim}{x}^2\}$.

7-27 $$E\{\underset{\sim}{x}_k\} = 0 \qquad E\{\underset{\sim}{x}_k^2\} = \sigma_k^2$$

$$E\left\{\left(\sum_{k=n_1}^{n_2} \underset{\sim}{x}_k\right)^2\right\} = \sum_{k=n_1}^{n_2} E\{\underset{\sim}{x}_k^2\}$$

If $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$, then given $\varepsilon > 0$, we can find n_o such that $\sum_{k=n+1}^{n+m} \sigma_k^2 < \varepsilon$ for any m and $n > n_o$. Thus

$$E\{(\underset{\sim}{y}_{n+m} - \underset{\sim}{y}_n)^2\} = E\left\{\left(\sum_{k=n+1}^{n+m} \underset{\sim}{x}_k\right)^2\right\} = \sum_{k=n+1}^{n+m} \sigma_k^2 < \varepsilon$$

This shows that (Cauchy), $\underset{\sim}{y}_k$ converges in the MS sense. The proof of the converse is similar.

7-28 If $f_i(x) = c\,e^{-cx}U(x)$ then $\Phi_i(s) = \dfrac{c}{c-s}$

$$\Phi(s) = \Phi_1(s) \cdots \Phi_n(s) = \frac{c^n}{(c-s)^n}$$

Hence (see Example 5-29) $f(x) = \dfrac{c^n x^{n-1}}{(n-1)!} e^{-cx}\, U(x)$

7-29 From Prob. 7-28 it follows that f(x) is the density of the sum $\underset{\sim}{x} = \underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_n$. Furthermore,

$$E\{\underset{\sim}{x}\} = \frac{n}{c} \qquad \sigma_x^2 = \frac{n}{c^2}$$

From the central limit theorem it follows, therefore, that for large n, the Erlang density is nearly equal to a normal curve with mean n/c and variance n/c^2.

7-30 $E\{\underset{\sim}{r}_i\} = 500 \qquad \sigma_i^2 = 50^2/3$

$\underset{\sim}{r} = \underset{\sim}{r}_1 + \underset{\sim}{r}_2 + \underset{\sim}{r}_3 + \underset{\sim}{r}_4 \qquad E\{\underset{\sim}{r}\} = 2{,}000 \qquad \sigma_r^2 = 10^4/3$

Thus, $\underset{\sim}{r}$ is approximately $N(2000;10^2/\sqrt{3})$

$P\{1900 \le \underset{\sim}{r} \le 2100\} = 2\,G\left(\frac{100\sqrt{3}}{100}\right) - 1 = 0.9169.$

7-31 The RVs $\underset{\sim}{x}_i$ are independent with (see Prob. 5-37)

$$f_i(x) = \frac{c_i}{\pi(c_i^2 + x^2)} \qquad \Phi_i(\omega) = e^{-c_i|\omega|}$$

In that case, (7-104) does not hold because

$$\int_{-\infty}^{\infty} x^\alpha f(x)\,dx = \frac{c_i}{\pi}\int_{-\infty}^{\infty} \frac{x^\alpha}{c_i^2 + x^2}\,dx = \infty \qquad \alpha > 2$$

In fact, the density of $\underset{\sim}{x} = \underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_n$ is Cauchy with parameter $c = c_1 + \cdots + c_n$ because

$$\Phi(\omega) = e^{-c_1|\omega|} \cdots e^{-c_n|\omega|} = e^{-(c_1 + \cdots + c_n)|\omega|}$$

7-32 In this problem, $\sigma_z^2 = E\{|\underset{\sim}{z}|^2\} = E\{\underset{\sim}{x}^2 + \underset{\sim}{y}^2\} = 2\sigma^2$

$$f_{\underset{\sim}{z}}(x) = f_x(x)f_y(y) = \frac{1}{2\pi\sigma^2}\,e^{-(x^2+y^2)/2\sigma^2} = \frac{1}{2\pi\sigma_z^2}e^{-|z|^2/\sigma_z^2}$$

$$\Phi_z(\Omega) = \Phi_x(u)\Phi_y(v) = \exp\left\{-\frac{1}{2}\sigma^2(u^2+v^2)\right\} = \exp\left\{-\frac{1}{4}\sigma_z^2|\Omega|^2\right\}$$

CHAPTER 8

8-1 (a) From (8-11) with γ=.95, u=.975, $z_{.975} \simeq 2$, σ=0.1, and n=9 we obtain

$$c = \frac{z_u \sigma}{\sqrt{n}} = 0.066$$

(b) From (8-11) with c=91.01-91=0.05mm:

$$z_u = \frac{c\sqrt{n}}{\sigma} = 1.5 \qquad u = .933 \qquad \gamma = .866$$

8-2 (a) From (8-11) with σ=1 and n=4: $\bar{x} \pm \sigma z_u/\sqrt{n} \simeq 203 \pm 1$mm

(b) From (8-12) with δ=.05: $c = \sigma/\sqrt{n\delta} = 2.236$mm

---

8-3 From (8-4) with γ=.9, u=.95: $\bar{x} \pm z_u\sigma/\sqrt{n} = 25{,}000 \pm 1{,}028$ miles

---

8-4 We wish to find n such that $P\{|\bar{x}-a|<0.2\} = 0.95$ where $a=E\{\bar{x}\}$. From (8-4) it follows with u=.975 and σ=0.1mm that

$$\frac{z_u \sigma}{\sqrt{n}} \le 0.2, \text{ hence, } n=1$$

8-5 In this problem, x is uniform with $E\{x\}=\theta$ and $\sigma^2=4/3$. We can use, however, the normal approximation for $\bar{x}$ because n=100. With γ=.95, (8-11) yields the interval

$$\bar{x} \pm z_{.975}\, \sigma\sqrt{n} = 30 \pm 0.227$$

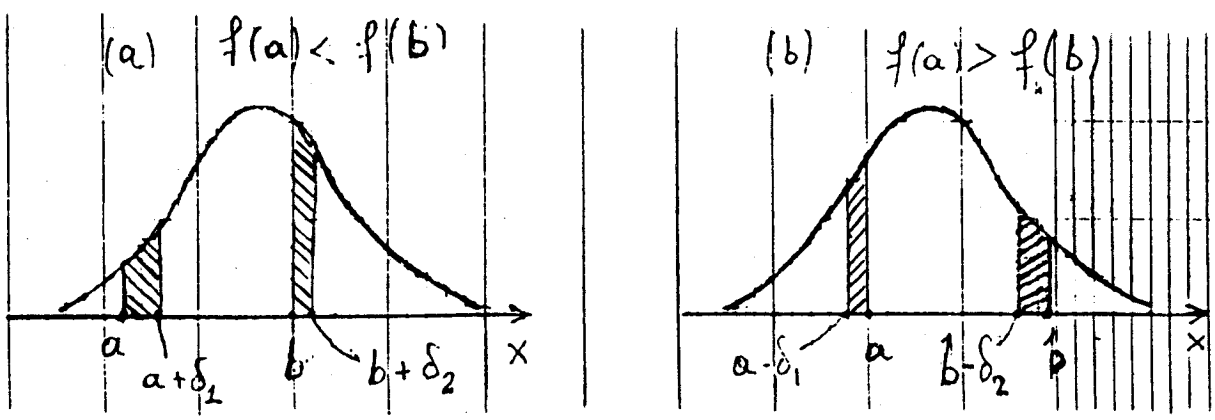
8-6 We shall show that if $f(x)$ is a density with a single maximum and $P\{a<x<b\}=\gamma$, then $b-a$ is minimum if $f(a)=f(b)$. The density $xe^{-x}U(x)$ is a special case. It suffices to show that $b-a$ is not minimum if $f(a)<f(b)$ or $f(a)>f(b)$.

Suppose first that $f(a)<f(b)$ as in figure (a). Clearly. $f'(a)>0$ and $f'(b)<0$, hence, we can find two constants $\delta_1>0$ and $\delta_2>0$ such that $P\{a+\delta_1 < x < b+\delta_2\} = \gamma$ and

$$f(a) < f(a+\delta_1) < f(b+\delta_2) < f(b)$$

From this it follows that $\delta_1 > \delta_2$, hence, the length of thenew interval $(a+\delta_1, b+\delta_2)$ is smaller than $b-a$.

If $f(a) > f(b)$, we form the interval $(a-\delta_1, b-\delta_2)$ (Fig. 8-6b) and proceed similarly.



<u>Special case.</u> If $f(x)=xe^{-x}$ then (see Problem 4-9) $F(x)=1-e^{-x}-xe^{-x}$, hence,

$$P\{a < x < b\} = e^{-a} + ae^{-a} - e^{-b} - be^{-b} = .95$$

And since $f(a)=f(b)$, the system

$$ae^{-a} = be^{-b} \qquad e^{-a} - e^{-b} = .95$$

results. Solving, we obtain $a \simeq 0.04$ $b \simeq 5.75$.

A numerically simpler solution results if we set

$$0.025 = P\{x \le a\} = F(a) \qquad 0.025 = P\{x > b\} = 1 - F(b)$$

as in (9-5). This yields the system

$$0.025 = 1 - e^{-a} - ae^{-a} \qquad 0.025 = e^{-b} + be^{-b}$$

Solving, we obtain $a=0.242$, $b=5.572$. However, the length $5.572-0.242=5.33$ of the resulting interval is larger than the length $4.75-0.04=4.71$ of the optimum interval.

8-7 We start with the general problem: We observe the n samples x_i of an $N(\eta,10)$ RV x and we wish to predict the value x of x at a future trial in terms of the average $\bar{x}$ of the observations. If η is known, we have an ordinary prediction problem. If it is unknown, we must first estimate it. To do so, we form the RV $w=x-\bar{x}$. This RV is

$N(0,\sigma_w)$ where $\sigma_w^2=\sigma_x^2+\sigma_{\bar{x}}^2=\sigma^2+\sigma^2/n$. With $c=z_{.975}\sigma_w\$$ it follows that $P\{|w| < c\}=.95$. Hence

$$P\{\bar{x} - c < x < \bar{x} + c\} = 0.95$$

For n=20 and σ=10 the above yields σ_w=10.25 and $c \simeq 20.5$. Thus, we can expect with .95 confidence coefficient that our bulb will last at least 80-20.5=59.5 and at most 80+20=100.5 hours.

8-8 The time of arrival of the 40th patient is the sum $x_1 + \cdots + x_n$ of n=39 RVs with exponential distribution. We shall estimate the mean $\eta=1/\theta$ of x in terms of its sample mean $\bar{x}=240/39=6.15$ minutes using two methods. The first is approximate (large n) and is based on (8-11).

<u>Normal approximation.</u> With $\lambda=\eta$ and $z_{.975}/\sqrt{39}=0.315$:

$$P\left\{\frac{\bar{x}}{1.315} < \eta < \frac{\bar{x}}{0.685}\right\} = .95 \qquad 4.68 < \eta < 8.98 \text{ minutes}$$

<u>Exact solution.</u> The RVs $\underset{\sim}{x}_i$ are i.i.d. with exponential distribution. From this and (7-52) it follows that their sum $\underset{\sim}{y}=\underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_n = n\underset{\sim}{\bar{x}}$ has an Erlang distribution:

$$\Phi_y(s) = \frac{\theta^n}{(\theta-s)^n} \qquad f_y(y) = \frac{\theta^n}{(n-1)!} y^{n-1} e^{-\theta y} U(y)$$

and the RV $\underset{\sim}{z}=2\theta\underset{\sim}{y} = 2n\theta\underset{\sim}{\bar{x}}$ has a $\chi^2(2n)$ distribution:

$$f_z(z) = \frac{1}{2\theta} f_y\left(\frac{z}{2\theta}\right) U(z) = \frac{z^{n-1}}{2^n(n-1)!} e^{-z/2} U(z)$$

Hence,

$$P\left\{\chi^2_{\delta/2}(2n) < \frac{2n\bar{x}}{\eta} < \chi^2_{1-\delta/2}(2n)\right\} = \gamma = 1-\delta$$

Since $\chi^2_{.025}(78) = 54.6$, $\chi^2_{.975}(78)=104.4$, and $2n\bar{x}=480$, this yields the interval

$$4.60 < \eta < 8.79 \text{ minutes}$$

8-9 From (8-19) with $\bar{x}=2{,}550/200=12.75$ $n=200$ and $z_u \simeq 2$

$$\lambda^2 - 25.52\,\lambda + 12.75^2 = 0 \qquad \lambda_1 = 12.255 < \lambda < 13.265 = \lambda_2$$

8-10 From (8-21) with $\bar{x}=2{,}080/4000=0.52$, $n=4{,}000$ and $z_u \simeq 2.326$.

$$p_{1,2} \approx \bar{x} \pm z_u \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = .52 \pm .018$$

Hence, $.502 < p < .538$.

8-11 (a) In this problem, $\bar{x}=0.40$, $n=900$ and $z_u \simeq 2$. From (8-21): Margin of error

$$\pm\ 100\ z_u\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = \pm\ 3.27\%$$

(b) We wish to find z_u. From (9-21) and Table 1a:

$$100 z_u\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = 2 \qquad z_u = 1.225 \qquad u = .89$$

This yields the confidence coefficient $\gamma = 2u - 1 = .78$

8-12 From (8-21) with $\bar{x}=0.29$ and $z_u=2$:

$$z_u\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = 0.04 \qquad\qquad n > \frac{\bar{x}(1-\bar{x})}{.04^2}\ z^2_u = 515$$

8-13 The problem is to find n such that [see (8-20)] $z_u\sqrt{\frac{p(1-p)}{n}} \le .02$ *for every p.* Since $z_u \simeq 2$ and $p(1-p) \le 1/4$, this is the case if

$$z_u\sqrt{1/4n} \le .02 \qquad n \ge 2{,}500$$

8-14 From (8-36) with k=1

$$f(p) = \begin{cases} 5 & .4 < p < .6 \\ 0 & \text{otherwise} \end{cases} \qquad P\{k=1\} = 5\int_{.4}^{.6} p\,dp = .5 = \frac{1}{\gamma}$$

$$f_p(p|1) = \begin{cases} 10p & .4 < p < .6 \\ 0 & \text{otherwise} \end{cases} \qquad \hat{p} = 10\int_{.4}^{.6} p^2 dp = .5067$$

8-15 From Prob. 8-8: $f_{\bar{x}}(\bar{x}|\theta) = \frac{(\theta n)^n}{(n-1)!}\bar{x}^{n-1}e^{-n\theta\bar{x}}$

From (8-32): $$f_\theta(\theta\,|\bar{x}) = \frac{(c+n\bar{x})^{n+1}}{n!}\theta^n e^{-(c+n\bar{x})\theta}$$

From (8-31): $$\hat{\theta} = \frac{(c+n\bar{x})^{n+1}}{n!}\int_0^\infty \theta^{n+1}e^{-(c+n\bar{x})\theta}d\theta = \frac{n+1}{c+n\bar{x}}$$

8-16 The sum $n\bar{x}$ is a Poisson RV with mean $n\theta$ (see Prob. 8-8). In the context of Bayesian estimation, this means that

$$f_{\bar{x}}(\bar{x}|\theta) = e^{-n\theta}\frac{(n\theta)^k}{k!} \qquad k = n\bar{x} = 0,1,\ldots$$

Inserting into (8-32), we obtain [see (4-76)]

$$f_\theta(\theta|\bar{x}) = \frac{(n+c)^{n\bar{x}+b+1}}{\Gamma(n\bar{x}+b+a)}\theta^{n\bar{x}+b}e^{-(n+c)\theta}$$

and (8-31) yields

$$\hat{\theta} = \frac{(n+c)^{n\bar{x}+b+1}}{\Gamma(n\bar{x}+b+1)}\frac{\Gamma(n\bar{x}+b+2)}{(n+c)^{n\bar{x}+b+2}} = \frac{n\bar{x}+b+1}{n+c} \xrightarrow[n\to\infty]{} \bar{x}$$

8-17 From (8-17) with $t_{.95}(9)=2.26$

$$\bar{x} \pm \frac{t_u s}{\sqrt{n}} = 90 \pm 3.57 \qquad 86.43 < \eta \cdot 93.57$$

From (8-24) with $\chi^2_{.975}(9)=19.02$, $\chi^2_{.025}(9)=2.70$.

$$\frac{9 \times 5^2}{19.02} = 11.83 < \sigma^2 < \frac{9 \times 5^2}{2.70} = 83.33 \qquad 3.44 < \sigma < 9.13$$

8-18 The RVs x_i/σ are $N(0,1)$, hence, the sum $z=(x_1^2 + \cdots + x^2_{10})/\sigma^2$ has a $\chi^2(10)$ distribution. This yields

$$P\{\chi^2_{.025}(10) < z < \chi^2_{.975}(10)\} = .95$$

$$\chi^2_{.025}(10) = 3.25 < \frac{4}{\sigma^2} < \chi^2_{.975}(10) = 20.48$$

$$0.442 < \sigma < 1.109$$

8-19 From (8-23) with $n=4, \chi^2_{.025}(4)=0.48,\ \chi^2_{.975}(4)=11.14$

$$n\hat{v} = .1^2 + .15^2 + .05^2 + .04^2 = .0366$$

$$\frac{.0366}{.048} > \sigma^2 > \frac{.0366}{11.14} \qquad 0.276 > \sigma > 0.057$$

8-20 In this problem $n=3,\ x_1+x_2+x_3=9.8$

$$f(x,c) \sim c^4x^3e^{-cx} \qquad f(X,c) = c^{4n}(x_1 \ldots x_n)^{3n}e^{-cn\bar{x}}$$

$$\frac{\partial f(X,c)}{\partial c} = \left(\frac{4n}{c} - n\bar{x}\right)f(X,\theta) = 0 \qquad \hat{c} = \frac{4}{\bar{x}} = 1.224$$

8-21 The joint density

$$f(X,c) = c^n e^{-cn(\bar{x}-x_0)} \qquad x_i > x_0$$

has an interior maximum if

$$\frac{\partial f(X,c)}{\partial c} = 0 \qquad \hat{c} = \frac{1}{\bar{x}-x_0}$$

8-22 The probability

$$p = 1 - F_x(200) = e^{-200c}$$

of the event $\{x > 200\}$ is a monoton decreasing function of c. To find the ML estimate $\hat{c}$ of c it suffices to find the ML estimate $\hat{p}$ of p. From Example 8-28 it follows with k=62 and n=80 that

$$\hat{p} = \frac{62}{80} = .775 \text{ hence}$$

$$\hat{c} = -\frac{1}{200} \ln \hat{p} = 0.0013$$

8-23 The samples of x are the integers x_i and the joint density of the RVs x_i equals

$$f(X,\theta) = e^{-n\theta} \prod \frac{\theta^{x_i}}{x_i!} = e^{-n\theta} \frac{\theta^{n\bar{x}}}{\prod x_i!}$$

Hence, $f(X,\theta)$ is maximum if $-n + n\bar{x}/\theta = 0$. This yields $\hat{\theta} = \bar{x}$

8-24 If $L = \ln f(x,\theta)$ then

$$\frac{\partial L}{\partial \theta} = \frac{1}{f}\frac{\partial f}{\partial \theta} \qquad \frac{\partial^2 L}{\partial \theta^2} = \frac{1}{f}\frac{\partial^2 f}{\partial \theta^2} - \frac{1}{f^2}\left(\frac{\partial f}{\partial \theta}\right)^2 \qquad \frac{\partial^2 L}{\partial \theta^2} + \left(\frac{\partial L}{\partial \theta}\right)^2 = \frac{1}{f}\frac{\partial^2 f}{\partial \theta^2}$$

But

$$E\left\{\frac{1}{f}\frac{\partial^2 f}{\partial \theta^2}\right\} = \int_R \frac{1}{f}\frac{\partial^2 f}{\partial \theta^2} f dX = 0 \quad \text{hence} \quad E\left\{\frac{\partial^2 L}{\partial \theta^2} + \left(\frac{\partial L}{\partial \theta}\right)^2\right\} = 0$$

8-25 (a) From (8-307): Critical region

$$\bar{x} > c = \eta_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}} = 8+2.326 \times \frac{2}{8} = 8.58$$

If η=8.7, then $\eta_q = \frac{8.7-8}{218} = 2.8$

$$\beta(\eta) = G(2.36 - 2.8) = .32$$

(b) We assume that α=.01, β(8.7)=.05 and wish to find n and c.

$$G(z_{1-\alpha}-\eta_q) = \beta \qquad z_{1-\alpha}-\eta_q = z_\beta$$

$$\eta_q = z_{.99} - z_{.05} = 4.97 = \frac{8.7-8}{2/\sqrt{n}}$$

$$n = 129 \qquad c = 8 + \frac{2}{\sqrt{129}} z_{.99} = 8.41$$

8-26 Our objective is to test the composite null hypothesis $\eta \geq \eta_0$=28 against the hypothesis $\eta < \eta_0$. Consider first the simple null hypothesis $\eta = \eta_0$=28. In this case, we can use (8-301) with

$$q = \frac{\bar{x}-\eta_0}{s/\sqrt{n}} \qquad \bar{x} = \frac{1}{17}\sum x_i = 27.67 \qquad s^2 = \frac{1}{16}\sum(x_i-\bar{x}) = 17.6$$

This yields s=4.2 and q=-0.33. Since

$$q_u = t_u(n-1) = t_{0.05}(16) = -1.95 < -0.33$$

we conclude that the evidence does not support the rejection of the hypothesis η=28. The resulting OC function $\beta_0(\eta)$ is determined from (9-60c).

If η_0>28, then the corresponding value of q is larger than -0.33. From this it follows that the evidence does not support the

hypothesis η_0 for any $\eta_0>28$. We note, however, that the corresponding OC function $\beta(\eta)$ is smaller than the function $\beta_0(\eta)$ obtained from (8-301) with $\eta_0=28$.

8-27 From (8-297) with $q_u=t_u(n-1)$: Critical region $|\bar{x}-\eta_0| > t_{1-\alpha/2}(n-1)s/\sqrt{n}$

1. $\underline{\alpha = .1}$ $\quad t_{.95}(63) = 1.67 \quad |\bar{x}-8| > 1.67 \times 1.5/8 = 0.313$

Since $\bar{x}=7.7$ is in the interval 8 ± 0.317, we accept H_0

2. $\underline{\alpha = .01}$ $\quad t_{.995}(63) = 2.62 \quad |\bar{x}-8| > 2.62 \times 1.5/8 = 0.49$

Since $\bar{x}=7.7$ is outside the interval 8 ± 0.49, we reject H_0.

8-28 We assume that the RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are normal and independent. We form the difference $\underset{\sim}{w}=\underset{\sim}{\bar{x}}-\underset{\sim}{\bar{y}}$ of their sample means

$$\underset{\sim}{\bar{x}} = \frac{1}{16}\sum_{i=1}^{16} \underset{\sim}{x}_1 \qquad \underset{\sim}{\bar{y}} = \frac{1}{26}\sum_{i=1}^{26} \underset{\sim}{y}_i$$

and use as test statistic the ratio

$$\underset{\sim}{q} = \frac{\underset{\sim}{w}}{\sigma_w} \qquad \sigma_w^2 = \frac{\sigma_x^2}{16} + \frac{\sigma_y^2}{26}$$

The RV $\underset{\sim}{q}$ is normal with $\sigma_q=1$ and under hypothesis H_0, $E\{\underset{\sim}{q}\}=0$. We can,

therefore, use (8-307) because $q_u=z_u$. To find q, we must determine σ_w. Since σ_x and σ_y are not specified, we shall use the approximations $\sigma_x \simeq s_x=1.1$ and $\sigma_y \simeq s_y=0.9$. This yields

$$\sigma_w^2 \simeq \frac{1.1^2}{16} + \frac{0.9^2}{26} = 0.107 \qquad q = \frac{\bar{x}-\bar{y}}{\sigma_w} = \frac{0.4}{0.327} = 1.223$$

Since $z_{0.95}=1.645 > 1.223$, we accept H_0.

8-29 (a) In this problem, n=64, k=22, $p_0=q_0=0.5$

$$q = \frac{k-np_0}{\sqrt{np_0q_0}} = 2.5 \qquad z_{\alpha/2} = -z_{1-\alpha/2} \simeq -2$$

Since 2.5 is outside the interval (2, -2), we reject the fair coin hypothesis [see (8-313)].

(b) From (8-313) with n=16, $p_0=q_0=0.5$:

$$\frac{k_1-np_0}{\sqrt{np_0q_0}} = z_{\alpha/2} \qquad \frac{k_2-np_0}{\sqrt{np_0q_0}} = -z_{\alpha/2}$$

This yields $k_1=8-2\times2=4$, $k_2=8+2\times2=12$

8-30 We shall use as test statistic the sum

$$\underset{\sim}{q} = \underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_m \qquad n = 22$$

The critical region of the test is $q<q_\alpha$ where $q=x_1+\cdots+x_n=90$ [see (8-301)].

The RV $\underset{\sim}{q}$ is Poisson distributed with parameter $n\lambda$. Under hypothesis H_0, $\lambda=\lambda_0=5$; hence, $\eta_q=n\lambda_0=110=\sigma_q^2$. To find q_α we shall use the normal approximation. With $\alpha=0.05$ this yields

$$q_\alpha = n\lambda_0 + z_\alpha \sqrt{n\lambda_0} = 90-17.25 = 72.75$$

Since 90 > 72.75, we accapt the hypothesis that $\lambda=5$.

8-31 From (9-75) with n=102 and $p_{0i}=1/6$

$$q = \sum_{i=1}^{6} \frac{(k_i-17)^2}{17} = 2 \qquad \chi^2_{.95}(5) \simeq 11$$

Since 2<11, we accept the fair die hypothesis.

8-32 Uniformly distributed integers from 0 to 9 means that they have the same probability of appearing. With m=10, $p_{01}=.1$, and n=1,000, it follows from (8-325) that

$$q = \sum_{j=0}^{9} \frac{(n_j-100)^2}{100} = 17.76 \qquad \chi^2_{.95}(9) = 16.92$$

Since 17.76 > 16.92, we reject the uniformity hypothesis.

8-33 In this problem

$$f(x,\theta) = e^{-\theta}\frac{\theta^x}{x!} \qquad f(X,\theta) = \frac{e^{-n\theta}\,\theta^{n\bar{x}}}{x_1!\cdots x_n!}$$

$f(X,\theta)$ is maximum for $\theta=\theta_m=\bar{x}$. And since $\theta_{m0}=\theta_0$ we conclude that

$$\lambda(X) = \frac{e^{-n\theta_0}\theta_0^{n\bar{x}}}{e^{-n\bar{x}}\bar{x}^{n\bar{x}}} \qquad w = -2\ln\lambda = 2n(\theta_0-\bar{x}) + \bar{x}\ln(\bar{x}/\theta_0)$$

With $n=50$, $\theta_0=20$, $\bar{x}=1{,}058/50=21.16$, this yields $w=3$. Since $m_0=1$, $m=1$, and $\chi^2_{.95}(1)=3.84>3$, we accept H_0.

8-34 We form the RVs

$$\underset{\sim}{z} = \sum_{i=1}^{m}\left(\frac{x_i-\eta_x}{\sigma_x}\right)^2 \qquad \underset{\sim}{w} = \sum_{i=1}^{n}\left(\frac{y_i-\eta_y}{\sigma_y}\right)^2$$

These RVs are $\chi^2(m)$ and $\chi^2(n$ respectively. If $\sigma_x=\sigma_y$, then

$$\underset{\sim}{q} = \frac{z/m}{\underset{\sim}{w}/n}$$

Hence (see Prob. 6-23), $\underset{\sim}{q}$ has a Snedecor distribution. To test the hypothesis $\sigma_x=\sigma_y$, we use (8-297) where $q_u=F_u(m,n)$ is the tabulated u percentile of the Snedecor distribution. This yields the following test:

$$\text{Accept } H_o \text{ iff} \qquad F_{\alpha/2}(m,n) < q < F_{1-\alpha/2}(m,n).$$

8-35 If $\underset{\sim}{x}$ has a student-t distribution, then $f(-x)=f(x)$, hence (see Prob. 6-75)

$$E\{\underset{\sim}{x}\} = 0 \qquad \sigma_x^2 = E\{\underset{\sim}{x}^2\} = \frac{n}{n-2}$$

8-36 (a) Suppose that the probability P(A) that player A wins a set equals $p=1-q$. He wins the match in five sets if he wins two of the first four sets and the fifth set. Hence, the probability $p_5(A)$ that he wins in five equals $6p^3q^2$. Similarly, the probability $p_5(B)$ that player B wins in five equals $6p^2q^3$. Hence,

$$p_5 = p_5(A) + p_5(B) = 6p^3q^2 + 6p^2q^3 = 6p^2q^2$$

is the probability that the match lasts five sets. If p=q=1/2, then p_5=3/8.

(b) Suppose now that P(A) = $\underset{\sim}{p}$ is an RV with density f(p). In this case,

$$\underset{\sim}{p}_5 = 6\underset{\sim}{p}^2(1-\underset{\sim}{p}^2)$$

is an RV. We wish to find its best bayesian estimate. Using the MS criterion, we obtain

$$\hat{p}_5 = E\{\underset{\sim}{p}_5\} = \int_0^1 6p^2(1-p^2)f(p)dp$$

If f(p)=1, then $\hat{p}_5 = 1/5$.

8-37 Given

$$f_v(v) \sim e^{-v^2/2\sigma^2} \qquad f_\theta(\theta) \sim e^{-(\theta-\theta_o)^2/2\sigma_o^2}$$

To show that

$$f_\theta(\theta|x) \sim e^{-(\theta-\theta_1)^2/2\sigma_1^2}$$

where

$$\frac{1}{\sigma_1^2} \equiv \frac{1}{\sigma_o^2} + \frac{n}{\sigma^2} \qquad \theta_1 \equiv \frac{\sigma_1^2}{\sigma_o^2}\theta_o + \frac{n\sigma_1^2}{\sigma^2}\bar{x}$$

<u>Proof</u>

$$f_x(x|\theta) = f_v(x-\theta) \sim \exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\}$$

$$f(X)|\theta \sim \exp\left\{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2\right\}$$

Since $\sum(x_i-\theta)^2 = \sum(x_i-\bar{x})^2 + n(\bar{x}-\theta)^2$, we conclude from (8-32) omitting factors that do not depend on θ that

$$f(\theta|X) \sim \exp\left\{-\frac{1}{2}\left[\frac{(\theta-\theta_o)^2}{\sigma_o^2} + \frac{n(\bar{x}-\theta)^2}{\sigma^2}\right]\right\}$$

The above bracket equals

$$\left(\frac{1}{\sigma_o^2} + \frac{n}{\sigma^2}\right)\theta^2 - 2\left(\frac{\theta_o}{\sigma_o^2} + \frac{n\bar{x}}{\sigma^2}\right)\theta + \cdots = \frac{1}{\sigma_1^2}(\theta^2 - 2\theta\theta_1) + \cdots$$

and (i) follows.

8-38 The likelihood function of X equals

$$f(X,\theta) = \frac{1}{(\sqrt{2\pi\theta})^n} \exp\left\{-\frac{1}{2\theta}\sum (x_i-\eta)^2\right\}$$

where $\theta = \sigma^2$ is the unknown parameter. Hence

$$L(X,\theta) = -\frac{n}{2}\ell n\,(2\pi\theta) - \frac{1}{2\theta}\sum (x_i-\eta)^2$$

$$\frac{\partial L(X,\theta)}{\partial\theta} = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum (x_i-\theta)^2 = 0 \qquad \hat{\theta} = \frac{1}{n}\sum (x_i-\eta)^2$$

8-39 The estimators $\hat{\underset{\sim}{\theta}}_1$ and $\hat{\underset{\sim}{\theta}}_2$ have the same variance because otherwise one or the other would not be best. Thus

$$E\{\hat{\underset{\sim}{\theta}}_1\} = E\{\hat{\underset{\sim}{\theta}}_2\} = \theta \qquad \text{var}\,\hat{\underset{\sim}{\theta}}_1 = \text{var}\,\hat{\underset{\sim}{\theta}}_2 = \sigma^2$$

If $\hat{\underset{\sim}{\theta}} = \frac{1}{2}(\hat{\underset{\sim}{\theta}}_1 + \hat{\underset{\sim}{\theta}}_2)$, then

$$E\{\hat{\underset{\sim}{\theta}}\} = \theta \qquad \sigma_{\hat{\underset{\sim}{\theta}}}^2 = \frac{1}{2}(\sigma^2 + \sigma^2 + 2r\sigma^2) = \frac{1}{2}(1+r)\sigma^2$$

where σ is the correlation coefficient of $\hat{\underset{\sim}{\theta}}_1$ and $\hat{\underset{\sim}{\theta}}_2$. If $r<1$ then $\sigma_{\hat{\theta}}<\sigma$ which is impossible. Hence, $r=1$ and $\hat{\underset{\sim}{\theta}}_1 = \hat{\underset{\sim}{\theta}}_2$ (see Prob. 6-53).

8-40 $k_1+k_2-np_1-np_2 = n-n(p_1+p_2) = 0$; Hence, $|k_1-np_1| = |k_2-np_2|$

$$\frac{(k_1-np_1)^2}{np_1} + \frac{(k_2-np_2)^2}{np_2} = (k_1-np_1)^2\left(\frac{1}{np_1} + \frac{1}{np_2}\right) = \frac{(k_1-np_1)^2}{np_1p_2}$$

8.41 It is given that

$$E\{T(X)\} = \int_{-\infty}^{\infty} T(X)\, f(X;\theta)\, dx = \psi(\theta),$$

so that after differentiating and making use of (8-81) we get

$$\int_{-\infty}^{\infty} T(X)\, \frac{\partial f(X;\theta)}{\partial \theta}\, dx = \psi^{'}(\theta) \qquad (8.41-1)$$

Also using (8-80)

$$\int_{-\infty}^{\infty} \psi(\theta)\, \frac{\partial f(X;\theta)}{\partial \theta}\, dx = 0, \qquad (8.41-2)$$

and the above two expressions give

$$\int_{-\infty}^{\infty} [T(X) - \psi(\theta)]\, \frac{\partial f(X;\theta)}{\partial \theta}\, dx = \psi^{'}(\theta) \qquad (8.41-3)$$

But

$$\frac{\partial f(X;\theta)}{\partial \theta} = \frac{1}{f(X;\theta)}\, \frac{\partial \log f(X;\theta)}{\partial \theta}$$

so that (8.41-3) simplifies to

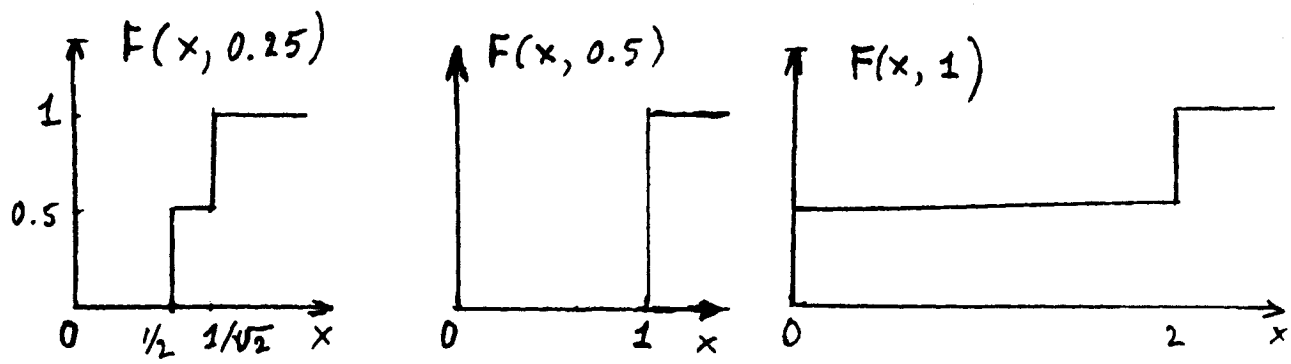
$$\int_{-\infty}^{\infty} \left[\{T(X) - \psi(\theta)\}\sqrt{f(X;\theta)}\right] \left[\sqrt{f(X;\theta)}\ \frac{\partial \log f(X;\theta)}{\partial \theta}\right] dx = \psi^{'}(\theta)$$

and application of Cauchy-Schwarz inequality as in (8-89)-(8-92), Text gives

$$E\left[\{T(X) - \psi(\theta)\}^2\right] \geq \frac{[\psi^{'}(\theta)]^2}{E\left\{\left(\frac{\partial \log f(X;\theta)}{\partial \theta}\right)^2\right\}}$$

9-1 (a) $E\{\underset{\sim}{x}(t)\} = t + 0.5 \sin \pi t$

$$\underset{\sim}{x}(t, \text{heads}) = \sin \pi t = \begin{cases} 1/\sqrt{2} & t = 0.25 \\ 1 & t = 0.5 \\ 0 & t = 1 \end{cases} \qquad \underset{\sim}{x}(t, \text{tails}) = 2t = \begin{cases} 0.5 \\ 1 \\ 2 \end{cases}$$



9-2

$$\underset{\sim}{x}(t) = e^{\underset{\sim}{a}t}$$

$$\eta(t) = \int_{-\infty}^{\infty} e^{at} f_a(a)da \qquad R(t_1,t_2) = \int_{-\infty}^{\infty} e^{at_1} e^{at_2} f_a(a)da$$

From (5-16) with $x = g(a) = e^{ta}$ $\quad g'(a) = t\, e^{ta} = tx$

$$f(x,t) = \frac{1}{x|t|} f_a\left(\frac{1}{t} \ln x\right)U(x)$$

9-3 As we know, $E\{\underset{\sim}{x}(t)\} = \lambda t$ and var $\underset{\sim}{x}(t) = \lambda^2 t^2$ [see (9-18)]. But $E\{\underset{\sim}{x}(9) = 6\}$ by assumption, hence, $\lambda = 2/3$

(a) $E\{\underset{\sim}{x}(8)\} = 24$ var $\underset{\sim}{x}^2(t) = 24^2$

(b) The RV $\underset{\sim}{x}(2)$ is Poisson distributed with parameter $2\lambda = 6$. Hence,

$$P\{\underset{\sim}{x}(2) \le 3\} = e^{-2\lambda} \sum_{k=0}^{3} \frac{(2\lambda)^k}{k!}$$

(c) The RVs $\underset{\sim}{z} = \underset{\sim}{x}(2)$ and $\underset{\sim}{w} = \underset{\sim}{x}(4) - \underset{\sim}{x}(2)$ are independent and Poisson distributed with parameter 2λ. Hence,

$$P\{\underset{\sim}{z}=k\} = e^{-2\lambda} \frac{(2\lambda)^k}{k!} \qquad P\{\underset{\sim}{z} = k, \underset{\sim}{w} = m\} = e^{-4\lambda} \frac{(2\lambda)^k}{k!} \frac{(2\lambda)^m}{m!}$$

$$P\{\underset{\sim}{x}(4) \le 5 \mid \underset{\sim}{x}(2) \le 3\} = \frac{P\{z\le 3,\ w\le 5-z\}}{P\{\underset{\sim}{z}\le 3\}} \qquad P\{\underset{\sim}{z} \le 3\} = \sum_{k=0}^{3} p\{\underset{\sim}{z}=k\}$$

$$P\{\underset{\sim}{z} \le 3, \underset{\sim}{w} \le 5 - \underset{\sim}{z}\} = \sum_{k=0}^{3} \sum_{m=0}^{5-k} P\{\underset{\sim}{z} = k, \underset{\sim}{w} = m\}$$

9-4 $\underset{\sim}{x}(t) = U(t - \underset{\sim}{c}) \qquad \underset{\sim}{y}(t) = \delta(t - \underset{\sim}{c}) = \underset{\sim}{x}'(t)$

For t_1 or $t_2 < 0$, $R(t_1,t_2) = 0$; for t_1 and $t_2 > T$, $R(t_1,t_2) = 1$. Otherwise,

$$R_x(t_1,t_2) = \frac{1}{T}\min(t_1,t_2) \qquad \frac{\partial R_x}{\partial t_1} = \frac{1}{T} U(t_1 - t_2) \qquad -\frac{\partial^2 R_x}{\partial t_1 \partial t_2} = \frac{1}{T}\,\delta(t_1 - t_2)$$

From this and (9-105) it follows that $T\,R_y(t_1 - t_2) = \delta(t_1 - t_2)$ for $0 < t_1, t_2 < T$ and 0 otherwise.

9-5 $\underset{\sim}{a} - \underset{\sim}{b}\,t = 0$ iff $t = \underset{\sim}{t}_1 = \underset{\sim}{a}/\underset{\sim}{b}$. Setting $\sigma_1 = \sigma_2 = \sigma$ and $r = 0$ in (6-63), we obtain

$$P\{0 < \underset{\sim}{t}_1 < T\} = \frac{1}{2} + \frac{1}{\pi}\arctan T - \left(\frac{1}{2} + \frac{1}{\pi}\arctan 0\right)$$

9-6 The equations

$$\underset{\sim}{w}''(t) = \underset{\sim}{v}(t)U(t) \qquad \underset{\sim}{w}(0) = \underset{\sim}{w}'(0) = 0$$

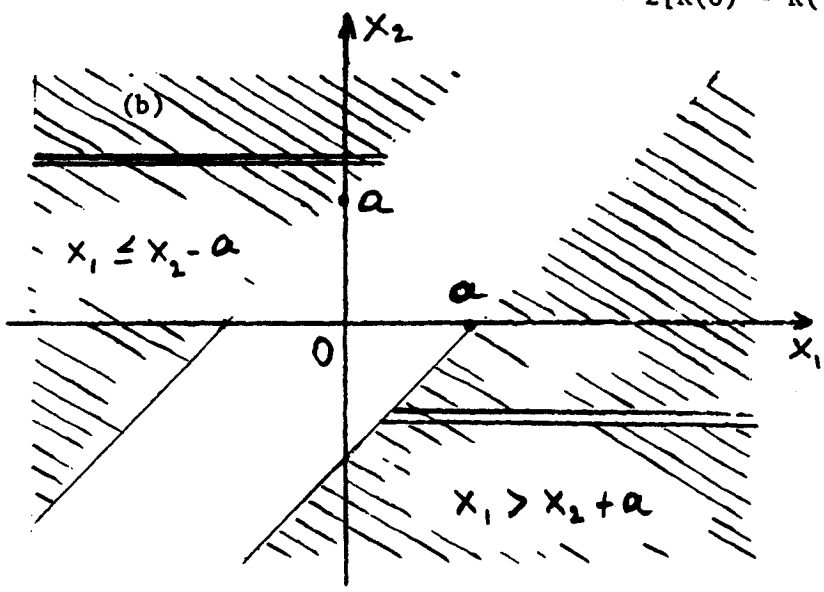
specify a system with input $\underset{\sim}{v}(t)U(t)$ and impulse response $h(t) = t\,U(t)$. Hence [see (9-100)]

$$E\{\underset{\sim}{w}^2(t)\} = q(t)U(t) * t^2U(t) = \int_0^t (t-\tau)^2 q(\tau)\,d\tau$$

9-7 (a) From (5-88) with $\underset{\sim}{x} = \underset{\sim}{x}(t+\tau) - \underset{\sim}{x}(t)$, and (8-101):

$$P\{|\underset{\sim}{x}(t+\tau) - \underset{\sim}{x}(t)| \geq a\} \leq \frac{E\{[\underset{\sim}{x}(t+\tau) - \underset{\sim}{x}(t)]^2\}}{a^2}$$

$$= 2[R(0) - R(\tau)]/a^2$$



The above probability equals the mass in the regions (shaded) $x_2 - x_1 > a$ and $x_2 - x_1 < -a$ Hence,

$$P\{|\underset{\sim}{x}(t+\tau) - \underset{\sim}{x}(t)| \geq a\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x_2 - a} f(x_1, x_2; \tau)\,dx_1\,dx_2 + \int_{-\infty}^{\infty}\int_{x_2 + a}^{\infty} f(x_1, x_2; \tau)\,dx_1\,dx_2$$

9-8 (a) The RV $\underset{\sim}{x}(t)$ is normal with zero mean and variance $E\{\underset{\sim}{x}^2(t)\} = R(0)=4$, hence it is $N(0,2)$ and $P\{\underset{\sim}{x}(t)\le 3\} = F(3) = G(1.5) = 0.933$

(b) $E\{[\underset{\sim}{x}(t+1) - \underset{\sim}{x}(t-1)]\} = 2[R(0)-R(2)] = 8(1-e^{-4})$

9-9 If $\underset{\sim}{x}(t) = \underset{\sim}{c}e^{j(\omega t+\theta)}$ and $\eta_c = 0$ then

$$\eta_x(t) = \eta_c e^{j(\omega t+\theta)} = 0 \qquad R_{xx}(t+\tau,t) = \sigma_c^2 e^{j\omega\tau}$$

hence, $\underset{\sim}{x}(t)$ is WSS. We shall prove the converse:

If the process $\underset{\sim}{x}(t) = \underset{\sim}{c}w(t)$ is WSS, then $\eta_c=0$ and $w(t) = e^{j(\omega t+\theta)}$ within a constant factor.

<u>Proof</u> $\eta_x(t) = \eta_c w(t)$ is independent of t; hence, $\eta_c=0$. The function $R_{xx}(t_1,t_2) = \sigma_c^2 w(t_1)w^*(t_2)$ depends only on $\tau=t_1-t_2$; hence, $w(t+\tau)w^*(t)=g(\tau)$. With $\tau=0$ this yields

$$|w(t)|^2 = g(0) = \text{constant} \qquad w(t) = ae^{j\phi(t)}$$

$$w(t+\tau)w^*(t) = a^2 e^{j[\phi(t+\tau)-\phi(t)]}$$

Hence the difference $\phi(t+\tau)-\phi(t)$ depends only on τ:

$$\phi(t+\tau)-\phi(t) = f(\tau) \tag{i}$$

From this it follows that, if $\phi(t)$ is continuous then, $\phi(t)$ is a linear function of t. To simplify the proof, we shall assume that $\phi(t)$ is differentiable. Differentiating with respect to t, we obtain $\phi'(t+\tau) = \phi'(t)$ for every τ. With t=0 this yields

$$\phi'(\tau) = \phi'(0) = \text{constant} \qquad \phi(\tau) = a\tau+b$$

9-10 We shall show that if $\underset{\sim}{x}(t)$ is a normal process with zero mean and $\underset{\sim}{z}(t) = \underset{\sim}{x}^2(t)$, then $C_{zz}(\tau) = 2C_{xx}^2(\tau)$.

From (7-61): If the RVs $\underset{\sim}{x}_k$ are normal and $E\{\underset{\sim}{x}_k\}=0$, then

$$E\{\underset{\sim}{x}_1\underset{\sim}{x}_2\underset{\sim}{x}_3\underset{\sim}{x}_4\} = E\{\underset{\sim}{x}_1\underset{\sim}{x}_2\}\, E\{\underset{\sim}{x}_3\underset{\sim}{x}_4\} + E\{\underset{\sim}{x}_1\underset{\sim}{x}_3\}\, E\{\underset{\sim}{x}_2\underset{\sim}{x}_4\} + E\{\underset{\sim}{x}_1\underset{\sim}{x}_4\}\, E\{\underset{\sim}{x}_2\underset{\sim}{x}_3\}$$

With $\underset{\sim}{x}_1=\underset{\sim}{x}_2=\underset{\sim}{x}(t+\tau)$ and $\underset{\sim}{x}_3 = \underset{\sim}{x}_4 = \underset{\sim}{x}(t)$, we conclude that the autocorrelation of $\underset{\sim}{z}(t)$ equals

$$E\{\underset{\sim}{x}^2(t+\tau)\underset{\sim}{x}^2(t)\} = E^2\{\underset{\sim}{x}^2(t+\tau)\} + 2E^2\{\underset{\sim}{x}(t+\tau)\underset{\sim}{x}(t)\} = R_{xx}^2(0) + 2R_{xx}^2(\tau)$$

And since $R_{xx}(\tau)=C_{xx}(\tau)$, and $E\{\underset{\sim}{z}(t)\} = R_{xx}(0)$, the above yields

$$C_{zz}(\tau) = R_{zz}(\tau) - E^2\{\underset{\sim}{z}(t)\} = 2C_{xx}^2(\tau)$$

9-11 $\underset{\sim}{y}''(t) + 4\underset{\sim}{y}'(t) + 13\underset{\sim}{y}(t) = \underset{\sim}{x}(t)$ all t

The process $\underset{\sim}{y}(t)$ is the response of a system with input $\underset{\sim}{x}(t) = 26 + \underset{\sim}{\nu}(t)$ and

$$H(s) = \frac{1}{s^2+45+13} \qquad h(t) = \frac{1}{3}\, e^{-2t}\sin 3t U(t)$$

Since $\eta_x = 26$, this yields $\eta_y = \eta_x H(0) = 2$. The centered process $\underset{\sim}{\tilde{y}}(t) = \underset{\sim}{y}(t)-\eta_y$ is the response due to $\underset{\sim}{\nu}(t)$. Hence [see (9-100)]

$$E\{\underset{\sim}{\tilde{y}}^2(t)\} = q\int_0^\infty h^2(t)dt = \frac{10}{104}$$

With b=4 and c=13 it follows that (see Example 9-276)

$$R_{\underset{\sim}{y}\underset{\sim}{y}}(\tau) = \frac{10}{104}e^{-2|\tau|}\left(\cos 3\tau - \frac{2}{3}\sin 3|\tau|\right) + 4$$

If $\underset{\sim}{\nu}$ is normal, then $\underset{\sim}{y}(t)$ is normal with mean 2 and variance $R_{yy}(0) - 4 = 10/104$; hence,

$$P\{\underset{\sim}{y}(t) \le 3\} = G\left(\frac{3-2}{0.31}\right) = G(3.24)$$

9-12 $E\{\underset{\sim}{y}(t)\} = 0 \qquad R_{yy}(t_1,t_2) = \dfrac{R_{xx}(t_1,t_2)}{f(t_1)f(t_2)} = w(t_1-t_2)$

$$E\{\underset{\sim}{z}(t)\} = 0 \qquad R_{zz}(t_1,t_2) = \frac{R_{xx}(t_1,t_2)}{\sqrt{q(t_1)}\,\sqrt{q(t_2)}} = \delta(t_1-t_2)$$

because $q(t_1)\delta(t_1-t_2) = \sqrt{q(t_1)}\,\sqrt{q(t_2)}\,\delta(t_1-t_2)$.

9-13 From (9-181) and the identity $4ab \le (a+b)^2$ it follows that

$$|R_{xy}(\tau)|^2 \le R_{xx}(0)R_{yy}(0) \le \frac{1}{4}\,[R_{xx}(0) + R_{yy}(0)]^2$$

9-14 Clearly (stationarity assumption)

$$E\{|\underset{\sim}{x}^*(t) - \underset{\sim}{y}^*(t)|^2\} = E\{|\underset{\sim}{x}(0) - \underset{\sim}{y}(0)|^2\} = 0$$

Furthermore,

$$E\{\underset{\sim}{x}(t+\tau)[\underset{\sim}{x}^*(t) - \underset{\sim}{y}^*(t)]\} = R_{xx}(\tau) - R_{xy}(\tau)$$

and [see (9-177)]

$$|E\{\underset{\sim}{x}(t+\tau)[\underset{\sim}{x}^*(t) - \underset{\sim}{y}^*(t)]\}|^2 \le E\{|\underset{\sim}{x}(t+\tau)|^2\}E\{|\underset{\sim}{x}^*(t) - \underset{\sim}{y}^*(t)|^2\} = 0$$

Hence, $R_{xx}(\tau) - R_{xy}(\tau) = 0$; similarly, $R_{yy}(\tau) = R_{xy}(\tau)$

9-15 $E\{|\underset{\sim}{x}(t+\tau) - \underset{\sim}{x}(t)|^2\} = E\{[\underset{\sim}{x}(t+\tau) - \underset{\sim}{x}(t)][\underset{\sim}{x}^*(t+\tau) - \underset{\sim}{x}^*(t)]\}$

$= R(0) - R(\tau) - R^*(\tau) + R(0) = 2R(0) - 2\,\underline{\mathrm{Re}}\,R(\tau)$

9-16 From $\Phi(1) = \Phi(2) = 0$ it follows that

$$E\{\cos \underset{\sim}{\phi}\} = E\{\sin \underset{\sim}{\phi}\} = E\{\cos 2\underset{\sim}{\phi}\} = E\{\sin 2\underset{\sim}{\phi}\} = 0$$

Hence, $E\{\underset{\sim}{x}(t)\} = \cos\omega t\; E\{\cos \underset{\sim}{\phi}\} - \sin\omega t\; E\{\sin \underset{\sim}{\phi}\} = 0$

and as in Example 9-14

$$2\cos[\omega(t+\tau) + \underset{\sim}{\phi}]\cos(\omega t + \underset{\sim}{\phi}) = \cos\omega\tau + \cos(2\omega t + \omega\tau + 2\underset{\sim}{\phi})$$

$$2R_x(\tau) = \cos\omega\tau$$

If $\underset{\sim}{\phi}$ is uniform in $(-\pi,\pi)$, then

$$\Phi(\lambda) = \frac{\sin\pi\omega}{\pi\omega} \qquad \Phi(1) = \Phi(2) = 0$$

9-17 (a) $\underset{\sim}{x}(t_1)\underset{\sim}{x}(t_2) = [\underset{\sim}{x}(t_1) - \underset{\sim}{x}(0)][\underset{\sim}{x}(t_2) - \underset{\sim}{x}(t_1) + \underset{\sim}{x}(t_1) - x(0)]$

$$R(t_1,t_2) = E\{[\underset{\sim}{x}(t_1) - \underset{\sim}{x}(0)]^2\} = E\{\underset{\sim}{x}^2(t_1)\} = R(t_1,t_1)$$

(b) If $t_1 + \varepsilon < t_2$, then $R_y(t_1,t_2) = 0$; if

$t_1 < t_2 < t_1 + \varepsilon$ then

$$E\{[\underset{\sim}{x}(t_1+\varepsilon) - \underset{\sim}{x}(t_1)][\underset{\sim}{x}(t_2+\varepsilon) - \underset{\sim}{x}(t_2)]\} = q(t_1+\varepsilon - t_2)$$

Hence, $\varepsilon^2 R_y(\tau) = q(\varepsilon - |\tau|)$ for $|\tau| = |t_2 - t_1| \le \varepsilon$

9-18

$$E\{\underset{\sim}{x}(t)\underset{\sim}{y}(t)\} = \int_{-\infty}^{\infty} E\{\underset{\sim}{x}(t)\underset{\sim}{x}(t-\tau)\}h(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} R_{xx}(t,t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} q(t)\delta(\tau)h(\tau)d\tau = h(0)q(t)$$

9-19 As in Prob. 5-14, $g(x) = 6 + 3\,F_x(x)$. In this case,

$E\{\underset{\sim}{x}^2(t)\} = 4$, hence, $\underset{\sim}{x}(t)$ is $N(0,2)$ and $F_x(x) = G(x/2)$

9-20 $\underset{\sim}{x}(t)$ is SSS, hence, $P\{x(t) \le y\} = F_x(y)$ does not depend on t. The RVs $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{x}(t)$ are independent, hence, [see (6-238)]

$$F_y(y) = P\{\underset{\sim}{x}(t - \underset{\sim}{\varepsilon}) \le y \mid \underset{\sim}{\varepsilon} = \varepsilon\} = P\{\underset{\sim}{x}(t-\varepsilon) \le y \mid \underset{\sim}{\varepsilon} = \varepsilon\}$$

$$= P\{\underset{\sim}{x}(t-\varepsilon) < y\} = F_x(y)$$

is independent of t. Similarly for higher order distributions.

9-21 $E\{\underset{\sim}{x}(t)\} = \eta =$ constant, hence, [see (9-102)] $E\{\underset{\sim}{x}'(t)\} = 0$

Furthermore, $R_{xx}(-\tau) = R_{xx}(\tau)$. hence, $R'_{xx}(0) = 0$ and (10-97) yields

$$E\{\underset{\sim}{x}(t)\underset{\sim}{x}'(t)\} = R_{xx'}(0) = 0$$

9-22 (a) $E\{\underset{\sim}{z}\,\underset{\sim}{w}\} = R_x(2) = 4e^{-4}$ $\qquad E\{\underset{\sim}{z}^2\} = E\{\underset{\sim}{w}^2\} = R_x(0) = 4$

$$E\{(\underset{\sim}{z}+\underset{\sim}{w})^2\} = R_x(0) + R_x(0) + 2R_x(2) = 8(1+e^{-4})$$

(b) $\underset{\sim}{z}$ is $N(0,2)$ $\qquad P\{\underset{\sim}{z} < 1\} = F_z(1) = G(1/2)$

$r_{zw} = e^{-4}$, $\quad f_{zw}(z,w) : N(0,0;2,2;e^{-4})$

9-23 The RV $\underset{\sim}{x}'(t)$ is normal with zero mean and variance

$$E\{|\underset{\sim}{x}'(t)|^2\} = R_{x'x'}(0) = -R''(0)$$

Hence, $P\{\underset{\sim}{x}'(t) \le a\} = F_{x'}(a) = G[a/\sqrt{|R''(0)|}]$

9-24 The function arc sin x is odd, hence, it can be expanded into a sine series in the interval (-1,1):

$$\alpha(x) \equiv \text{arc sin } x = \sum_{n=1}^{\infty} b_n \sin n\pi x \qquad |x| \le 1$$

$$b_n = \int_{-1}^{1} \alpha(x)\sin n\pi x\,dx = -\frac{1}{n\pi}\int_{-1}^{1} \alpha(x)\,d\cos n\pi x$$

$$= -\frac{\alpha(x)\cos n\pi x}{n\pi}\Bigg|_{-1}^{1} + \frac{1}{n\pi}\int_{-1}^{1} \cos n\pi x\,d\alpha(x)$$

$$= -\frac{\cos n\pi}{n} + \frac{1}{n\pi}\int_{-\pi/2}^{\pi/2} \cos(n\pi \sin x)\,dx$$

and the result follows because [see (9-81)]

$$R_y(\tau) = \frac{2}{\pi}\text{ arc sin }\frac{R_x(\tau)}{R_x(0)} \qquad J_o(z) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(z\sin x)\,dx$$

9-25 As we know [see (5-100) and (6-193)]

$$E\{e^{j\omega \underset{\sim}{x}(t)}\} = \exp\{-\frac{R(0)}{2}\ \omega^2\}$$

$$E\{e^{j[\omega_1 \underset{\sim}{x}(t+\tau)+\omega_2 \underset{\sim}{x}(t)]}\} = \exp\{-\frac{1}{2}[R(0)\omega_1^2 + 2R(\tau)\omega_1\omega_2 + R(0)\omega_2^2]\}$$

Hence, with $j\omega = a$

$$E\{I\, e^{a\underset{\sim}{x}(t)}\} = \exp\{\frac{a^2}{2} R_x(0)\}\, I$$

$$E\{I\, e^{a\underset{\sim}{x}(t+\tau)} I\, e^{a\underset{\sim}{x}(t)}\} = I^2 \exp\{a\ [R_x(0) + R_x(\tau)]\}$$

9-26 (a) $R_y(\tau) = a^2 E\{\underset{\sim}{x}[c(t+\tau)]\underset{\sim}{x}(ct)] = a^2 R(c\tau)$

(b) If $\underset{\sim}{z}_\varepsilon(t) = \sqrt{\varepsilon}\ \underset{\sim}{x}(\varepsilon t)$ then $R_{z_\varepsilon}(\tau) = \varepsilon R_x(\varepsilon\tau)$ [as in (a)].

If $\delta > 0$ is sufficiently small and $\phi(t)$ is continuous at the origin, then

$$\int_{-\delta}^{\delta} R_{z_\varepsilon}(\tau)\phi(\tau)d\tau \simeq \phi(0) \int_{-\delta}^{\delta} \varepsilon R_x(\varepsilon\tau)\, d\tau$$

$$= \phi(0) \int_{-\varepsilon\delta}^{\varepsilon\delta} R(\tau)d\tau \xrightarrow[\varepsilon\to\infty]{} \phi(0) \int_{-\infty}^{\infty} R(\tau)d\tau = q\ \phi(0)$$

Hence, $R_{z_\varepsilon}(\tau) \longrightarrow q\ \delta(\tau)$ as $\varepsilon \to \infty$.

9-27

$$\underset{\sim}{y}(t) = \int_{t-T}^{t} \underset{\sim}{x}(\tau)h(t-\tau)d\tau$$

Hence, $\underset{\sim}{y}(t_1)$ and $\underset{\sim}{y}(t_2)$ depend linearly on the values of $\underset{\sim}{x}(t)$ in the intervals $(t_1 - T, t_1)$ and $(t_2 - T, t_2)$ respectively. If $|t_1 - t_2| > T$ then these intervals do not overlap and since $E\{\underset{\sim}{x}(\tau_1)\underset{\sim}{x}(\tau_2)\} = 0$ for $\tau_1 \neq \tau_2$, it follows that $E\{\underset{\sim}{y}(t_1)\underset{\sim}{y}(t_2)\} = 0$.

9-28 (a)

$$I(t) = E\left\{\int_0^t \int_0^t h(t,\alpha)\underset{\sim}{x}(\alpha)h(t,\beta)\underset{\sim}{x}(\beta)\, d\alpha d\beta\right\}$$

$$= \int_0^t\int_0^t h(t,\alpha)h(t,\alpha)q(\alpha)\delta(\alpha-\beta)d\alpha d\beta = \int_0^t h^2(t,\alpha)q(\alpha)d\alpha$$

(b) If $\underset{\sim}{y}'(t) + c(t)\underset{\sim}{y}(t) = \underset{\sim}{x}(t)$, then $\underset{\sim}{y}(t)$ is the output of a linear time-varying system as in (a) with impulse response $h(t,\alpha)$ such that

$$\frac{\partial h(t,\alpha)}{\partial t} + c(t)h(t,\alpha) = \delta(t-\alpha) \qquad h(\alpha^-,\alpha) = 0$$

or equivalently

$$\frac{\partial h(t,\alpha)}{\partial t} + c(t)h(t,\alpha) = 0 \qquad t > 0 \quad h(\alpha^+,\alpha) = 1$$

This yields

$$h(t,\alpha) = e^{-\int_\alpha^t c(\tau)d\tau}$$

Hence, if

$$I(t) = \int_0^t h^2(t,\alpha)q(\alpha)d\alpha \quad \text{then} \quad I'(t) + 2c(t)I(t) = q(t)$$

because the impulse response of this equation equals

$$e^{-2\int_\alpha^t c(\tau)d\tau} = h^2(t,\alpha)$$

9-29 (a) If $\underset{\sim}{y}'(t) + 2\underset{\sim}{y}(t) = \underset{\sim}{x}(t)$, then $\underset{\sim}{y}(t) = \underset{\sim}{x}(t)*h(t)$

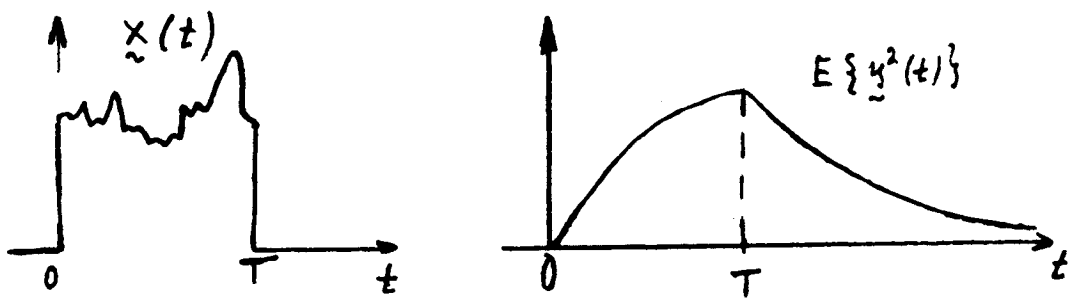
where $h(t) = e^{-2t}U(t)$ and with $q(t) = 5$, (10-90) yields

$$E\{\underset{\sim}{y}^2(t)\} = 5*e^{-4t}U(t) = 5\int_0^{\infty} e^{-4\tau}d\tau = \frac{5}{4}$$

(b) As in (a) with $q(t) = 5U(t)$. Hence, for $t > 0$

$$E\{\underset{\sim}{y}^2(t)\} = 5U(t)*e^{-4t}U(t) = 5\int_0^{t} e^{-4\tau}d\tau = \frac{5}{4}(1-e^{-4t})$$

9-30



From (9-90) with $q(t) = N[U(t) - U(t-T)]$

$$E\{\underset{\sim}{y}^2(t)\} = \begin{cases} AN\int_0^{t} e^{-2\alpha(t-\tau)}d\tau = \frac{AN}{2\alpha}(1-e^{-2\alpha t}) & 0 \le t < T \\ AN\int_0^{T} e^{-2\alpha(t-\tau)}d\tau = \frac{AN}{2\alpha}(e^{2\alpha T}-1)e^{-2\alpha t} & t > T \end{cases}$$

9-31 Since $\underset{\sim}{x}(t)$ is WSS, the moments of S equal the moments of

$$\underset{\sim}{z} = \int_{-5}^{5} \underset{\sim}{x}(t)dt$$

Hence, (see Fig. 9-5)

$$E\{\underset{\sim}{s}^2\} = \int_{-5}^{5}\int_{-5}^{5} R_x(t_1-t_2)dt_1dt_2 = \int_{-10}^{10}(10-|\tau|)R_x(\tau)d\tau$$

$$E\{\underset{\sim}{s}\} = 80 \qquad \sigma_s^2 = 2\int_0^{10}(10-\tau)10e^{-2\tau}d\tau$$

9-32 $\underset{\sim}{y}(t) = \underset{\sim}{x}(t)*h(t) \qquad h(t) = e^{-2t}U(t)$

(a) $E\{\underset{\sim}{y}^2(t)\} = 5*e^{-4t}\,U(t) = 5/4$

$$R_{xy}(t_1,t_2) = 5\,\delta(t_1-t_2)*e^{-2t_2}\,U(t_2) = 5e^{-2(t_2-t_1)}U(t_2-t_1)$$

$$R_{yy}(t_1,t_2) = 5e^{-2(t_2-t_1)}U(t_2-t_1)*e^{-2t_1}\,U(t_1)$$

$$= \frac{5}{4}\,e^{-2|t_1-t_2|}$$

The first equation follows from (9-100) with $q(t) = 5$; the second from (9-94) with $R_{xx}(t_1,t_2) = 5\delta(t_1-t_2)$, and the third from (9-96).

(b) With $R_{xx}(t_1,t_2) = 5\delta(t_1-t_2)U(t_1)U(t_2)$, (9-94) and (9-96) yield the following: For t_1 or $t_2 < 0$, $R_{xy}(t_1,t_2) = R_{yy}(t_1,t_2) = 0$. For $0 < t_1 < t_2$

$$R_{xy}(t_1,t_2) = 5\delta(t_1-t_2)*e^{-2t_2} = 5\,e^{-2t_2}$$

$$R_{yy}(t_1,t_2) = \int_0^{t_1} 5\,e^{-2(t_1-\tau)}\,e^{-2(t_1-\tau)}\,d\tau = \frac{5}{4}\,e^{-2(t_2-t_1)}(1-e^{-4t_1})$$

9-33 $$\int_{-\infty}^{\infty} e^{-\alpha\tau^2} e^{-s\tau} d\tau = e^{s^2/4\alpha} \int_{-\infty}^{\infty} e^{-\alpha(\tau + s/2\alpha)^2} d\tau = \sqrt{\frac{\pi}{\alpha}}\, e^{s^2/4\alpha}$$

This yields

$$e^{-\alpha\tau^2} \longleftrightarrow \sqrt{\frac{\pi}{\alpha}}\, e^{-\omega^2/4\alpha}$$

$$e^{-\alpha\tau^2} \cos \omega_o\tau \longleftrightarrow \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \left[e^{-(\omega-\omega_o)^2/4\alpha} + e^{-(\omega+\omega_o)^2/4\alpha} \right]$$

9-34 $$G(x_1,x_2;\omega) = \int_{-\infty}^{\infty} f(x_1,x_2;\tau) e^{-j\omega\tau} d\tau$$

$$R(\tau) = E\{\underset{\sim}{x}(t+\tau)\underset{\sim}{x}(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f(x_1,x_2;\tau) dx_1 dx_2$$

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} e^{-j\omega\tau} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f(x_1,x_2;\tau) dx_1\, dx_2 d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 \int_{-\infty}^{\infty} e^{-j\omega\tau}\, f(x_1,x_2;\tau) d\tau dx_1 dx_2$$

9-35 The process $\underset{\sim}{y}(t) = \underset{\sim}{x}(t+a) - \underset{\sim}{x}(t-a)$ is the output of a system with input $\underset{\sim}{x}(t)$ and system function

$$H(\omega) = e^{ja\omega} - e^{-ja\omega} = 2j \sin a\omega$$

Hence [see (9-150)]

$$S_y(\omega) = 4 \sin^2 a\omega\, S_x(\omega) = (2 - e^{j2a\omega} - e^{-j2a\omega}) S_x(\omega)$$

$$R_y(\tau) = 2\, R_x(\tau) - R_x(\tau + 2a) - R_x(\tau - 2a)$$

9-36 Since $S(\omega) \geq 0$, we conclude with (9-136) that

$$R(0) - R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)(1 - \cos \omega\tau)\, d\omega$$

$$\geq \frac{1}{8\pi} \int_{-\infty}^{\infty} S(\omega)(1 - \cos 2\omega\tau)\, d\omega = \frac{1}{4}\,[R(0) - R(2\tau)]$$

and the result follows for $n = 1$. Repeating the above, we obtain the general result.

9-37 From (6-197)

$$E\{\underset{\sim}{x}^2(t+\tau)\underset{\sim}{x}^2(t)\} = E\{\underset{\sim}{x}^2(t+\tau)\}E\{\underset{\sim}{x}^2(t)\} + 2E^2\{\underset{\sim}{x}^2(t+\tau)\underset{\sim}{x}^2(t)\}$$

Hence,

$$R_y(\tau) = R_x^2(0) + 2\,R_x^2(\tau) = I^2(1 + e^{-2\alpha|\tau|} + e^{-2\alpha|\tau|}\cos 2\beta\tau)$$

$$S_y(\omega) = \left[2\pi\delta(\omega) + \frac{4\alpha}{4\alpha^2 + \omega^2} + \frac{2\alpha}{4\alpha^2 + (\omega - 2\beta)^2} + \frac{2\alpha}{4\alpha^2 + (\omega + 2\beta)^2}\right]$$

Furthermore,

$$\eta_y = E\{\underset{\sim}{x}^2(t)\} = R_x(0) \qquad C_y(\tau) = 2R_x^2(\tau)$$

9-38

$$\int_{-\infty}^{\infty} S(\omega)\left|\sum_i a_i\, e^{j\omega\tau_i}\right|^2 d\omega = \int_{-\infty}^{\infty} S(\omega) \sum_{i,k} a_i a_k^*\, e^{j\omega(\tau_i - \tau_k)}\, d\omega$$

$$= \sum_{i,k} a_i a_k^*\, R(\tau_i - \tau_k) \geq 0$$

9-39 (a) $S(s) = \dfrac{1}{1+s^4} = \dfrac{1}{(s^2+\sqrt{2}s+1)(s^2-\sqrt{2}s+1)}$

A special case of example 9-27b with $b = \sqrt{2}$, $c = 1$. Hence,

$$R(\tau) = \frac{1}{2\sqrt{2}} e^{-|\tau|/\sqrt{2}} \left(\cos\frac{\tau}{\sqrt{2}} + \sin\frac{|\tau|}{\sqrt{2}}\right)$$

(b) From the pair $e^{-2|\tau|} \leftrightarrow 4/(4+\omega^2)$ and the convolution theorem it follows that

$$e^{-2|\tau|} * e^{-2|\tau|} \leftrightarrow \frac{16}{(4+\omega^2)^2}$$

Hence, for $\tau > 0$

$$16\,R(\tau) = \int_{\infty}^{\infty} e^{-2|x|}\, e^{-2|\tau-x|} dx = \int_{-\infty}^{0} e^{2x}\, e^{-2(\tau-x)} dx$$

$$+ \int_{0}^{\tau} e^{-2x}\, e^{-2(\tau-x)}\, dx + \int_{\tau}^{\infty} e^{-2x}\, e^{2(\tau-x)}\, dx = \frac{1}{2} e^{-2\tau}(1+2\tau)$$

And since $R(-\tau) = R(\tau)$, the above yields

$$e^{-2|\tau|}\,\frac{1+2|\tau|}{32} \longleftrightarrow \frac{1}{(4+\omega^2)^2}$$

9-40 $H^*(-s^*)\Big|_{s=j\omega} = H^*(j\omega)$ $\qquad H^*(1/z^*)\Big|_{z=e^{j\omega T}} = H^*(e^{j\omega T})$

Hence

$$H(s)H^*(-s^*)\Big|_{s=j\omega} = |H(j\omega)|^2 \qquad H(z)H^*(1/z^*)\Big|_{z=j\omega T} = |H(e^{j\omega T})|^2$$

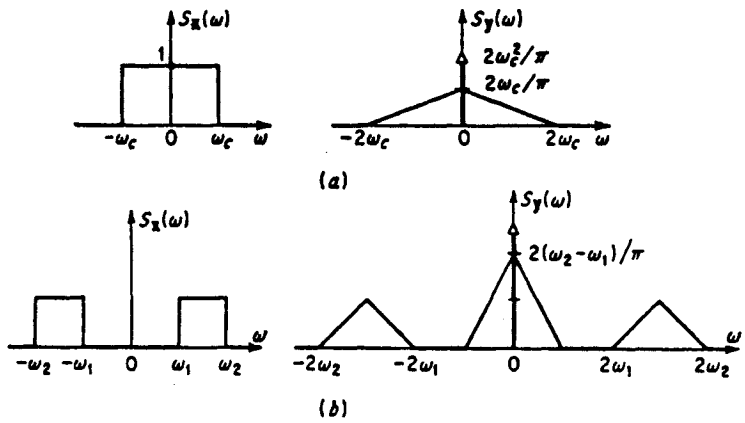
9-41 From (6-197)

$$R_y(\tau) = E\{\underset{\sim}{x}^2(t+\tau)\underset{\sim}{x}^2(t)\}$$

$$= E\{\underset{\sim}{x}^2(t+\tau)\}E\{\underset{\sim}{x}^2(t)\} + 2\ E^2\{\underset{\sim}{x}(t+\tau)\underset{\sim}{x}(t)\} = R_x^2(0) + 2\ R_x^2(\tau)$$

From the above and the frequency convolution theorem it follows that

$$S_y(\omega) = 2\pi R_x^2(0)\delta(\omega) + \frac{1}{\pi} S_x(\omega) * S_x(\omega)$$



9-42 $\underset{\sim}{y}(t) = 2\underset{\sim}{x}(t) + 3\underset{\sim}{x}'(t) \qquad \eta_x = 5 \qquad C_{xx}(\tau) = 4e^{-2|\sigma|}$

The process $\underset{\sim}{y}(t)$ is the output of the system $H(s) = 2+3s$ with input $\underset{\sim}{x}(t)$. Hence, $\eta_y=5H(0)=10$

$$S_{yy}^c(\omega) = S_{xx}^c(\omega)|2+3j\omega|^2 = \frac{16}{4+\omega^2}(4+9\omega^2) = 144 - \frac{512}{4+\omega^2} = S_{yy}(\omega) - 2\pi\eta_y^2\delta(\omega)$$

9-43 (a) $\underset{\sim}{y}'(t) + 3\underset{\sim}{y}(t) = \underset{\sim}{x}(t)$, $R_{xx}(\tau) = 5\delta(\tau)$. The process $\underset{\sim}{y}(t)$ is the output of the system

$$H(s) = \frac{1}{s+3} \qquad h(t) = e^{-3t}U(t)$$

Hence, [see (9-100) and (9-150)]

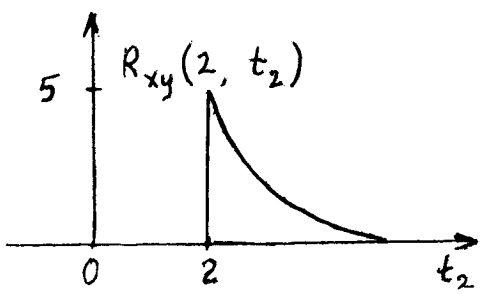
$$E\{\underset{\sim}{y}^2(t)\} = 5\int_0^\infty e^{-6t}dt = \frac{5}{6}$$

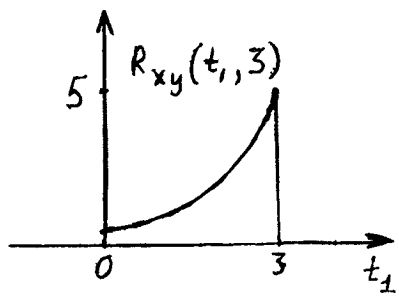
$$S_{yy}(\omega) = \frac{5}{\omega^2+9} \qquad R_{yy}(\tau) = \frac{5}{6}e^{-3|\tau|}$$

(b) As in Example 9-18:

$$E\{\underset{\sim}{y}^2(t)\} = 5\int_0^t e^{-6\alpha}d\alpha = \frac{5}{6}(1-e^{-6t}) \qquad t > 0$$

$$R_{xy}(t_1,t_2) = 5e^{-2|t_2-t_1|}U(t_1)U(t_2)U(t_2-t_1)$$





9-44 We shall show that: If $\underset{\sim}{x}(t)$ is a complex process with autocorrelation $R(\tau)$ and $|R(\tau_1)|=R(0)$ for some τ_1, then $R(\tau)=e^{j\omega_0\tau}w(\tau)$ where $w(\tau)$ is a periodic function with period τ_1. Furthermore, the process $\underset{\sim}{y}(t) = e^{-j\omega_0 t}\underset{\sim}{x}(t)$ is MS periodic.

<u>Proof</u> Clearly, $R(\tau_1) = R(0)e^{j\phi}$. With $\omega_0 = \phi/\tau_1$,

$$R_{yy}(\tau) = E\{\underset{\sim}{x}(t+\tau)e^{-j\omega_0(t+\tau)}\underset{\sim}{x}^*(t)e^{j\omega_0 t}\} = R(\tau)e^{-j\omega\tau}$$

Hence, $R_{yy}(\tau_1)=e^{-j\omega_0\tau_1}R(\tau_1) = R(0) = R_{yy}(0)$. From this and (10-168) it follows that the function $w(\tau) = R_{yy}(\tau)$ is periodic.

9-45 (a) The cross spectrum $S_{\check{x}x}(\omega) = -j\,\mathrm{sgn}\omega\, S_{xx}(\omega)$ is an odd function. Hence,

$$E\{\underset{\sim}{x}(t)\check{\underset{\sim}{x}}'(t)\} = \frac{-j}{2\pi}\int_{\infty}^{\infty} \mathrm{sgn}\omega S_{xx}(\omega)d\omega = 0$$

(b) The process $\check{\underset{\sim}{x}}(t)$ is the output of the system

$$(-j\,\mathrm{sgn}\omega)(-j\,\mathrm{sgn}\omega) = -1$$

with input $\underset{\sim}{x}(t)$. Hence, $\check{\check{\underset{\sim}{x}}}(t) = -\underset{\sim}{x}(t)$.

9-46 In general

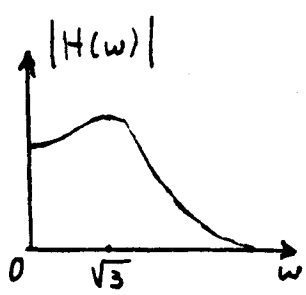
$$E\{\underset{\sim}{y}^2(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)|H(\omega)|^2 d\omega$$

$$\le |H(\omega_m)|^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)d\omega = E\{\underset{\sim}{x}^2(t)\}|H(\omega_m)|^2$$

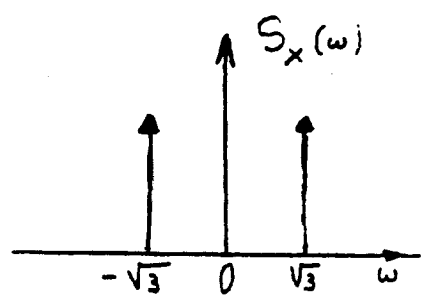
where $|H(\omega_m)|$ is the maximum of $|H(\omega)|$. In our case,

$$|H(\omega)|^2 = \frac{1}{(5-\omega)^2 + 4\omega^2} \quad \text{is maximum for } \omega = \sqrt{3}$$

and $|H(\omega_m)|^2 = 1/16$. Hence $E\{\underset{\sim}{y}^2(t)\} \le 10/16$ with equality if $R_x(10) = 10\cos\sqrt{3}\,\tau$ (Fig. b).



(a)



(b)

9-47 If $R_x(\tau) = e^{j\omega_o\tau}$, then $S_x(\omega) = 2\pi\delta(\omega-\omega_o)$, hence, the integral of $S_x(\omega)$ equals zero in any interval not including the point $\omega = \omega_o$. From (9-182) it follows that the same is true for the integral of $S_{xy}(\omega)$. This shows that $S_{xy}(\omega)$ is a line at $\omega = \omega_o$ for any $\underset{\sim}{y}(t)$.

9-48 (a) As in (9-147) and (9-149)

$$R_{yx}(\tau) = R_{xx}(\tau) * h(\tau) = \int_{-\infty}^{\infty} e^{j\alpha(\tau-\gamma)} h(\gamma)\, d\gamma = e^{j\alpha\tau} H(\alpha)$$

$$R_{yy}(\tau) = R_{xx}(\tau) * \rho(\tau) = \int_{-\infty}^{\infty} e^{j\alpha(\tau-\gamma)} \rho(\gamma)\, d\gamma = e^{j\alpha\tau} |H(\alpha)|^2$$

(b) As in (9-94) and (9-95)

$$R_{yx}(t_1,t_2) = e^{-j\beta t_2} \int_{-\infty}^{\infty} e^{j\alpha(t_1-\gamma)} h(\gamma)\, d\gamma = e^{j(\alpha t_1-\beta t_2)} H(\alpha)$$

$$R_{yy}(t_1,t_2) = e^{-j\alpha t_1} H(\alpha) \int_{-\infty}^{\infty} e^{-j\beta(t_2-\gamma)} h(\gamma)\, d\gamma = e^{j(\alpha t_1-\beta t_2)} H(\alpha) H^*(\beta)$$

because $h(t)$ is real and $H(-\beta) = H^*(\beta)$.

9-49 If $S_{xx}(\omega)S_{yy}(\omega) \equiv 0$ then $S_{xx}(\omega) = 0$ or $S_{yy}(\omega) = 0$ in any interval (a,b). From this and (10-168) it follows that the integral of $S_{xy}(\omega)$ in any interval equals zero, hence, $S_{xy}(\omega) \equiv 0$.

9-50 This is the discrete-time version of theorem (9-162). From (9-163)

$$E^2\{(\underset{\sim}{x}[n+m+1] - \underset{\sim}{x}[n+m])\underset{\sim}{x}[n]\} \le E\{|\underset{\sim}{x}[n+m+1] - \underset{\sim}{x}[n+m]|^2\}E\{|\underset{\sim}{x}[n]|^2\}$$

$$(R[m+1] - R[m])^2 \le 2(R[0] - R[1])R[0] = 0$$

Hence, $R[m+1] = R[m]$ for any m.

9-51 We shall show that

$$2\,\frac{R^2[1]}{R[0]} - R[0] \le R[2] \le R[0] \qquad \text{(i)}$$

The covariance matrix of the RVs $\underset{\sim}{x}[n]$, $\underset{\sim}{x}[n+1]$, and $\underset{\sim}{x}[n+2]$ is nonnegative [see (7-29)]:

$$\begin{vmatrix} R[0] & R[1] & R[2] \\ R[1] & R[0] & R[1] \\ R[2] & R[1] & R[0] \end{vmatrix} \ge 0$$

This yields

$$R[0]R^2[2] - 2\,R^2[1]R[2] - R^3[0] + 2R[0]R^2[1] \le 0$$

The above is a quadratic in R[2] with roots

$$R[0] \text{ and } -R[0] + 2\,R^2[1]/R[0]$$

Since it is nonpositive, R[2] must be between the roots as in (i)

9-52 If $\underset{\sim}{x}[n] = Ae^{jn\underset{\sim}{\omega}T}$ then

$$R_x[m] = A^2E\{e^{j(m+n)\underset{\sim}{\omega}T}e^{-jn\underset{\sim}{\omega}T}\} = A^2\int_{-\sigma}^{\sigma} e^{jm\omega T}\,f(\omega)\,d\omega$$

But [see (9-194)]

$$R[m] = \frac{1}{2\sigma}\int_{\sigma}^{\sigma} S_x(\omega)e^{jm\omega T}d\omega$$

hence, $A^2f(\omega) = S_x(\omega)/2\sigma$

9-53 (a) If $\underset{\sim}{y}(0) = \underset{\sim}{y}'(0) = 0$, then $\underset{\sim}{y}(t)$ is the output of a system with input $\underset{\sim}{x}(t)U(t)$ and impulse response $h(t)$ such that

$$h''(t) + 7h'(t) + 10h(t) = \delta(t) \qquad h(0^-) = h'(0^-) = 0$$

$$h(t) = \frac{1}{3}\left(e^{-2t} - e^{-5t}\right)U(t)$$

and with $q(t) = 5\,U(t)$, (9-100) yields

$$E\{\underset{\sim}{y}^2(t)\} = \frac{5}{9}\int_0^t \left(e^{-2\tau} - e^{-5\tau}\right)^2 d\tau$$

(b) If $\underset{\sim}{y}[-1] = \underset{\sim}{y}[-2] = 0$, then $\underset{\sim}{y}[n]$ is the output of a system with input $\underset{\sim}{x}[n]U[n]$ and delta response $h[n]$ such that

$$8h[n] - 6h[n-1] + h[n-2] = \delta[n] \qquad h[-1] = h[-2] = 0$$

$$h[n] = \left(\frac{1}{2^{n+2}} - \frac{1}{2^{2n+3}}\right)U[n]$$

and with $q[n] = 5\,U[n]$, (10-176) yields

$$E\{\underset{\sim}{y}^2[n]\} = 5\sum_{k=0}^{n}\left(\frac{1}{2^{k+2}} - \frac{1}{2^{2k+3}}\right)^2$$

9-54 $\underset{\sim}{y}[n] = \underset{\sim}{x}[n]*h[n] \qquad h[n] = 2^{-n}U[n]$

(a) $E\{\underset{\sim}{y}^2[n]\} = 5*2^{-2n}U[n] = 0$

$$R_{xy}[m_1,m_2] = 5\delta[m_1-m_2]*2^{-m_2}U[m_2] = 5\,2^{-(m_2-m_1)}U[m_2-m_1]$$

$$R_{yy}[m_1,m_2] = 5\times 2^{-(m_2-m_1)}U[m_2-m_1]*2^{-m_1}U[m_1]$$

$$= \frac{20}{3}\times 2^{-|m_1-m_2|}$$

The first equation follows from (9-190) with $q[n] = 5$; the second and third from (9-191) with $R_{xx}[m_1,m_2] = 5\,\delta[m_1-m_2]$.

(b) With $R_{xx}[m_1,m_2] = 5\,\delta[m_1-m_2]U[m_1]U[m_2]$, Prob. 9-25a yields the following: For m_1 or $m_2 < 0$, $R_{xy}[m_1,m_2] = R_{yy}[m_1,m_2] = 0$. For $0 < m_1 < m_2$

$$R_{xy}[m_1,m_2] = 5\,\delta[m_1-m_2]*2^{-m_2} = 5\times 2^{-m_2}$$

$$R_{yy}[m_1,m_2] = \sum_{k=0}^{m_1} 5\times 2^{-(m_2-k)}\,2^{-(m_1-k)} = \frac{5}{3}\,2^{-(m_2-m_1)}\left(4 - 2^{-2m_1}\right)$$

9-55

(a) $R_x[m_1,m_2] = q[m_1]\delta[m_1-m_2]$

$$E\{\underset{\sim}{s}^2\} = \sum_{n=0}^{N}\sum_{k=0}^{N} a_n a_k \; E\{\underset{\sim}{x}[n]\underset{\sim}{x}[k]\}$$

$$= \sum_{n=0}^{N}\sum_{k=0}^{N} a_n a_k \; q[n]\delta[n-k] = \sum_{n=0}^{N} a_n^2 \; q[n]$$

(b) $R_x(t_1,t_2) = q(t_1)\delta(t_1-t_2)$

$$E\{s^2\} = \int_0^T\int_0^T a(t)a(\tau) \; E\{x(t)x(\tau)\}d\tau dt$$

$$= \int_0^T\int_0^T a(t)a(\tau)q(t)\delta(t-\tau)d\tau dt = \int_0^T a^2(t)q(t)dt$$

CHAPTER 10

10-1

(a) If $\underset{\sim}{x}(t)$ is a Poisson process as in Fig. 9-3a, then for a fixed t, $\underset{\sim}{x}(t)$ is a Poisson RV with parameter λt. Hence [see (5-119)] its characteristic function equals $\exp\{\lambda t(e^{j\omega}-1)\}$.

(b) If $\underset{\sim}{x}(t)$ is a Wiener process then $f(x,t)$ is $N(0,\sqrt{\alpha t})$. Hence [see (5-100)] its first order characteristic function equals $\exp\{-\alpha t\omega^2/2\}$.

10-2 For large $\underset{\sim}{t}$, $\underset{\sim}{x}(t)$ and $\underset{\sim}{y}(t)$ can be approximated by two independent Wiener processes as in (10-52):

$$f_x(x,t) = \frac{1}{\sqrt{2\pi\alpha t}}\, e^{-x^2/2\alpha t} \qquad f_y(y,t) = \frac{1}{\sqrt{2\pi\alpha t}}\, e^{-y^2/2\alpha t}$$

Hence, $\underset{\sim}{z}(t)$ has a Rayleigh density [see (6-70)]. [Note. Exactly, $\underset{\sim}{z}(t)$ is a discrete-type RV taking the values $s\sqrt{m^2+n^2}$ where m and n are integers]. The product $f_z(z,t)dz$ equals approximately the probability that $\underset{\sim}{z}(t)$ is between z and $z+dz$ provided that $dz \gg T$.

10-3 The voltage $\underset{\sim}{y}(t)$ is the output of a system with input $\underset{\sim}{n}_e(t)$ and system function

$$H_1(s) = \frac{1}{LCs^2 + RCs + 1}$$

Hence,

$$S_v(\omega) = S_{n_e}(\omega)|H_1(j\omega)|^2 = \frac{2kTR}{(1-\omega^2 LC)^2 + R^2C^2\omega^2}$$

Furthermore,

$$Z_{ab}(s) = \frac{R + Ls}{LCs^2 + RCs + 1} \qquad \underline{Re}\ Z_{ab}(j\omega) = \frac{R}{(1-\omega^2 LC)^2 + R^2C^2\omega^2}$$

in agreement with (10-75).

The current $\underset{\sim}{i}(t)$ is the output of a system with input $\underset{\sim}{n}_e(t)$ and system function

$$H_2(s) = \frac{1}{R + Ls}$$

Hence,

$$S_i(\omega) = S_{n_e}(\omega)|H_2\ (j\omega)|^2 = \frac{2kTR}{R^2 + \omega^2 L^2}$$

Furthermore (short circuit admittance)

$$Y_{ab}(s) = \frac{1}{R + LS} \qquad \underline{Re}Y_{ab}(j\omega) = \frac{2kTR}{R^2 + L^2\omega^2}$$

in agreement with (10-78).

10-4 The equation $m\underset{\sim}{x}''(t) + f\underset{\sim}{x}'(t) = \underset{\sim}{F}(t)$ specifies a system with

$$H(s) = \frac{1}{ms^2 + fs} \qquad h(t) = \frac{1}{f}(1 - e^{-ft/m})U(t)$$

and (9-100) yields

$$E\{\underset{\sim}{x}^2(t)\} = \frac{2kTf}{f^2}\int_0^t (1 - e^{-2\alpha\tau})^2\, d\tau \qquad \alpha = \frac{f}{2m}$$

10-5 As in Example 12-2, a and b are such that

$$\underset{\sim}{x}(\tau) - a\,\underset{\sim}{x}(0) - b\underset{\sim}{v}(0) \perp \underset{\sim}{x}(0),\ \underset{\sim}{v}(0)$$

This yields

$$R_{xx}(\tau) = aR_{xx}(0) + b\,R_{xv}(0) \qquad \text{(i)}$$

$$R_{xv}(\tau) = aR_{xv}(0) + b\,R_{vv}(0)$$

where [see (10-163)]

$$R_{xx}(\tau) = A\,e^{-\alpha\tau}(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta\tau) \qquad \tau > 0$$

$$R_{xv}(\tau) = -R'_{xx}(\tau) = A\,e^{-\alpha\tau}(\sin\beta\tau)\,\frac{\alpha^2+\beta^2}{\beta}$$

$$R_{vv}(\tau) = R'_{xv}(\tau) = A\,e^{-\alpha\tau}(\cos\beta\tau - \frac{\alpha}{\beta}\sin\beta\tau)\,\frac{\alpha^2+\beta^2}{\beta^2}$$

Inserting into (i) and solving, we obtain

$$a = e^{-\alpha\tau}(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta\tau)$$

$$b = \frac{1}{\beta}e^{-\alpha\tau}\sin\beta\tau$$

Finally,

$$P = E\{[\underset{\sim}{x}(t) - a\,\underset{\sim}{x}(0) - b\,\underset{\sim}{v}(0)]\underset{\sim}{x}(t)\} = R_{xx}(0) - a\,R_{xx}(t) - b\,R_{xv}(t)$$

$$= \frac{2kTf}{m^2}\left[1 - e^{-2\alpha t}(1 + \frac{2\alpha^2}{\beta}\sin^2\beta t + \frac{\alpha}{\beta}\sin 2\beta t)\right]$$

10-6 If $\underset{\sim}{x}(t) = \underset{\sim}{w}(t^2)$ then [see (10-70)]

$$R_x(t_1,t_2) = E\{\underset{\sim}{w}(t_1^2)\underset{\sim}{w}(t_2^2)\} = \alpha t_1^2$$

If $\underset{\sim}{y}(t) = \underset{\sim}{w}^2(t)$ then [see (6-197)]

$$R_y(t_1,t_2) = E\{\underset{\sim}{w}^2(t_1)\underset{\sim}{w}^2(t_2)\}$$

$$= E\underset{\sim}{w}^2(t_1)E\{\underset{\sim}{w}^2(t_2)\} + 2\,E^2\{\underset{\sim}{w}(t_1)\underset{\sim}{w}(t_2)\} = \alpha^2 t_1 t_2 + 2\alpha^2 t_1^2$$

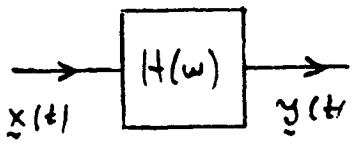
10-7 From (10-112):

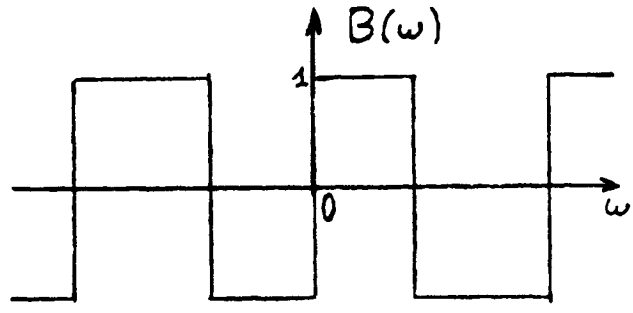
$$\eta_s = 3\int_0^{10} 2\,dt = 60 \qquad \sigma_s^2 = 3\int_0^{10} 4dt = 120 \qquad E\{\underset{\sim}{s}^2\} = 3720$$

$\underset{\sim}{s}(7) = 0$ if there are no points in the interval (7-10, 7). The number of points in this interval is a Poission RV with parameter $10\lambda = 30$. Hence, $P\{\underset{\sim}{s}(7) = 0\} = e^{-30}$.

10-8

$H(\omega) = j\,B(\omega)$





From the assumption: $S_{xx}(\omega) = S_{yy}(\omega) \qquad S_{xy}(-\omega) = -S_{xy}(\omega)$

From (9-148): $S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2 \qquad S_{xy}(\omega) = S_{xx}(\omega)H^*(\omega)$

Combining, we obtain

$$|H(\omega)|^2 = 1 \qquad H(-\omega) = -H(\omega)$$

Since h(t) is real, the second equation yields $H(\omega) = jB(\omega)$ and from the first it follows that

$$|B(\omega)| = 1$$

as in the figure.

10-9 With $\underset{\sim}{i}(t) = \underset{\sim}{a}(t)$, $\underset{\sim}{q}(t) = \underset{\sim}{b}(t)$, (11-63) yields

$$S_i(\omega) = S_q(\omega) \qquad S_{iq}(\omega) = -S_{qi}(\omega) = S_{qi}(-\omega)$$

Hence [see (11-75) and (11-82)],

$$S_w(\omega) = 2\,S_i(\omega) + 2j\,S_{qi}(\omega)$$

$$S_w(-\omega) = 2\,S_i(\omega) - 2j\,S_{qi}(\omega)$$

Adding and subtracting, we obtain

$$4\,S_i(\omega) = S_w(\omega) + S_w(-\omega) \qquad 4j\,S_{iq}(\omega) = S_w(-\omega) - S_w(\omega)$$

10-10 From (10-133)

$$\underset{\sim}{x}(t) = \underline{\mathrm{Re}}\,[\underset{\sim}{w}(t)e^{j\omega_0 t}]$$

$$\underset{\sim}{x}(t-\tau) = \underline{\mathrm{Re}}[\underset{\sim}{w}_\tau(t)e^{j\omega_0 t}] = \underline{\mathrm{Re}}[\underset{\sim}{w}(t-\tau)e^{j\omega_0(t-\tau)}]$$

$$\underset{\sim}{w}_\tau(t) = \underset{\sim}{w}(t-\tau)e^{-j\omega_0\tau}$$

10-11 $R''_x(\tau) \longleftrightarrow -\omega^2 S_x(\omega)$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 S_x(\omega)\,d\omega = -R''_x(0)$$

and with ω_0 the optimum carrier frequency, (10-150) yields

$$E\{|\underset{\sim}{w}'(t)|^2\} = \frac{M}{2\pi} = -2R''_x(0) - 2\omega_0^2 R_x(0)$$

10-12 From the stationarity of the process $\underset{\sim}{x}(t)\cos\omega t + \underset{\sim}{y}(t)\sin\omega t$ it follows that [see (10-130)]

$$C_{xx}(\tau) = C_{yy}(\tau) \qquad C_{xy} = -C_{yx}(\tau) \tag{i}$$

Using these identities, we shall express the joint density f(X,Y) of the 2n RVs

$$\underset{\sim}{X} = [\underset{\sim}{x}(t_1), \ldots, \underset{\sim}{x}(t_n)] \qquad \underset{\sim}{Y} = [\underset{\sim}{y}(t_1), \ldots, \underset{\sim}{y}(t_n)]$$

in terms of the covariance matrix C_{ZZ} of the complex vector $\underset{\sim}{Z} = \underset{\sim}{X} + j\underset{\sim}{Y}$. From (i) it follows that

$$E\{x(t_i)\underset{\sim}{x}(t_j)\} = E\{\underset{\sim}{y}(t_i)\underset{\sim}{y}(t_j)\} \qquad E\{\underset{\sim}{x}(t_i)\underset{\sim}{y}(t_j)\} = -E\{\underset{\sim}{y}(t_i)\underset{\sim}{x}(t_j)\}$$

This yields

$C_{XX} = C_{YY}$, and $C_{XY} = -C_{YX}$; hence, f(X,Y) is given by (8-62).

10-13 The signal $\underset{\sim}{c}(t) = f(t)$ is an extreme case of a cyclostationary process as in (10-178) with

$$h(t) = \begin{cases} f(t) & 0 \le t < T \\ 0 & \text{otherwise} \end{cases} \longleftrightarrow H(\omega) = \int_0^T f(t)e^{-j\omega t}dt$$

and $c_m = 1$, $R[m] = 1$. Hence [see (10A-2)]

$$\sum_{m=-\infty}^{\infty} R_m e^{-jm\omega T} = \sum_{m=-\infty}^{\infty} e^{-jm\omega T} = T\sum_{m=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T}m)$$

From the above and (10-180) it follows that the process $\underset{\sim}{x}(t) = f(t - \underset{\sim}{\theta})$ is stationary with power spectrum

$$S(\omega) = \left|\int_0^T f(t)e^{-j\omega t}dt\right|^2 \sum_{m=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T}m)$$

10-14 The process

$$\underset{\sim}{y}_N(t) = \underset{\sim}{x}(t+\tau) - \sum_{n=-N}^{N} \underset{\sim}{x}(t+nT)\,\frac{\sin\sigma(\tau-nT)}{\sigma(\tau-nT)}$$

is the output of a system with input $\underset{\sim}{x}(t)$ and system function

$$H_N(\omega) = e^{j\omega\tau} - \sum_{n=-N}^{N} \frac{\sin\sigma(\tau-nT)}{\sigma(\tau-nT)}\, e^{jnT\omega}$$

Furthermore, $\underset{\sim}{\varepsilon}_N(\tau) = \underset{\sim}{y}_N(0)$, hence [see (9-153)]

$$E\{\underset{\sim}{\varepsilon}_N^2(\tau)\} = E\{\underset{\sim}{y}^2(0)\} = \frac{1}{2\pi}\int_{\infty}^{\infty} S(\omega)\,|H_N(\omega)|^2 d\omega \qquad \text{(i)}$$

The function $H_N(\omega)$ is the truncation error in the Fourier series expansion of $e^{j\omega\tau}$ in the interval $(-\sigma,\sigma)$. Hence, for $N > N_0$

$$|H_N(\omega)| < \varepsilon \qquad\qquad |\omega| < \sigma$$

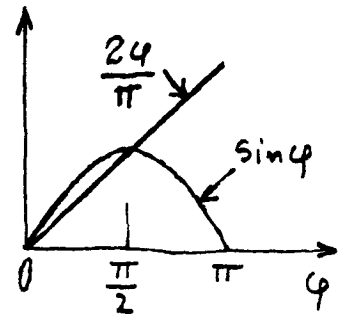
From this and (i) it follows that, if $S(\omega) = 0$ for $|\omega| < \sigma$, then

$$E\{\underset{\sim}{\varepsilon}_N^2(\tau)\} = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)\,|H_N(\omega)|^2 d\omega < \varepsilon\, R(0) \qquad N > N_0$$

10-15 [see after (10-195)]

$$R(0) - R(\tau) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)(1-\cos\omega\tau)\,d\omega$$

$$\le \frac{\tau^2}{4\pi}\int_{-\sigma}^{\sigma} \omega^2 S(\omega)\,d\omega = \frac{-\tau^2}{2} R''(0)$$



Furthermore, since

$$\sin\phi \ge \frac{2\phi}{\pi} \qquad 0 \le \phi \le \frac{\pi}{2}$$

we obtain

$$R(0) - R(\tau) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)\, 2\sin^2\frac{\omega\tau}{2}\, d\omega$$

$$\ge \frac{2\tau^2}{\pi^2}\,\frac{1}{2\pi}\int_{-\sigma}^{\sigma} \omega^2 S(\omega)\,d\omega = \frac{-2\tau^2}{\pi^2} R''(0)$$

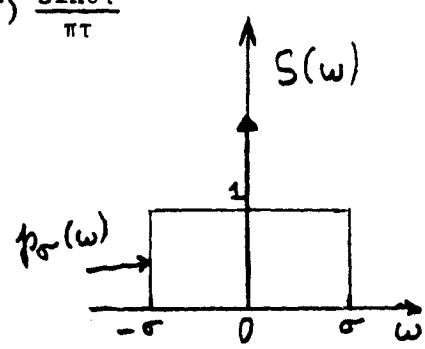
10-16 With $T = \pi/\sigma$

$$R(mT) = E\{\underset{\sim}{x}(nT+mT)\underset{\sim}{x}(nT)\} = \begin{cases} I & m = 0 \\ \eta^2 & m \ne 0 \end{cases}$$

Hence [see (10-196)]

$$R(\tau) = \sum_{m=-\infty}^{\infty} R(mT)\,\frac{\sin\sigma(\tau-mT)}{\sigma(\tau-mT)} = \eta^2 + (I-\eta^2)\,\frac{\sin\sigma\tau}{\pi\tau}$$

$$S(\omega) = 2\pi\eta^2\delta(\omega) + 2\pi(I-\eta^2)p_\sigma(\omega)$$

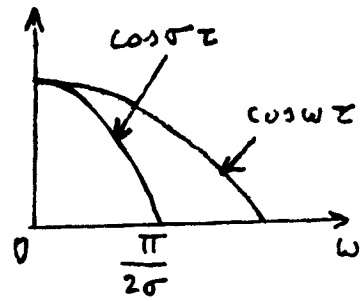


10-17 Given $E\{\underset{\sim}{x}(n+m)\underset{\sim}{x}(n)\} = N\delta[m]$

This is a special case of Prob. 10-16 with $\eta = 0$, $I = N$.

10-18 If $|\tau| < \pi/2\sigma$, then

$$\cos\omega\tau \geq \cos\sigma\tau \qquad |\omega| \leq \sigma$$



$$R(\tau) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)\cos\omega\tau\, d\omega$$

$$\geq \frac{\cos\sigma\tau}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)\, d\omega = R(0)\cos\sigma\tau$$

10-19 From (10-133) with $c = \sigma$

$$P_1(\omega,\tau) + j\omega\, P_2(\omega,\tau) = 1$$

$$P_1(\omega,\tau) + j(\omega+\tau)P_2(\omega,\tau) = e^{j\sigma\tau}$$

Hence,

$$P_1(\omega,\tau) = 1 - \frac{\omega}{\sigma}(e^{j\sigma\tau} - 1) \qquad P_2(\omega,\tau) = \frac{1}{j\sigma}(e^{j\sigma\tau} - 1)$$

Inserting into (11-141), we obtain

$$p_1(\tau) = \frac{4\sin^2(\sigma\tau/2)}{\sigma^2\tau^2} \qquad p_2(\tau) = \frac{4\sin^2(\sigma\tau/2)}{\sigma^2\tau}$$

and with $t = 0$, the desired result follows from (10-206) because $\bar{T} = 2T$ and

$$\sin^2\frac{\sigma(\tau-2nT)}{2} = \sin^2\left(\frac{\sigma\tau}{2} - n\pi\right) = \sin^2\frac{\sigma\tau}{2}$$

10-20 As in (10-213)

$$\underset{\sim}{P}(\omega) = \frac{1}{\lambda} \int_{-a}^{a} \cos\omega t \; \underset{\sim}{z}(t) \cos \omega_c t \, dt$$

$$E\{\underset{\sim}{P}(\omega)\} = \int_{-a}^{a} \cos\omega t \cos \omega_c t \, dt$$

$$\sigma^2_{P(\omega)} = \frac{1}{\lambda} \int_{-a}^{a} \cos^2\omega_c t_2 \cos^2 \omega t_2 \, d t_2$$

10-21 We shall show that if

$$\underset{\sim}{X}_c(\omega) = \frac{1}{\lambda} \sum_{|\underset{\sim}{t}_i|<c} \underset{\sim}{x}(t_i) e^{-j\omega \underset{\sim}{t}_i} = \frac{1}{\lambda} \int_{-a}^{a} \underset{\sim}{x}(t) \underset{\sim}{z}(t) e^{-j\omega t} dt$$

where $\underset{\sim}{z}(t) = \sum \delta(t-\underset{\sim}{t}_i)$ is a Poisson impulse train, then

$$E\{|\underset{\sim}{X}_c(\omega)|^2\} \simeq 2cS_x(\omega) + \frac{2c}{\lambda} R_x(0)$$

<u>Proof</u>

Since $R_z(\tau) = \lambda^2 + \lambda\delta(\tau)$, it follows that

$$E\left\{|\underset{\sim}{X}_c(\omega)|^2\right\} = \frac{1}{\lambda^2} \int_{-c}^{c} \int_{-c}^{c} R_x(t_1-t_2) e^{-j\omega(t_1-t_2)} dt_1 dt_2$$

$$= \int_{-c}^{c} e^{j\omega t_2} \int_{-c}^{c} R_x(t_1-t_2) e^{-j\omega t_1} dt_1 dt_2 + \frac{1}{\lambda} \int_{-c}^{c} R_x(0) dt_2$$

If $\int_{-\infty}^{\infty} |R_x(\tau)| < \infty$ then for sufficient large c, the inner integral on the right is nearly equal to $S_x(\omega)^{-j\omega t_2}$ and (i) follows.

10-22 $E\{\underset{\sim}{z}(t)\} = g(t)$ $\qquad E\{\underset{\sim}{w}(t)\} = g(t) - g(T)t/T = g(t)$

$$\underset{\sim}{w}(t) = (1 - \frac{t}{T}) \int_0^t \underset{\sim}{x}(\alpha)d\alpha - \frac{t}{T} \int_t^T \underset{\sim}{x}(\alpha)d\alpha$$

The above two integrals are uncorrelated because $\underset{\sim}{n}(t)$ is white noise. Hence, as in Example 9-5

$$\sigma_w^2 = (1 - \frac{t}{T})^2 Nt + \frac{t^2}{T^2} N(T - t) = Nt(1 - \frac{t}{T})$$

<u>Note</u> The above shows that the information that $g(T) = 0$ can be used to improve the estimate of $g(t)$. Indeed, if we use $\underset{\sim}{w}(t)$ instead of $\underset{\sim}{z}(t)$ for the estimate of $g(t)$ in terms of the data $\underset{\sim}{x}(t)$, the variance is reduced from Nt to $Nt(1-t/T)$.

10-23 (a) Since $|\sum_i a_i b_i| \le \sum |a_i||b_i|$, it suffices to assume that the numbers a_i and b_i are real. The quadratic

$$I(z) = \sum_i (a_i - z\, b_i)^2 = z^2 \sum_i b_i^2 - 2z \sum_i a_i b_i + \sum_i a_i^2$$

is nonnegative for every real z, hence, its discriminant cannot be positive. This yields (i).

(b) With $f[n]$ and $R_\nu[m] = S_o \delta[m]$ as in Prob. 10-24a (white noise)

$$y_f[n_0] = \sum h[n] f[n_0 - n] \qquad \underset{\sim}{y}_\nu[n] = \sum h[n] \underset{\sim}{\nu}[n]$$

$$E\{\underset{\sim}{y}_\nu^2[n]\} = S_0 R[0] = S_o \sum |h[n]|^2$$

[see (9-213)] And (i) yields

$$\frac{y_f^2[n_0]}{E\{\underset{\sim}{y}_\nu^2[n]\}} = \frac{|\sum h[n] f[n_0 - n]|^2}{S_0 \sum h^2[n]} \le \frac{1}{S_0} \sum |h[n]|^2$$

with equality iff $h[n] = kf^*[n_0 - n]$.

10-24 (a) Given $F(z)$ and $S_\nu(\omega) = S_0 \equiv$ constant . The z transform of $y_f[n]$ equals $F(z)H(z)$. Hence, [see (9-109)]

$$y_f[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\omega T})H(e^{j\omega T})e^{jn\omega T}d\omega$$

$$\frac{y_f^2[n]}{E\{\underset{\sim}{y}_\nu^2[n]\}} = \frac{\left|\int_{-\pi}^{\pi} F(e^{j\omega T})H(e^{j\omega T})d\omega\right|^2}{S_0\int_{-\pi}^{\pi}\left|H(e^{j\ T})\right|^2 d\omega}$$

$$\leq \frac{1}{S_0}\int_{\pi}^{\pi}\left|F(e^{j\omega T})\right|^2 d\omega$$

The last inequality follows from Schwarz's inequality with equality iff

$$H(e^{j\omega T}) = kF^*(e^{j\omega T}) = kF(e^{-j\omega T}), \text{ i.e., iff } H(z) = kF(z^{-1})$$

(b) Given arbitrary $R_\nu[m]$, $F(z)$, and the form of $H(z)$ (FIR); to find the coefficients a_m of $H(z)$. In this case

$$y_f[n] = a_0 f[n] + a_1 f[n-1] + \cdots + a_N f[n-N]$$

$$\underset{\sim}{y}_\nu[n] = a_0\underset{\sim}{\nu}[n] + a_1\underset{\sim}{\nu}[n-1] + \cdots + a_N\underset{\sim}{\nu}[n-N]$$

To maximize the signal-to-noise ratio it suffices to minimize

$$E\{\underset{\sim}{y}_\nu^2[n]\} = \sum_{k,r=0}^{N} a_k a_r R_\nu[k-r]$$

subject to the constraint that the sum

$$y_f[0] = a_0 f[0] + a_1 f[-1] + \cdots + a_N f[-N]$$

is constant. With λ a constant (Lagrange multiplier), we minimize the sum

$$I = \sum_{k,r=0}^{N} a_k a_r R\,[k-r] - \lambda\left[\sum_{k=0}^{N} a_k f[-k] - y_f[0]\right]$$

this yields the system

$$\frac{\partial I}{\partial a_k} = 0 = \sum_{r=0}^{N}\left[a_r R_\nu[k-r] - \lambda f[-k]\right] \qquad k = 0,\ldots,N$$

whose solution yields a_k.

10-25 $B = A|H(\omega_0)| = \dfrac{A}{\sqrt{\alpha^2+\omega_0^2}}$ $\qquad S_{y_n}(\omega) = \dfrac{N}{\alpha^2+\omega^2}$

$$R_{y_n}(\tau) = \frac{N}{2\alpha}\, e^{-\alpha|\tau|} \qquad E\{y_n^2(t)\} = R_{y_n}(0) = \frac{N}{2\alpha}$$

$$\frac{B^2}{E\{y_n^2(t)\}} = \frac{2A^2}{N}\,\frac{\alpha}{\alpha^2+\omega_0^2} \qquad \text{Max. if } \alpha = \omega_0$$

10-26 Since $H(\omega)$ is determined within a constant factor, we can sssume that the response $y_f(t_o)$ of the optimum $H(\omega)$ due to $f(t)$ is constant:

$$y_f(t_o) = \sum_{i=0}^{m} a_i\, f(t_o - iT) = c \qquad \text{(i)}$$

Our problem is to minimize the variance

$$V = E\{\underset{\sim}{y}_\nu^2(t)\} = \sum_{n=0}^{m} a_n \sum_{i=0}^{m} a_i\, R(nT-iT) \qquad \text{(ii)}$$

of $\underset{\sim}{y}_\nu(t)$ subject to the constraint (i). This yields the system

$$\frac{\partial V}{\partial a_n} = \sum_{i=0}^{m} a_i\, R(nT-iT) - kf(t_o - nT) = 0$$

where k is a constant (lagrange multiplier). With a_n so determined, we conclude from (ii) that

$$V = \sum_{n=0}^{m} ka_n\, f(t_o - nT) = ky_f(t_o) \qquad r^2 = \frac{y_f^2(t_o)}{ky_f(t_o)}$$

10-27 $R_{yyy}(\mu,\nu) = E\{\underset{\sim}{x}(t+\mu)+c[\,[\underset{\sim}{x}(t+\nu)+c]\,[\underset{\sim}{x}(t)+c]\} = R(\mu,\nu) + cR(\mu) + cR(\nu) + cR(\mu-\nu) + c^3$

because $E\{\underset{\sim}{x}(t)\} = 0$. Furthermore,

$$R(\mu) \Leftrightarrow 2\pi S(u)\delta(v) \qquad R(\nu) = 2\pi\delta(u)S(v) \qquad c^3 \Leftrightarrow 4\pi^2\delta(u)\delta(v)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(\mu-\nu)e^{-j(u\mu+v\nu)}d\mu d\nu = \int_{-\infty}^{\infty} R(\tau)e^{-ju\tau}d\tau \int_{-\infty}^{\infty} e^{-j(u+v)\nu}d\nu = 2\pi S(u)\delta(u+v)$$

10-28 We shall use the equations $E\{\tilde{x}(t)\} = 0$, $E\{\tilde{x}^2(t)\} = \lambda t$. Suppose that $t_1 < t_2 < t_3$. Clearly,

$$\tilde{x}(t_2) = \tilde{x}(t_1) + [\tilde{x}(t_2)-\tilde{x}(t_1)]$$

$$\tilde{x}(t_3) = \tilde{x}(t_1) + [\tilde{x}(t_2) - \tilde{x}(t_1)] + [\tilde{x}(t_3)-\tilde{x}(t_2)] \tag{i}$$

Inserting into the product $\tilde{x}(t_1)\tilde{x}(t_2)\tilde{x}(t_3)$ and using the identity $E\{\tilde{x}(t_i)-\tilde{x}(t_j)\} = 0$ and the independence of the three terms on the right of (i), we obtain

$$E\{x(t_1)x(t_2)x(t_3)\} = E\{x^3(t_1)\} = \lambda t_1 = \lambda \min(t_1, t_2, t_3)$$

Since $\tilde{z}(t) = \tilde{x}'(t)$, we conclude from (9-120)-(9-122) that

$$R_{\tilde{z}\tilde{z}\tilde{z}}(t_1,t_2,t_3) = \frac{\partial^3 R_{xxx}(t_1,t_2,t_3)}{\partial t_1 \partial t_2 \partial t_3} = \lambda \frac{\partial^3 \min(t_1,t_2,t_3)}{\partial t_1 \partial t_2 \partial t_3}$$

It suffices therefore to show that the right side equals $\lambda\delta(t_1-t_2)\delta(t_1-t_2)$. This is a consequence of the following:

$$\frac{\partial \min(t_1,t_2,t_3)}{\partial t_3} = t_1 U(t_2-t_1)\delta(t_3-t_1) + t_2\, U(t_1-t_2)\,\delta(t_3-t_2)$$

$$+ U(t_1-t_3)U(t_2-t_3) - t_3\delta(t_1-t_3)U(t_2-t_3) - t_3 U(t_1-t_3)\delta(t_2-t_3)$$

$$= U(t_1-t_3)U(t_2-t_3)$$

because $t_i\delta(t_i-t_j) = t_j\delta(t_j-t_i)$. Hence,

$$\frac{\partial^2 \min(t_1,t_2,t_3)}{\partial t_2 \partial t_3} = U(t_1-t_3)\delta(t_2-t_3) \qquad \frac{\partial^2 \min(t_1,t_2,t_3)}{\partial t_1 \delta t_2 \partial t_3} = \delta(t_1-t_2)\delta(t_1-t_3)$$

10-29 See outline given in text.

CHAPTER 11

11-1 $$S_x(z) = \frac{5 - 2(z + 1/z)}{10 - 3(z + 1/z)} = \frac{2}{3} + \frac{5/9}{10/3 - (z + 1/z)}$$

$$R[m] = \frac{2}{3} + \frac{5}{18}\, 3^{-|m|} \qquad \Gamma(z) = \frac{3z - 1}{2z - 1}$$

11-2 $$S_x(s) = \frac{s^4 + 64}{s^4 - 10s^2 + 9} = \frac{s^2 + 4s + 8}{s^2 + 4s + 3}\;\frac{s^2 - 4s + 8}{s^2 - 4s + 3}$$

$$L(s) = \frac{s^2 + 4s + 8}{s^2 + 4s + 3}$$

11-3 First proof

$$\underset{\sim}{s}[n] = \sum_{k=0}^{\infty} \ell[n]\underset{\sim}{i}[n-k] \qquad E\{\underset{\sim}{x}^2[n]\} = \sum_{k=0}^{\infty} \ell^2[k]$$

Second proof

$$S(z) = L(z)L(1/z) \qquad R[m] = \ell[m] * \ell[-m] = \sum_{k=0}^{\infty} \ell[k]\ell[k-m]$$

$$R[0] = \sum_{k=0}^{\infty} \ell^2[k]$$

11-4 (a) This is a special case of (11-22) and (11-23).

(b) From (a) it follows that

$$R''_{yx}(\tau) + 3\,R'_{yx}(\tau) + 2\,R_{yx}(\tau) = q\delta(\tau)$$

Since $R_{xx}(\tau) = 0$ for $\tau < 0$, the above shows that

$$R_{yx}(\tau) = 0 \text{ for } \tau \leq 0^- \qquad R'_{yx}(0^-) = 0$$

Furthermore,

$$S_{yx}(s) = \frac{q}{s^2 + 3s + 2}$$

hence (initial value theorem)

$$R_{yx}(0^+) = \lim_{s \to \infty} s\;S_{yx}(s) = 0 \qquad R'_{yx}(0^+) = \lim_{s \to \infty} s^2\;S_{yx}(s) = q$$

Similarly,

$$R''_{yy}(\tau) + 3\,R'_{yy}(\tau) + 2\,R_{yy}(\tau) = R_{xy}(\tau) = R_{yx}(-\tau) = 0 \quad \text{for } \tau > 0$$

$$S_{yy}(s) = \frac{q}{(s^2 + 3s + 2)(s^2 - 3s + 2)} = \frac{qs/12 + q/4}{s^2 + 3s + 2} + \frac{-qs/12 + q/4}{s^2 - 3s + 2}$$

$$S^+_{yy}(s) = \frac{qs/12 + q/4}{s^2 + 3s + 2}$$

$$R^+_{yy}(0^+) = R_{yy}(0) = \lim_{s \to \infty} s^2\;S^+_{yy}(s) = \frac{q}{12}$$

$$R'_{yy}(0) = \lim_{s \to \infty} s\;[s\;S^+_{yy}(s) - \frac{q}{12}] = 0$$

11-5 $$S_x(z) = S_s(z) + S_\nu(z) = \frac{1}{D(z)} + q = \frac{1 + qD(z)}{D(z)}$$

If $R_s[m] = 2^{-|m|}$ and $S_\nu(z) = 5$, then (see Example 9-31)

$$S_s(z) = \frac{1.5}{2.5 - (z^{-1} + z)} \qquad S_x(z) = \frac{5 - 14z^{-1} + 5z^{-2}}{1 - 2.5z^{-1} + z^{-2}}$$

11-6 The process

$$\underset{\sim}{y}[n] = \frac{1}{n}\sum_{k=1}^{n} \underset{\sim}{x}(nT+kT)$$

is the output of a system with input $\underset{\sim}{x}[n]$ and system function

$$H(z) = \frac{1}{n}\sum_{k=1}^{n} z^k$$

Furthermore, $\underset{\sim}{s} = \underset{\sim}{y}[0]$ and

$$n^2\left|H(e^{j\omega T})\right|^2 = \left|\sum_{k=1}^{n} e^{jk\omega T}\right|^2$$

$$= \left|\frac{e^{j\omega T} - e^{j(n+1)\omega T}}{1 - e^{j\omega T}}\right|^2 = \frac{\sin^2 n\omega T/2}{\sin^2 \omega T/2}$$

Hence [see (9-51)]

$$E\{\underset{\sim}{s}^2\} = R_y[0] = \frac{1}{2\pi n^2}\int_{-\infty}^{\infty} S_x(\omega)\,\frac{\sin^2 n\omega T/2}{\sin^2 \omega T/2}\,d\omega$$

11-7 Since $R(t_1,t_2) = e^{-c|t_1-t_2|}$, (12-58) yields

$$\int_{-a}^{t_1} e^{-c(t_1-t_2)}\phi(t_2)dt_2 + \int_{t_1}^{a} e^{c(t_1-t_2)}\phi(t_2)dt_2 = \lambda\phi(t_1) \qquad \text{(i)}$$

Differentiating twice and using (i) we obtain (omitting details)

$$\lambda\phi''(t) + (2c - \lambda c^2)\phi(t) = 0$$

Hence,

$$\phi(t) = \beta\cos\omega t \quad \text{and} \quad \phi(t) = \beta'\cos\omega' t$$

To determine ω, we insert into (i). This yields

$$\frac{2c}{c^2+\omega^2} + \frac{\omega\sin a\omega - c\cos a\omega}{c^2+\omega^2}\,e^{-ac}\,(e^{ct} + e^{-ct}) = 2c\,\lambda\cos\omega t$$

This yields

$$\omega_n \sin a\omega_n - c \cos a\omega_n = 0 \qquad \lambda_n = \frac{2c}{c^2 + \omega_n^2}$$

The constants β_n are determined from (normalization)

$$1 = \int_{-a}^{a} \beta_n^2 \cos^2 \omega_n t \, dt \qquad \beta_n^2 = \frac{1}{a + c\,\lambda_n}$$

Similarly for $\beta_n' \sin \omega_n' t$.

11-8 As in (9-60)

$$E\{|\underset{\sim}{X}_T(\omega)|^2\} = \int_{-T/2}^{T/2} R(t_1 - t_2) e^{-j\omega(t_1 - t_2)} dt_1 dt_2$$

$$= \int_{-T}^{T} (T - |\tau|) R(\tau)\, e^{-j\omega\tau} d\tau$$

Differentiating with respect to T and using the fact that if

$$\phi(t) = \int_{-t}^{t} f(x;t)\,dx$$

then

$$\frac{d\phi(t)}{dt} = f(t;t) - f(-t,t) + \int_{-t}^{t} \frac{\partial f}{\partial t}(x,t)\,dx$$

we obtain

$$\frac{\partial E\{|\underset{\sim}{X}_T(\omega)|^2\}}{\partial T} = \int_{-T}^{T} R(\tau) e^{-j\omega\tau} d\tau = E\{\frac{\partial}{\partial T} |\underset{\sim}{X}_T(\omega)|^2\}$$

The above approaches $S(\omega)$ as $T \to \infty$.

11-9 $$E\{\underset{\sim}{X}(\omega)\} = \int_{-a}^{a} 5\cos 3t\, e^{-j\omega t}dt = \frac{5\sin a(\omega-3)}{\omega-3} + \frac{5\sin a(\omega+3)}{\omega+3}$$

Var. $\underset{\sim}{X}(\omega) = 2qa = 4a.$

11-10 $$E\{\underset{\sim}{X}(u)\underset{\sim}{X}(v)\} = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \sigma_n^2\delta[n-k]\; e^{-j(nu-kv)T}$$

$$= \sum_{n=-\infty}^{\infty} \sigma_n^2\; e^{-jn(u-v)T}$$

11-11 Shifting the origin, we set

$$\underset{\sim}{c}_n = \frac{1}{T}\int_{-T/2}^{T/2} \underset{\sim}{x}(t)e^{-jn\omega_o t}dt \qquad \beta_n(\alpha) = \frac{1}{T}\int_{-T/2}^{T/2} R(\tau-\alpha)e^{-jn\omega_o\tau}d\tau$$

(a) We shall show that if

$$\hat{\underset{\sim}{x}}(t) = \sum_{n=-\infty}^{\infty} \underset{\sim}{c}_n e^{jn\omega_o t} \text{ then } E\{|\underset{\sim}{x}(t)-\hat{\underset{\sim}{x}}(t)|^2\} = 0 \text{ for } |t| < T/2 \qquad \text{(i)}$$

<u>Proof</u> $$E\{\underset{\sim}{c}_n\underset{\sim}{x}^*(\alpha)\} = \frac{1}{T}\int_{-T/2}^{T/2} E\{\underset{\sim}{x}(t)\underset{\sim}{x}^*(\alpha)\}e^{-jn\omega_o t}dt = \beta_n(\alpha)$$

The functions $\beta_n(\alpha)$ are the coefficients of the Fourier expansion of $R(\tau-\alpha)$:

$$R(\tau-\alpha) = \sum_{n=-\infty}^{\infty} \beta_n(\alpha)e^{jn\omega_o\tau} \qquad |\tau| < T/2 \qquad \text{(ii)}$$

Hence

$$E\{\hat{\underset{\sim}{x}}(t)\underset{\sim}{x}^*(t)\} = \sum_{n=-\infty}^{\infty} E\{\underset{\sim}{c}_n\underset{\sim}{x}^*(t)\}^{jn\omega_o t} = \sum_{n=-\infty}^{\infty} \beta_n(t)e^{jn\omega_o t}$$

From (ii) it follows with $\tau = \alpha = t$ that the last sum equals R(0). Similarly, $E\{\hat{\underset{\sim}{x}}^*(t)\underset{\sim}{x}(t)\} =$ R(0) and (i) results.

(b) $E\{\underset{\sim}{c}_n\underset{\sim}{c}_m^*\} = \frac{1}{T}\int_{-T/2}^{T/2} E\{c_n\underset{\sim}{x}^*(t)\}e^{jn\omega_o t}dt = \frac{1}{T}\int_{-T/2}^{T/2} \beta_n(t)e^{jn\omega_o t}dt$

(c) If T is sufficiently large, then

$$T\beta_n(\alpha) = \int_{-T/2}^{T/2} R(\tau-\alpha)e^{-jn\omega_o\tau}d\tau \simeq S(n\omega_o)e^{-jn\omega_o\alpha}$$

$$E\{c_n c_m^*\} = \frac{S(n\omega_o)}{T^2}\int_{-T/2}^{T/2} e^{j(m-n)\omega_o\alpha}d\alpha \simeq \begin{cases} S(n\omega_o)/T & m=n \\ 0 & m\neq n \end{cases}$$

Thus, for large T, the coefficients $\underset{\sim}{c}_n$ of an arbitrary WSS process are nearly orthogonal.

11-12 $$E\{\underset{\sim}{x}(t_1)\underset{\sim}{x}^*(t_2)\} = \frac{1}{4\pi^2} E\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\{\underset{\sim}{X}(u)\underset{\sim}{X}^*(v)\}e^{j(ut_1-vt_2)}dudv\right.$$

$$= \frac{1}{4\pi^2} E\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(u)\delta(u-v)e^{j(ut_1-vt_2)}dudv\right. = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} Q(u)e^{ju(t_1-t_2)}du$$

This depends only on $\tau = t_1-t_2$:

$$R_{xx}(\tau) = \frac{1}{x_2}\int_{-\infty}^{\infty} Q(u)e^{ju\tau}du \qquad S_{xx}(\omega) = \frac{Q(\omega)}{2\pi}$$

11-13 Equations (11-79) can be written in the following form:

$$E\{\underset{\sim}{A}(u)\underset{\sim}{A}(v)\} = Q(u)\delta(u-v) = E\{\underset{\sim}{B}(u)\underset{\sim}{B}(v)\} \qquad E\{\underset{\sim}{A}(u)\underset{\sim}{B}(v)\} = 0$$

for $u \geq 0$, $v \geq 0$. We shall show that if the above is true and $E\{\underset{\sim}{A}(\omega)\} = E\{\underset{\sim}{B}(\omega)\} = 0$, then the process

$$\underset{\sim}{x}(t) = \frac{1}{\pi}\int_0^{\infty}\left[\underset{\sim}{A}(\omega)\cos\omega t - \underset{\sim}{B}(\omega)\sin\omega t\right] d\omega$$

is WSS.

<u>Proof</u> Clearly, $E\{\underset{\sim}{x}(t)\} = 0$ and

$$E\{\underset{\sim}{x}(t+\tau)\underset{\sim}{x}(t)\}$$

$$= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty E\{\underset{\sim}{A}(u)\cos u(t+\tau) - \underset{\sim}{B}(u)\sin u(t+\tau)\}\,[\underset{\sim}{A}(v)\cos vt - B(v)\sin vt]\,du\,dv$$

$$= \frac{1}{\pi}\int_0^\infty\int_0^\infty Q(u)\delta(u-v)\,[\cos u(t+\tau)\cos vt + \sin u(t+\tau)\sin v(t)\,du\,dv\,]\,dv\,du$$

$$= \frac{1}{\pi^2}\int_0^\infty Q(u)\,[\cos u(t+\tau)\cos u\,t + \sin u(t+\tau)\sin ut]\,du$$

$$= \frac{1}{\pi^2}\int_0^\infty Q(u)\cos u\tau\,du$$

From this and (9-136) it follows that $\underset{\sim}{x}(t)$ is WSS with $S_{xx}(\omega) = Q(\omega)/\pi$.

11-14 $\quad E\{\underset{\sim}{v}(t)\} = 0 \qquad E\{\underset{\sim}{X}_T(\omega)\} = \int_{-T}^{T} f(t)e^{-j\omega t}dt$

The above integral is the transform of the product $f(t)p_T(t)$, hence (frequency convolution theorem), it equals $F(\omega)*\sin T\omega/\pi\omega$.

$$\text{Var}\,\underset{\sim}{X}_T(\omega) = E\left\{\left|\int_{-T}^{T}\underset{\sim}{\nu}(t)e^{-j\omega t}dt\right|^2\right\}$$

The integral is the transform of the nonstationary white noise $\underset{\sim}{\nu}(t)p_T(t)$. The autocorrelation of this process equals $q(t_1)\delta(t_1-t_2)$ where $q(t) = qp_T(t)$. Hence, [see (11-69)]

$$\text{Var}\,\underset{\sim}{X}_T(\omega) = Q(0) = \int_{-T}^{T} q\,dt = 2qT$$

14-1 It suffices to show that [see (14-41)]

$$H(A \cdot B|B_j) = H(A|B_j)$$

Since

$$A_iB_kB_j = \begin{cases} A_iB_j & k = j \\ \{\emptyset\} & k \neq j \end{cases} \qquad \text{and } P(A_iB_j|B_j) = P(A_i|B_j)$$

(14-40) yields

$$H(A \cdot B|B_j) = -\sum_{i,k} P(A_iB_k|B_j) \log P(A_iB_k|B_j)$$

$$= -\sum_i P(A_i|B_j) \log P(A_i|B_j) = H(A|B_j)$$

14-2 If $\alpha < \beta$, then $\phi'(\alpha) > \phi'(\beta)$ because

$$\phi'(\alpha) - \phi'(\beta) = \log(\beta/\alpha) > 0. \quad \text{Hence,}$$

$$\int_a^b \phi'(\alpha)d\alpha > \int_{a+c}^{b+c} \phi'(\alpha)d\alpha \qquad c > 0$$

This yields

$$\phi(p_1+p_2) - \phi(p_1) = \int_{p_1}^{p_1+p_2} \phi'(\alpha)d\alpha < \int_0^{p_2} \phi'(\alpha)d\alpha = \phi(p_2)$$

Similarly

$$\phi(p_1+\varepsilon) - \phi(p_1) - \phi(p_2) + \phi(p_2-\varepsilon)$$

$$= \int_{p_1}^{p_1+\varepsilon} \phi'(\alpha)d\alpha - \int_{p_2-\varepsilon}^{p_2} \phi'(\alpha)d\alpha > 0$$

14-3 Applying the identity

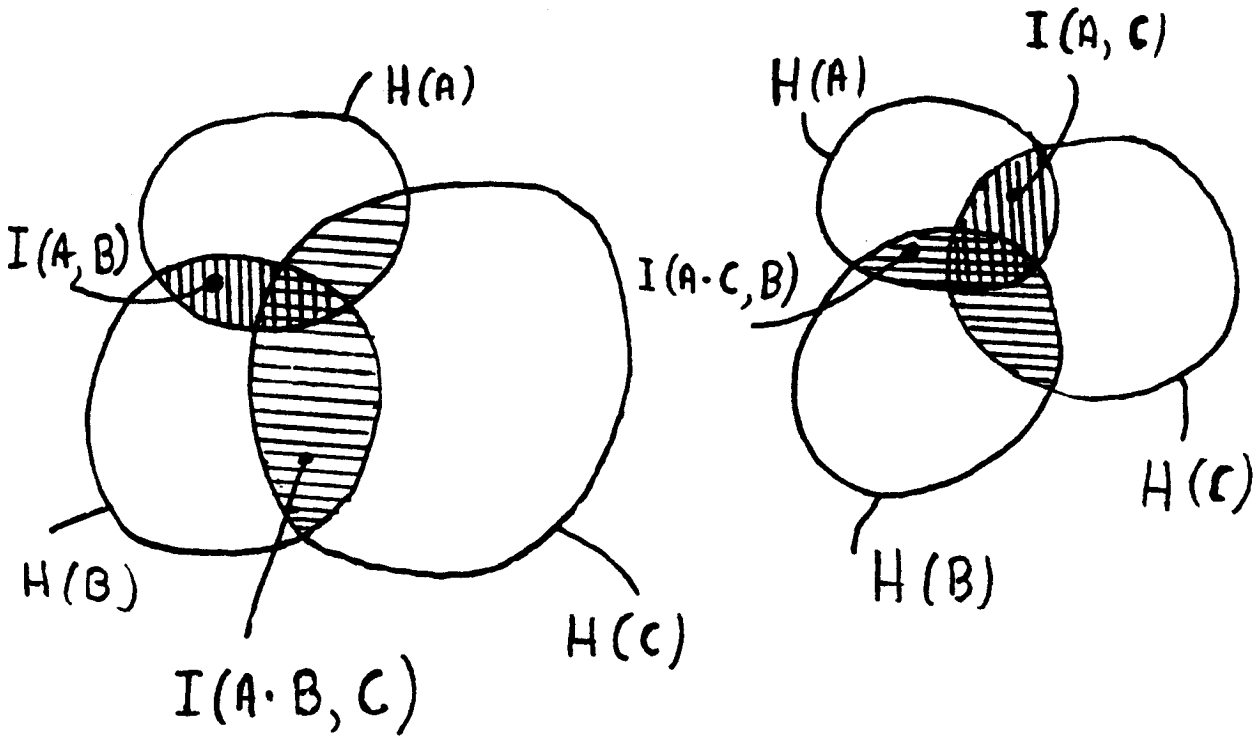
$$H(A_1 \cdot A_2) = H(A_1) + H(A_2|A_1) \tag{i}$$

to the partitions $A_1 = A$, $A_2 = B \cdot C$ and $A_1 = A \cdot B$, $A_2 = C$, we obtain the first line. The second line follows from the first [see (i)]. The third line is a consequence of the first two.

14-4 It follows if we apply the identity

$$I(A_1, A_2) = H(A_1) + H(A_2) - H(A_1 \cdot A_2)$$

to the partitions $A_1 = A \cdot B$, $A_2 = C$.



14-5 **(a)** **From** (14-53)

$$I(A,B \cdot C) = H(A) + H(B \cdot C) - H(A \cdot B \cdot C)$$

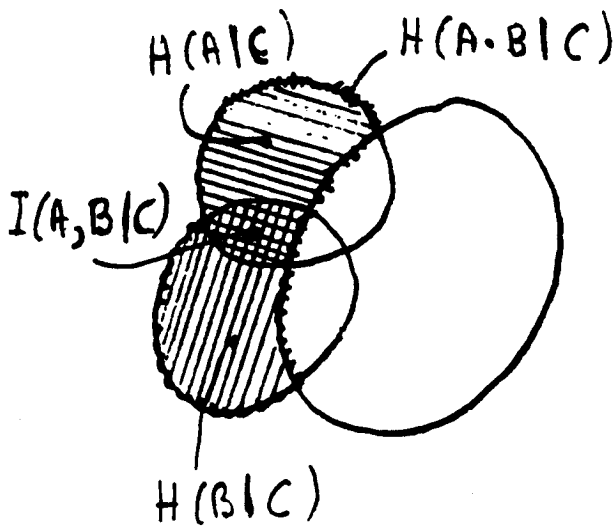
$$I(A,C) = H(A) + H(C) - H(A \cdot C)$$

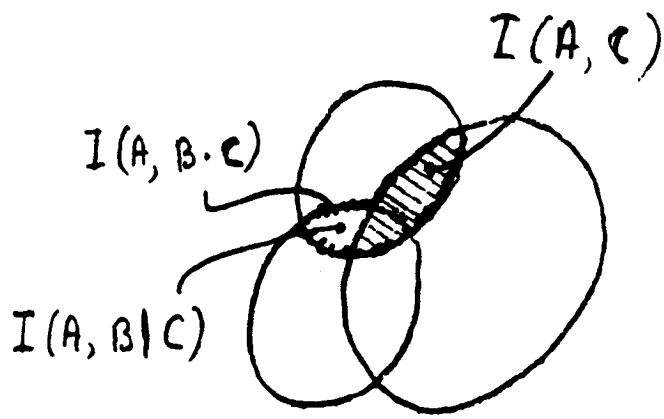
and since (see Prob. 14-4)

$$H(A \cdot B \cdot C) - H(A \cdot C) = H(A \cdot B|C) - H(A|C)$$

we conclude with (14-49) **that**

$$I(A,B \cdot C) - I(A \cdot C) = H(B|C) + H(A|C) - H(A \cdot B|C)$$





(b) If $B \cdot C$ is observed, then the resulting prediction in the uncertainty of A equals $I(A, B \cdot C)$. But, if $B \cdot C$ is observed, then C is observed, hence, the reduction in the uncertainty of A is at least $I(A,C)$. Hence

$$I(A,B \cdot C) \geq I(A,C)$$

with equality only if $I(A,B|C) = 0$, i.e., if in the subsequence of trials in which C occurred, knowledge of the occurrence of B gives no information about A.

14-6 The partition $H(A^3)$ has eight elements with respective probabilities

$$p^3, p^2q, p^2q, p^2q, pq^2, pq^2, pq^2, q^3$$

Hence

$$H(A^3) = -p^3\log p^3 - 3p^2q\log p^2q - 3pq^2\log pq^2 - q^3\log q^3$$

$$= -3p(p^2+2pq+q^2)\log p - 3q(p^2+2pq+q^2)\log q$$

$$= -3p\log p - 3q\log q = 3H(A)$$

14-7 The density of the RV $\underset{\sim}{w} = \underset{\sim}{x} + a$ equals $f_x(w-a)$. Hence,

$$H(\underset{\sim}{x}+a) = -\int_{-\infty}^{\infty} f_x(w-a)\log f_x(w-\alpha)dw$$

$$= -\int_{-\infty}^{\infty} f_x(x)\log f_x(x)dx = H(\underset{\sim}{x})$$

The joint density of the RVs $\underset{\sim}{x}$ and $\underset{\sim}{z} = \underset{\sim}{x} + \underset{\sim}{y}$ equals $f_{xy}(x,z-x)$. Hence [see (14-90)]

$$H(\underset{\sim}{z}|\underset{\sim}{x}) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,z-x)\log f_{xy}(x,z-x)/f_x(x)dxdz$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\log f_{xy}(x,y)/f_x(x)dxdy = H(\underset{\sim}{y}|\underset{\sim}{x})$$

14-8 The RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ take the values x_i and y_j respectively *then* $\underset{\sim}{z} = x_i + y_j$ iff $\underset{\sim}{x} = x_i$ and $\underset{\sim}{y} = y_j$ (assumption). Hence,

$$\{\underset{\sim}{z} = x_i + y_j\} = \{\underset{\sim}{x} = x_i\} \cap \{\underset{\sim}{y} = y_j\}$$

This shows that $A_z = A_x \cdot B_y$. Furthermore, since the RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are independent, the events $\{\underset{\sim}{x} = x_i\}$ and $\{\underset{\sim}{y} = y_j\}$ are also independent. This shows that the partitions A_x and B_y are independent and [see (14-44) and Prob. 14-1]

$$H(A_z|A_x) = H(A_x \cdot A_y|A_x) = H(A_y|A_x) = H(A_y)$$

From this it follows that $H(\underset{\sim}{z}|\underset{\sim}{x}) = H(\underset{\sim}{y})$ because [see (14-88) and (14-41)]

$$H(\underset{\sim}{z}|\underset{\sim}{x}) = H(A_z|A_x)$$

14-9 As we see from (14-80)

$H(\underset{\sim}{x}) = \ln a$ where we assume that $a = N\delta$. The RV $\underset{\sim}{y}$ takes the values $0, \delta, \ldots, (N-1)\delta$ with probability $1/N$. The conditional density of $\underset{\sim}{x}$ assuming $\underset{\sim}{y} = k\delta$ is uniform in the interval $(k\delta, k\delta + \delta)$. Hence,

$$H(\underset{\sim}{x}|\underset{\sim}{y} = k\delta) = -\int_{k\delta}^{k\delta+\delta} f(x|\underset{\sim}{y} = k\delta) \ln f(x|\underset{\sim}{y} = k\delta)\,dx = \ln \delta$$

And as in (14-41)

$$H(\underset{\sim}{x}|\underset{\sim}{y}) = \sum_{k=0}^{N} H(\underset{\sim}{x}|\underset{\sim}{y} = k\delta) P\{\underset{\sim}{y} = k\delta\} = \ln \delta$$

Finally [see (14-95)]

$$I(\underset{\sim}{x}, \underset{\sim}{y}) = H(\underset{\sim}{x}) - H(\underset{\sim}{x}|\underset{\sim}{y}) = \ln a - \ln \delta$$

14-10 If $y_i = g(x_i)$, $y_j = g(x_j)$ and $y_i = y_j$ then $x_i = x_j$. Hence,

$$p_{ij} = \begin{cases} p_i & i = j \\ 0 & i \neq j \end{cases} \qquad p_i = P\{\underset{\sim}{x} = x_i\}$$

and

$$H(\underset{\sim}{x}, \underset{\sim}{y}) = -\sum_{i,j} p_{ij} \log p_{ij} = -\sum_i p_i \log p_i = H(\underset{\sim}{x})$$

14-11 From Prob. 10-10 it follows with $g(x) = x$ that $H(\underset{\sim}{x},\underset{\sim}{x}) = H(\underset{\sim}{x})$. And since [see (14-103)] $H(\underset{\sim}{x},\underset{\sim}{x}) = H(\underset{\sim}{x}|\underset{\sim}{x}) + H(\underset{\sim}{x})$ we conclude that $H(\underset{\sim}{x}|\underset{\sim}{x}) = 0$. From Prob. 14-3 it follows that

$$H(\underset{\sim}{y},\underset{\sim}{x}|\underset{\sim}{x}) = H(A_y \cdot A_x|A_x) = H(A_x \cdot A_x) + H(A_y|A_x \cdot A_x)$$

$$= H(A_y|A_x) = H(\underset{\sim}{y}|\underset{\sim}{x})$$

because $A_x \cdot A_x = A_x$ and $H(A_x \cdot A_x) = H(\underset{\sim}{x},\underset{\sim}{x}) = 0$.

14-12 $E\{\underset{\sim}{x}_n\} = 0 \qquad E\{\underset{\sim}{x}_n^2\} = 5 \qquad E\{\underset{\sim}{y}_n\} = 0$

$$E\{\underset{\sim}{y}_n^2\} = \sum_{k=0}^{\infty} 2^{-2k}\, E\{\underset{\sim}{x}_{n-k}^2\} = \frac{20}{3} \qquad E\{\underset{\sim}{x}_n \underset{\sim}{y}_n\} = E\{\underset{\sim}{x}_n^2\} = 5$$

(a) From (14-95), (14-84), and (15-86) with $\mu_{11} = 5$, $\mu_{22} = 20/3$, and $\mu_{12} = 5$

$$H(\underset{\sim}{x}) = \ln\sqrt{10\pi e} \qquad H(\underset{\sim}{y}) = \ln\sqrt{40\pi e/3} \qquad H(\underset{\sim}{x},\underset{\sim}{y}) = \ln 10\pi e/\sqrt{3}$$

$$I(\underset{\sim}{x},\underset{\sim}{y}) = \ln 2$$

(b) The process $\underset{\sim}{y}(t)$ is the output of the system

$$L(z) = \frac{1}{1 - 0.5\, z^{-1}} \qquad \ell_o = 1$$

with input $\underset{\sim}{x}_n$. Since $\bar{H}(\underset{\sim}{x}) = H(\underset{\sim}{x})$ and [see (12A-1)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|L(e^{j\phi})|\,d\phi = \ln \ell_o = 0$$

(14-133) yields $\bar{H}(\underset{\sim}{y}) = \bar{H}(\underset{\sim}{x}) = H(\underset{\sim}{x}) = \ln\sqrt{10\pi e}$.

14-13

$$\bar{H}(\underset{\sim}{x}) = H(\underset{\sim}{x}) = -\frac{1}{2}\int_4^6 \ln\frac{1}{2}\,dx = \ln 2$$

And as in Prob. 14-12 with $\ell_o = 5$,

$$\bar{H}(\underset{\sim}{y}) = \bar{H}(x) + \ln 5 = \ln 10$$

14-14 Given that $f(x) = 0$ for $|x|>1$ and $E\{\underset{\sim}{x}\} = 0.3$, find $f(x)$. With $g(x) = x$, (14-143) yields $f(x) = Ae^{-\lambda x}$ where

$$A\int_{-1}^{1} e^{-\lambda x}\,dx = \frac{A}{\lambda}(e^{\lambda} - e^{-\lambda}) = 1$$

$$A\int_{-1}^{1} xe^{-\lambda x}\,dx = \frac{A}{\lambda^2}(e^{\lambda} - e^{-\lambda}) - \frac{A}{\lambda}(e^{\lambda} - e^{-\lambda}) = 0.31$$

Solving, we obtain $A \simeq 0.425$, $\lambda \simeq -1$

14-15 $f(x) = Ae^{-\lambda x}$ for $1<x<5$ and 0 otherwise,

$$A\int_1^5 e^{-\lambda x}\,dx = 0.31 \qquad A\int_1^5 xe^{-\lambda x}dx = 3\,\frac{37}{60}$$

Hence, $A \simeq 1.06$, $\lambda \simeq 0.5$

14-16 From (14-151) with $x_k=k$, $g_1(x_k) = g_1(k) = k$, $k=1, \ldots, 6$

$$g_2(x_k)=\begin{cases} 0 & k=1,3,5 \\ 1 & k=2,4,6 \end{cases} \qquad p_k=\begin{cases} Ae^{-\lambda_1 k} & k=1,3,5 \\ Ae^{-\lambda_1 x-\lambda_2} & k=2,4,6 \end{cases}$$

Since $p_1 + p_3 + p_5 = 0.5$ and $E\{\underset{\sim}{x}\} = 4.44$, we conclude with $z = e^{-\lambda_2}$ and $w = e^{-\lambda_2}$ that

$$A(z+z^3+z^5) = Aw(z^2+z^4z^6)$$

$$A(\underset{\sim}{z}+3z^3+5z^5) + Aw(2z^2+4z^4+6z^6) = 4.44$$

This yields $A \simeq 0.0437$, $\underset{\sim}{z} = 1/w \simeq 1.468$

14-17 (a) The transformation $\underset{\sim}{y} = 3\underset{\sim}{x}$ is one-to-one, hence, $H(\underset{\sim}{y}) = H(\underset{\sim}{x})$

(b) From (14-113) with $g(x) = 3x$: $H(\underset{\sim}{y}) = H(\underset{\sim}{x}) + \ln 3$

14-18 (a) For fair dice, $P\{7\} = \frac{1}{6}$, $P(11) = \frac{1}{18}$, $P\{\text{neither 7 nor 11}\} = \frac{14}{18}$

$$H(A) = -\left(\frac{1}{6}\ln\frac{1}{6} + \frac{1}{18}\ln\frac{1}{18} + \frac{14}{18}\ln\frac{14}{18}\right) = 0.655$$

(b) From (14-10) with n=100 and N=3:

$$n_T \simeq e^{nH(A)} \simeq 2.79 \times 10^{28} \qquad n_a \simeq N^n \simeq 5.16 \times 10^{47}$$

14-19 The process $\underset{\sim}{x}_n$ is WSS with entropy rate $\overline{H}(x)$. Show that, if

$$\underset{\sim}{w}_n = \sum_{k=0}^{n} \underset{\sim}{x}_{n-k}\, \ell_k$$

then

$$\lim_{n\to\infty} \frac{1}{n+1} H(\underset{\sim}{w}_o,\ldots,\underset{\sim}{w}_n) = \overline{H}(x) + \ln|\ell_o| \qquad \text{(i)}$$

<u>Proof.</u> The RVs $\underset{\sim}{w}_o,\ldots,\underset{\sim}{w}_n$ are linear transformations of the RVs $\underset{\sim}{x}_o,\ldots,\underset{\sim}{x}_n$ and the transformation matrix equals

$$\begin{bmatrix} \ell_0 & 0 & \cdots & 0 \\ \ell_1 & \ell_0 & \cdots & 0 \\ \hline \ell_n & \ell_{n-1} & \cdots & 0 \end{bmatrix}$$

Since the determinant of this transformation equals $|\ell_o|^{n+1}$, (14-115) yields

$$H(\underset{\sim}{w}_o,\ldots,\underset{\sim}{w}_n) = H(\underset{\sim}{x}_o,\ldots\underset{\sim}{x}_n) + (n+1)\ln|\ell_o|$$

Dividing by (n+1) we obtain (i) as $n\to\infty$.

14-20 As in Example 14-19, $f(p) = A\, e^{-\lambda p}$. To find λ, we use the λ-η curve of Fig. 14-16. This yields

$$\lambda \approx -1.23 \qquad f(p) \approx 0.51\, e^{1.23p}$$

14-21 As in Example 14-22, $p_k = A\, e^{-\lambda k}$. To find λ, we use the w-η curve of Fig. 14-17. This yields (see also Jaynes)

$$w \approx 1.449 \qquad \lambda \approx -0.371$$

p_1	p_2	p_3	p_4	p_5	p_6
0.054	0.079	0.114	0.165	0.240	0.348

14-22 The unknown density is normal as in (14-157) where

$$\Delta = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & m_{23} \\ 1 & m_{23} & 4 \end{vmatrix} = -4m_{23}^2 + 2m_{23} + 56$$

The moment $m_{23} = E\{\underset{\sim}{x}_2 \underset{\sim}{x}_3\}$ must be such as to maximize Δ. This yields $m_{23} = 0.25$.

14-23

Shannon

$L = 2.7$

p_i	0.3	0.2	0.15	0.15	0.1	0.06	0.04	
	$\frac{1}{4} \le p_i < \frac{1}{2}$	$\frac{1}{8} \le p_i < \frac{1}{4}$			$\frac{1}{16} \le p_i < \frac{1}{8}$	$\frac{1}{32} \le p_i < \frac{1}{16}$		$\sum_{i=1}^{7} \frac{1}{2^{n_i}}$
n_i	2	3	3	3	4	5	5	0.75
	2	3	3	3	4	4	4	0.8125
	2	3	3	3	3	4	4	0.875
	2	3	3	3	3	3	4	0.9375
	2	3	3	3	3	3	3	1
x_i	00	010	011	100	101	110	111	

Fano

L = 2.6

p_i	0.3	0.2	0.15	0.15	0.1	0.06	0.04
	A_0 0.5		A_1 0.5				
	A_{00} 0.3	A_{01} 0.2	A_{10} 0.3		A_{11} 0.2		
			A_{100} 0.15	A_{101} 0.15	A_{110} 0.1	A_{111} 0.1	
						A_{1110}	A_{1111}
x_i	00	01	100	101	110	1110	1111

Huffman

L = 2.6

	1	2	3	4	5	6	7
	1	2	3	4	5	6 0	7 1
	1	2	5 0	6 10	7 11	3	4
	1	3 0	4 1	2	5 0	6 10	7 11
	2 0	5 10	6 110	7 111	1	3 0	4 1
	1 0	3 10	4 11	2 0	5 10	6 110	7 111
	1 00	3 010	4 011	2 10	5 110	6 1110	7 1111
x_i	00	10	010	011	110	1110	1111

14-24 If $\underset{\sim}{x}_n = 0$, then $\bar{x}_n = 000$ and $\underset{\sim}{y}_n = 1$ iff $\bar{y}_n$ consists of one 0 or no zeros. The probability of one and only one zero equals $3\beta^2(1-\beta)$ [see (3-13)]; the probability of no zeros equals β^3. Hence,

$$P\{\underset{\sim}{y}_n = 1 | \underset{\sim}{x}_n = 0\} = 3\beta^2(1-\beta) + \beta^3$$

Thus, the redundantly coded channel of Example 14-29 is symmetrical as in (14-191) with probability of error $\beta_1 = \beta^2$.

14-25 If the received information is always wrong, then

$$P\{\underset{\sim}{y}_n = 1 | \underset{\sim}{x}_n = 0\} = \beta = 1, \text{ hence } C = 1 - r(\beta) = 1$$